Caspian Journal of Applied Mathematics, Ecology and Economics
V. 4, No 2, 2016, December
ISSN 1560-4055

H-Monotone Operators with an Application to a System of Nonlinear Implicit Variational Inclusions

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Abstract. In this paper, we introduce and study a system of variational inclusions called system of nonlinear implicit variational inclusion problem involving A-monotone and H-monotone operators in semi-inner product spaces. We prove that the resolvent operator associated with A-monotone and H-monotone operators is Lipschitz continuous. Further, we prove the existence and uniqueness of solutions for this system of variational inclusions. Furthermore, we suggest an iterative algorithm for finding the approximate solution of this system and discuss the convergence criteria of the sequences generated by the iterative algorithm under some suitable conditions.

Key Words and Phrases: System of nonlinear implicit variational inclusion problem, A-monotone operator, H-monotone operator, Semi-inner product space, 2-uniformly smooth Banach space, Resolvent operator technique, Iterative algorithm, Convergence analysis.

2010 Mathematics Subject Classifications: 49J40, 47H10

1. Introduction

Variational inequality problems have a wide range of applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Variational inequality problems have been generalized and extended in different directions using the novel and innovative techniques. A useful generalization of variational inequality is a variational inclusion. There are a number of numerical methods, including projection methods, Wiener-Hopf equations, descent and decomposition for solving variational inequalities. For further details of the approximation solvability of variational inclusions, we refer to [1,4,6,9,11,13-15].

The projection method and its generalizations such as resolvent operators have been widely used to solve variational inequalities/inclusions and their generalizations, see for example [1,4-7,11,13,15,17]. It is known that the monotonicity of the underlying operator plays a prominent role in solving different classes of variational inequality problems. In 2003, Fang and Huang [5] introduced and studied a new class of variational inclusions

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involving H-monotone operators in a Hilbert space. They have obtained a new algorithm for solving the associated class of variational inclusions using resolvent operator technique.

A considerable research in approximation solvability and A-monotone operators and H-η-accretive operators has been carried out by Mohsen *et al.* [19], He *et al.* [10], Lan *et al.* [16], Verma [25,27]. Fang *et al.* [6] have considered a class of variational inclusions and discussed its solvability using H-η-accretive operators.

Motivated and inspired by the work going on in this direction, in this paper we give the existence and Lipschitz continuity of the resolvent operators. As an application, we consider a class of system of nonlinear implicit variational inclusions involving A-monotone and H-monotone operators in semi-inner product spaces. Furthermore, we prove the existence and uniqueness of solution of the system of nonlinear implicit variational inclusions. Moreover, using resolvent operator, we suggest an iterative algorithm for approximating the solution of this system and discuss the convergence analysis of the sequences generated by the iterative algorithm. The results presented in this paper generalize and improve many known results in the literature, see for example [7,9,14,23] and the related references cited therein.

2. Resolvent Operator and Formulation of Problem

We need the following definitions and results from the literature.

Definition 1 *[18]. Let X be a vector space over the field F of real or complex numbers. A functional $[.,.] : X \times X \to F$ is called a semi-inner product if it satisfies the following:*

(i) $[x + y, z] = [x, z] + [y, z], \quad \forall x, y, z \in X;$

(ii) $[\lambda x, y] = \lambda[x, y], \quad \forall \lambda \in F$ *and* $x, y \in X;$

(iii) $[x, x] > 0$, *for* $x \neq 0;$

(iv) $|[x, y]|^2 \leq [x, x][y, y].$

The pair $(X, [.,.])$ *is called a semi-inner product space.*

We observe that $||x|| = [x, x]^{\frac{1}{2}}$ is a norm on X. Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Giles [8] had proved that if the underlying space X is a uniformly convex smooth Banach space then it is possible to find a semi-inner product, uniquely. Also the unique semi-inner product has the following nice properties:

(i) $[x, y] = 0$ if and only if y is orthogonal to x, that is if and only if $||y|| \leq ||y + \lambda x||, \quad \forall$ scalars λ.

(ii) Generalized Riesz representation theorem: If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x, y], \quad \forall x \in X$.

(iii) The semi-inner product is continuous, that is for each $x, y \in X$, we have $\text{Re}[y, x + \lambda y] \to \text{Re}[y, x]$ as $\lambda \to 0$.

The sequence space $l^p,\ p > 1$ and the function space $L^p,\ p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces, uniquely.

Example 2 *The real sequence space l^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[x, y] = \frac{1}{||y||_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}, \quad x, y \in l^p.$$

Example 3 *[8]. The real Banach space $L^p(X, \mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[f, g] = \frac{1}{||g||_p^{p-2}} \int_X f(x)|g(x)|^{p-1} sgn(g(x)) d\mu, \quad f, g \in L^p.$$

Definition 4 *[29]. Let X be a real Banach space. Then:*

(i) The modulus of smoothness of X is defined as

$$\rho_X(t) = \sup\left\{ \frac{||x + y|| + ||x - y||}{2} - 1 : ||x|| = 1, ||y|| = t,\ t > 0 \right\}.$$

(ii) X is said to be uniformly smooth if $\lim\limits_{t \to 0} \frac{\rho_X(t)}{t} = 0$.

(iii) X is said to be p-uniformly smooth if there exists a positive real constant k such that $\rho_X(t) \leq kt^p,\ p > 1$. Clearly, X is 2-uniformly smooth if there exists a positive real constant k such that $\rho_X(t) \leq kt^2$.

Lemma 5 *[29]. Let $p > 1$ be a real number and X be a smooth Banach space. Then the following statements are equivalent:*

(i) X is 2-uniformly smooth.

(ii) There is a constant $k > 0$ such that for every $x, y \in X$, the following inequality holds

$$||x + y||^2 \leq ||x||^2 + 2\langle y, f_x \rangle + k||y||^2, \tag{1}$$

where $f_x \in J(x)$ and $J(x) = \{x^\star \in X^\star : \langle x, x^\star \rangle = ||x||^2 \text{ and } ||x^\star|| = ||x||\}$ is the normalized duality mapping.

Remark 6. *Every normed linear space is a semi-inner product space (see[18]). In fact by Hahn Banach theorem, for each $x \in X$, there exists atleast one functional $f_x \in X^\star$ such that $\langle x, f_x \rangle = ||x||^2$. Given any such mapping f from X into $X^\star$, we can verify that $[y, x] = \langle y, f_x \rangle$ defines a semi-inner product. Hence we can write the inequality (1) as*

$$||x + y||^2 \leq ||x||^2 + 2[y, x] + k||y||^2, \quad \forall x, y \in X. \tag{2}$$

The constant k is chosen with best possible minimum value. We call k, as the constant of smoothness of X.

