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The Existence of Global Solutions of a Semi Linear Parabolic Equation with a Singular Potential

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Abstract. In the domain $Q_R^{'} = \{x; |x| > R\} \times (0; +\infty)$ we consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} = -\Delta^2 u + \frac{C_0}{|x|^4} u + |x|^\sigma |u|^q \\ u|_{t=0} = u_0(x) \geq 0 \\ \int\limits_0^\infty \int\limits_{\partial B_R} u dx dt \geq 0, \quad \int\limits_0^\infty \int\limits_{\partial B_R} \Delta u dx dt \leq 0. \end{cases}$$

Nonexistence of global solutions is analyzed.

Key Words and Phrases: Semilinear parabolic equation, biharmonic operator, global solution, singular potential, critical exponent, method of test functions.

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1. Introduction

Let us introduce the following denotations:$x = (x_1, ..., x_n) \in R^n$, $n > 4$, $r = |x| = \sqrt{x_1^2 + ... + x_n^2}$, $B_R = \{x; |x| < R\}$, $B_R^{'} = \{x; |x| > R\}$, $B_{R_1,R_2} = \{x; R_1 < |x| < R_2\}$, $Q_R = B_R \times (0; +\infty)$, $Q_R^{'} = B_R^{'} \times (0; +\infty)$, $\partial B_R = \{x; |x| = R\}$, $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$, $C_{x,t}^{4,1}\left(Q_R^{'}\right)$ is the set of functions four times continuously differentiable with respect to x and continuously differentiable with respect to t.

In the domain $Q_R^{'}$ consider the following problem:

$$\frac{\partial u}{\partial t} = -\Delta^2 u + \frac{C_0}{|x|^4} u + |x|^\sigma |u|^q \tag{1.1}$$

$$u|_{t=0} = u_0(x) \geq 0 \tag{1.2}$$

$$\int\limits_0^\infty \int\limits_{\partial B_R} u dx dt \geq 0, \qquad \int\limits_0^\infty \int\limits_{\partial B_R} \Delta u dx dt \leq 0, \tag{1.3}$$

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where $q > 1$, $0 \le C_0 \le \left(\frac{n(n-4)}{4}\right)^2$, $\sigma > -4$, $u_0(x) \in C\left(B_R^{'}\right)$, $\Delta^2 u = \Delta(\Delta u)$, $\Delta u = \sum\limits_{i=1}^{n} \frac{\partial^2 u}{\partial x_t^2}$.

We will study the existence of non-negative global solutions of problem (1.1)-(1.3). We will understand the solution of the problem in the classic sense. The function $u(x,t) \in C_{x,t}^{4,1}(Q_R^{'}) \cap C(B_R^{'} \times [0,+\infty))$ will be said the solution of problem (1.1)-(1.3) if $u(x,t)$ satisfies equation (1.1) at each point of $Q_R^{'}$, condition (1.2) for $t = 0$ and condition (1.3) for $|x| = R$.

The problems of non-existence of global solutions for different classes of differential equations and inequalities play a key role in theory and applications. Therefor, they are at constant attention of mathematicians and a great number of works were devoted to them. Survey of such results are in the monograph [1]. In the classical paper [2] Fujita considered the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u^p, (x,t) \in R^n \times (0,+\infty), \\ u|_{t=0} = u_0(x), x \in R^n. \end{cases} \tag{1.4}$$

And it is proved that positive global solutions of problem (1.4) do not exist for $1 < p < p^* = 1 + \frac{2}{n}$, and for $p > p*$ for small $u_0(x)$ there are positive global solutions.The case $p = p*$ was investigated in [3], [4] and it is proved that in this case there also do not exist positive global solutions.The results of Fujita's work [2] aroused great interest in the problem of the absence of global solutions, and they were expanded in several directions.For example, instead of R^n, various bounded and unbounded domains are considered, or more general operators were considered than the Laplace operator and nonlinearities of a different type.A survey of such papers is available in [5], in the monograph [1] and in the book [6]. Weakly nonlinear equations with a biharmonic operator were considered by many authors. In the paper [7] for $c = 0$ problem (1.1)-(1.3) is considered in the domain Q_R and it is proved that if $\sigma \le -4$, $q > 1$, then the solution is absent. In this paper we consider problem (1.1)-(1.3) for $0 \le c \le \frac{(n-2)^2}{4}$, $\sigma > -4$ and also in the papers [7], using the technique of test function, worked out Mitidieri and Pohozaev in the papers [1],[8], find an exact exponent of absence of a global solutions.

