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Global Bifurcation of Solutions for the Problem of Population Modeling

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Abstract. We consider nonlinear Sturm-Liouville problem with indefinite weight function which arise from population modeling. We show the existence of two families of continua of solutions corresponding to the usual nodal properties and emanating from zero and infinity.

Key Words and Phrases: nonlinear Sturm-Liouville problem, indefinite weight, population modeling, bifurcation point, eigenvalue, oscillatory properties of eigen functions.

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1. Introduction

We consider the following nonlinear Sturm-Liouville equation

$$(\ell y)(x) \equiv -(p(x)y'(x))' + q(x)y(x) = \lambda\rho(x)y(x) + g(x, y(x), y'(x), \lambda),\ x \in (0, 1), \tag{1}$$

subject to the boundary conditions

$$\alpha_0 y(0) - \beta_0 y'(0) = 0, \tag{2}$$

$$\alpha_1 y(1) + \beta_1 y'(1) = 0, \tag{3}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $p(x)$ is a positive and continuously differentiable function on $[0, 1]$, $q(x)$ and $\rho(x)$ are real-valued continuous functions on $[0, 1]$, α_i, β_i, $i = 0, 1$, are real constants such that $|\alpha_i| + |\beta_i| > 0$, $i = 0, 1$. We also assume that the nonlinear term g is continuous function on $[0, 1] \times \mathbb{R}^3$ satisfying the condition:

$$g(x, u, s, \lambda) = o(|u| + |s|),\ \text{as } |u| + |s| \to 0, \tag{4}$$

or

$$g(x, u, s, \lambda) = o(|u| + |s|),\ \text{as } |u| + |s| \to \infty, \tag{5}$$

uniformly in $x \in [0, 1]$ and $\lambda \in \Lambda$ for any bounded interval $\Lambda \subset \mathbb{R}$.

Nonlinear Sturm-Liouville eigenvalue problems arise in many applications, for example, the problem (1)-(3) with indefinite weight arise from population modeling. In this model, weight function ρ changes sign corresponding to the fact that the intrinsic population growth rate is positive at same points and is negative at others, for details, see [5, 7].

If condition (4) holds then we can consider bifurcation from zero, i.e., bifurcation of nontrivial solutions from the set of trivial solutions $\mathcal{R} = \mathbb{R} \times \{0\}$. Problem (1)-(3) in the case $\rho > 0$ has been considered in [10]. This paper prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1[0,1]$ emanating from bifurcation point (in $\mathcal{R}$) corresponding to the eigenvalues of the linear problem, obtained from (1)-(3) by setting $F \equiv 0$. Similar problems for the nonlinear eigenvalue problems of ordinary differential equations of second and fourth order with definite weight function have been considered in [1-4, 12].

If condition (5) holds then the problem (1)-(3) is said to be asymptotically linear (see [9]) and we consider bifurcation from infinity, i.e., the existence of solutions of problem (1)-(3) having arbitrarily large norm. In the case $\rho > 0$ the existence of solutions of problem (1)-(3) with large norm (bifurcating from infinity) is considered in [11] and [12]. In these papers the bifurcation problem from infinity is transformed to a problem involving bifurcation from zero for the eigenvalues of the corresponding linear problem and then the global bifurcation theorems from [11] is applied.

In the investigation of bifurcation from zero and infinity for the problem (1)-(3) with indefinite weight function ρ, the main difficulty is connected with the fact that the eigenfunctions of the linear problem corresponding to the positive and negative eigenvalues with the same serial numbers have same number of zeros. Therefore in this case the standard global bifurcation results from [10] and [11] are not directly applicable. However, by using the results of [6], [10] and [11], we shall establish the global bifurcation results from zero and infinity for the problem (1)-(3) with indefinite weight function ρ. We prove the existence of global continua of solutions bifurcating from zero and infinity which are similar to those obtained in [10], [11] and [12].