Example 7. *The function space L^p is 2-uniformly smooth for $p \geq 2$ and it is p-uniformly smooth for $1 < p < 2$. If $2 \leq p < \infty$, then we have for all $x, y \in L^p$,*

$$||x + y||^2 \leq ||x||^2 + 2[y, x] + (p - 1)||y||^2.$$

Here the constant of smoothness is $p - 1$.

Definition 8. *Let X be a real 2-uniformly smooth Banach space. A mapping $T : X \to X$ is said to be:*

(i) r-strongly monotone if there exists a positive constant r such that

$$[Tx - Ty, x - y] \geq r||x - y||^2, \quad \forall x, y \in X.$$

(ii) m-relaxed monotone if there is a positive constant m such that

$$[Tx - Ty, x - y] \geq (-m)||x - y||^2, \quad \forall x, y \in X.$$

Let $M : X \to 2^X$ be a set-valued map. We denote both the mapping and its graph by M, that is $M = \{(x, y) : y \in M(x)\}$. The domain of M is defined by

$$\text{Dom}(M) = \{x \in X : \exists\, y \in X : (x, y) \in M\}.$$

The range of M is defined by

$$\text{Range}(M) = \{y \in X : \exists\, x \in X : (x, y) \in M\}.$$

The inverse M^{-1} of M is $\{(y, x) : (x, y) \in M\}$.
For any two set-valued mappings N and M, and any real number ρ, we define

$$N + M = \{(x, y + z) : (x, y) \in N, (x, z) \in M\},$$

$$\rho M = \{(x, \rho y) : (x, y) \in M\}.$$

For a map $A : X \to X$ and a set-valued map $M : X \to 2^X$, we define

$$A + M = \{(x, y + z) : Ax = y \quad \text{and} \quad (x, z) \in M\}.$$

Definition 9. *Let X be a real 2-uniformly smooth Banach space. The mapping $M : X \to 2^X$ is said to be*

(i) Monotone if

$$[x - y, u - v] \geq 0, \quad \forall (x, u), (y, v) \in M;$$

(ii) r-strongly monotone if there exists a positive constant $r > 0$ such that

$$[x - y, u - v] \geq r||u - v||^2, \quad \forall (x, u), (y, v) \in M;$$

(iii) m-relaxed monotone if there exists a positive constant m such that

$$[x-y,u-v] \geq (-m)||u-v||^2, \quad \forall (x,u),(y,v) \in M.$$

Definition 10. *Let X be a real 2-uniformly smooth Banach space. Let $A : X \to X$ be a single-valued mapping and $M : X \to 2^X$ be a set-valued mapping on X. The map M is said to be A-monotone if*

(i) M is m-relaxed monotone;

(ii) $(A+\rho M)(X) = X$, where $\rho > 0$ is a positive real number.

Definition 11. *The resolvent operator $J^M_{\rho,A} : X \to X$ is defined by $J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u) \ \ \forall u \in X$.*

Definition 12. *Let $H : X \to X$ be an r-strongly monotone operator. The map $M : X \to 2^X$ is said to be H-monotone if*

(i) M is monotone;

(ii) $(H+\rho M)(X) = X$, where ρ is a positive real number.

Definition 13. *The resolvent operator $J^M_{\rho,H} : X \to X$ is defined by $J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u) \ \ \forall u \in X$.*

Graph convergence plays a crucial role in variational problems, optimization problems and approximation theory. For details on graph convergence one may refer to Aubin and Frankowska [2], Rockafellar and Wets [21] and Verma [26].

Definition 14 *[28]. Let $A : X \to X$ be an r-strongly monotone and s-Lipschitz continuous operator. Let $\{M^n\}, M^n : X \to 2^X$ be a sequence of A-monotone set-valued mappings for $n = 0,1,2,\cdots$. The sequence $\{M^n\}$ is graph convergent to M, denoted by $M^n \xrightarrow{AG} M$, if for every $(x,y) \in$ graph (M), there exists a sequence $\{(x_n,y_n)\} \in$ graph(M^n) such that $x^n \to x$ and $y^n \to y$ as $n \to \infty$.*

Lemma 15 *[28]. Let $A : X \to X$ be s-Lipschitz continuous and r-strongly monotone operator. Let $\{M^n\}, \ M^n : X \to 2^X$ be a sequence of A-monotone set-valued mappings. Then the sequence $M^n \xrightarrow{AG} M$ if and only if $J^{M^n}_{\rho,A}(u) \to J^M_{\rho,A}(u) \ \ \forall u \in X$ and $\rho > 0$, where $J^M_{\rho,A} = (A+\rho M)^{-1}$.*

Definition 16. *Let $H : X \to X$ be an r-strongly monotone and s-Lipschitz continuous operator. Let $\{M^n\}, M^n : X \to 2^X$ be a sequence of H-monotone set-valued mappings for $n = 0,1,2,\cdots$. The sequence $\{M^n\}$ is graph convergent to M, denoted by $M^n \xrightarrow{HG} M$, if for every $(x,y) \in$ graph (M), there exists a sequence $\{(x_n,y_n)\} \in$ graph(M^n) such that $x^n \to x$ and $y^n \to y$ as $n \to \infty$.*

Lemma 17 *[22]. Let $H : X \to X$ be s-Lipschitz continuous and r-strongly monotone. Let $\{M^n\}, \ M^n : X \to 2^X$ be a sequence of H-monotone set-valued maps. Then the* *sequence* $M^n \overset{HG}{\longrightarrow} M$ *if and only if* $J^{M^n}_{\rho,H}(u) \to J^{M}_{\rho,H}(u) \quad \forall u \in X$ *and* $\rho > 0$, *where* $J^{M}_{\rho,H} = (H + \rho M)^{-1}$.

Lemma 18 *[17]. Let* $\{\xi_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *be sequences of non-negative real numbers that satisfy: there exists a positive integer* n_0 *such that for* $n \geq n_0$,

$$\xi_{n+1} \leq (1 - \lambda_n)\xi_n + \beta_n \lambda_n + \gamma_n,$$

where $\lambda_n \in [0, 1]$, $\quad \sum\limits_{n=0}^{\infty} \lambda_n = +\infty$, $\lim\limits_{n \to \infty} \beta_n = 0$ *and* $\sum\limits_{n=0}^{\infty} \gamma_n < \infty$. *Then* $\sum\limits_{n=0}^{\infty} \xi_n = 0$.

