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Constructive Method for Solving the External Dirichlet Boundary – Value Problem for the Helmholtz Equation

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Abstract. This work presents the justification of collocation method for the boundary integral equation of the external Dirichlet boundary – value problem for the Helmholtz equation. Besides, the sequence of approximate solutions is built which converges to the exact solution of the original problem and the estimate for the rate of convergence is obtained.

Key Words and Phrases: Helmholtz equation, external Dirichlet boundary – value problem, surface integral, cubature formula, collocation method.
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1. Introduction and Problem Statement

It is known that one of the methods for solving the external Dirichlet boundary – value problem for the Helmholtz equation is its reduction to the boundary integral equation (BIE). Integral equation methods play a central role in the study of boundary – value problems associated with the scattering of acoustic or electromagnetic waves by bounded obstacles. This is primarily due to the fact that the mathematical formulation of such problems leads to equations defined over unbounded domains, and hence their reformulation in terms of boundary integral equations not only reduces the dimensionality of the problem, but also allows one to replace a problem over an unbounded domain by one over a bounded domain. Since BIE is solved only in very rare cases, it is therefore of paramount importance to develop approximate methods for solving BIE with an appropriate theoretical justification.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a twice continuously differentiable boundary S. Consider the external Dirichlet boundary – value problem for the Helmholtz equation: us to find a function u which is twice continuously differentiable in $\mathbb{R}^3 \backslash \bar{D}$ and continuous on S, satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^3 \backslash \bar{D}$, the Sommerfeld radiation condition

$$\left(\frac{x}{|x|}, \, grad\, u\,(x) \right) - i\, k\, u\,(x) = o\, \left(\frac{1}{|x|} \right), \quad |x| \to \infty,$$

and the boundary condition

$$u\,(x) = f\ (x)\ \text{ on } S,$$

where k is a wave number with $Im\,k \geq 0$, and f is a given continuous function on S.

It is proved in [1] that the potential of double layer

$$u\ (x) = \int\limits_S \frac{\partial\Phi_k\,(x,y)}{\partial\vec{n}\ (y)}\,\varphi\,(y)\ dS_y, \quad x \in \mathbb{R}^3 \backslash \bar{D},$$

is a solution of the external Dirichlet boundary – value problem for the Helmholtz equation if the density φ is a solution of BIE

$$\varphi + K\,\varphi = 2f, \tag{1}$$

where

$$(K\varphi)\,(x) = \ 2\int\limits_S \frac{\partial\Phi_k\,(x,y)}{\partial\vec{n}\ (y)}\,\varphi\,(y)\ dS_y, \quad x \in S,$$

and $\Phi_k\,(x,y)$ is fundamental solution the Helmholtz equation, i.e.

$$\Phi_k\,(x,y) = \ e^{i\,k\,|x-y|}/\ (4\pi\ |x-y|)\ ,\ \ x,y \in \mathbb{R}^3,\ x \neq y.$$

Let us note that the integral equations of boundary – value problems for the Helmholtz equation in the two – dimensional case were first considered by Kupradse [2-4]. In the present paper, we study an approximate solution of the external Dirichlet boundary – value problem for the Helmholtz equation by the integral equations method (1).

2. Main Results

Divide S into elementary domains $S = \bigcup\limits_{l=1}^{N} S_l^N$ in such a way that:

(1) for every $l = \overline{1,\,N}$ the domain S_l^N is closed and the set of its internal points $\overset{0}{S_l^N}$ with respect to S is nonempty, with $mes\,\overset{0}{S_l^N} = mes S_l^N$ and $\overset{0}{S_l^N} \cap \overset{0}{S_j^N} = \varnothing$ for $j \in \{1,\,2,\,...N\}\,,\ j \neq l$;

(2) for every $l = \overline{1,\,N}$ the domain S_l^N is a connected piece of the surface S with a continuous boundary;

(3) for every $l = \overline{1\,,\,N}$ there exists a so-called control point $x_l \in S_l^N$ such that:

(3.1) $r_l(N) \sim R_l(N)$ $(r_l(N) \sim R_l(N) \ \ \Leftrightarrow\ \ C_1 \leq r_l\,(N)\,/R_l\,(N) \leq C_2$, C_1 and C_2 are positive constants independent of N), where $r_l(N) = \min\limits_{x\in\partial S_l^N} |x - x_l|$ and $R_l(N) = \max\limits_{x\in\partial S_l^N} |x - x_l|$;

(3.2) $R_l(N) \leq d/2$, where d is the radius of a standard sphere (see [5]);

(3.3) for every $j = \overline{1\,,\,N}$, $r_j(N) \sim r_l(N)$.