2. Auxiliary facts

Let us consider in $R^n \backslash \{0\}$ the linear equation

$$\Delta^2 u - \frac{C_0}{|x|^4} u = 0. \tag{2.1}$$

If $u(x) = u(r)$ is a radial solution of equation (2.1), then

$$\Delta^2 u - \frac{C_0}{|x|^4} u = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r}\right) - \frac{C_0}{|x|^4} u =$$

$$= \frac{\partial^4 u}{\partial r^4} + \frac{2(n-1)}{r}\frac{\partial^3 u}{\partial r^3} + \frac{(n-1)(n-3)}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{(n-1)(n-3)}{r^3}\frac{\partial u}{\partial r} - \frac{C_0}{r^4}u = 0. \quad (2.2)$$

This is the Euler equation. Its characteristically equation has the following form:

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2(n-1)\lambda(\lambda-1)(\lambda-2) +$$
$$+ (n-1)(n-3)\left(\lambda^2 - 2\lambda\right) - C_0 = 0. \quad (2.3)$$

Make the substitution $\lambda - 1 = t$. Then we get

$$t(t-2)\left(t^2-1\right) + 2(n-1)\left(t^2-1\right)t + (n-1)(n-3)\left(t^2-1\right) - C_0 = 0. \quad (2.4)$$

Hence

$$t^4 + 2(n-2)t^3 + \left((n-2)^2 - 2\right)t^2 - 2(n-2)t - (n-2)^2 + 1 - C_0 = 0,$$

$$\left(t^2 + (n-2)t - 1\right)^2 - \left((n-2)^2 + C_0\right) = 0,$$

$$\left(t^2 + (n-2)t - 1 - \sqrt{(n-2)^2 + C_0}\right)\left(t^2 + (n-2)t - 1 + \sqrt{(n-2)^2 + C_0}\right) = 0.$$

So,

$$t = -\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 + \sqrt{(n-2)^2 + C_0}},$$

$$t = -\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 - \sqrt{(n-2)^2 + C_0}}.$$

are the all roots of equation (2.4).

Hence,

$$\lambda = -\frac{n-4}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 + \sqrt{(n-2)^2 + C_0}},$$

$$\lambda = \frac{n-4}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 - \sqrt{(n-2)^2 + C_0}}.$$

all the roots of equation (2.3).

For brevity of notation we denote:

$$(n-2)^2 + C_0 = D, \quad \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 \pm \sqrt{(n-2)^2 + C_0}} = \alpha_\pm.$$

We consider the function

$$\xi(|x|) = \frac{1}{2}\left(1 + \frac{\sqrt{D} - \alpha_+}{\alpha_-}\right)|x|^{-\frac{n-4}{2} + \alpha_-} +$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)|x|^{-\frac{n-4}{2}-\alpha_-}-|x|^{-\frac{n-4}{2}-\alpha_+}.$$

Obviously, $\xi\left(|x|\right)$ is a radial solution of equation (2.1) in $R^n\backslash\{0\}$.
Show that $\xi\left(x\right)$ satisfies the following conditions:

$$\xi|_{|x|=1}=0,\quad \left.\frac{\partial\xi}{\partial r}\right|_{|x|=1}\geq 0,\quad \Delta\xi|_{|x|=1}=0,\quad \left.\frac{\partial\left(\Delta\xi\right)}{\partial r}\right|_{|x|=1}\leq 0. \tag{2.5}$$

$$\xi\left(x\right)|_{|x|=1}=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)-1=0,$$

$$\left.\frac{\partial\xi}{\partial r}\right|_{|x|=1}=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}+\alpha_-\right)+$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}-\alpha_-\right)-\left(-\frac{n-4}{2}-\alpha_+\right)=$$

$$=\frac{1}{2}\left(\alpha_-+\sqrt{D}-\alpha_+\right)-\frac{1}{2}\left(\alpha_--\sqrt{D}+\alpha_+\right)+\alpha_+=\sqrt{D}\geq 0.$$

$$\Delta\xi|_{|x|=1}=\left.\left(\frac{\partial^2\xi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\xi}{\partial r}\right)\right|_{|x|=1}=$$