Definition 19. *The Hausdorff metric* $\mathcal{H}(\cdot, \cdot)$ *on* $CB(X)$, *is defined by*

$$\mathcal{H}(A, B) = \max\left\{\sup_{u \in A} \inf_{v \in B} d(u, v),\ \sup_{v \in B} \inf_{u \in A} d(u, v)\right\},\ A, B \in CB(X),$$

where $d(\cdot, \cdot)$ *is the induced metric on* X *and* $CB(X)$ *denotes the family of all nonempty closed and bounded subsets of* X.

Definition 20 *[3]. A set-valued mapping* $T : X \to CB(X)$ *is said to be* γ-$\mathcal{H}$-*Lipschitz continuous, if there exists a constant* $\gamma > 0$ *such that*

$$\mathcal{H}(T(x), T(y)) \leq \gamma \|x - y\|, \quad \forall x, y \in X.$$

Theorem 21 *(Nadler [20]). Let* $T : X \to CB(X)$ *be a set-valued mapping on* X *and* (X, d) *be a complete metric space. Then:*

(i) For any given $\xi > 0$ *and for any given* $u, v \in X$ *and* $x \in T(u)$, *there exists* $y \in T(v)$ *such that*

$$d(x, y) \ \leq \ (1 + \xi)\mathcal{H}(T(u), T(v));$$

(ii) If $T : X \to C(X)$, *then (i) holds for* $\xi = 0$, *(where* $C(X)$ *denotes the family of all nonempty compact subsets of* X*).*

The following lemmas play crucial role in for the proof of main results.

Lemma 22. *If* $H : X \to X$ *is* r-*strongly monotone and* $M : X \to 2^X$ *is* H-*monotone, then the resolvent operator* $J^{M}_{\rho,H}$ *is* $\frac{1}{r}$-*Lipschitz continuous.*

Proof. For any $x, y \in X$, we have

$$J^{M}_{\rho,H}(x) = (H + \rho M)^{-1}(x),$$

$$J^{M}_{\rho,H}(y) = (H + \rho M)^{-1}(y).$$

This implies that

$$\frac{1}{\rho}\Big(x - H(J^{M}_{\rho,H}(x))\Big) \in M(J^{M}_{\rho,H}(x)),$$

$$\frac{1}{\rho}\Big(y - H(J^M_{\rho,H}(y))\Big) \in M(J^M_{\rho,H}(y)).$$

Since H is r-strongly monotone and M is H-monotone, we have

$$\begin{aligned}||x-y||\ ||J^M_{\rho,H}(x) - J^M_{\rho,H}(y)|| &\geq [x-y, J^M_{\rho,H}(x) - J^M_{\rho,H}(y)]\\ &= [x-y-(HJ^M_{\rho,H}(x) - HJ^M_{\rho,H}(y)), J^M_{\rho,H}(x) - J^M_{\rho,H}(y)]\\ &\quad + [HJ^M_{\rho,H}(x) - HJ^M_{\rho,H}(y), J^M_{\rho,H}(x) - J^M_{\rho,H}(y)]\\ &\geq 0 + [HJ^M_{\rho,H}(x) - HJ^M_{\rho,H}(y), J^M_{\rho,H}(x) - J^M_{\rho,H}(y)]\\ &\geq r||J^M_{\rho,H}(x) - J^M_{\rho,H}(y)||^2.\end{aligned}$$

This implies that

$$||J^M_{\rho,H}(x) - J^M_{\rho,H}(y)|| \leq \frac{1}{r}||x-y||.$$

Lemma 23. *If $A : X \to X$ be r-strongly monotone and $M : X \to 2^X$ be A-monotone. Then the resolvent operator $J^M_{\rho,A} : X \to X$ is $\frac{1}{r-\rho m}$-Lipschitz continuous for $0 < \rho < \frac{r}{m}$, where r, ρ and m are positive constants.*

Proof. For any $x, y \in X$, we have

$$J^M_{\rho,A}(x) = (A + \rho M)^{-1}(x),$$

$$J^M_{\rho,A}(y) = (A + \rho M)^{-1}(y).$$

This implies that

$$\frac{1}{\rho}\Big(x - A(J^M_{\rho,A}(x))\Big) \in M(J^M_{\rho,A}(x)),$$

$$\frac{1}{\rho}\Big(y - A(J^M_{\rho,A}(y))\Big) \in M(J^M_{\rho,A}(y)).$$

M is A-monotone, implies, M is m-relaxed monotone. Hence we have

$$\begin{aligned}&\frac{1}{\rho}[(x - A(J^M_{\rho,A}(x))) - (y - A(J^M_{\rho,A}(y))), J^M_{\rho,A}(x) - J^M_{\rho,A}(y)]\\ &\qquad = \frac{1}{\rho}[x-y-(AJ^M_{\rho,A}(x) - AJ^M_{\rho,A}(y)), J^M_{\rho,A}(x) - J^M_{\rho,A}(y)]\\ &\qquad \geq (-m)||J^M_{\rho,A}(x) - J^M_{\rho,A}(y)||^2.\end{aligned}$$

Now we have

$$||x-y||\ ||J^M_{\rho,A}(x) - J^M_{\rho,A}(y)|| \geq [x-y, J^M_{\rho,A}(x) - J^M_{\rho,A}(y)]$$

$$= [x - y - (AJ^M_{\rho,A}(x) - AJ^M_{\rho,A}(y)), J^M_{\rho,A}(x) - J^M_{\rho,A}(y)]$$

$$+ [AJ^M_{\rho,A}(x) - AJ^M_{\rho,A}(y), J^M_{\rho,A}(x) - J^M_{\rho,A}(y)]$$

$$\geq -\rho m||J^M_{\rho,A}(x) - J^M_{\rho,A}(y)||^2 + r||J^M_{\rho,A}(x) - J^M_{\rho,A}(y)||^2$$

$$= (r - \rho m)||J^M_{\rho,A}(x) - J^M_{\rho,A}(y)||^2.$$

This implies that

$$||x - y|| \geq (r - \rho m)||J^M_{\rho,A}(x) - J^M_{\rho,A}(y)||,$$

or,

$$||J^M_{\rho,A}(x) - J^M_{\rho,A}(y)|| \leq \frac{1}{r - \rho m}||x - y||, \quad 0 < \rho < \frac{r}{m}.$$

