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Existence and Uniqueness Results for Non-linear Fractional Integro-differential Equation with Non-local Boundary Conditions

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Abstract. In this paper, we shall establish sufficient and necessary conditions for the existence and uniqueness of solutions for a first order boundary value problem including a fractional integro-differential equations. This will be accomplished by Banach and Kransnoselskii fixed- point theorems.

Key Words and Phrases: Boundary value problem, fractional differential equation, fixed point theorem

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1. Introduction

In recent years a considerable interest has been shown in the so-called fractional calculus, which allows us to consider integration and differentiation of any order, not necessarily integer [12]. In fact the fractional calculus can be considered an old and yet novel topic. Starting from some speculations of Leibniz and Euler, followed by the works of other eminent mathematicians including Laplace, Fourier, Abel, Liouville and Riemann, it has undergone a rapid development especially during the past two decades. (see [9, 10]). Some results for fractional differential inclusions can be found in the book by Plotnikov et al [13]. For most details, we refer to the books by Podlubny [14] and Kilbass [11].

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering [3]. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [2, 4, 5, 7, 11]).

On the other hand some generalization of fractional order differential equations have been done on time scales by authors [1] and also the well-posed of BVP and IVP for fractional order differential equations has been discussed with respect to the number of boundary conditions [8].

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One of the emerging branches of this study is the theory of fractional quasi-linear equations, i.e. quasi-linear equations where the integer derivative with respect to time is replaced by a derivative of fractional order.

In this paper we consider the fractional boundary value problem including a quasi-linear equation with integral term in right hand side

$$^{c}D^{\alpha}y(t) = f(t, y(s)) + \int_0^t g(t, s, y(s))ds, \qquad t \in J = [0, T], \tag{1}$$

$$ay(0) + by(T) = c, \tag{2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, and a, b, c are real constants. Some existence results were given for the problem (1)-(2) with $g = 0$ by Benchohra and et all in [3] and initial value problem with Riemann-Liouville fractional operator by Furati and et all in [15]. We shall establish sufficient and necessary conditions for the existence of solutions for that boundary value problem for fractional differential equation. This will be accomplished by Banach and Kransnoselskii fixed-point theorems.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper. At first, we use the notation of $\mathcal{C}$ as a Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|y\|_\infty := \sup\{|y(t)| : t \in J\}$. For measurable function $m : J \to \mathbb{R}$, define the norm

$$\|m\|_{L^p(J,\mathbb{R})} := \begin{cases} \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \\ \inf\limits_{\mu(\overline{J})=0} \{ \sup\limits_{t \in J - \overline{J}} |m(t)|\}, & p = \infty, \end{cases}$$

where $\mu(\overline{J})$ is the Lebesgue measure of $\overline{J}$. Let $L^p(J, \mathbb{R})$ be the Banach space of all Lebesgue measurable functions $m : J \to \mathbb{R}$ with $\|m\|_{L^p(J,\mathbb{R})} < \infty$.
We need some basic definitions and properties of fractional calculus theory which are used in this paper. For more details, see [10, 14].

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a Lebesgue-measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by (the Abel-integral operator)*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)ds, \tag{3}$$

provided that the integral exist.

Definition 2. *The fractional derivative (in the sense of Caputo) of order $0 < \alpha < 1$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as the left inverse of the fractional integral of f*

$$^{c}D^{\alpha}f(t) = I^{1-\alpha}\frac{d}{dt}f(t). \tag{4}$$

That is

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f'(s)ds, \tag{5}$$

provided that the right side exists.

For proving the existence and uniqueness solution of problem (1), we need some fixed point theorems.

Theorem 1. *(Kransnoselskii) Let $\mathcal{M}$ be a closed convex non-empty subset of a Banach space $\mathcal{S}$. Suppose that A and B map $\mathcal{M}$ to $\mathcal{S}$ and that*

1. *$Ax + By \in \mathcal{M}$ $(\forall x, y \in \mathcal{M}$)*
2. *A is a contraction mapping*
3. *B is a compact and continuous.*

Then there exists $z \in \mathcal{M}$ such that $Az + Bz = z$

Proof. See [6].