$$=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}+\alpha_-\right)\left(\frac{n}{2}+\alpha_-\right)+$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}-\alpha_-\right)\left(\frac{n}{2}-\alpha_-\right)-\left(-\frac{n-4}{2}-\alpha_+\right)\left(\frac{n}{2}-\alpha_+\right)=$$

$$=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n\left(n-4\right)}{4}+\alpha_-^2+2\alpha_-\right)+$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n\left(n-4\right)}{4}+\alpha_-^2-2\alpha_-\right)-\left(-\frac{n\left(n-4\right)}{4}+\alpha_+^2-2\alpha_+\right)=$$

$$=-\frac{n\left(n-4\right)}{4}\left(\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)-1\right)+$$

$$+\alpha_-^2-\alpha_+^2+\alpha_-+\sqrt{D}-\alpha_+-\alpha_-+\sqrt{D}-\alpha_++2\alpha_+=$$

$$=-\sqrt{D}-\sqrt{D}+\sqrt{D}+\sqrt{D}=0.$$

$$\frac{\partial}{\partial r}(\Delta\xi)\Big|_{|x|=1} = \left(\frac{\partial^3\xi}{\partial r^3} + \frac{n-1}{r}\frac{\partial^2\xi}{\partial r^2} - \frac{n-1}{r^2}\frac{\partial\xi}{\partial r}\right)\Big|_{|x|=1} =$$

$$= \frac{1}{2}\left(1 + \frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left[\left(-\frac{n-4}{2}+\alpha_-\right)\left(-\frac{n-4}{2}+\alpha_- - 1\right)\left(-\frac{n}{2}+\alpha_-\right) + \right.$$

$$\left. + (n-1)\left(-\frac{n-4}{2}+\alpha_+\right)\left(-\frac{n}{2}+\alpha_-\right)\right] +$$

$$+\frac{1}{2}\left(1 - \frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left[\left(-\frac{n-4}{2}-\alpha_-\right)\left(-\frac{n-4}{2}-\alpha_- - 1\right)\left(-\frac{n}{2}-\alpha_-\right) + \right.$$

$$\left. + (n-1)\left(-\frac{n-4}{2}-\alpha_-\right)\left(-\frac{n}{2}-\alpha_-\right)\right] -$$

$$-\left[\left(-\frac{n-4}{2}-\alpha_+\right)\left(-\frac{n-4}{2}-\alpha_+ - 1\right)\left(-\frac{n}{2}-\alpha_+\right) + \right.$$

$$\left. + (n-1)\left(-\frac{n-4}{2}-\alpha_+\right)\left(-\frac{n}{2}-\alpha_+\right)\right] =$$

$$= \frac{1}{2}\left(1 + \frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}+\alpha_-\right)\left(-\frac{n}{2}+\alpha_-\right)\left(\frac{n}{2}+\alpha_-\right) +$$

$$+\frac{1}{2}\left(1 - \frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}-\alpha_-\right)\left(-\frac{n}{2}-\alpha_-\right)\left(\frac{n}{2}-\alpha_+\right) -$$

$$-\left(-\frac{n-4}{2}-\alpha_+\right)\left(-\frac{n}{2}-\alpha_+\right)\left(\frac{n}{2}-\alpha_+\right) =$$

$$= \frac{1}{2}\left(1 + \frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}+\alpha_-\right)\left(\alpha_-^2 - \frac{n^2}{4}\right) +$$

$$+\frac{1}{2}\left(1 - \frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}-\alpha_-\right)\left(\alpha_-^2 - \frac{n^2}{4}\right) - \left(-\frac{n-4}{2}-\alpha_+\right)\left(\alpha_+^2 - \frac{n^2}{4}\right) =$$

$$= \left(2-n-\sqrt{D}\right)\left(-\frac{n-4}{2} + \frac{1}{2}\left(\alpha_- + \sqrt{D} - \alpha_+ - \alpha_- + \sqrt{D} - \alpha_+\right)\right) +$$

$$+ \left(2-n+\sqrt{D}\right)\left(\frac{n-4}{2}+\alpha_+\right) = \left(2-n-\sqrt{D}\right)\left(-\frac{n-4}{2} + \sqrt{D} - \alpha_+\right) +$$

$$+ \left(2-n+\sqrt{D}\right)\left(\frac{n-4}{2}+\alpha_+\right) = -\left(2-n-\sqrt{D}\right)\left(\frac{n-4}{2}+\alpha_+\right) + (2-n)\sqrt{D} - D +$$