Taking $A = I$, the identity operator, we immediately have the following corollary:

Corollary 24. *Let $M : X \to 2^X$ be m-relaxed monotone. Then the resolvent operator $J^M_{\rho,I} = (I + \rho M)^{-1} : X \to X$ is $\frac{1}{1-\rho m}$-Lipschitz continuous for $0 < \rho < \frac{1}{m}$, where ρ and m are positive constants and I is the identity mapping.*

For each $i = 1, 2, 3$, let X be a real 2-uniformly smooth Banach space and $M_i : X \to 2^X$ be a set-valued mapping. Let $N_i : X \times X \to X$ be any single-valued mapping. Let $g_i : X \to X$ be any mapping such that $\text{Range}(g_i) \cap \text{Dom}(M_i) \neq \emptyset$. Let $B_i, C_i : X \to C(X)$ be multi-valued mappings. We consider the following system of nonlinear implicit variational inclusion problem (in short, SNVIP):
Given $\theta_i \in X$, find $(x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3)$ where $x_i \in X, u_i \in B_i(x_i), v_i \in C_i(x_i)$ such that

$$\begin{aligned} &\theta_1 \in N_1(u_1, v_1) + M_1(g_1(x_1)), \\ &\theta_2 \in N_2(u_2, v_2) + M_2(g_2(x_2)), \qquad (3) \\ &\theta_3 \in N_3(u_3, v_3) + M_3(g_3(x_3)). \end{aligned}$$

Some Special Cases:
I. For $\theta_3 = 0, N_3(u_3, v_3) \equiv 0, M_3(g_3(x_3)) \equiv 0, \ \forall x_3 \in X$. Then above problem (3) reduces to the following problem:

$$\begin{aligned} &\theta_1 \in N_1(u_1, v_1) + M_1(g_1(x_1)), \\ &\theta_2 \in N_2(u_2, v_2) + M_2(g_2(x_2)). \qquad (4) \end{aligned}$$

Problem (4) is the set-valued generalization of the variational inclusion problem considered and studied by Fang *et al* [7].

II. For $\theta_1 = \theta_2 = \theta_3 = \theta, \ \forall \theta \in X, \ N_1(u_1, v_1) = N_2(u_2, v_2) = N_3(u_3, v_3) = S(u) - T(u), \ \forall u \in X$ and $M_1(g_1(x_1)) = M_2(g_2(x_2)) = M_3(g_3(x_3)) = M(g(u)), \ \forall u \in X$ where $M : X \to 2^X$ is a set-valued mapping, $S, T : X \to X$ are single-valued mappings and

$g : X \to X$ is any mapping such that $\text{Range}(g) \cap \text{Dom}(M)$ is nonempty. Then problem (3) reduces to the following problem:
For a given element $\theta \in X$, find an element $u \in X$ such that

$$\theta \in S(u) - T(u) + M(g(u)). \tag{5}$$

Problem (5) has been considered and studied by Sahu *et al.* [23].

We remark that for suitable choices of different mappings M_i, N_i, g_i, B_i, C_i and the underlying space X, we obtain different classes of known and new classes f variational inequalities/inclusions from SNVIP(3), see for example [5,9,14] and the related references cited therein.

3. Existence of Solution

We give the following theorem which guarantees the existence of solution of SNVIP (3).

Theorem 25. *Let X be a real 2-uniformly smooth Banach space. Suppose for each $i = 1, 2, 3$, $A_i : X \to X$ be r_i-strongly monotone and $M_i : X \to 2^X$ be A_i-monotone. Let $N_i : X \times X \to X$ and $g_i : X \to X$ be any mapping such that $Dom(M_i) \cap Range(g_i) \neq \emptyset$. Then $(x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3)$ is a solution of SNVIP (3) where $x_i \in X, u_i \in B_i(x_i), v_i \in C_i(x_i)$ if and only if it satisfies*

$$g_i(x_i) = J^{M_i}_{\rho_i, A_i}\Big(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i \theta_i\Big),$$

where ρ_i is a positive real constant.

Proof. Suppose that for each $i = 1, 2, 3$, (x_i, u_i, v_i) is a solution of SNVIP (3). Then we have

$$\theta_i \in N_i(u_i, v_i) + M_i(g_i(x_i)).$$

This implies that

$$\begin{aligned} &\rho_i\theta_i - \rho_i N_i(u_i, v_i) \in \rho_i M_i(g_i(x_i)), \\ &\Rightarrow A_i(g_i(x_i)) + \rho_i\theta_i - \rho_i N_i(u_i, v_i) \in A_i(g_i(x_i)) + \rho_i M_i(g_i(x_i)), \\ &\Rightarrow J^{M_i}_{\rho_i, A_i}\Big(A_i(g_i(x_i)) + \rho_i\theta_i - \rho_i N_i(u_i, v_i)\Big) = J^{M_i}_{\rho_i, A_i}(A_i + \rho_i M_i)(g_i(x_i)), \\ &\Rightarrow g_i(x_i) = J^{M_i}_{\rho_i, A_i}\Big(A_i(g_i(x_i)) + \rho_i\theta_i - \rho_i N_i(u_i, v_i)\Big). \end{aligned}$$

Conversely, assume that

$$\begin{aligned} &g_i(x_i) = J^{M_i}_{\rho_i, A_i}\Big(A_i(g_i(x_i)) + \rho_i\theta_i - \rho_i N_i(u_i, v_i)\Big), \\ &\Rightarrow A_i(g_i(x_i)) + \rho_i\theta_i - \rho_i N_i(u_i, v_i) \in (A_i + \rho_i M_i)(g_i(x_i)), \\ &\Rightarrow \rho_i\theta_i - \rho_i N_i(u_i, v_i) \in \rho_i M_i(g_i(x_i)), \end{aligned}$$

$\Rightarrow \theta_i \in N_i(u_i, v_i) + M_i(g_i(x_i))$.

