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On Some Properties of Harmonic Functions from Hardy-Morrey type Classes

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Abstract. In this paper Morrey-Poisson class of harmonic functions in the unit circle is introduced, the Dirichlet problem with the boundary value from the Morrey Lebesgue space is considered.

Key Words and Phrases: Dirichlet problem, Morrey-Poisson class, maximal function, Morrey-Lebesgue space.

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1. Introduction

Let $\omega = \{z \in C : |z| < 1\}$ be the unit disk on the complex plane C and $\gamma = \partial\omega$ be its circumference.

Consider the following Dirichlet problem for the Laplace equation

$$\left.\begin{array}{l} \Delta u = 0, \ \ in \ \ \omega \,, \\ u\Big/_{\gamma} = f \,, \end{array}\right\} \tag{1}$$

where $f : \gamma \to R$ some real function. Assume $u_r(t) = u\left(re^{it}\right)$ and let

$$h_p = \left\{ u : \Delta u = 0 \ \ in \ \ \omega \,, \ \ and \ \ \|u\|_{h_p} < +\infty \right\},$$

where

$$\|u\|_{h_p} = \sup_{0<r<1} \|u_r\|_p \,,$$

$$\|g\|_p = \left(\int_{-\pi}^{\pi} |g(t)|^p \, dt \right)^{\frac{1}{p}}, 1 \leq p < +\infty.$$

By $P_z(\varphi)$ denote a Poisson kernel for the unit circle

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$$P_z(\varphi) = Re\frac{e^{i\varphi} + re^{it}}{e^{i\varphi} - re^{it}} = \frac{1 - r^2}{1 - 2r\cos(t - \varphi) + r^2}, z = re^{it} \in \omega.$$

If $f \in L_p(\gamma) =: L_p$, then the problem (1) is solvable in class h_p, and its solution can be represented as a Poisson-Lebesgue integral

$$u\left(re^{it}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_z(\varphi) f(\varphi)\, d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(t - \varphi) + r^2} f(\varphi)\, d\varphi,$$

wherein a boundary value $u\big/_{\gamma} = f$ in (1) is understood in the sense that nontangential values on γ:

$$u\left(e^{it}\right) = \lim_{z \to e^{it}} u(z),$$

exist and a.e. on γ coincides with $f\left(e^{it}\right)$, i.e.

$$u\left(e^{it}\right) = f\left(e^{it}\right), \text{ a.e. } t \in (-\pi, \pi), \tag{2}$$

and moreover

$$\lim_{r \to 1-0} \|u_r(\cdot) - f(\cdot)\|_p = 0. \tag{3}$$

These results are well known and illuminated, e.g., in the monograph I.I.Danilyuk [27] .

It should be noted that the concept of Morrey space was introduced by C. Morrey [1] in 1938 in the study of qualitative properties of the solutions of elliptic type equations with BMO (Bounded Mean Oscillations) coefficients (see also [2, 3]). This space provides a large class of weak solutions to the Navier-Stokes system [4]. In the context of fluid dynamics, Morrey-type spaces have been used to model the fluid flow in case where the vorticity is a singular measure supported on some sets in R^n [5]. There appeared lately a large number of research works which considered many problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc. in Morrey-type spaces (for more details see [2-26]). It should be noted that the matter of approximation in Morrey-type spaces has only started to be studied recently (see, e.g., [11, 12, 16, 17]), and many problems in this field are still unsolved.

In the present paper non-tangential maximal function is considered and it is estimated from above a maximum operator, and the proof is carried out for the Poisson-Stieltjes integral, when the density belongs to the corresponding Morrey-Lebesgue space.

It should be noted that similar problems with respect to the analytical functions from Hardy classes were considered in [16, 17, 31].

2. Needful Information

We will need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$. All the constants throughout this paper (can be different in different places) will be denoted by c.

By Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0 < \alpha \leq 1$, $p \geq 1$, we mean the normed space of all measurable functions $f(\cdot)$ on Γ with the finite norm

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_B \left(\left| B \bigcap \Gamma \right|_\Gamma^{\alpha-1} \int_{B \bigcap \Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty,$$

where sup is taken all over the balls B with the centre on Γ. $L^{p,\alpha}(\Gamma)$ is a Banach space with $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$. Similarly we define the weighted Morrey-Lebesgue space $L^{p,\alpha}_\mu(\Gamma)$ with the weight function $\mu(\cdot)$ on Γ equipped with the norm

$$\|f\|_{L^{p,\alpha}_\mu(\Gamma)} = \|f\mu\|_{L^{p,\alpha}(\Gamma)}, \, f \in L^{p,\alpha}_\mu(\Gamma).$$

The inclusion $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 < \alpha_1 \leq \alpha_2 \leq 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall \alpha \in (0,1]$, $\forall p \geq 1$. For $\Gamma = \gamma$ we will use the notation $L^{p,\alpha}(\gamma) = L^{p,\alpha}$ and the spaces $L^{p,\alpha}(\gamma)$ and $L^{p,\alpha}(-\pi,\pi)$ we will identify by usual method.

More details on Morrey-type spaces can be found in [2-26].

We will use the following concepts. Let $\Gamma \subset C$ be some bounded rectifiable curve, $t = t(\sigma)$, $0 \leq \sigma \leq 1$, be its parametric representation with respect to the arc length σ, and l be the length of Γ. Let $d\mu(t) = d\sigma$, i.e. let $\mu(\cdot)$ be a linear measure on Γ. Let

$$\Gamma_t(r) = \{\tau \in \Gamma : |\tau - t| < r\}, \Gamma_{t(s)}(r) = \{\tau(\sigma) \in \Gamma : |\sigma - s| < r\}.$$

It is absolutely clear that $\Gamma_{t(s)}(r) \subset \Gamma_t(r)$.

Definition 1. Curve Γ is said to be Carleson if $\exists c > 0$:

$$\sup_{t \in \Gamma} \mu(\Gamma_t(r)) \leq cr, \forall r > 0.$$

Curve Γ is said to satisfy the chord-arc condition at the point $t_0 = t(s_0) \in \Gamma$ if there exists a constant $m > 0$ independent of t such that $|s - s_0| \leq m |t(s) - t(s_0)|$, $\forall t(s) \in \Gamma$. Γ satisfies a chord-arc condition uniformly on Γ if $\exists m > 0 : |s - \sigma| \leq m |t(s) - t(\sigma)|$, $\forall t(s), t(\sigma) \in \Gamma$.

Let's recall some facts about the homogeneous Morrey-type spaces from the work [10]. Let $(X; d; \nu)$ be a homogeneous space equipped with the quasi-distance $d(\cdot;\cdot)$ and the measure $\nu(\cdot)$. Recall that the quasi-distance $d : X^2 \to R_+$ is a function which satisfies the following conditions:

i) $d(x;y) \geq 0$ & $d(x;y) = 0 \Leftrightarrow x = y$; $\forall x, y \in X$;

ii) $d(x;y) \leq c(d(x;z) + d(z;y))$, $\forall x, y \in X$.

Let $B_r(x)$ be an open ball

$$B_r(x) = \{y \in X : d(x;y) < r\}.$$

Set

$$\nu\left(B_r\left(x\right)\right)=\int_{B_r(x)}1\,d\nu.$$

Assume that X has a constant homogeneous dimension æ >0, i.e. $\exists c_1;\, c_2>0$:

$$c_1 r^{æ}\le \nu\left(B_r\left(x\right)\right)\le c_2 r^{æ}, \forall x\in X, \forall r>0. \qquad (æ)$$

In this case, the Morrey space $L^{p,\,\lambda}\left(X\right)$ is defined by means of the norm

$$\|f\|_{L^{p,\,\lambda}(X)}=\sup_{x\in X,\,r>0}\left\{\frac{1}{r^{\lambda}}\int_{B_r(x)}\left|f\left(y\right)\right|^p d\nu\left(y\right)\right\}^{1/p}.$$