$$+\left(2-n+\sqrt{D}\right)\left(\frac{n-4}{2}+\alpha_+\right)=\sqrt{D}\frac{n-4}{2}-\alpha_+\left(2-n\right)+\sqrt{D}\alpha_++\left(2-n\right)\sqrt{D}-D+$$

$$+\sqrt{D}\frac{n-4}{2}+\left(2-n\right)\alpha_++\sqrt{D}\alpha_+=2\sqrt{D}\alpha_+-2\sqrt{D}-D=$$

$$=\sqrt{D}\left(2\alpha_+-\sqrt{D}-2\right)\leq 0.$$

Indeed, as $C_0\geq 0$, then

$$(n-2)^2\leq (n-2)^2+C_0=D.$$

Then

$$4+4\sqrt{D}+(n-2)^2\leq 4+4\sqrt{D}+D.$$

Hence

$$4\left(\left(\frac{n-2}{2}\right)^2+1+\sqrt{D}\right)\leq\left(2+\sqrt{D}\right)^2$$

$$4\alpha_+^2\leq\left(2+\sqrt{D}\right)^2,$$

$$2\alpha_+\leq 2+\sqrt{D}.$$

So,

$$2\alpha_+-\sqrt{D}-2\leq 0.$$

3. Formulation of the basic result and proof.

The following theorem is the basic result of this paper.

Theorem. Let $n>4$, $\sigma>-4$, $0\leq C_0\leq(\frac{n(n-4)}{4})^2$ and $1<q\leq 1+\frac{\sigma+4}{\frac{n+4}{2}+\alpha_-}$. If $u(x,t)$ is the solution of problem (1.1)-(1.3), then $u(x,t)\equiv 0$.

Proof.

For simplicity of notation we take $R=1$. Assume that $u(x)\geq 0$ is the solution of problem (1.1)-(1.3) in Q_R' .

Let us consider the following functions:

$$\varphi\left(x\right)=\begin{cases}1, \text{ for } 1\leq|x|\leq\rho\\ \left(2-\frac{|x|}{\rho}\right)^\beta, \text{ for } \rho\leq|x|\leq 2\rho\\ 0, \text{ for } |x|\geq 2\rho\end{cases}$$

$$T_\rho\left(t\right)=\begin{cases}1, \text{ for } 0\leq t\leq\rho^\chi\\ \left(2-\rho^{-\chi}t\right)^\mu, \text{ for } \rho^\chi\leq t\leq 2\rho^\chi\\ 0, \text{ for } t\geq 2\rho^\chi\end{cases}$$

where β,μ are larger positive numbers, moreover β is such that for $|x| = 2\rho$

$$\psi = \frac{\partial\psi}{\partial r} = \frac{\partial^2\psi}{\partial r^2} = \frac{\partial^3\psi}{\partial r^3} = 0, \tag{3.1}$$

and χ will be defined later.

Multiply, equation (1.1) by the function

$$\psi(x,t) = T_\rho(t)\,\xi(x)\,\varphi(x)$$

and integrate in domain Q_1'.

After integrating by parts, we get

$$\int\limits_{Q_1'} u^q |x|^\sigma T_\rho \xi\varphi dxdt = -\int\limits_{Q_1'} u\xi\varphi \frac{dT\rho}{dt} dxdt +$$
$$+\int\limits_{Q_1'} uT_\rho \Delta^2(\xi\varphi)\,dxdt - \int\limits_{Q_1'} \frac{C_0}{|x|^4} uT_\rho \xi\varphi dxdt -$$
$$-\int\limits_{B_1'} u_0(x)\,\xi(x)\,\varphi(x)\,dx + \int\limits_0^\infty T_\rho(t)\,dt \times$$
$$\times\left[\int\limits_{\partial B_{1,2\rho}} \frac{\partial(\Delta u)}{\partial\nu}\xi\varphi ds - \int\limits_{\partial B_{1,2\rho}} \Delta u \frac{\partial(\xi\varphi)}{\partial\nu} ds + \right.$$
$$\left.+\int\limits_{\partial B_{1,2\rho}} \frac{\partial u}{\partial\nu}\Delta(\xi\varphi)\,ds - \int\limits_{\partial B_{1,2\rho}} u\frac{\partial}{\partial\nu}(\Delta(\xi\varphi))\,ds\right] \tag{3.2}$$