It is clear that $r(N) \sim R(N)$ and $\lim\limits_{N\to\infty} r(N) = \lim\limits_{N\to\infty} R(N) = 0$, where $R(N) = \max\limits_{l=\overline{1,N}} R_l(N)$, $r(N) = \min\limits_{l=\overline{1,N}} r_l(N)$.

Such a partition, as well as the partition of the unit sphere into elementary parts, has been carried out earlier in [6].

Let $S_d(x)$ and $\Gamma_d(x)$ be the parts of the surface S and the tangential plane $\Gamma(x)$, respectively, at the point $x \in S$, contained inside the sphere $B_d(x)$ of radius d centered at the point x. Besides, let $\tilde{y} \in \Gamma(x)$ be the projection of the point $y \in S$. Then

$$|x-\tilde{y}| \le |x-y| \le C_1(S)\ |x-\tilde{y}| \text{ and } mes S_d(x) \le C_2(S)\, mes\Gamma_d(x), \tag{1}$$

where $C_1(S)$ and $C_2(S)$ are positive constants that depend only on S (if S is a sphere, then $C_1(S) = \sqrt{2}$ and $C_2(S) = 2$).

Lemma 2.1 ([6]). *There exist the constants $C_0' > 0$ and $C_1' > 0$, independent of N, such that for $\forall\ l, j \in \{1, 2, ..., N\}$, $j \ne l$, and $\forall\ y \in S_j^N$ the inequality $C_0'\ |y-x_l| \le |x_j - x_l| \le C_1'\ |y-x_l|$ holds.*

For a continuous function $\varphi(x)$ on S, we introduce the modulus of continuity, which has the following form:

$$\omega(\varphi,\, \delta) = \delta \sup_{\tau \ge \delta} \frac{\overline{\omega}(\varphi,\, \tau)}{\tau}, \quad \delta > 0,$$

where $\bar{\omega}(\varphi,\, \tau) = \max\limits_{\substack{|x-y|\le\tau \\ x,\, y\in S}} |\varphi(x) - \varphi(y)|$.

Let

$$k_{lj} = 2\ |sgn\,(l-j)|\ \frac{\partial\Phi_k\,(x_l, x_j)}{\partial\vec{n}\ (x_j)}\, mes\, S_j^N \text{ for } l\,, j = \overline{1\,,\, N}.$$

It is proved in [7] that the expression

$$\left(K^N\varphi\right)\,(x_l) = \sum_{j=1}^{N} k_{l\,j}\,\varphi\,(x_j)$$

are cubature formula at the points x_l, $l = \overline{1,\, N}$, for the integral $(K\varphi)\,(x)$, with

$$\max_{l=\overline{1,\,N}} \left|\,(K\varphi)\,(x_l) - \left(K^N\varphi\right)\,(x_l)\,\right| \le M^*\,(\,\|\varphi\|_\infty\, R\,(N)\ |\ln R\,(N)\,| + \omega\,(\varphi, R\,(N))) \tag{2}$$

Let $\mathbb{C}^N -$ be a space of vectors $z^N = \left(z_1^N, z_2^N, \ldots, z_N^N\right)^{\mathrm{T}}$, $z_l^N \in \mathbb{C}$, $l = \overline{1,N}$, equipped with the norm $\left\|\,z^N\right\| = \max\limits_{l=\overline{1,N}} \left|z_l^N\right|$, and

$$K_l^N\, z^N = \sum_{j=1}^{N} k_{l\,j}\, z_j^N\,,\ \ l = \overline{1,N}\,, \qquad K^N\, z^N = \left(K_1^N\, z^N,\ K_2^N\, z^N\,,\, \ldots, K_N^N\, z^N\right)\,.$$

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*Here and after, M denotes positive constants which can be different in different inequalities.