3. Existence Results

Before starting and proving main result, let introduce the following hypotheses.

H1 The function $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times J \times \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable respect to t in J.

H2 There exists a constant $\alpha_1 \in [0, \alpha)$ and real-valued function $m(t) \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}_+)$ such that

$$|f(t, u_1) - f(t, u_2)| \leq m(t)|u_1 - u_2|, \quad \text{for each } t \in J \text{ and all } u_1, u_2 \in \mathbb{R}.$$

H3 There exists a constant $\alpha_2 \in [0, \alpha)$ and real-valued integrable function $\varphi(s, t)L^{\frac{1}{\alpha_2}}(J \times J, \mathbb{R}_+)$ such that

$$|g(s, t, u_1) - g(s, t, u_2)| \leq \varphi(s, t)|u_1 - u_2|, \quad \text{for each } t \in J \text{ and all } u_1, u_2 \in \mathbb{R}.$$

H4 There exists a constant $\alpha_3 \in [0, \alpha)$ and real-valued integrable function $h(t) \in L^{\frac{1}{\alpha_3}}(J, \mathbb{R}_+)$ such that

$$|f(t, u)| \leq h(t), \text{for each } t \in J \text{ and all } u \in \mathbb{R}.$$

H5 There exists a constant $\alpha_4 \in [0,\alpha)$ and real-valued integrable function $k(s,t) \in L^{\frac{1}{\alpha_4}}(J \times J, \mathbb{R}_+)$ such that

$$|g(s,t,u)| \leq k(s,t), \text{for each } s,t \in J \text{ and all } u \in \mathbb{R}.$$

H6 There exists constant $\lambda \in [0, 1-\frac{1}{\alpha_1}]$ for some $1 < \lambda < \frac{1}{1-\alpha}$ and $N > 0$ such that

$$f(t,u) \leq N(1+u^{\lambda})\text{for each} t \in J \text{and} u \in \mathbb{R}$$

For convenience, let

$$M = ||m(t)||_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}_+)}, \Phi = ||\varphi(s,t)||_{L^{\frac{1}{\alpha_2}}(J\times J,\mathbb{R}_+)},$$
$$H = ||h(t)||_{L^{\frac{1}{\alpha_3}}(J,\mathbb{R}_+)}, K = ||k(s,t)||_{L^{\frac{1}{\alpha_4}}(J\times J,\mathbb{R}_+)}.$$

Our first result is based on the Banach fixed point theorem.

Theorem 2. *Assume that **H1-H5** hold. If*

$$\begin{aligned} & \Omega_{\alpha,\alpha_1,\alpha_2,T}(t) \\ \leq\ & \frac{1}{\Gamma(\alpha)}\left[\frac{M}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\left(t^{\alpha-\alpha_1}+\frac{|b|T^{\alpha-\alpha_1}}{|a+b|}\right)+\frac{\Phi}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\left(t^{\alpha-\alpha_2}+\frac{|b|T^{\alpha-\alpha_2}}{|a+b|}\right)\right] \\ \leq\ & \rho < 1, \end{aligned}$$

the system (1) *has unique solution.*

Proof. For each $t \in J$, we have

$$\begin{aligned} \int_0^t |(t-s)^{\alpha-1} f(s,y(s))|ds &\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_3}} ds\right)^{1-\alpha_3} \left(\int_0^t h(s)^{\frac{1}{\alpha_3}} ds\right)^{\alpha_3} \\ &\leq \left(\int_0^T (t-s)^{\frac{\alpha-1}{1-\alpha_3}} ds\right)^{1-\alpha_3} \left(\int_0^T h(s)^{\frac{1}{\alpha_3}} ds\right)^{\alpha_3} \\ &\leq \frac{T^{\alpha-\alpha_3}}{(\frac{\alpha-\alpha_3}{1-\alpha_3})^{1-\alpha_3}} H, \end{aligned}$$