Estimate the integrals in the square bracket, taking into account (2.5), (3.1) and condition (1.3), we get:

$$\int\limits_{\partial B_{1,2\rho}} \frac{\partial(\Delta u)}{\partial\nu}\xi\varphi ds = 0,$$

$$-\int\limits_{\partial B_{1,2\rho}} \Delta u \frac{\partial(\xi\varphi)}{\partial\nu} ds = -\int\limits_{|x|=1} \Delta u \frac{\partial(\xi\varphi)}{\partial\nu} ds - \int\limits_{|x|=2\rho} \Delta u \frac{\partial(\xi\varphi)}{\partial\nu} ds =$$
$$= \int\limits_{|x|=1} \Delta u \left(\frac{\partial\xi}{\partial r}\varphi + \xi\frac{\partial\varphi}{\partial r}\right) ds - \int\limits_{|x|=2\rho} \Delta u \left(\frac{\partial\xi}{\partial r}\varphi + \xi\frac{\partial\varphi}{\partial r}\right) ds =$$
$$= \int\limits_{|x|=1} \Delta u \frac{\partial\xi}{\partial r} ds = \sqrt{D} \int\limits_{|x|=1} \Delta u ds \le 0,$$

$$\int\limits_{\partial B_{1,2\rho}} \frac{\partial u}{\partial \nu}\Delta\left(\xi\varphi\right)ds = \int\limits_{\partial B_{1,2\rho}} \frac{\partial u}{\partial \nu}\left(\Delta\xi\varphi + 2\left(\nabla\xi,\nabla\varphi\right) + \xi\Delta\varphi\right)ds =$$

$$= -\int\limits_{|x|=1} \frac{\partial u}{\partial r}\Delta\xi ds = 0,$$

$$-\int\limits_{\partial B_{1,2\rho}} u\frac{\partial}{\partial \nu}\left(\Delta\left(\xi\varphi\right)\right)ds = -\int\limits_{|x|=1} u\frac{\partial}{\partial \nu}\left(\Delta\xi\varphi + 2\left(\nabla\xi,\nabla\varphi\right) + \xi\Delta\varphi\right)ds =$$

$$= \int\limits_{|x|=1} u\frac{\partial\left(\Delta\xi\right)}{\partial r}ds = \sqrt{D}\left(2\alpha_+ - \sqrt{D} - 2\right)\int\limits_{|x|=1} uds \le 0.$$

As $\int\limits_{B_1'} u_0\left(x\right)\zeta\left(x\right)\varphi\left(x\right)dx \ge 0$, $\int\limits_0^\infty T_\rho\left(t\right)dt > 0$, then from (3.2)

$$\int\limits_{Q_1'} u^q\left|x\right|^\sigma T_\rho\xi\varphi dxdt \le -\int\limits_{Q_1'} u\xi\varphi\frac{dT_\rho}{dt}dxdt +$$

$$+\int\limits_{Q_1'} uT_\rho\Delta^2\left(\xi\varphi\right)dxdt - \int\limits_{Q_1'} \frac{C_0}{\left|x\right|^4}uT_\rho\xi\varphi dxdt =$$

$$= -\int\limits_{Q_1'} u\xi\varphi\frac{dT_\rho}{dt}dxdt + \int\limits_{Q_1'} uT_\rho\varphi\left(\Delta^2\xi - \frac{C_0}{\left|x\right|^4}\xi\right)dxdt +$$

$$+\int\limits_{Q_1'} uT_\rho\left[4\left(\nabla\left(\Delta\xi\right),\nabla\varphi\right) + 4\left(\nabla\xi,\nabla\left(\Delta\varphi\right)\right) + 2\Delta\xi\Delta\varphi + \right.$$

$$\left. +4\sum_{i,j=1}^n \frac{\partial^2\xi}{\partial x_i\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right]dxdt \le$$

$$\le -\int\limits_{\rho^\varkappa}^{2\rho^\varkappa}\int\limits_{B_1'} u\xi\varphi\frac{dT_\rho}{dt}dxdt + \int\limits_0^{2\rho^\varkappa}\int\limits_{B_{\rho,2\rho}} uT_\rho J\left(\xi,\varphi\right)dxdt, \qquad (3.3)$$

where $J(\xi,\varphi)$ denotes the expression in the square bracket, i.e.