Theorem 26. *Let X be a real 2-uniformly smooth Banach space. Suppose for each $i = 1, 2, 3$, $A_i : X \to X$ be r_i-strongly monotone map and $M_i : X \to 2^X$ be A_i-monotone set-valued map. Let $g_i : X \to X$ be a map such that $Dom(M_i) \cap Range(g_i) \neq \emptyset$ and g_i be β_i-Lipschitz continuous and q_i-strongly monotone. Suppose that $N_i : X \times X \to X$ is ξ_i-Lipschitz continuous in the first argument, γ_i-Lipschitz continuous in the second argument and δ_i-strongly monotone w.r.t $A_i(g_i)$ in the first argument and that $A_i(g_i)$ be σ_i-Lipschitz continuous. Let $B_i, C_i : X \to C(X)$ be such that B_i is $L_{B_i} - \mathcal{H}$-Lipschitz continuous, C_i is $L_{C_i} - \mathcal{H}$-Lipschitz continuous. In addition if $r_i - \rho_i m_i > 0$, $1 - 2q_i + k\beta_i^2 > 0$ and $0 < \sqrt{1 - 2q_i + k\beta_i^2} + \dfrac{1}{\mu_i(r_i - \rho_i m_i)} < 1$, where ρ_i is a positive real constant and k is the constant of smoothness of the Banach space X. Then SNVIP (3) has a solution.*

Proof. Define the mapping $F_i : X \to X$ by

$$F_i(x_i) = x_i - g_i(x_i) + J^{M_i}_{\rho_i, A_i}\Big(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i\theta_i\Big). \tag{6}$$

Then for any elements $x_i, x_i' \in X$, we have

$$\begin{aligned}
||F_i(x_i)-&F_i(x_i')|| \\
&= ||\{x_i - g_i(x_i) + J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i\theta_i)\} \\
&\quad -\{x_i' - g_i(x_i') + J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i')) - \rho_i N_i(u_i', v_i') + \rho_i\theta_i)\}|| \\
&= ||(x_i - x_i') - (g_i(x_i) - g_i(x_i')) + J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i\theta_i) \\
&\quad -J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i')) - \rho_i N_i(u_i', v_i') + \rho_i\theta_i)|| \\
&\leq ||(x_i - x_i') - (g_i(x_i) - g_i(x_i'))|| + ||J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i\theta_i) \\
&\quad -J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i')) - \rho_i N_i(u_i', v_i') + \rho_i\theta_i)||.
\end{aligned} \tag{7}$$

Since X is 2-uniformly smooth Banach space, we have

$$\begin{aligned}
||(x_i - x_i') - (g_i(x_i) - g_i(x_i'))||^2 & \\
&\leq ||x_i - x_i'||^2 - 2[g_i(x_i) - g_i(x_i'), x_i - x_i'] \\
&\quad +k||g_i(x_i) - g_i(x_i')||^2 \\
&\leq ||x_i - x_i'||^2 - 2q_i||x_i - x_i'||^2 + k\beta_i^2||x_i - x_i'||^2 \\
&= (1 - 2q_i + k\beta_i^2)||x_i - x_i'||^2
\end{aligned}$$

$$\Rightarrow ||(x_i - x_i') - (g_i(x_i) - g_i(x_i'))|| \leq \sqrt{1 - 2q_i + k\beta_i^2}\ ||x_i - x_i'||. \tag{8}$$

Since the resolvent operator $J^{M_i}_{\rho_i,A_i}$ is $\dfrac{1}{(r_i - \rho_i m_i)}$-Lipschitz continuous, we have

$$\begin{aligned}
&||J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i\theta_i) - J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i')) - \rho_i N_i(u_i', v_i') + \rho_i\theta_i)|| \\
&\leq \frac{1}{(r_i - \rho_i m_i)}||(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i\theta_i) \\
&\quad -(A_i(g_i(x_i')) - \rho_i N_i(u_i', v_i') + \rho_i\theta_i)|| \\
&\quad -\frac{1}{(r_i - \rho_i m_i)}||(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i)) + \rho_i(N_i(u_i', v_i) - N_i(u_i', v_i)) \\
&\quad -(A_i(g_i(x_i')) - \rho_i N_i(u_i', v_i'))|| \\
&= \frac{1}{(r_i - \rho_i m_i)}\Big\{||(A_i(g_i(x_i)) - A_i(g_i(x_i'))) - \rho_i(N_i(u_i, v_i) - N_i(u_i', v_i))|| \\
&\quad +\rho_i||N_i(u_i', v_i) - N_i(u_i', v_i')||\Big\}.
\end{aligned} \tag{9}$$

Again, since $A_i(g_i)$ is σ_i-Lipschitz continuous, $N_i(.,.)$ is δ_i-strongly monotone w.r.t $A_i(g_i)$ in the first argument and ξ_i-Lipschitz continuous in the first argument, we have

$$\begin{aligned}
&||(A_i(g_i(x_i)) - A_i(g_i(x_i'))) - \rho_i(N_i(u_i, v_i) - N_i(u_i', v_i))||^2 \\
&\leq ||(A_i(g_i(x_i)) - A_i(g_i(x_i')))||^2 \\
&\quad -2\rho_i[N_i(u_i, v_i) - N_i(u_i', v_i), A_i(g_i(x_i)) - A_i(g_i(x_i'))] \\
&\quad +k||\rho_i(N_i(u_i, v_i) - N_i(u_i', v_i))||^2 \\
&\leq \sigma_i^2||x_i - x_i'||^2 - 2\rho_i\delta_i||x_i - x_i'||^2 + k\rho_i^2\xi_i^2||u_i - u_i'||^2 \\
&\leq \sigma_i^2||x_i - x_i'||^2 - 2\rho_i\delta_i||x_i - x_i'||^2 + k\rho_i^2\xi_i^2(\mathcal{H}(B_i(x_i), B_i(x_i')))^2 \\
&\leq \sigma_i^2||x_i - x_i'||^2 - 2\rho_i\delta_i||x_i - x_i'||^2 + k\rho_i^2\xi_i^2 L_{B_i}^2||x_i - x_i'||^2 \\
&= (\sigma_i^2 - 2\rho_i\delta_i + k\rho_i^2\xi_i^2 L_{B_i}^2)||x_i - x_i'||^2
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow ||(A_i(g_i(x_i)) - A_i(g_i(x_i'))) - \rho_i(N_i(u_i, v_i) - N_i(u_i', v_i))|| \\
&\qquad\qquad \leq \sqrt{\sigma_i^2 - 2\rho_i\delta_i + k\rho_i^2\xi_i^2 L_{B_i}^2}\ ||x_i - x_i'|| \\
&\qquad\qquad \leq \frac{1}{\lambda_i}||x_i - x_i'||,
\end{aligned} \tag{10}$$

where $\lambda_i = \left(\sqrt{\sigma_i^2 - 2\rho_i\delta_i + k\rho_i^2\xi_i^2 L_{B_i}^2}\right)^{-1}$.