also we have

$$\begin{aligned} \int_0^t |(t-s)^{\alpha-1} \int_0^s g(s,\tau,y(\tau))d\tau| ds &\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\right)^{1-\alpha_2} \left(\int_0^t \int_0^s \varphi(s,\tau)^{\frac{1}{\alpha_2}} d\tau ds\right)^{\alpha_2} \\ &\leq \left(\int_0^T (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\right)^{1-\alpha_2} \left(\int_0^T \int_0^s \varphi(s,\tau)^{\frac{1}{\alpha_2}} d\tau ds\right)^{\alpha_2} \\ &= \frac{T^{\alpha-\alpha_2}}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_3}} \Phi. \end{aligned}$$

Thus $|(t-s)^{\alpha-1}f(s,y(s))|$ and $|(t-s)^{\alpha-1}\int_0^s g(s,\tau,y(\tau))d\tau|$ are Lebesgue measurable with respect to $s\in[0,t]$ for all $t\in J$ and $y\in\mathcal{C}$, then $(t-s)^{\alpha-1}f(s,y(s))$ and $(t-s)^{\alpha-1}\int_0^s g(s,\tau,y(\tau))d\tau$ are Bochner integrable with respect to $s\in[0,t]$ for all $t\in J$. Hence, the fractional boundary value problem (1)-(2) is equivalent to the integral equation

$$\begin{aligned} y(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(f(s,y(s))+\int_0^s g(s,\tau,y(\tau))d\tau\right)ds \\ & -\frac{1}{a+b}\left[b\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left(f(s,y(s))+\int_0^s g(s,\tau,y(\tau))d\tau\right)ds-c\right]. \end{aligned} \tag{6}$$

Let

$$r\geq \frac{T^{\alpha-\alpha_3}H}{(\frac{\alpha-\alpha_3}{1-\alpha_3})^{1-\alpha_3}}+\frac{|b|}{|a+b|}\frac{T^{\alpha-\alpha_3}H}{(\frac{\alpha-\alpha_3}{1-\alpha_3})^{1-\alpha_3}}+\frac{T^{\alpha-\alpha_4}K}{(\frac{\alpha-\alpha_4}{1-\alpha_4})^{1-\alpha_4}}+\frac{|b|}{|a+b|}\frac{T^{\alpha-\alpha_4}K}{(\frac{\alpha-\alpha_4}{1-\alpha_4})^{1-\alpha_4}}+\frac{|c|}{|a+b|}.$$

We define the operator F on $B_r:=\{y\in\mathcal{C}:\|y\|_\infty\leq r\}$, as follows

$$\begin{aligned} F(y)(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(f(s,y(s))+\int_0^s g(s,\tau,y(\tau))d\tau\right)ds \\ & -\frac{b}{a+b}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left(f(s,y(s))+\int_0^s g(s,\tau,y(\tau))d\tau\right)ds \\ & +\frac{c}{a+b}. \end{aligned} \tag{7}$$

Proof will be continued in several steps
Step 1. $Fy\in B_r$ for every $y\in B_r$.
In fact for $y\in B_r$ and all $t\in J$, one can verify that F is continuous on J, i. e., $Fy\in C(J,\mathbb{R})$, and

$$\begin{aligned} & |(Fy)(t)|\leq \\ \leq\; & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,y(s))|ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\int_0^s|g(s,\tau,y(\tau))|d\tau ds \\ & +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T(T-s)^{\alpha-1}|f(s,y(s))|ds \\ & +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T(T-s)^{\alpha-1}\int_0^s|g(s,\tau,y(\tau))|d\tau ds+\frac{|c|}{|a+b|} \\ \leq\; & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\int_0^s k(s,\tau)d\tau ds \\ & +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T(T-s)^{\alpha-1}h(s)ds \end{aligned}$$