2. Preliminary

By (4) the linearization of problem (1)-(3) at $y = 0$ is the linear Sturm-Liouville problem

$$\begin{cases} -(p(x)y'(x))' + q(x)y(x) = \lambda\rho(x)y(x), \ x \in (0,\ 1), \\ y \in B.C., \end{cases} \tag{6}$$

where by $B.C.$ is the set of the boundary conditions (2)-(3). It is a classical result (see [8]) that the problem (6) in the case $\rho(x) > 0$, $x \in [0,1]$, possesses infinitely many real eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$, all of which are simple, and $\lim\limits_{k \to +\infty} \lambda_k = +\infty$. The eigenfunction $y_k(x)$ corresponding to eigenvalue λ_k, $k \in \mathbb{N}$, has exactly $k-1$ simple nodal zeros in the interval $(0,1)$ (by a nodal zero we mean the function changes sign at the zero and at a simple nodal zero, the derivative of the function is nonzero).

Let E be the Banach space of all continuously differentiable functions on $[0,1]$ which satisfy the conditions $B.C.$. E is equipped with its usual norm $||u||_1 = ||u||_\infty + ||u'||_\infty$, where $||u||_\infty = \max\limits_{x\in[0,1]} |u(x)|$. Let S_k^+ be the set of $u \in E$ which have exactly $k-1$ simple nodal zeroes on $(0,1)$ and which are positive for $0 \neq x$ near 0; then, $S_k^- = -S_k^+$ and $S_k = S_k^- \cup S_k^+$. The sets S_k^+, S_k^- and S_k are open in E. Moreover, if $u \in \partial S_k$, then u has at least one double zero in $[0,1]$.

Theorem 2.1. *(see [8; Ch. 10, § § 10·6, 10·61]). If ρ changes sign in the interval $(0,1)$ ($i.e.$ $\text{meas}\,\{x \in [0,1] : \sigma\rho(x) > 0\} > 0$, $\sigma \in \{+,-\}$), $q(x) \geq 0$, $x \in [0,1]$, and $\alpha_0\beta_0 \geq 0$, $\alpha_1\beta_1 \geq 0$, then the eigenvalues of problem (6) are all real and simple, and form a two sequences*

$$0 > \lambda_1^- > \lambda_2^- > ... > \lambda_k^- \mapsto -\infty \text{ and } 0 < \lambda_1^+ < \lambda_2^+ < ... < \lambda_k^+ \mapsto +\infty.$$

Moreover, for each $k \in \mathbb{N}$ and each $\sigma \in \{+,-\}$ the eigenfunction $y_k^\sigma(x)$ corresponding to eigenvalue λ_k^σ, has exactly $k-1$ simple nodal zeros in the interval $(0,1)$ (more precisely, $y_k^\sigma(x) \in S_k$).

Throughout the sequel we assume that the following conditions are satisfied:

$$\begin{aligned} &\text{meas}\,\{x \in [0,1] : \sigma\rho(x) > 0\} > 0,\, \sigma \in \{+,-\}, \\ &q(x) \geq 0, x \in [0,1], \text{ and } \alpha_i\beta_i \geq 0, i = 0,1. \end{aligned} \tag{7}$$

It follows from Theorem 2.1 that for each $k \in \mathbb{N}$ the eigenfunctions of $y_k^-(x)$ and $y_k^+(x)$, corresponding to the eigenvalues λ_k^- and λ_k^+, respectively, have exactly $k-1$ simple nodal zeros in the interval $(0,1)$. Hence, if the conditions (7) are satisfied, then first sight it seems that the continua which bifurcates from the point $(\lambda_k^+, 0)$ and is contained in $\mathbb{R} \times S_k$, will meet $(\lambda_k^-, 0)$ and this prevents the first alternative of [10, Theorem 1.3] occurring. But thanks to Dancer [6] we show that this is not happening.

3. Global bifurcation from zero for problem (1)-(3)

We denote by $\mathfrak{L}$ the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (1)-(3). The eigenfunction y_k^σ, $\sigma \in \{+,-\}$, corresponding to the eigenvalue λ_k^σ of problem (6) is made unique by requiring that $y_k^\sigma \in S_k^+$ and $||y_k^\sigma|| = 1$.

One of the main results is the following theorem.

Theorem 3.1. *For each $k \in \mathbb{N}$, each $\nu \in \{+,-\}$ and each $\sigma \in \{+,-\}$ there exists a continuum $(\mathfrak{L}_k^\sigma)^\nu$ of solutions of problem (1)-(3) in $(\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k^\sigma, 0)\}$ which meets $(\lambda_k^\sigma, 0)$ and ∞ in $\mathbb{R} \times E$.*

Proof. Let (λ, y) is a solution of problem (1)-(3) and $y \in \partial S_k^\nu$. Hence y has double zero in $[0,1]$. Then, using growth estimate on g near the double zero and linearity of ℓ and ρy and applying Gronwall's inequality we obtain that $y \equiv 0$ on $[0,1]$.

