Caspian Journal of Applied Mathematics, Ecology and Economics
V. 4, No 2, 2016, December
ISSN 1560-4055

Fredholm Property of the Boundary Problem for the Cauchy-Riemann Equation in the Curvelinear Strip

N.A. Aliyev*, M.B. Mursalova

Abstract. The paper deals with non-local boundary value problem for the homogeneous Cauchy-Riemann equation in the curvilinear strip. Considered problem reduces to the system of Fredholm integral equations of the second type with the regular kernels.
Key Words and Phrases: Cauchy-Riemann equation, Fredholm integral equation, mean relation, non-local boundary value problem.
2010 Mathematics Subject Classifications: Primary 34A12, 34B05, 35A08

1. Introduction

As it is known from the theory of ordinary differential equations in the boundary value problem the number of boundary conditions coincides with the order of the equation under consideration [1].

The boundary value problem for partial equations on the whole is considered for an elliptic type equations and the Laplace equation is a physical model of such equations [2]. Unlike boundary value problems for a linear ordinary differential equations, in the boundary value problem for partial equations the number of boundary conditions coincides with the half of the highest order derivative of the equation under consideration. We consider the Cauchy-Riemann equation which is equation of elliptic type of the first order and so local boundary conditions (Direchlet, Neymann, Puankare and etc.) aren't correct for this equation.

A lot of different non-local boundary problems for the Cauchy-Riemann equation in different boundary plane domains were investigated in [3] and Fredholm property of considered problems were proved in almost all of them.

Statement of the problem:

Let $D = \{x \backslash (x_1, x_2), x_1 \in R, x_2 = (0; \gamma_2(x_1))$ is convex with respect to x_2 curvelinear strip. Consider following non-local boundary value problem for the homogeneous Cauchy-Riemann equation

$$lu \equiv \frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} = 0, \quad x \in D \subset R^2, \tag{1}$$

*Corresponding author.

$$u\left(x_1,\gamma_2\left(x_1\right)\right)=\alpha\left(x_1\right)u\left(x_1,0\right)+\varphi\left(x_1\right),\qquad x_1\in R, \tag{2}$$

where $\alpha\left(x_1\right)$ is the complex-valued function, $\gamma_2\left(x_1\right)>0$ is the Lapunov type curveline, $\Gamma=\Gamma_1\cup\Gamma_2$ is the boundary of domain D:

$$\Gamma_1=\left\{x=\left(x_1,x_2\right):x_1\in R,\quad x_2=0\right\},$$

$$\Gamma_2=\left\{x=\left(x_1,x_2\right):x_2=\gamma_2\left(x_1\right)>0,\qquad x_1\in R\right\}.$$

As it is known [2], function

$$U\left(x-\xi\right)=\frac{1}{2\pi}\cdot\frac{1}{x_2-\xi_2+i\left(x_1-\xi_1\right)}, \tag{3}$$

is the fundamental solution of equation (1). Proceeding from (3) and (1) we obtain following main relation:

$$\int\limits_D luU\left(x-\xi\right)dx=\int\limits_D\frac{\partial u\left(x\right)}{\partial x_2}U\left(x-\xi\right)dx+i\int\frac{\partial u\left(x\right)}{\partial x_1}U\left(x-\xi\right)dx=0. \tag{4}$$

Applying the 2-nd Ostroqradsky-Gauss formula we get:

$$\int\limits_D\frac{\partial u\left(x\right)}{\partial x_2}U\left(x-\xi\right)dx+i\int\frac{\partial u\left(x\right)}{\partial x_1}U\left(x-\xi\right)dx=$$

$$=\int\limits_\Gamma u\left(x\right)U\left(x-\xi\right)\cos\left(\nu,x_2\right)dx-\int\limits_D u\left(x\right)\frac{\partial U\left(x-\xi\right)}{\partial x_2}dx+$$

$$+i\int\limits_\Gamma u\left(x\right)U\left(x-\xi\right)\cos\left(\nu,x_1\right)dx-i\int\limits_D u\left(x\right)\frac{\partial u\left(x\right)}{\partial x_1}dx=0,$$

where ν is an external normal to the boundary Γ.