$$
\begin{aligned}
& +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}\int_0^s k(s,\tau)d\tau ds+\frac{|c|}{|a+b|}\\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_3}}ds\right)^{1-\alpha_3}\left(\int_0^t h(s)^{\frac{1}{\alpha_3}}ds\right)^{\alpha_3}\\
& +\frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_4}}ds\right)^{1-\alpha_4}\left(\int_0^t\int_0^s k(s,t)^{\frac{1}{\alpha_4}}d\tau ds\right)^{\alpha_4}\\
& +\frac{|b|}{\Gamma(\alpha)|a+b|}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_3}}ds\right)^{1-\alpha_3}\left(\int_0^t h(s)^{\frac{1}{\alpha_3}}ds\right)^{\alpha_3}\\
& +\frac{|b|}{\Gamma(\alpha)|a+b|}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_4}}ds\right)^{1-\alpha_4}\left(\int_0^t\int_0^s k(s,t)^{\frac{1}{\alpha_4}}d\tau ds\right)^{\alpha_4}+\frac{|c|}{|a+b|}\\
\leq\ & \frac{T^{\alpha-\alpha_3}H}{(\frac{\alpha-\alpha_3}{1-\alpha_3})^{1-\alpha_3}}+\frac{|b|}{|a+b|}\frac{T^{\alpha-\alpha_3}H}{(\frac{\alpha-\alpha_3}{1-\alpha_3})^{1-\alpha_3}}+\frac{T^{\alpha-\alpha_4}K}{(\frac{\alpha-\alpha_4}{1-\alpha_4})^{1-\alpha_4}}+\frac{|b|}{|a+b|}\frac{T^{\alpha-\alpha_4}K}{(\frac{\alpha-\alpha_4}{1-\alpha_4})^{1-\alpha_4}}+\frac{|c|}{|a+b|}\\
\leq\ & r,
\end{aligned}
$$

which implies that $\|Fy\|_\infty \leq r$. Thus we can conclude that for all $y\in B_r, Fy\in B_r$.
Step 2. F is a contraction mapping on B_r.
For $x,y\in B_r$ and any $t\in J$, using **H2** and **H3** and Hölder inequality, we get

$$
\begin{aligned}
& |(Fx)(t)-(Fy)(t)|\leq\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds\\
& +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s |g(s,\tau,x(\tau))-g(s,\tau,y(\tau))|d\tau ds\\
& +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds\\
& +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}\int_0^s |g(s,\tau,x(\tau))-g(s,\tau,y(\tau))|d\tau ds\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)|x(s)-y(s)|ds\\
& +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s \varphi(s,\tau)|x(s)-y(s)|d\tau ds\\
& +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}m(s)|x(s)-y(s)|ds\\
& +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}\int_0^s \varphi(s,\tau)|x(s)-y(s)|d\tau ds\\
\leq\ & \frac{\|x-y\|_\infty}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^t m(s)^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}\\
& +\frac{\|x-y\|_\infty}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-2}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t\int_0^s \varphi(s,\tau)^{\frac{1}{\alpha_1}}d\tau ds\right)^{\alpha_2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\|x-y\|_\infty |b|}{\Gamma(\alpha)|a+b|}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^t m(s)^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}\\
&+\frac{\|x-y\|_\infty |b|}{\Gamma(\alpha)|a+b|}\left(\int_0^T (T-s)^{\frac{\alpha-2}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t\int_0^s \varphi(s,\tau)^{\frac{1}{\alpha_2}}d\tau ds\right)^{\alpha_2}\\
\leq\ &\frac{\|x-y\|_\infty}{\Gamma(\alpha)}\frac{t^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\|m\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}_+)}\\
&+\frac{\|x-y\|_\infty}{\Gamma(\alpha)}\frac{t^{\alpha-\alpha_2}}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\|\varphi\|_{L^{\frac{1}{\alpha_2}}(J\times J,\mathbb{R}_+)}\\
&+\frac{|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)}\frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\|m\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}_+)}\\
&+\frac{|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2}}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\|\varphi\|_{L^{\frac{1}{\alpha_2}}(J\times J,\mathbb{R}_+)}\\
\leq\ &\frac{1}{\Gamma(\alpha)}\left[\frac{M}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\left(t^{\alpha-\alpha_1}+\frac{|b|T^{\alpha-\alpha_1}}{|a+b|}\right)+\frac{\Phi}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\left(t^{\alpha-\alpha_2}+\frac{|b|T^{\alpha-\alpha_2}}{|a+b|}\right)\right]\\
&\times\|x-y\|_\infty.
\end{aligned}
$$