By (7) $\lambda = 0$ is not an eigenvalue of the spectral problem (6). Then using Green's function $h(x,t)$ of differential expression $\ell(y)$ together with the boundary conditions (2)-(3) problem (1)-(3) can be converted to the equivalent integral equation

$$y(x) = \lambda \int_0^\pi h\,(x,t)\,\rho(t) y(t) dt + \int_0^\pi h\,(x,t)\,g(t, y(t), y'(t), \lambda)\,dt. \tag{8}$$

Define $L : E \to E$ and $F : \mathbb{R} \times E \to E$ by

$$Ly(x) = \int_0^\pi h\,(x,t)\,\rho(t) y(t) dt \;\text{ and }\; F(\lambda, y(x)) = \int_0^\pi h\,(x,t)\,g(t, y(t), y'(t), \lambda)\,dt, \tag{9}$$

respectively.

Since $\rho(x)$ is continuous, it follows from the properties of $h(x,t)$ that $L : E \to E$ is a completely continuous operator. The operator G can be represented as the composition of the Fredholm operator L with $\rho(x) \equiv 1$ and the superposition operator $g(\lambda, y(x)) = g(x, y(x), y'(x), \lambda)$. Since g is continuous in $[0,l] \times \mathbb{R}^3$, it follows that $g : \mathbb{R} \times E \to C[0,1]$ is continuous. Hence $G : \mathbb{R} \times E \to E$ is completely continuous. By (4) we have

$$G(\lambda, y) = o(||y||) \;\text{ as } ||y|| \to 0, \tag{10}$$

uniformly with respect to $\lambda \in \Lambda$.

By virtue of (8)-(9) problem (1)-(3) can be written in the following equivalent form

$$y = \lambda L y + G(\lambda, y), \tag{11}$$

and therefore, it is enough to investigate the structure of the set of solutions of (1)-(3) in $\mathbb{R} \times E$.

Note that problem (11) is of the form (0.1) of [10]. The linearization of this problem at $y = 0$ is the spectral problem

$$y = \lambda L y. \tag{12}$$

Obviously, the problem (12) is equivalent to the spectral problem (6). Consequently, the eigenvalues of (6) are the characteristic values of (12) and are simple. Hence all eigenvalues λ_k^σ, $k \in \mathbb{N}$, $\sigma \in \{+,-\}$, satisfy the hypotheses of Theorem 1.3 from [10] and accordingly there exists a component $\mathfrak{L}_k^\sigma$ of $\mathfrak{L}$ with contains $(\lambda_k^\sigma, 0)$ and is either unbounded in $\mathbb{R} \times E$ or contains $(\lambda_j^\sigma, 0)$, where $j \neq k$. It follows from [10; Lemma 1.24] that if $(\lambda, y) \in \mathfrak{L}_k^\sigma$ and is near $(\lambda_k^\sigma, 0)$, then $y = \tau y_k^\sigma + w$ with $w = o(|\tau|)$. Since S_k is open in E and $y_k^\sigma \in S_k$, then

$$(\lambda, y) \in \mathbb{R} \times S_k \;\text{ and }\; ((\mathfrak{L}_k^\sigma \setminus \{(\lambda_k^\sigma, 0)\}) \cap B_\delta(\lambda_k^\sigma)) \subset \mathbb{R} \times S_k \tag{13}$$

for all $\delta > 0$ small, where $B_\delta(\lambda_k^\sigma)$ is a open ball in $\mathbb{R} \times E$ of radius δ centered at $(\lambda_k^\sigma, 0)$. By an above remark,

$$(\mathfrak{L}_k^\sigma \setminus \{(\lambda_k^\sigma, 0)\}) \cap (\mathbb{R} \times \partial S_k) = \emptyset. \tag{14}$$