$$J\left(\xi,\varphi\right) \equiv 4\left(\nabla\left(\Delta\xi\right),\nabla\varphi\right) + 4\left(\nabla\xi,\nabla\left(\Delta\varphi\right)\right) +$$

$$+2\Delta\xi\Delta\varphi + 4\sum_{i,j=1}^n \frac{\partial^2\xi}{\partial x_i\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j}$$

Using the Holder inequality, from (3.3) we get

$$\int\limits_{Q_1'} u^q |x|^\sigma T_\rho \xi\varphi dxdt \le \left(\int\limits_{\rho^\chi}^{2\rho^\chi} \int\limits_{B_1'} u^q |x|^\sigma T_\rho \xi\varphi dxdt \right)^{\frac{1}{q}} \times$$

$$\times \left(\int\limits_{\rho^\chi}^{2\rho^\chi} \int\limits_{B_1'} \frac{\left|\frac{dT_\rho}{dt}\right|^{q'} \xi\varphi}{T_\rho^{q'-1} |x|^{\sigma(q'-1)}} dxdt \right)^{\frac{1}{q'}} +$$

$$+ \left(\int\limits_0^{2\rho^\chi} \int\limits_{B_{\rho,2\rho}} u^q |x|^\sigma T_\rho \xi\varphi dxdt \right)^{\frac{1}{q}} \left(\int\limits_0^{2\rho^\chi} \int\limits_{B_{\rho,2\rho}} \frac{|J(\xi,\varphi)|^{q'} T_\rho}{\xi^{q'-1}\varphi^{q'-1} |x|^{\sigma(q'-1)}} dxdt \right)^{\frac{1}{q'}}, \qquad (3.4)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Hence we get

$$\int\limits_{Q_1'} u^q |x|^\sigma T_\rho \xi\varphi dxdt \le C_1 \int\limits_{\rho^\chi}^{2\rho^\chi} \int\limits_{B_1'} \frac{\left|\frac{dT_\rho}{dt}\right|^{q'} \xi\varphi}{T_\rho^{q'-1} |x|^{\sigma(q'-1)}} dxdt +$$

$$+ C_2 \int\limits_0^{2\rho^\chi} \int\limits_{B_{\rho,2\rho}} \frac{|J(\xi,\varphi)|^{q'}}{\xi^{q'-1}\varphi^{q'-1} |x|^{\sigma(q'-1)}} dxdt. \qquad (3.5)$$

Making the substitution $t = \rho^\chi \tau$, $r = \rho\theta$, $\widetilde{T}(\tau) = T_\rho(\rho^\chi \tau)$, $\widetilde{\xi}(\theta) = \xi(\rho\theta)$, $\widetilde{\varphi}(\theta) = \varphi(\rho\theta)$, we estimate the integrals in the right hand side of (3.5).

$$I_1 \equiv \int\limits_{\rho^\chi}^{2\rho^\chi} \int\limits_{B_1'} \frac{\left|\frac{dT_\rho}{dt}\right|^{q'} \xi\varphi}{T_\rho^{q'-1} |x|^{\sigma(q'-1)}} dxdt \le$$

$$\le \int\limits_{\rho^\chi}^{2\rho^\chi} \frac{\left|\frac{dT_\rho}{dt}\right|^{q'}}{T_\rho^{q'-1}} dt \int\limits_{B_{1,2\rho}} |x|^{-\sigma(q'-1)} \xi\varphi dx \le$$

$$\le C_3 \rho^{-\chi q' + \chi} \int\limits_1^2 \frac{\left|\frac{d\widetilde{T}}{d\tau}\right|^{q'}}{\widetilde{T}^{(q'-1)}} d\tau \int\limits_1^{2\rho} r^{-\frac{n-4}{2}+\alpha_-} r^{-\sigma(q'-1)} r^{n-1} dr \le$$

$$\le C_3 \rho^{\chi(1-q') - \frac{n-4}{2} + \alpha_- - \sigma(q'-1) + n} A_1\left(\widetilde{T}\right) =$$

$$= C_3 \rho^{\chi(1-q') + \frac{n+4}{2} + \alpha_- - \sigma(q'-1)} A_1\left(\widetilde{T}\right), \qquad (3.6)$$

where

$$A_1\left(\widetilde{T}\right)=\int_1^2\frac{\left|\frac{d\widetilde{T}}{d\tau}\right|^{q'}}{\widetilde{T}^{(q'-1)}}d\tau.$$