Now, since $N_i(.,.)$ is γ_i-Lipschitz continuous w.r.t second argument and C_i is L_{C_i}-$\mathcal{H}$-Lipschitz continuous, we have

$$\begin{aligned} ||N_i(u_i', v_i) - N_i(u_i', v_i')|| &\leq \gamma_i ||v_i - v_i'|| \\ &\leq \gamma_i \mathcal{H}(C_i(x_i), C_i(x_i')) \\ &\leq \gamma_i L_{C_i} ||x_i - x_i'||. \end{aligned} \tag{11}$$

Combining (9), (10) and (11), we get

$$\begin{aligned} ||J^{M_i}_{\rho_i,A_i}&(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i\theta_i) \\ &- J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i')) - \rho_i N_i(u_i', v_i') + \rho_i\theta_i)|| \\ &\leq \frac{1}{(r_i - \rho_i m_i)} \left(\frac{1}{\lambda_i} + \rho_i\gamma_i L_{C_i}\right) ||x_i - x_i'|| \\ &\leq \frac{1}{\mu_i(r_i - \rho_i m_i)} ||x_i - x_i'||, \end{aligned} \tag{12}$$

where $\mu_i = \left(\frac{1}{\lambda_i} + \rho_i\gamma_i L_{C_i}\right)^{-1}$.

Using (8) and (12) in (7), we get

$$\begin{aligned} ||F_i(x_i) - F_i(x_i')|| &\leq \sqrt{1 - 2q_i + k\beta_i^2}\ ||x_i - x_i'|| + \frac{1}{\mu_i(r_i - \rho_i m_i)} ||x_i - x_i'|| \\ &\leq \left\{\sqrt{1 - 2q_i + k\beta_i^2} + \frac{1}{\mu_i(r_i - \rho_i m_i)}\right\} ||x_i - x_i'||. \end{aligned}$$

Since $r_i - \rho_i m_i > 0,\ 1 - 2q_i + k\beta_i^2 > 0$ and $0 < \sqrt{1 - 2q_i + k\beta_i^2} + \frac{1}{\mu_i(r_i - \rho_i m_i)} < 1$. Hence the map $F_i : X \to X$ (defined by (6)) is a contraction and thus has a fixed point say $x_i \in X$. Hence we have $g_i(x_i) = J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i\theta_i)$. As a result of Theorem 25, SNVIP (3) has a solution.

When $X = L^p(R),\ 2 \leq p < \infty$, we have the following corollary:

Corollary 27. *Let for each $i = 1, 2, 3$, $A_i : L^p \to L^p$ be an r_i-strongly monotone map and $M_i : L^p \to 2^{L^p}$ be A_i-monotone set-valued map. Suppose that $g_i : L^p \to L^p$ is a map such that $Dom(M_i) \cap Range(g_i) \neq \emptyset$ and g_i is β_i-Lipschitz continuous and q_i-strongly monotone. Suppose that $N_i : L^p \times L^p \to L^p$ is ξ_i-Lipschitz continuous in the first argument, γ_i-Lipschitz continuous in the second argument and δ_i-strongly monotone w.r.t $A_i(g_i)$ in the first argument and that $A_i(g_i)$ be σ_i-Lipschitz continuous. Let $B_i, C_i : L^p \to C(L^p)$ be such that B_i is $L_{B_i} - \mathcal{H}$-Lipschitz continuous, C_i is $L_{C_i} - \mathcal{H}$-Lipschitz continuous. In addition if $r_i - \rho_i m_i > 0$, $1 - 2q_i + (p-1)\beta_i^2 > 0$ and $0 < \sqrt{1 - 2q_i + (p-1)\beta_i^2} + \frac{1}{\mu_i(r_i - \rho_i m_i)} < 1$, where ρ_i is a positive real constant and $(p-1)$ is the constant of smoothness of the function space L^p, then SNVIP (3) has a solution.*

When the set-valued map $M_i : X \to 2^X$ is H_i-monotone. Then we have the following corollary, the proof is similar to the proof of Theorem 26 but here we have to use Lemma 22 instead of Lemma 23.

Corollary 28. Let X be a real 2-uniformly smooth Banach space. Suppose for each $i = 1, 2, 3$, $H_i : X \to X$ be r_i-strongly monotone map and $M_i : X \to 2^X$ be H_i-monotone set-valued map. Let $g_i : X \to X$ be a map such that $\mathrm{Dom}(M_i) \cap \mathrm{Range}(g_i) \neq \emptyset$ and g_i be β_i-Lipschitz continuous and q_i-strongly monotone. Suppose that $N_i : X \times X \to X$ is ξ_i-Lipschitz continuous in the first argument, γ_i-Lipschitz continuous in the second argument and δ_i-strongly monotone w.r.t $H_i(g_i)$ in the first argument and that $H_i(g_i)$ be σ_i-Lipschitz continuous. Let $B_i, C_i : X \to C(X)$ be such that B_i is $L_{B_i} - \mathcal{H}$-Lipschitz continuous, C_i is $L_{C_i} - \mathcal{H}$-Lipschitz continuous. In addition if $1 - 2q_i + k\beta_i^2 > 0$ and $0 < \sqrt{1 - 2q_i + k\beta_i^2} + \dfrac{1}{\mu_i r_i} < 1$, where k is the constant of smoothness of the Banach space X, then SNVIP (3) has a solution.

4. Iterative Algorithm and Convergence Analysis

Based on Theorems 25 and 21, we give an iterative method for finding an approximate solution of SNVIP (3).

Iterative Algorithm 29. *For each $i = 1, 2, 3$, given (x_i^0, u_i^0, v_i^0), where $x_i^0 \in X, u_i^0 \in B_i(x_i^0)$ and $v_i^0 \in C_i(x_i^0)$ such that $B_i, C_i : X \to C(X)$, compute the sequences $\{x_i^n\}, \{u_i^n\}, \{v_i^n\}$ defined by the iterative schemes:*

$$x_i^{n+1} = (1 - \alpha^n)x_i^n + \alpha^n\{x_i^n - g_i(x_i^n) + J_{\rho_i, A_i}^{M_i^n}(A_i(g_i(x_i^n)) - \rho_i N_i(u_i^n, v_i^n) + \rho_i\theta_i)\}$$

$$u_i^n \in B_i(x_i^n) : ||u_i^{n+1} - u_i^n|| \leq \mathcal{H}(B_i(x_i^{n+1}), B_i(x_i^n))$$

$$v_i^n \in C_i(x_i^n) : ||v_i^{n+1} - v_i^n|| \leq \mathcal{H}(C_i(x_i^{n+1}), C_i(x_i^n))$$

where $M_i^n : X \to 2^X$ are A_i-monotone set-valued mappings for $n = 0, 1, 2, ...,$ $J_{\rho_i, A_i}^{M_i^n} = (A_i + \rho_i M_i^n)^{-1}$ and α^n be a sequence of real numbers such that $\alpha^n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha^n = +\infty$.