Next we decompose $\mathfrak{L}_k^\sigma$, $k \in \mathbb{N}$, $\sigma \in \{+,-\}$, into two subcontinua $(\mathfrak{L}_k^\sigma)^+$ and $(\mathfrak{L}_k^\sigma)^-$ in accordance with Dancer's construction (see [6, p. 1070-1071). Again writing $y = \tau y_k^\sigma + w$ for $(\lambda, y) \in (\mathfrak{L}_k^\sigma \backslash \{(\lambda_k^\sigma, 0)\})$ and near $(\lambda_k^\sigma, 0)$ we have $\tau y_k^\sigma \in \mathbb{R} \times S_k^\nu$ if $0 \neq \tau \in \mathbb{R}^\nu$. Therefore, by (13) we have

$$(((\mathfrak{L}_k^\sigma)^+ \backslash \{(\lambda_k^\sigma, 0)\}) \cap B_\delta(\lambda_k^\sigma)) \subset \mathbb{R} \times S_k^+ \text{ and } (((\mathfrak{L}_k^\sigma)^- \backslash \{(\lambda_k^\sigma, 0)\}) \cap B_\delta(\lambda_k^\sigma)) \subset \mathbb{R} \times S_k^-$$

for all $\delta > 0$ small. Moreover, it follows from (14) that

$$(\mathfrak{L}_k^\sigma)^+ \cap (\mathbb{R} \times \partial S_k^+) = \emptyset \text{ and } (\mathfrak{L}_k^\sigma)^- \cap (\mathbb{R} \times \partial S_k^-) = \emptyset.$$

It is clear from the last four relations that $(\mathfrak{L}_k^\sigma)^\nu \backslash \{(\lambda_k^\sigma, 0)\})$, $\nu \in \{+,-\}$, cannot leave $\mathbb{R} \times S_k^\nu$ outside of a neighborhood of $(\lambda_k^\sigma, 0)$. Since $S_k^+ \cap S_k^- = \emptyset$ it follows by remark to Theorem 2 from [6, p. 1073] that

$$((\mathfrak{L}_k^\sigma)^+ \backslash \{(\lambda_k^\sigma, 0)\}) \cap ((\mathfrak{L}_k^\sigma)^+ \backslash \{(\lambda_k^\sigma, 0)\}) = \emptyset.$$

Hence by [8, theorem 2] we have

$$(\mathfrak{L}_k^\sigma)^+ \subset ((\mathbb{R} \times S_k^+) \cup \{(\lambda_k^\sigma, 0)\}) \text{ and } (\mathfrak{L}_k^\sigma)^- \subset ((\mathbb{R} \times S_k^-) \cup \{(\lambda_k^\sigma, 0)\}),$$

and both are unbounded in $\mathbb{R} \times E$. The proof of this theorem is complete.

Remark 3.1. If the nonlinear term g has the form $g(x, y, s, \lambda) = \lambda\, g_1(x, y, s, \lambda)$, where g_1 is continuous function on $[0,\, 1] \times \mathbb{R}^3$ satisfying the condition (4), then the problem (1)-(3) does not have a solution of the form $(0, u)$ (this follows from the fact that 0 is not an eigenvalue of the corresponding linear problem (6)). In this case it is obvious that for each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ the continua $(\mathfrak{L}_k^+)^\nu$ and $(\mathfrak{L}_k^-)^\nu$ do not intersect and this again confirms the validity of Theorem 3.1

4. Global bifurcation from infinity for problem (1)-(3)

Now we consider problem (1)-(3) under condition (5). We say (λ, ∞) is a bifurcation point for (1)-(3) if every neighborhood of (λ, ∞) contains solutions of (1)-(3), i.e. there exists a sequence $\{(\lambda_n, u_n,)\}_{n=1}^\infty$ of solutions of this problem such that $\lambda_n \to \lambda$ and $u_n \to \infty$ as $n \to \infty$.

Theorem 4.1. *If* (5) *holds then, for each* $k \in \mathbb{N}$ *and each* $\sigma \in \{+\,,\,-\}$ *there exists an unbounded component* $\mathfrak{D}_k^\sigma$ *of* $\mathfrak{L} \cup (\lambda_k^\sigma \times \{\infty\})$*, containing* $I_k \times \{\infty\}$*. Moreover, if* $\Lambda \subset \mathbb{R}$ *is an interval such that* $\Lambda \cap \{\lambda_k^\sigma\}_{k=1}^\infty = \lambda_k^\sigma$ *and* $\mathcal{M}$ *is a neighborhood of* $I_k^\sigma \times \{\infty\}$ *whose projection on* $\mathbb{R}$ *lies in* Λ *and whose projection on* E *is bounded away from* 0*, then either*