So we obtain

$$\|Fx-Fy\|\leq \Omega_{\alpha,\alpha_1,\alpha_2,T}(t)\|x-y\|_\infty.$$

Thus F is a contraction mapping due to the condition 6.
By Banach contraction principle, we can deduce that F has an unique fixed point which is just the unique solution of the fractional BVP 1-2.

Theorem 3. *Assume that H1, H2, H3, H5 and H6 are hold. if*

$$\Omega'_{\alpha,\alpha_2,T}=\frac{1}{\Gamma(\alpha)}\left[\frac{\Phi}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\left(t^{\alpha-\alpha_2}+\frac{|b|T^{\alpha-\alpha_2}}{|a+b|}\right)\right]\leq\rho'<1.$$

Then fractional BVP 1-2 has at least one solution on J.

Proof. We shall use Krasnoselskii's fixed point theorem to prove that F defined by (7) has at least one fixed point. For this we can rewrite F as $(Fy)(t)=(Ay)(t)+(By)(t)$, such that

$$
\begin{aligned}
Ay(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s g(s,\tau,y(\tau))d\tau ds\\
&-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\int_0^s g(s,\tau,y(\tau))d\tau ds\right],
\end{aligned}
$$

and

$$By(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,y(s))}{(t-s)^{1-\alpha}}-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T \frac{f(s,y(s))}{(T-s)^{1-\alpha}}ds-c\right].$$

Let r be the same in theorem 2 and also B_r. As is seen in Theorem 2, the operator F is an into B_r. The proof will be presented in several steps.
Step1. A is a contraction mapping

$$\begin{aligned}
& |(Ax)(t)-(Ay)(t)| \leq \\
& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s |g(s,\tau,x(\tau))-g(s,\tau,y(\tau))|d\tau ds \\
& +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}\int_0^s |g(s,\tau,x(\tau))-g(s,\tau,y(\tau))|d\tau ds \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s \varphi(s,\tau)|x(s)-y(s)|d\tau ds \\
& +\frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}\int_0^s \varphi(s,\tau)|x(s)-y(s)|d\tau ds \\
\leq\ & \frac{\|x-y\|_\infty}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-2}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t\int_0^s \varphi(s,\tau)^{\frac{1}{\alpha_1}}d\tau ds\right)^{\alpha_2} \\
\leq\ & \frac{\|x-y\|_\infty}{\Gamma(\alpha)}\frac{t^{\alpha-\alpha_2}}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\|\varphi\|_{L^{\frac{1}{\alpha_2}}(J\times J,\mathbb{R}_+)} \\
& +\frac{|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)}\frac{T^{\alpha-\alpha_2}}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\|\varphi\|_{L^{\frac{1}{\alpha_2}}(J\times J,\mathbb{R}_+)} \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left[\frac{\Phi}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\left(t^{\alpha-\alpha_2}+\frac{|b|T^{\alpha-\alpha_2}}{|a+b|}\right)\right]\|x-y\|_\infty.
\end{aligned}$$