So we get on Γ_1:

$$\int\limits_{\Gamma_1}u\left(x\right)U\left(x-\xi\right)\left[\cos\left(\nu,x_2\right)+i\cos\left(\nu,x_1\right)\right]dx=$$

$$=\frac{1}{2\pi}\int\limits_R\frac{u\left(x_1,0\right)}{-\xi_2+i\left(x_1-\xi_1\right)}\left[\cos\left(\nu,x_2\right)+i\cos\left(\nu,x_1\right)\right]dx_1=$$

$$=-\frac{1}{2\pi}\int\limits_R\frac{u\left(x_1,0\right)}{-\xi_2+i\left(x_1-\xi_1\right)}dx_1. \tag{5}$$

By analogy, we get on Γ_2

$$\frac{1}{2\pi}\int\limits_{\Gamma_2}\frac{u\left(x_1,\gamma_2\left(x_1\right)\right)}{x_2-\xi_2+i\left(x_1-\xi_1\right)}\left[\cos\left(\nu,x_2\right)+i\cos\left(\nu,x_1\right)\right]dx=$$

$$= \frac{1}{2\pi} \int\limits_{\Gamma_2} \frac{u\left(x_1, \gamma_2\left(x_1\right)\right)}{\gamma_2\left(x_1\right) - \xi_2 + i\left(x_1 - \xi_1\right)} \left[\cos\left(x_1, \tau\right) - i \sin\left(x_1, \tau\right)\right] dx =$$

$$= \frac{1}{2\pi} \int\limits_{R} \frac{u\left(x_1, \gamma_2\left(x_2\right)\right) \left[1 - i\gamma_2'\left(x_1\right)\right]}{\gamma_2\left(x_1\right) - \xi_2 + i\left(x_1 - \xi_1\right)} dx_1, \tag{6}$$

here $\tau \perp \nu$.

Finally, we get

$$\frac{1}{2\pi} \int\limits_{R} \frac{-u\left(x_1, 0\right)}{-\xi_2 + i\left(x_1 - \xi_1\right)} dx_1 + \frac{1}{2\pi} \int\limits_{R} \frac{u\left(x_1, \gamma_2\left(x_1\right)\right)}{\gamma_2\left(x_1\right) - \xi_2 + i\left(x_1 - \xi_1\right)} \left[1 - i\gamma_2'\left(x_1\right)\right] dx_1 =$$

$$= \begin{cases} u\left(\xi_1, \xi_2\right) & \text{if} \quad \xi \in D, \\ \frac{1}{2} u\left(\xi_1, \xi_2\right), & \text{if} \quad \xi \in \Gamma. \end{cases} \tag{7}$$

In the main relation (7) the first correlation is the solution, and the second is the necessary condition. This we established

Theorem 1. *Let $D \in R^2$ is convex with respect to x_2 curvelinear strip with the Lyapunov type boundary Γ_2 in the upper half-plane. Then each solution of equation (1) determined in D satisfies mean relation (7).*

From (7) we get following expression for the necessary conditions:

$$u\left(\xi_1, 0\right) = \frac{i}{\pi} \int\limits_{R} \frac{u\left(x_1, 0\right)}{x_1 - \xi_1} dx_1 +$$

$$+ \frac{1}{\pi} \int\limits_{R} \frac{u\left(x_1, \gamma_2\left(x_1\right)\right)}{\gamma_2\left(x_1\right) + i\left(x_1 - \xi_1\right)} \left[1 - i\gamma_2'\left(x_1\right)\right] dx_1 \tag{8}$$