Now, we give the convergence analysis of the sequences generated by the Iterative Algorithm 29.

Theorem 30. *Let X be a real 2-uniformly smooth Banach space. Suppose for each $i = 1, 2, 3$, $A_i : X \to X$ be r_i-strongly monotone map and s_i-Lipschitz continuous. Let $M_i^n : X \to 2^X$ be a sequence of A_i-monotone set-valued maps such that $M_i^n \xrightarrow{AG} M_i$ as $n \to \infty$. Suppose that $g_i : X \to X$ is q_i-strongly monotone and β_i-Lipschitz continuous and $N_i : X \times X \to X$ is ξ_i-Lipschitz continuous in the first argument, γ_i-Lipschitz continuous in the second argument and δ_i-strongly monotone w.r.t $A_i(g_i)$ in the first argument and that $A_i(g_i)$ be σ_i-Lipschitz continuous. Let $B_i, C_i : X \to C(X)$ be such that B_i is $L_{B_i} - \mathcal{H}$-Lipschitz continuous, C_i is $L_{C_i} - \mathcal{H}$-Lipschitz continuous. In addition if $r_i - \rho_i m_i > 0$,*

$1-2q_i+k\beta_i^2>0$ *and* $0<\sqrt{1-2q_i+k\beta_i^2}+\frac{1}{\mu_i(r_i-\rho_i m_i)}<1$, *where* ρ_i *is a positive real constant and* k *is the constant of smoothness of the Banach space* X. *Then for each* $i=1,2,3$, *the sequences* $\{x_i^n\},\{u_i^n\},\{v_i^n\}$ *generated by Iterative Algorithm 29 converges strongly to* x_i,u_i,v_i, *where* $(x_1,x_2,x_3,u_1,u_2,u_3,v_1,v_2,v_3)$ *is a solution of SNVIP (3).*

Proof. Let x_i be a solution of SNVIP (3). Then by Iterative Algorithm 29, we have

$$\begin{aligned}
||x_i^{n+1}-x_i|| &= ||(1-\alpha^n)x_i^n+\alpha^n\{x_i^n-g_i(x_i^n)+J_{\rho_i,A_i}^{M_i^n}(A_i(g_i(x_i^n)) \\
&\quad -\rho_i N_i(u_i^n,v_i^n)+\rho_i\theta_i)\}-(1-\alpha^n)x_i-\alpha^n x_i|| \\
&= ||(1-\alpha^n)x_i^n+\alpha^n\{x_i^n-g_i(x_i^n)+J_{\rho_i,A_i}^{M_i^n}(A_i(g_i(x_i^n)) \\
&\quad -\rho_i N_i(u_i^n,v_i^n)+\rho_i\theta_i)\}-(1-\alpha^n)x_i-\alpha^n\{x_i-g_i(x_i) \\
&\quad +J_{\rho_i,A_i}^{M_i}(A_i(g_i(x_i))-\rho_i N_i(u_i,v_i)+\rho_i\theta_i)\}|| \\
&\le (1-\alpha^n)||x_i^n-x_i||+\alpha^n||x_i^n-x_i-(g_i(x_i^n)-g_i(x_i))|| \\
&\quad +\alpha^n||J_{\rho_i,A_i}^{M_i^n}(A_i(g_i(x_i^n))-\rho_i N_i(u_i^n,v_i^n)+\rho_i\theta_i) \\
&\quad -J_{\rho_i,A_i}^{M_i}(A_i(g_i(x_i))-\rho_i N_i(u_i,v_i)+\rho_i\theta_i)|| \\
&\le (1-\alpha^n)||x_i^n-x_i||+\alpha^n||x_i^n-x_i-(g_i(x_i^n)-g_i(x_i))|| \\
&\quad +\alpha^n||J_{\rho_i,A_i}^{M_i^n}(A_i(g_i(x_i^n))-\rho_i N_i(u_i^n,v_i^n)+\rho_i\theta_i) \\
&\quad -J_{\rho_i,A_i}^{M_i^n}(A_i(g_i(x_i))-\rho_i N_i(u_i,v_i)+\rho_i\theta_i)|| \\
&\quad +\alpha^n||J_{\rho_i,A_i}^{M_i^n}(A_i(g_i(x_i))-\rho_i N_i(u_i,v_i)+\rho_i\theta_i) \\
&\quad -J_{\rho_i,A_i}^{M_i}(A_i(g_i(x_i))-\rho_i N_i(u_i,v_i)+\rho_i\theta_i)||. \qquad (13)
\end{aligned}$$

Using Lemma 23, σ_i-Lipschitz continuity of $A_i(g_i)$, δ_i-Strongly monotonocity of $N_i(.,.)$ w.r.t $A_i(g_i)$ in the first argument, ξ_i-Lipschitz continuity of $N_i(.,.)$ in the first argument and γ_i-Lipschitz continuity of $N_i(.,.)$ in the second argument, we have

$$\begin{aligned}
&||J_{\rho_i,A_i}^{M_i^n}(A_i(g_i(x_i^n))-\rho_i N_i(u_i^n,v_i^n)+\rho_i\theta_i)-J_{\rho_i,A_i}^{M_i^n}(A_i(g_i(x_i))-\rho_i N_i(u_i,v_i)+\rho_i\theta_i)|| \\
&\le \frac{1}{r_i-\rho_i m_i}||(A_i(g_i(x_i^n))-\rho_i N_i(u_i^n,v_i^n)+\rho_i\theta_i)-(A_i(g_i(x_i))-\rho_i N_i(u_i,v_i)+\rho_i\theta_i)|| \\
&\le \frac{1}{\mu_i(r_i-\rho_i m_i)}||x_i^n-x_i||. \qquad (14)
\end{aligned}$$