Theorem 1 ([10]). *Let $(X;\,d;\,\nu)$ be a homogeneous space equipped with the quasi-metrics d and the measure ν with $\nu\left(X\right)=+\infty$, and the condition* (æ) *be true. Then the maximal operator $\left(\left|B_r\left(x\right)\right|_{\nu}=:\nu\left(B_r\left(x\right)\right)\right)$:*

$$M_{\nu}f\left(x\right)=\sup_{r>0}\frac{1}{\left|B_r\left(x\right)\right|_{\nu}}\int_{B_r(x)}\left|f\left(y\right)\right|d\nu\left(y\right),$$

is bounded in $L^{p,\,\lambda}\left(X\right)$ for $1<p<+\infty$, $0\le\lambda<$ æ.

3. Weighted Morrey-type space $h_{\rho}^{p,\,\alpha}$ and Hardy-Littlewood operator

Let $\rho:[-\pi,\,\pi]\to R_+=(0,\,+\infty)$, be some weight function. Consider the weighted Morrey-type space $h_{\rho}^{p,\,\alpha}$ of harmonic functions in ω furnished with the norm

$$\|u\|_{h_{\rho}^{p,\,\alpha}}=\sup_{0<r<1}\left\|u_r\left(\cdot\right)\rho\left(\cdot\right)\right\|_{p,\,\alpha},$$

where

$$u_r\left(t\right)=u\left(re^{it}\right)=u\left(r\cos t;\,r\sin t\right).$$

Assume that the weight $\rho\left(\cdot\right)$ satisfies the following condition

$$\rho^{-1}\in L_q, \frac{1}{p}+\frac{1}{q}=1. \qquad (4)$$

Applying Hölder inequality we obtain

$$\int_{-\pi}^{\pi}\left|u_r\left(\cdot\right)\right|dt\le\left(\int_{-\pi}^{\pi}\left|u_r\left(\cdot\right)\rho\left(\cdot\right)\right|^p dt\right)^{1/p}\left(\int_{-\pi}^{\pi}\rho^{-q}\left(t\right)dt\right)^{1/q}\le$$

$$\le\left(2\pi\right)^{\frac{1-\alpha}{p}}\sup_{I\in[-\pi,\pi]}\left(\frac{1}{\left|I\right|^{1-\alpha}}\int_I\left|u_r\rho\right|^p dt\right)^{1/p}\left\|\rho^{-1}\right\|_{L_q}=$$

$$= (2\pi)^{\frac{1-\alpha}{p}} \left\| \rho^{-1} \right\|_{L_q} \|u_r\|_{h^{p,\,\alpha}_{\rho}} .$$

It follows immediately that if the condition (4) is true, then $u \in h_1$. Consequently, every function $u \in h^{p,\,\alpha}_{\rho}$ has nontangential boundary values $u^+ \left(e^{it}\right)$ on γ. Then, by Fatou's lemma (see e.g. [28, 29, 30]) we have $u_r \left(e^{it}\right) \to u^+ \left(e^{it}\right)$ as $r \to 1-0$ a.e. in $[-\pi, \pi]$. Applying Fatou theorem on passage to the limit, we obtain

$$\int_I \left| u^+ \left(e^{it}\right) \rho(t) \right|^p dt \le \varliminf_{r \to 1-0} \int_I \left| u_r \left(e^{it}\right) \rho(t) \right|^p dt \le$$

$$\le \|u\|^p_{h^{p,\,\alpha}_{\rho}} |I|^{1-\alpha} ,$$

because

$$\left| u_r \left(e^{it}\right) \rho(t) \right| \to \left| u^+ \left(e^{it}\right) \rho(t) \right| , r \to 1-0, \quad \text{for a.e.}\, t \in [-\pi, \pi] .$$