$$u\left(\xi_1, \gamma_2\left(\xi_1\right)\right) = -\frac{i}{\pi} \int\limits_{R} \frac{u\left(x_1, \gamma_2\left(x_1\right)\right)}{x_1 - \xi_1} dx_1 +$$

$$+ \frac{i}{\pi} \int\limits_{R} \frac{u\left(x_1, \gamma_2\left(x_1\right)\right)}{\gamma_2'\left(\sigma_2\right) + i} \cdot \frac{\gamma_2'\left(\sigma_2\right) - \gamma_2'\left(x_1\right)}{x_1 - \xi_1} dx_1 -$$

$$- \frac{1}{\pi} \int\limits_{R} \frac{u\left(x_1, 0\right)}{-\gamma_2\left(\xi_1\right) + i\left(x_1 - \xi_1\right)} dx_1. \tag{9}$$

Proceeding from boundary condition (2) by means of necessary conditions (8), (9) we construct following linear combination:

$$u\left(\xi_1, \gamma_2\left(\xi_1\right)\right) + \alpha\left(\xi_1\right) u\left(\xi_1, 0\right) = -\frac{i}{\pi} \int\limits_{R} \frac{u\left(x_1, \gamma_2\left(x_1\right)\right)}{x_1 - \xi_1} dx_1 +$$

$$+\frac{i}{\pi}\int\limits_R \frac{u\left(x_1,\gamma_2\left(x_1\right)\right)}{\gamma_2'\left(\sigma_2\right)+i}\left(\frac{\gamma_2'\left(\sigma_2\right)-\gamma_2'\left(x_1\right)}{x_1-\xi_1}\right)dx_1+$$

$$+\frac{1}{\pi}\int\limits_R \frac{u\left(x_1,0\right)dx_1}{\gamma_2\left(\xi_1\right)-i\left(x_1-\xi_1\right)}+\alpha\left(\xi_1\right)\left[\frac{i}{\pi}\int\limits_R \frac{u\left(x_1,0\right)}{x_1-\xi_1}dx_1\right]+$$

$$+\frac{\alpha\left(\xi_1\right)}{\pi}\int\limits_R \frac{u\left(x_1,\gamma_2\left(x_1\right)\right)}{\gamma_2\left(x_1\right)+i\left(x_1-\xi_1\right)}\cdot\left[1-i\gamma_2'\left(x_1\right)\right]dx_1,$$

$$u\left(\xi_1,\gamma_2\left(\xi_1\right)\right)-\alpha\left(\xi_1\right)u\left(\xi_1,0\right)=\varphi\left(\xi_1\right). \tag{10}$$

From this system we get

$$u(\xi_1,\gamma_2(\xi_1))=\frac{1}{2\pi}\int\limits_R \frac{u\left(x_1,0\right)dx_1}{\gamma_2\left(\xi_1\right)-i(x_1-\xi_1)}+$$

$$+\frac{i}{2\pi}\int\limits_R \frac{u\left(x_1,\gamma_2\left(x_1\right)\right)dx_1}{\gamma_2'\left(\sigma_2\right)+i}\left(\frac{\gamma_2'\left(\sigma_2\right)-\gamma_2'\left(z_1\right)}{x_1-\xi_1}\right)dx_1+$$

$$+\frac{\alpha\left(\xi_1\right)}{2\pi}\int\limits_R \frac{u(x_1,\gamma_2\left(x_1\right))}{\gamma_2\left(x_1\right)+i\left(x_1-\xi_1\right)}\left[1-i\gamma_2'\left(x_1\right)\right]dx_1-$$

$$-\frac{i}{2\pi}\int\limits_R \left(u\left(x_1,\gamma\left(x_1\right)-\alpha\left(\xi_1\right)u\left(x_1,0\right)\right)\right)\frac{dx_1}{x_1-\xi_1}+\frac{\varphi\left(\xi_1\right)}{2}. \tag{11}$$