Step 2. B is continuous.
Let y_n be a sequence such that $y_n \to y$ in $\mathcal{C}$. Then for each $t\in J$, we have

$$\begin{aligned}
& |(By_n)(t)-(By)(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds \\
& +\frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)|y_n(s)-y(s)|ds \\
& +\frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}m(s)|y_n(s)-y(s)|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|y_n - y\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s)ds \\
&\quad + \frac{|b|\|y_n - y\|_\infty}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m(s)ds \\
&\leq \frac{\|y_n - y\|_\infty}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds\right)^{1-\alpha_1} \left(\int_0^t m(s)^{\frac{1}{\alpha_1}} ds\right)^{\alpha_1} \\
&\quad + \frac{|b|\|y_n - y\|_\infty}{|a+b|\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_1}} ds\right)^{1-\alpha_1} \left(\int_0^T m(s)^{\frac{1}{\alpha_1}} ds\right)^{\alpha_1} \\
&\leq \frac{\|y_n - y\|_\infty}{\Gamma(\alpha)} \frac{t^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}_+)} \\
&\quad + \frac{|b|\|y_n - y\|_\infty}{|a+b|\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}_+)} \\
&\leq \frac{MT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} (1 + \frac{|b|}{|a+b|}) \|y_n - y\|_\infty.
\end{aligned}
$$

Thus

$$
\|By_n - By\|_\infty \leq \frac{MT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left(1 + \frac{|b|}{|a+b|}\right) \|y_n - y\|_\infty \to 0 \text{ as } n \to \infty.
$$

Step 3. B maps bounded set into bounded set in $\mathcal{C}$
Indeed, it is enough to show that for any $\eta > 0$, there exists a $\ell > 0$ such that for each $y \in B_\eta$ where $B_\eta = \{y \in \mathcal{C} : \|y\|_\infty \leq \eta\}$, we have $\|By\|_\infty \leq \ell$

$$
\begin{aligned}
& |(By)(t)| \leq \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y(s))| ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s, y(s))| ds + \frac{|c|}{|a+b|} \\
&\leq \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (1 + |y(s)|^\lambda) ds \\
&\quad + \frac{|b|N}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} (1 + |y(s)|^\lambda) ds + \frac{|c|}{|a+b|} \\
&\leq \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y(s)|^\lambda ds + \frac{|c|}{|a+b|} \\
&\quad + \frac{|b|N}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds + \frac{|b|N}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |y(s)| ds \\
&\leq \frac{NT^\alpha}{\Gamma(\alpha+1)} + \frac{NT^\alpha|b|}{|a+b|\Gamma(\alpha+1)} + \frac{NT^\alpha(\eta)^\lambda}{\Gamma(\alpha+1)} + \frac{NT^\alpha|b|(\eta)^\lambda}{|a+b|\Gamma(\alpha+1)} + \frac{|c|}{|a+b|} := \ell.
\end{aligned}
$$

Step 4. B maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$. Let $0 \leq t_1 < t_2 \leq T, y \in B_\eta$

$$
\begin{aligned}
& |(By)(t_2) - (By)(t_1)| = \\
= & \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))ds\right| \\
\leq & \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]f(s,y(s))ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds\right| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]|f(s,y(s))|ds \\
& + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,y(s))|ds \\
\leq & \frac{N}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}](1+|y(s)|^\lambda)ds \\
& + \frac{N}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(1+|y(s)|^\lambda)ds \\
\leq & \frac{N(1+(\eta)^\lambda)}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds \\
& + \frac{N(1+(\eta)^\lambda)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
\leq & \frac{N(1+(\eta)^\lambda)}{\Gamma(\alpha)}[|t_1^\alpha - t_2^\alpha| + 2(t_2-t_1)^\alpha] \\
\leq & \frac{3N(1+(\eta)^\lambda)(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}
$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero, therefore B is equicontinuous. As a consequence of Step 2-4 together with Arzela-Ascoli theorem, we can conclude that B is continuous and completely continuous. As a consequence of Krasnoselskii's fixed point theorem we deduce that F has a fixed point which is a solution of the fractional BVP (1).

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