Then the BIE (1) by the system of algebraic equations with respect to z_l^N, approximate values of $\varphi(x_l)$, $l = \overline{1, N}$, stated as follows:

$$z^N + K^N z^N = 2p^N f, \tag{3}$$

where $p^N f = (f(x_1), f(x_2), \ldots, f(x_N))$.

To justify the collocation method, we will use Vainikko's convergence theorem for linear operator equations (see [8]). To formulate that theorem, we need some definitions and a theorem from [8].

Definition 2.1 ([8]). *A system $Q = \{q^N\}$ of operators $q^N : C(S) \to \mathbb{C}^N$ is called a connecting system for $C(S)$ and $\mathbb{C}^N$ if*

$$\|q^N\varphi\| \to \|\varphi\|_\infty \text{ as } N \to \infty, \quad \forall\varphi \in C(S);$$

$$\|q^N(a\varphi + a'\varphi') - (a\,q^N\varphi + a'q^N\varphi')\| \to 0 \text{ as } N \to \infty, \quad \forall\varphi, \varphi' \in C(S), \quad a, a' \in \mathbb{C}.$$

Definition 2.2 ([8]). *A sequence $\{\varphi_N\}$ of elements $\varphi_N \in \mathbb{C}^N$ is called $Q-$convergent to $\varphi \in C(S)$ if $\|\varphi_N - q^N\varphi\| \to 0$ as $N \to \infty$. We denote this fact by $\varphi_N \overset{Q}{\to} \varphi$.*

Definition 2.3 ([8]). *A sequence $\{\varphi_N\}$ of elements $\varphi_N \in \mathbb{C}^N$ is called $Q-$compact if every subsequence of it $\{\varphi_{N_m}\}$ contains a $Q-$convergent subsequence $\left\{\varphi_{N_{m_k}}\right\}$.*

Proposition 2.1 ([8]). *Let $q^N : C(S) \to \mathbb{C}^N$ be linear and bounded. Then the following conditions are equivalent:*

(1) *the sequence $\{\varphi_N\}$ is $Q-$compact and the set of its $Q-$limit points is compact in $C(S)$;*

(2) *there exists a relatively compact sequence $\{\varphi^{(N)}\} \subset C(S)$ such that*

$$\left\|\varphi_N - q^N\varphi^{(N)}\right\| \to 0 \text{ as } N \to \infty.$$

Definition 2.4 ([8]). *A sequence of operators $\mathrm{E}^N : \mathbb{C}^N \to \mathbb{C}^N$ is called $QQ-$convergent to the operator $\mathrm{E} : C(S) \to C(S)$ if for every $Q-$convergent sequence $\{\varphi_N\}$ the relation $\varphi_N \overset{Q}{\to} \varphi \Rightarrow \mathrm{E}^N\varphi_N \overset{Q}{\to} \mathrm{E}\varphi$ holds. We denote this fact by $\mathrm{E}^N \overset{QQ}{\to} \mathrm{E}$.*

Definition 2.5 ([8]). *We say that a sequence of linear bounded operators $\mathrm{E}^N : \mathbb{C}^N \to \mathbb{C}^N$ converges compactly to the linear bounded operator $\mathrm{E} : C(S) \to C(S)$ if $\mathrm{E}^N \overset{QQ}{\to} \mathrm{E}$ and the following compactness condition holds:*

$$\varphi_N \in \mathbb{C}^N, \quad \|\varphi_N\| \le M \quad \Rightarrow \{\mathrm{E}^N\varphi_N\} \textit{ is } Q-\textit{compact}.$$

Theorem 2.3 ([8]). *Let the following conditions hold:*

1) $Ker\ (I + \mathrm{E}) = \{0\}$;

2) $I^N + \mathrm{E}^{N'}s \quad (N \ge N_0)$ *are Fredholm operators of index zero;*

3) $\vartheta_N \overset{Q}{\to} \vartheta, \quad \vartheta_N \in \mathbb{C}^N, \quad \vartheta \in C(S)$;