Consider the last integral in (11):

$$-\frac{i}{2\pi}\int\limits_R \left[u\left(x_1,\gamma_2\left(x_1\right)\right)-\left\{\alpha\left(\xi_1\right)-\alpha\left(x_1\right)+\alpha\left(x_1\right)\right\}u\left(x_1,0\right)\right]\frac{dx_1}{x_1-\xi_1}=$$

$$=\frac{i}{2\pi}\int\limits_R \frac{\alpha\left(\xi_1\right)-\alpha\left(x_1\right)}{x_1-\xi_1}u\left(x_1,0\right)dx_1-$$

$$-\frac{i}{2\pi}\int\limits_R \left[u\left(x_1,\gamma_2\left(x_1\right)\right)-\alpha\left(x_1\right)u\left(x_1,0\right)\right]\cdot\frac{dx_1}{x_1-\xi_1}=$$

$$=\frac{i}{2\pi}\int\limits_R \frac{\alpha\left(\xi_1\right)-\alpha\left(x_1\right)}{x_1-\xi_1}u\left(x_1,0\right)dx_1-\frac{i}{2\pi}\int\limits_R \frac{\varphi\left(x_1\right)}{x_1-\xi_1}dx_1. \tag{12}$$

After term-by-term integrating we get for the last integral in (12):

$$-\frac{i}{2\pi}\int\limits_R \varphi\left(x_1\right)d\ln\left|x_1-\xi_1\right|=-\frac{i}{2\pi}\left[\varphi\left(x_1\right)\ln\left|x_1-\xi_1\right|\Big|_{-\infty}^{+\infty}-\right.$$

$$-\int_{-\infty}^{+\infty} \varphi'(x_1)\ln|x_1-\xi_1|\,dx_1\Bigg]. \tag{13}$$

So we obtain

$$u(\xi_1,\gamma_2(\xi_1)) = \frac{1}{2\pi}\int_R \frac{u(x_1,0)\,dx_1}{\gamma_2(\xi_1)-i(x_1-\xi_1)} +$$

$$+\frac{i}{2\pi}\int_R \frac{u(x_1,\gamma_2(x_1))}{\gamma_2'(\sigma_2)+i}\,\frac{\gamma_2'(\sigma_2)-\gamma_2'(x_1)}{x_1-\xi_1}dx_1+$$

$$+\frac{\alpha(\xi_1)}{2\pi}\int_R \frac{u(x_1,\gamma_2(x_1))}{\gamma_2(x_1)+i(x_1-\xi_1)}\left[1-i\gamma_2'(x_1)\right]dx_1+$$

$$+\frac{i}{2\pi}\left[\int_R \frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}u(x_1,0)\,dx_1-\varphi(x_1)\ln|x_1-\xi_1|\Big|_{-\infty}^{+\infty}+\right.$$

$$\left.+\int_R \varphi'(x_1)\ln|x_1-\xi_1|\,dx_1\right]+\frac{\varphi(\xi_1)}{2}. \tag{14}$$

From system (10) we get

$$u(\xi_1,0) = \frac{1}{2\alpha(\xi_1)}\left[\frac{1}{\pi}\int_R \frac{u(x_1,0)}{\gamma_2(\xi_1)-i(x_1-\xi_1)}dx_1 +\right.$$

$$+\frac{i}{\pi}\int_R \frac{u(x_1,\gamma_2(x_1))}{\gamma_2'(\sigma_2)+i}\cdot\frac{\gamma_2'(\sigma_2)-\gamma_2'(x_1)}{x_1-\xi_1}dx_1+$$

$$\left.+\frac{\alpha(\xi_1)}{\pi}\int_R \frac{u(x_1,\gamma_2(x_1))}{\gamma_2(x_1)+i(x_1-\xi_1)}\left(1-i\gamma_2'(x_1)\right)dx_1\right]-$$

$$-\frac{\varphi(\xi)}{2\alpha(\xi_1)}+\frac{i}{2\alpha(\xi_1)\pi}\left\{\int_R \frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}\cdot u(x_1,0)\,dx_1-\right.$$

$$\left.-\varphi(x_1)\ln|x_1-\xi_1|\Big|_{-\infty}^{+\infty}+\int_R \varphi'(x_1)\ln|x_1-\xi_1|\,dx_1\right\}. \tag{15}$$