1^o. $\mathfrak{D}_k^\sigma \backslash \mathcal{M}$ *is bounded in* $\mathbb{R} \times E$ *in which case* $D_k \backslash M$ *meets* $\mathcal{R} = \{(\lambda, 0) \,|\, \lambda \in \mathbb{R}\}$ *or*

2^o. $D_k \backslash \mathcal{M}$ *is unbounded.*

If 2^o *occurs and* $\mathfrak{D}_k \backslash \mathcal{M}$ *has a bounded projection on* $\mathbb{R}$*, then* $\mathfrak{D}_k \backslash \mathcal{M}$ *meets* $\lambda_j^\sigma \times \{\infty\}$ *for some* $j \neq k$*.*

The proof of this theorem is similar to that of [11, Theorems 1.6 and 2.4] with the use of Theorem 3.1.

By using Theorems 3.1, 3.4 and [11, Corollary 1.8 and Theorem 2.4] we can prove the following theorem.

Theorem 4.2. *If* (5) *holds, then for each* $k \in \mathbb{N}$ *and each* $\sigma \in \{+, -\}$ *there are two subcontinua* $(\mathfrak{D}_k^\sigma)^+$ *and* $(\mathfrak{D}_k^\sigma)^-$*, consisting of the bifurcation branch* $\mathfrak{D}_k^\sigma$*, which satisfy the alternates of Theorem 4.1. Moreover, there exists a neighborhood* $\mathcal{N} \subset \mathcal{M}$ *of* $\lambda_k^\sigma \times \{\infty\}$ *such that* $((\mathfrak{D}_k^\sigma)^\nu \cap \mathcal{N}) \subset ((\mathbb{R} \times S_k^\nu) \cup (\lambda_k^\sigma \times \{\infty\}))$ *for each* $\nu \in \{+, -\}$.

Next, if conditions (4) and (5) both hold then we can improve Theorems 3.1, 4.1 and 4.2 as follows.

Theorem 4.3. *If* (4) *and* (5) *hold then, for each* $k \in \mathbb{N}$*, each* $\sigma \in \{+\ -\}$ *and each* $\nu \in \{+, -\}$*, we have* $(\mathfrak{D}_k^\sigma)^\nu \subset \mathbb{R} \times S_k^\nu$*, and alternative* 2^o *of Theorem 4.1 cannot hold. Furthermore, if* $(\mathfrak{D}_k^\sigma)^\nu$ *meets* $\mathcal{R}$ *for some* λ*, then* $\lambda = \lambda_k^\sigma$*. Similarly, if* $(\mathfrak{L}_k^\sigma)^\nu$ *meets* $\{(\lambda, \infty) : \lambda \in \mathbb{R}\}$*, then* $\lambda = \lambda_k^\sigma$*.*

Proof. If (4) holds, then it follows from the proof of Theorem 3.1 that $\mathfrak{L} \cap (\mathbb{R} \times \partial S_k^\nu) = \emptyset$. Hence the sets $\mathfrak{L} \cap (\mathbb{R} \times S_k^\nu)$ and $\mathfrak{L} \backslash (\mathbb{R} \times S_k^\nu)$ are mutually separated in $\mathbb{R} \times E$. Then by virtue of [13, Corollary 26.6] every component of $\mathfrak{L}$ must be a subset of one or another of these sets. Since for each $\sigma \in \{+, -\}$ the set $(\mathfrak{D}_k^\sigma)^\nu$ is the component of $\mathfrak{L}$ which intersect $\mathbb{R} \times S_k^\nu$, this component must be a subset of $\mathbb{R} \times S_k^\nu$, i.e. $(\mathfrak{D}_k^\sigma)^\nu \subset (\mathbb{R} \times S_k^\nu)$. Hence by the second assertion of Theorem 4.2 it follows that alternative 2^0 of Theorem 4.1 cannot hold. Then from Theorem 2.1 and 3.1 implies that $(\mathfrak{D}_k^\sigma)^\nu$ can only meet $\mathcal{R}$ if $\lambda = \lambda_k^\sigma$. In a similar way, by Theorems 4.1 and 4.2, $(\mathfrak{L}_k^\sigma)^\nu$ can only meet (λ, ∞) if $\lambda = \lambda_k^\sigma$. The proof of this theorem is complete.

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