$$I_2\equiv\int_0^{2\rho^{\chi}}\int_{B_{\rho,2\rho}}\frac{|J(\xi,\varphi)|^{q'}T_\rho}{\xi^{q'-1}\varphi^{q'-1}|x|^{\sigma(q'-1)}}dxdt=$$

$$=\int_0^{2\rho^{\chi}}T_\rho(t)\,dt\int_{B_{\rho,2\rho}}\frac{|J(\xi,\varphi)|^{q'}}{\xi^{q'-1}\varphi^{q'-1}|x|^{\sigma(q'-1)}}dx \tag{3.7}$$

Estimate each addend of $J(\xi,\varphi)$ separately

$$|(\nabla(\Delta\xi),\nabla\varphi)|=\left|\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial^2\xi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\xi}{\partial r}\right)\frac{\partial\varphi}{\partial x_i}\right|=$$

$$=\left|\left(\frac{\partial^3\xi}{\partial r^3}+\frac{n-1}{r}\frac{\partial^2\xi}{\partial r^2}-\frac{n-1}{r^2}\frac{\partial\xi}{\partial r}\right)\frac{\partial\varphi}{\partial r}\right|\leq$$

$$\leq C_4r^{-\frac{n-4}{2}+\alpha_--3}\left|\frac{\partial\varphi}{\partial r}\right|$$

$$|\Delta\xi\Delta\varphi|=\left|\left(\frac{\partial^2\xi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\xi}{\partial r}\right)\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\varphi}{\partial r}\right)\right|\leq$$

$$\leq C_5r^{-\frac{n-4}{2}+\alpha_--2}\left|\frac{\partial^2\varphi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\varphi}{\partial r}\right|,$$

$$|(\nabla\xi,\nabla(\Delta\varphi))|\leq$$

$$\leq C_6r^{-\frac{n-4}{2}+\alpha_--1}\left(\left|\frac{\partial^3\varphi}{\partial r^3}\right|+\frac{n-1}{r}\left|\frac{\partial^2\varphi}{\partial r^2}\right|+\frac{n-1}{r^2}\left|\frac{\partial\varphi}{\partial r}\right|\right),$$

$$\left|\sum_{i,j=1}^{n}\frac{\partial^2\xi}{\partial x_i\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|=$$

$$=\left|\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial\xi}{\partial r}\frac{x_i}{r}\right)\frac{\partial}{\partial x_j}\left(\frac{\partial\varphi}{\partial r}\frac{x_i}{r}\right)\right|=$$

$$=\left|\sum_{i,j=1}^{n}\left(\frac{\partial^2\xi}{\partial r^2}\frac{x_ix_j}{r^2}+\frac{\partial\xi}{\partial r}\left(\frac{\delta_{ij}}{r}-\frac{x_ix_j}{r^3}\right)\right)\times\right.$$

$$\left.\times\left(\frac{\partial^2\varphi}{\partial r^2}\frac{x_ix_j}{r^2}+\frac{\partial\varphi}{\partial r}\left(\frac{\delta_{ij}}{r}-\frac{x_ix_j}{r^3}\right)\right)\right|\leq$$

$$\leq C_7\left(\left|\frac{\partial^2\xi}{\partial r^2}\right|+\frac{1}{r}\left|\frac{\partial\xi}{\partial r}\right|\right)\left(\left|\frac{\partial^2\varphi}{\partial r^2}\right|+\frac{1}{r}\left|\frac{\partial\varphi}{\partial r}\right|\right)\leq$$

$$\leq C_8 r^{-\frac{n-4}{2}+\alpha_--2}\left(\left|\frac{\partial^2\varphi}{\partial r^2}\right|+\frac{1}{r}\left|\frac{\partial\varphi}{\partial r}\right|\right).$$

Using all of these ones, from (3.7) we get

$$I_2\leq C_9\rho^{\chi}\int_{\rho}^{2\rho}\frac{r^{\left(-\frac{n-4}{2}+\alpha_--4\right)q'}\left(r\left|\frac{\partial\varphi}{\partial r}\right|+r^2\left|\frac{\partial^2\varphi}{\partial r^2}\right|+r^3\left|\frac{\partial^3\varphi}{\partial r^3}\right|\right)^{q'}r^{n-1}}{r^{\left(-\frac{n-4}{2}+\alpha_-\right)(q'-1)+\sigma(q'-1)}\varphi^{q'-1}}dr.$$