4) $\mathrm{E}^N \to \mathrm{E}$ *compactly.*

Then the equation $(I+\mathrm{E})\,\varphi\;=\;\vartheta$ *has a unique solution* $\tilde{\varphi}\in C(S)$*, the equation* $\left(I^N+\mathrm{E}^N\right)\,\varphi_N=\vartheta_N\quad(N\geq N_0)$ *has a unique solution* $\tilde{\varphi}_N\in\mathbb{C}^N$*, and* $\tilde{\varphi}_N\overset{Q}{\to}\tilde{\varphi}$ *with*

$$c_1\,\left\|\,\left(I^N+\mathrm{E}^N\right)\,q^N\tilde{\varphi}-\vartheta_N\right\|\leq\left\|\tilde{\varphi}_N-q^N\tilde{\varphi}\,\right\|\leq c_2\,\left\|\,\left(I^N+\mathrm{E}^N\right)\,q^N\tilde{\varphi}-\vartheta_N\,\right\|\,,$$

where

$$c_1\;=1/\sup_{N\geq N_0}\,\left\|\,I^N+\mathrm{E}^N\right\|>0\,,\;\;c_2\;=\;\sup_{N\geq N_0}\,\left\|\,\left(I^N+\mathrm{E}^N\right)^{-1}\right\|<+\infty.$$

Theorem 2.2. *Let* $Im\,k>0$*, then the equations (1) and (3) have unique solutions* $\varphi_*\in C(S)$ *and* $z_*^N\in\mathbb{C}^N\quad(N\geq N_0)$*, respectively, and* $\left\|\,z_*^N-p^N\varphi_*\right\|\to 0$ *as* $N\to\infty$ *with the following estimate for the rate of convergence:*

$$\left\|\,z_*^N-p^N\varphi_*\right\|\leq M\;\left[\;\|f\|_\infty\,R(N)\;\;|\ln R(N)\,|+\omega\;(f,R(N))\,\right].$$

Proof. Let's verify that the conditions of Theorem 2.1 are satisfied. It is proved in [1] that if $Im\,k>0$, then $Ker\;(I+K)=\{\,0\,\}$. Obviously, the operators I^N+B^N are Fredholm operators of index zero and the system operators $P=\left\{\,p^N\right\}$ is a connecting system for the spaces $C(S)$ and $\mathbb{C}^N$. Then $I^N+K^N\overset{PP}{\to}I+K$. By Definition 2.5, it remains only to verify the compactness condition, which in view of Proposition 2.1 is equivalent to the following one: $\forall\left\{z^N\right\}$, $z^N\in\mathbb{C}^N$, $\left\|z^N\right\|\leq M$, there exists a relatively compact sequence $\left\{\,K_N\,z^N\right\}\subset C(S)$ such that

$$\left\|\,K^N\,z^N-p^N\left(\;K_N\,z^N\right)\,\right\|\to 0\,\text{as}\;N\to\infty.$$

As $\left\{K_N\,z^N\right\}$, we choose the sequence

$$\left(K_N\,z^N\right)\;(x)=2\sum_{j=1}^{N}z_j^N\int\limits_{S_j^N}\frac{\partial\Phi_k\,(x,y)}{\partial\vec{n}\,(y)}dS_y.$$

Take arbitrary points $x',\;x''\in S$ such that $|x'-x''|=\delta<d/2$. Then

$$\left|\,\left(K_N\,z^N\right)\;(x')-\left(K_N\,z^N\right)\;(x'')\,\right|\leq\;M\;\left\|z^N\right\|\;\int\limits_{S}\left|\frac{\partial\Phi_k\,(x',y)}{\partial\vec{n}\,(y)}-\frac{\partial\Phi_k\,(x'',y)}{\partial\vec{n}\,(y)}\right|dS_y\leq$$

$$M\;\left\|z^N\right\|\int\limits_{S_{\delta/2}(x')}\left|\frac{\partial\Phi_k\,(x',y)}{\partial\vec{n}\,(y)}\right|dS_y+M\;\left\|z^N\right\|\int\limits_{S_{\delta/2}(x'')}\left|\frac{\partial\Phi_k\,(x'',y)}{\partial\vec{n}\,(y)}\right|dS_y+$$

$$M\;\left\|z^N\right\|\int\limits_{S_{\delta/2}(x')}\left|\frac{\partial\Phi_k\,(x'',y)}{\partial\vec{n}\,(y)}\right|dS_y+M\;\left\|z^N\right\|\int\limits_{S_{\delta/2}(x'')}\left|\frac{\partial\Phi_k\,(x',y)}{\partial\vec{n}\,(y)}\right|dS_y+$$

$$M\;\left\|z^N\right\|\int\limits_{S\backslash\left(S_{\delta/2}(x')\cup S_{\delta/2}(x'')\right)}\left|\frac{\partial\Phi_k\,(x',y)}{\partial\vec{n}\,(y)}-\frac{\partial\Phi_k\,(x'',y)}{\partial\vec{n}\,(y)}\right|dS_y.$$