It follows immediately that $u^+ \in L^{p,\,\alpha}_{\rho}$ and

$$\left\| u^+ \right\|_{p,\,\alpha;\,\rho} \le \|u\|_{h^{p,\,\alpha}_{\rho}} .$$

If the relation

$$\rho^{-1} \in L_{q+0} (-\pi, \pi) , \;\; \text{i.e.}\, \exists \varepsilon > 0 : \rho^{-1} \in L_{q+\varepsilon} (-\pi, \pi) , \tag{5}$$

true, then we have

$$\int_{\pi}^{\pi} |u_r(\cdot)|^{1+\delta} dt \le \left(\int_{-\pi}^{\pi} |u_r(\cdot)\, \rho(\cdot)|^p dt \right)^{\frac{1+\delta}{p}} \left(\int_{-\pi}^{\pi} |\rho(\cdot)|^{-\frac{pq}{p-q\delta}} dt \right)^{\frac{1}{q} - \frac{\delta}{p}} \le c_\delta \|u\|^{1+\delta}_{h^{p,\,\alpha}_{\rho}} ,$$

where $\delta > 0$ is a sufficiently small number, and c_δ is a constant depending only on δ. Then, in view of the classical results, the representation

$$u\left(re^{it}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u^+(s)\, P(r;\, s-t)\, ds, \tag{6}$$

is true, where $u^+(s) =: u^+\left(e^{is}\right)$, $s \in [-\pi, \pi]$, and $P_z(\varphi) =: P(r;\, \theta - \varphi)$ is a Poisson kernel for the unit disk

$$P_z(\varphi) = P_r(\theta - \varphi) = P(r;\, \theta - \varphi) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} , \; z = re^{i\theta}.$$

Thus, if $u \in h^{p,\,\alpha}_{\rho}$ and $\rho(\cdot)$ satisfies the condition (5), then $u^+ \in L^{p,\,\alpha}_{\rho}$ and the relation (6) holds.

Now let's prove the converse. In other words, let's prove that if $u^+ \in L^{p,\,\alpha}_{\rho}$ and the representation (6) holds, then $u \in h^{p,\,\alpha}_{\rho}$. To do so, we need some auxiliary facts.

Consider the arbitrary nontangential internal angle θ_0 with a vertex at the point $z = e^{it} \in \gamma$, $t \in [-\pi, \pi]$. Denote by $M_\mu f(t)$ the Hardy-Littlewood type maximal function (or Hardy-Littlewood operator) of the function $f(\cdot)$:

$$M_\mu f(x) = \sup_{I \ni x} \frac{1}{\mu(I)} \int_I |f(t)| \, d\mu(t),$$

where sup is taken over all intervals $I \subset [-\pi, \pi]$ which contain x, and $\mu(\cdot)$ is a Borel measure on $[-\pi, \pi]$, which satisfies the condition

$$\mu(I) > 0, \text{ for } \forall I : |I| > 0.$$

It is shown that there exists a positive constant C_{θ_0}, depending only on θ_0 such that

$$\sup_{z \in \theta_0} |u_\mu(z)| \leq C_{\theta_0} M_\mu f(t), \forall t \in [-\pi, \pi],$$

where

$$u_\mu(z) = u\left(re^{it}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r; s-t) \, u^+(s) \, d\mu(s).$$

For a usual maximal operator, this fact was established in [29, p.237] and [30, p.30].

Consider the Poisson kernel $P_z(t)$ in the upper half-plane

$$P_z(t) =: P_y(x-t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}, z = x + iy, y > 0.$$

Let $f \in L_1\left(\frac{d\mu(t)}{1+t^2}\right)$ and consider the Poisson integral

$$u_\mu(x; y) = \int_R P_y(x-s) \, f(s) \, d\mu(s).$$

The following main lemma is proved.