Combining (13) and (14), we get

$$||x_i^{n+1}-x_i||\le(1-\alpha^n)||x_i^n-x_i||+\alpha^n||x_i^n-x_i-(g_i(x_i^n)-g_i(x_i))||$$

$$
\begin{aligned}
&+\frac{\alpha^n}{\mu_i(r_i-\rho_i m_i)}||x_i^n - x_i|| + \alpha^n||J^{M_i^n}_{\rho_i,A_i}(A_i(g_i(x_i)) - \rho_i N_i(u_i,v_i) + \rho_i\theta_i) \\
&-J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i)) - \rho_i N_i(u_i,v_i) + \rho_i\theta_i)|| \\
= \;& (1-\alpha^n)||x_i^n - x_i|| + \alpha^n||x_i^n - x_i - (g_i(x_i^n) - g_i(x_i))|| \\
&+\frac{\alpha^n}{\mu_i(r_i-\rho_i m_i)}||x_i^n - x_i|| + \alpha^n f_i^n,
\end{aligned} \tag{15}
$$

where

$$f_i^n = ||J^{M_i^n}_{\rho_i,A_i}(A_i(g_i(x_i)) - \rho_i N_i(u_i,v_i) + \rho_i\theta_i) - J^{M_i}_{\rho_i,A_i}(A_i(g_i(x_i)) - \rho_i N_i(u_i,v_i) + \rho_i\theta_i)||,$$

and $f_i^n \to 0$ as $n \to \infty$.

Again, since g_i is q_i-strongly monotone and β_i-Lipschitz continuous, we have

$$||x_i^n - x_i - (g_i(x_i^n) - g_i(x_i))|| \leq \sqrt{1 - 2q_i + k\beta_i^2}\; ||x_i^n - x_i||. \tag{16}$$

Using (16) in (15), we get

$$
\begin{aligned}
||x_i^{n+1} - x_i|| &\leq (1-\alpha^n)||x_i^n - x_i|| + \alpha^n\sqrt{1 - 2q_i + k\beta_i^2}\; ||x_i^n - x_i|| \\
&\quad +\frac{\alpha^n}{\mu_i(r_i-\rho_i m_i)}||x_i^n - x_i|| + \alpha^n f_i^n \\
&= \left(1 - \alpha^n\left\{1 - \sqrt{1 - 2q_i + k\beta_i^2} - \frac{1}{\mu_i(r_i-\rho_i m_i)}\right\}\right)||x_i^n - x_i|| + \alpha^n f_i^n \\
&= (1 - \alpha^n(1-h_i))||x_i^n - x_i|| + \alpha^n f_i^n,)
\end{aligned} \tag{17}
$$

where $h_i := \sqrt{1 - 2q_i + k\beta_i^2} + \dfrac{1}{\mu_i(r_i-\rho_i m_i)}$ and $h_i < 1$ by assumption. Hence

$$||x_i^{n+1} - x_i|| \leq (1 - \alpha^n(1-h_i))||x_i^n - x_i|| + \alpha^n(1-h_i)\frac{f_i^n}{(1-h_i)}. \tag{18}$$

If $m_i^n = ||x_i^n - x_i||$, $n_i^n = \dfrac{f_i^n}{(1-h_i)}$ and $t_i^n = \alpha^n(1-h_i)$, then we have

$$m_i^{n+1} \leq (1-t_i^n)m_i^n + t_i^n n_i^n.$$

Using Lemma 18, we have $m_i^n \to 0$ as $n \to \infty$ and thus $x_i^n \to x_i$ as $n \to \infty$. Hence $\{x_i^n\}$ converges strongly to a solution of SNVIP (3).

Since B_i is $L_{B_i} - \mathcal{H}$-Lipschitz continuous, it follows from Iterative Algorithm 29 that

$$||u_i^n - u_i|| \leq \mathcal{H}(B_i(x_i^n), B_i(x_i)) \leq L_{B_i}||x_i^n - x_i||.$$

This implies $u_i^n \to u_i$ as $n \to \infty$.
Further, we claim that $u_i \in B_i(x_i)$

$$\begin{aligned} d(u_i, B_i(x_i)) &\leq ||u_i - u_i^n|| + d(u_i^n, B_i(x_i)) \\ &\leq ||u_i - u_i^n|| + \mathcal{H}(B_i(x_i^n), B_i(x_i)) \\ &\leq ||u_i - u_i^n|| + L_{B_i}||x_i^n - x_i|| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since, $B_i(x_i)$ is compact, we have $u_i \in B_i(x_i)$. Similarly, we can prove that $v_i \in C_i(x_i)$. Thus the approximate solution (x_i^n, u_i^n, v_i^n) generated by Iterative Algorithm 29 converges strongly to (x_i, u_i, v_i) a solution of SNVIP (3). This completes the proof.

Similar results can be obtained for H_i-monotone operators. For the sake of completeness, we state the following result for H_i-monotone operators.

Corollary 31. *Let X be a real 2-uniformly smooth Banach space. Suppose for each $i = 1, 2, 3$, $H_i : X \to X$ be r_i-strongly monotone map and s_i-Lipschitz continuous. Let $M_i^n : X \to 2^X$ is a sequence of H_i-monotone set-valued maps such that $M_i^n \overset{HG}{\to} M_i$ as $n \to \infty$. Suppose that $g_i : X \to X$ is q_i-strongly monotone and β_i-Lipschitz continuous and $N_i : X \times X \to X$ is ξ_i-Lipschitz continuous in the first argument, γ_i-Lipschitz continuous in the second argument and δ_i-strongly monotone w.r.t $H_i(g_i)$ in the first argument and that $H_i(g_i)$ be σ_i-Lipschitz continuous. Let $B_i, C_i : X \to C(X)$ be such that B_i is $L_{B_i} - \mathcal{H}$-Lipschitz continuous, C_i is $L_{C_i} - \mathcal{H}$-Lipschitz continuous. In addition if $1 - 2q_i + k\beta_i^2 > 0$ and $0 < \sqrt{1 - 2q_i + k\beta_i^2} + \dfrac{1}{\mu_i r_i} < 1$, where k is the constant of smoothness of the Banach space X. Then for each $i = 1, 2, 3$, the sequences $\{x_i^n\}, \{u_i^n\}, \{v_i^n\}$ generated by Iterative algorithm 29 converges strongly to x_i, u_i, v_i where $(x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3)$ is a solution of SNVIP (3).*

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Received 03 March 2016
Accepted 19 November 2016