Using the inequality

$$\left| \frac{\partial \Phi_k (x, y)}{\partial \vec{n}\,(y)} \right| \le \frac{M}{|x-y|}, \quad \forall\, x\,, y \in S,\ x \ne y,$$

and the formula for reducing surface integral to a double integral, we obtain:

$$\int\limits_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k (x', y)}{\partial \vec{n}\,(y)} \right| dS_y \le M \int\limits_{S_{\delta/2}(x')} \frac{1}{|x'-y|}\, dS_y \le\ M\,\delta,$$

$$\int\limits_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k (x'', y)}{\partial \vec{n}\,(y)} \right| dS_y \le\ M\,\delta.$$

Besides, taking into account the inequalities $|x''-y| \ \ge\ \delta/2\,,\ \ \forall y \in S_{\delta/2}\,(x')$ and $|x'-y| \ge\ \delta/2\,,\ \ \forall y \in S_{\delta/2}\,(x'')$, we have:

$$\int\limits_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k (x'', y)}{\partial \vec{n}\,(y)} \right| dS_y \le M \int\limits_{S_{\delta/2}(x')} \frac{1}{|x''-y|}\, dS_y \le\ \frac{2M}{\delta} mes\left(S_{\delta/2}\,(x')\right) \le M\,\delta,$$

$$\int\limits_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k (x', y)}{\partial \vec{n}\,(y)} \right| dS_y \le M\,\delta.$$

It is easy to show that

$$\left| \frac{\partial \Phi_k (x', y)}{\partial \vec{n}\,(y)} - \frac{\partial \Phi_k (x'', y)}{\partial \vec{n}\,(y)} \right| \le \frac{M\,\delta}{|x'-y|^2}, \quad \forall y \in S\backslash \left(S_{\delta/2}\left(x'\right) \cup S_{\delta/2}\left(x''\right)\right).$$

Hence we find

$$\int\limits_{S\backslash\left(S_{\delta/2}(x') \cup S_{\delta/2}(x'')\right)} \left| \frac{\partial \Phi_k (x', y)}{\partial \vec{n}\,(y)} - \frac{\partial \Phi_k (x'', y)}{\partial \vec{n}\,(y)} \right| dS_y \le M\,\delta\ |\ln \delta|\,.$$

Then

$$\left|\ \left(K_N\, z^N\right)\ \left(x'\right) - \left(K_N\, z^N\right)\ \left(x''\right)\ \right| \le M\ \left\|z^N\right\|\ \delta\ \ |\ln \delta|\,, \tag{4}$$

and, consequently, $\left\{K_N\, z^N\right\} \subset C\,(S)$.

The relative compactness of the sequence $\left\{K_N\, z^N\right\}$ follows from the Arzela theorem. In fact, the uniform boundedness follows directly from the condition $\left\|z^N\right\| \le M$, and the equicontinuity follows from the estimate (4). Then, applying Theorem 2.1 we obtain that the equations (1) and (3) have unique solutions $\varphi_* \in C\,(S)$ and $z_*^N \in \mathbb{C}^N$ $(N \ge N_0)$, respectively, with

$$c_1\,\delta_N \le \left\|\,z_*^N - p^N \varphi_*\right\| \le\ c_2\,\delta_N,$$

where

$$c_1 = 1/\sup_{N\geq N_0} \left\| I^N + K^N \right\| > 0\,, \quad c_2 = \sup_{N\geq N_0} \left\| \left(I^N + K^N \right)^{-1} \right\| < +\infty,$$

$$\delta_N = \max_{l=\overline{1,N}} \left| (K\varphi_*) \ (x_l) - \left(K^N \varphi_* \right) (x_l) \right|.$$