Lemma 1. *Let $\mu(\cdot)$ be a Borel measure on R with*

$$\mu(I) > 0, \forall I : |I| > 0; \sup_{y>0;\, x \in R} \int_R P_y(s - |x|) \, d\mu < +\infty.$$

Then, for $f \in L_1\left(\frac{d\mu(t)}{1+t^2}\right)$, the function

$$u_\mu(x; y) = \int_R P_y(x-s) \, f(s) \, d\mu(s),$$

which is harmonic on the upper half-plane, satisfies the relation

$$\sup_{z \in \Gamma_{\mu;\, \alpha_0}(t)} |u_\mu(z)| \leq A_{\alpha_0} M_\mu f(t), t \in R,$$

where M_μ is the Hardy-Littlewood type maximal function

$$M_\mu f(x) = \sup_{I \ni x} \frac{1}{\mu(I)} \int_I |f(t)| \, d\mu(t),$$

$$\Gamma_{\mu;\alpha_0}(t) = \{(x;\, y) \in C : \mu((-|x-t|,\, |x-t|)) < \alpha_0 y;\, y > 0\},\, \alpha_0 > 0,$$

and A_{α_0} is a constant depending only on α_0.

By M we denote the usual Hardy-Littlewood operator, i.e.

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)| \, dt,$$

where $|I|$ is a Lebesgue measure of the interval $I \subset [-\pi, \pi]$.

It is not difficult to see that the Lebesgue measure on R satisfies all the conditions of Lemma 1.

Let's go back to Theorem 1 [10] . Let the condition (æ) be fulfilled. Note that Theorem 1 [10] is true in case $\mu(X) < +\infty$, too. Because its proof is based on the Fefferman-Stein inequality which is true also in case $\mu(X) < +\infty$. Let's apply this theorem to our case. In our case we have $X = R$, $d(x;\, y) = |x-y|$ and æ $= 1$. So, if the measure $\mu(\cdot)$ satisfies the conditions of Theorem 1 [10] in our case, then we have

$$\int_I |M_\mu f|^p \, d\mu \le c\, |I|^{1-\alpha},$$

where $|I|$ is a Lebesgue measure of the set $I \subset R$. Then from (**??**) it directly follows that $u_\mu \in h^{p,\alpha}(d\mu)$, where $h^{p,\alpha}(d\mu)$ is a class of harmonic functions on the upper half-plane equipped with the norm

$$\|u_\mu\|_{h^{p,\alpha}(d\mu)} = \sup_{y>0} \sup_{I \subset R} \left(\frac{1}{|I|^{1-\alpha}} \int_I |u_\mu(x;\, y)|^p \, d\mu(x) \right)^{1/p}.$$

So we get the validity of the following theorem.

Theorem 2. *Assume that the measure $\mu(\cdot)$ satisfies the conditions (Iis an interval)*

$$\mu(I) \sim |I|,\, \forall I \subset R;\ \sup_{y>0;\, x \in R} \int_R P_y(s - |x|) \, d\mu(s) < +\infty.$$

Let

$$u_\mu(z) = u_\mu(x;\, y) = \int_R P_y(x-t) f(t) \, d\mu(t),\, f \in L^{p,\alpha}(d\mu),\, 0 \le 1-\alpha < 1,$$

where $L^{p,\alpha}(d\mu)$ is a Morrey space equipped with the norm

$$\|f\|_{p,\alpha;\, d\mu} = \sup_{I \subset R} \left\{ \frac{1}{|I|^{1-\alpha}} \int_I |f(y)|^p \, d\mu(y) \right\}^{1/p}.$$

Then for $\forall\alpha_0 > 0$, $\exists A_{\alpha_0} > 0$*:*

$$\sup_{(x;\,y)\in\Gamma_{\alpha_0}(t)} |u_\mu\,(x;y)| \leq A_{\alpha_0} M_\mu f\,(t)\,, \forall t \in R, \tag{7}$$

and $u_\mu^* \in h^{p,\,\alpha}\,(d\mu)$*:*

$$\left\|u_\mu^*\right\|_{h^{p,\,\alpha}(d\mu)} \leq A_{\alpha_0}\,\|f\|_{p,\,\alpha;\,d\mu}\,, \tag{8}$$

where $u_\mu^*\,(\cdot)$ *is a nontangential maximal function for* u*:*

$$u_\mu^*\,(t) = \sup_{z\in\Gamma_{\alpha_0}(t)} |u_\mu\,(z)|\,, t \in R.$$

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