Hence

$$I_2\leq C_{10}\rho^{\chi-\frac{n-4}{2}+\alpha_--4q'-\sigma(q'-1)+n}\int_1^2\frac{\left(\theta\left|\frac{\partial\widetilde{\varphi}}{\partial\theta}\right|+\theta^2\left|\frac{\partial^2\widetilde{\varphi}}{\partial\theta^2}\right|+\theta^3\left|\frac{\partial^3\widetilde{\varphi}}{\partial\theta^3}\right|\right)^{q'}}{\theta^{\frac{n-4}{2}-\alpha_-+4q'+\sigma(q'-1)-n+1}\widetilde{\varphi}^{(q'-1)}}d\theta\leq$$

$$\leq C_{10}\rho^{\chi+\frac{n+4}{2}+\alpha_--4q'-\sigma(q'-1)}A_2\left(\widetilde{\varphi}\right), \tag{3.8}$$

where $A_2\left(\widetilde{\varphi}\right)$ denotes the last integral.

Obviously, for large μ and β, $A_1\left(\widetilde{T}\right)<\infty$, $A_2\left(\widetilde{\varphi}\right)<\infty$.

We take χ so that

$$\chi-4q'-\sigma\left(q'-1\right)=\chi-\left(1-q'\right)-\sigma\left(q'-1\right).$$

Hence $\chi=4$.

Using (3.6), (3.8) and (3.5) we get

$$\int_{Q_1'}|u|^q|x|^{\sigma}T_\rho\xi\varphi dxdt\leq\left(C_{11}A_1\left(\widetilde{T}\right)+C_{12}A_2\left(\widetilde{\varphi}\right)\right)\rho^{\frac{n+4}{2}+\alpha_--(\sigma+4)(q'-1)}. \tag{3.9}$$

Let now $(\sigma+4)\left(q'-1\right)-\frac{n+4}{2}-\alpha_->0$.

Then

$$(\sigma+4)\frac{1}{q-1}>\frac{n+4}{2}+\alpha_-$$

and

$$q<1+\frac{\sigma+4}{\frac{n+4}{2}+\alpha_-}.$$

In this case, tending ρ to $+\infty$ from (3.9) we get, that

$$\int_{Q_1'}|u|^q|x|^{\sigma}\xi dxdt\leq 0.$$

This means that $u \equiv 0$.

Let now $(\sigma+4)(q'-1)-\frac{n+4}{2}-\alpha_-=0$.

Then from (3.6), (3.8) we get $I_1<C$, $I_2<C$ and therefore

$$\int\limits_{Q_1'} u^q |x|^\sigma \xi dxdt < C.$$

From the property of the integral we get

$$\int\limits_0^\infty \int\limits_{B_{\rho,2\rho}} u^q |x|^\sigma \xi dxdt \to 0, \tag{3.10}$$

and

$$\int\limits_{\rho^4}^{2\rho^4} \int\limits_{B_1'} u^q |x|^\sigma \xi dxdt \to 0. \tag{3.11}$$

Then from (3.4)

$$\int\limits_{Q_1'} u^q |x|^\sigma T_\rho \xi\varphi dxdt \le \left(\int\limits_{\rho^4}^{2\rho^4}\int\limits_{B_1'} u^q |x|^\sigma \xi T_\rho\varphi dxdt\right)^{\frac{1}{q}} I_1^{\frac{1}{q'}} +$$

$$+\left(\int\limits_0^{2\rho^4}\int\limits_{B_{\rho,2\rho}} u^q |x|^\sigma T_\rho\xi\varphi dxdt\right)^{\frac{1}{q}} I_2^{\frac{1}{q'}} \le$$

$$\le \left(\int\limits_{\rho^4}^{2\rho^4}\int\limits_{B_1'} u^q |x|^\sigma \xi dxdt\right)^{\frac{1}{q}} I_1^{\frac{1}{q'}} + \left(\int\limits_0^{\infty}\int\limits_{B_{\rho,2\rho}} u^q |x|^\sigma T_\rho\xi\varphi dxdt\right)^{\frac{1}{q}} I_2^{\frac{1}{q'}} \to 0$$

as $\rho \to +\infty$ by (3.9), (3.10).

Hence it follows that $u \equiv 0$.

This completely proves the theorem.

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