Using the inequality (2), we obtain:

$$\delta_N \leq M \ \left[\|\varphi_*\|_\infty R(N) \ |\ln R(N)| + \omega \ (\varphi_*, R(N)) + \right.$$
$$\left. \|f\|_\infty R(N) \ |\ln R(N)| + \omega \ (f, R(N)) \right].$$

As $\varphi_* = 2\ (I+K)^{-1} f$, we have

$$\|\varphi_*\|_\infty \leq 2 \left\| (I+K)^{-1} \right\| \ \|f\|_\infty .$$

Besides, taking into account the estimate

$$\omega \ (K\varphi_*, R\,(N)) \leq M \ \|\varphi_*\|_\infty \ R\,(N) \ |\ln R\,(N)\,|\,,$$

we obtain:

$$\omega \ (\varphi_*\,,\, R\,(N)) = \omega \ (2f - K\varphi_*, R\,(N)) \leq 2\omega \ (f, R\,(N)) + \omega \ (K\varphi_*, R\,(N)) \leq$$
$$M \ \|\,f\,\|_\infty \ R\,(N) \ |\ln R\,(N)\,|\,,$$

consequently

$$\delta_N \leq M \ [\ \|\,f\,\|_\infty \ R\,(N) \ |\ln R\,(N)\,| + \omega \ (f, R\,(N))\,]\,.$$

Theorem is proved.

Let's state the main result of this work.

Theorem 2.3. *Let* $Im\,k > 0, x_0 \in \mathbb{R}^3\backslash\bar{D}$ *and* $z_*^N = (z_1^*, z_2^*, \ldots, z_N^*)^{\mathrm{T}}$ *be a solution of the system of algebraic equations (3). Then the sequence*

$$u_N(x_0) = \sum_{j=1}^{N} \frac{\partial\Phi_k\,(x_0, x_j)}{\partial\vec{n}\,(x_j)} \ z_j^* \, mesS_j^N$$

converges to the value of the solution $u\,(x)$ *of the external Dirichlet boundary - value problem for the Helmholtz equation at the point* x_0*, with*

$$|u_N\,(x_0) - u\,(x_0)| \leq M \ [\ \|\,f\,\|_\infty \ R\,(N) \ |\ln R\,(N)\,| + \omega \ (f, R\,(N))\ ]\,.$$

Proof. Let the function $\varphi_* \in C\,(S)$be a solution of the equation (1). Then, as is known, the function

$$u\,(x) = \int_S \frac{\partial\Phi_k\,(x,y)}{\partial\vec{n}\,(y)} \varphi_*\,(y) \ dS_y\,, \quad x \in \mathbb{R}^3\backslash\bar{D},$$

is a solution of the external Dirichlet boundary – value problem for the Helmholtz equation. Evidently,

$$u(x_0) - u_N(x_0) = \sum_{j=1}^{N} \int_{S_j^N} \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \left(\varphi_*(x_j) - z_j^*\right) dS_y +$$

$$\sum_{j=1}^{N} \int_{S_j^N} \left(\frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} - \frac{\partial \Phi_k(x_0, x_j)}{\partial \vec{n}(x_j)} \right) \varphi_*(y)\ dS_y +$$

$$\sum_{j=1}^{N} \int_{S_j^N} \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \left(\varphi_*(y) - \varphi_*(x_j)\right) dS_y +$$

$$\sum_{j=1}^{N} \int_{S_j^N} \left(\frac{\partial \Phi_k(x_0, x_j)}{\partial \vec{n}(x_j)} - \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \right) \left(\varphi_*(x_j) - z_j^*\right)\ dS_y +$$

$$\sum_{j=1}^{N} \int_{S_j^N} \left(\frac{\partial \Phi_k(x_0, x_j)}{\partial \vec{n}(x_j)} - \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \right) \left(\varphi_*(y) - \varphi_*(x_j)\right)\ dS_y \quad .$$

As $x_0 \notin S$, then

$$\left| \frac{\partial \Phi_k(x_0, x_j)}{\partial \vec{n}(x_j)} - \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \right| \leq M\, R(N)\,, \quad \forall\, y \in S_j^N.$$

As a result, taking into account Theorem 2.2, we obtain the proof of Theorem 2.3.

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