Assuming(*):
-$\alpha(x_1)\neq 0$, belongs to the Holder days with the index $\mu\leq 1$
-function $\varphi(x_1)$ is continuously differentiable and $\varphi(+\infty)=\varphi(-\infty)=0$.

Then we obtain in (14) (15)

$$u(\xi_1,\gamma_2(\xi_1)) = \frac{1}{2\pi}\int_R \frac{u(x_1,0)\,dx_1}{\gamma_2(\xi_1)-i(x_1-\xi_1)}+$$

$$+\frac{i}{2\pi}\int_R \frac{u(x_1,\gamma_2(x_1))\,dx_1}{\gamma_2'(\sigma_2)+i}\cdot\frac{\gamma_2'(\sigma_2)-\gamma_2'(x_1)}{x_1-\xi_1}dx_1+$$

$$+\frac{\alpha(\xi_1)}{2\pi}\int_R \frac{u(x_1,\gamma_2(x_1))}{\gamma_2(x_1)+i(x_1-\xi_1)}\cdot\left[1-i\gamma_2'(x_1)\right]dx_1+$$

$$+\frac{i}{2\pi}\left[\int_R \frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}u(x_1,0)\,dx_1+\right.$$

$$\left.+\int_R \varphi'(x_1)\ln|x_1-\xi_1|\,dx_1\right]+\frac{\varphi(\xi_1)}{2}, \qquad (16)$$

$$u(\xi_1,0) = \frac{1}{2\alpha(\xi_1)}\left[\frac{1}{\pi}\int_R \frac{u(x_1,0)\,dx_1}{\gamma_2(\xi_1)-i(x_1-\xi_1)}+\right.$$

$$\left.+\frac{i}{\pi}\int_R \frac{u(x_1,\gamma_2(x_1))}{\gamma_2'(\sigma_2)+i}\cdot\frac{\gamma_2'(\sigma_2)-\gamma_2'(x_1)}{x_1-\xi_1}dx_1\right]+$$

$$+\frac{1}{2\pi}\int_R \frac{u(x_1,\gamma_2(x_1))}{\gamma_2(x_1)+i(x_1-\xi_1)}\cdot\left(1-i\gamma_2'(x_1)\right)dx_1-$$

$$-\frac{\varphi(\xi_1)}{2\alpha(\xi_1)}+\frac{i}{2\alpha(\xi_1)\pi}\int_R \frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}u(x_1,0)\,dx_1+$$

$$+\frac{i}{2\alpha(\xi_1)\pi}\int_R \varphi'(x_1)\ln|x_1-\xi_1|\,dx_1.$$

So following theorem holds

Theorem 2. *Under conditions of Theorem 1 and (*) the problem (1),(2) reduces to the system (16) of Fredholm integral equations of the second type with the regular kernels.*

References

[1] Tricomi F. *Differential equation*, III, Moscow, 1962,361 p.

[2] Vladimirov V.S. *Equation of the mathematical physics*, Nauka, Moscow, 1971.

[3] Aliyev N.A., Mursalova M.B. *Fredholm property of the boundary problem for the Cauchy-Riemann equation in the curvelinear strip*, International Workshop on Non-harmonic Analysis and Differential Operators, 25-27 may, 2016, Azerbaijan.

Nihan Aliyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az1141, Baku, Azerbaijan
E-mail: nihan@aliev.info

Metanet Mursalova
Baku State University, Baku, Azerbaijan
E-mail: metanet.mursalova@mail.ru

Received 05 September 2016
Accepted 12 November 2016