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On Some New Fejér Type Inequalities

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Abstract. In this paper, we establish some new inequalities for differentiable mappings whose derivatives in absolute value are convex. These results are connected with Fejér's inequality holding for convex functions.

Key Words and Phrases: Fejér Inequality, Hermite-Hadamard inequality, convex functions.
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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers, $a, b \in I$ and $a < b$. The following double inequality is well known in the literature as Hadamard's inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave.

For some results which generalize, improve and extend Hermite-Hadamard inequality and trapezoidal inequality, see [3]-[8], [10]-[12].

Let real function f be defined on a nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [9], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1):

Theorem 1. *Let $f : I \to \mathbb{R}$ be convex on I and let $a, b \in I$ with $a < b$. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative and symmetric to $\frac{a+b}{2}$.

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If $g=1$, then we are talking about the Hermite-Hadamard inequalities. More about those inequalities can be found in a number of papers and monographs. For recent results and generalizations concerning Fejér inequality (2) see [2], [13]-[18].

In [8], Dragomir proved the following Lemma for Hadamard type inequalities:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. Then, one has the identity:*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}\int_a^b p(x)f'(x)dx, \tag{3}$$

where

$$p(x) = \begin{cases} x-a, & x \in \left[a, \frac{a+b}{2}\right) \\ x-b, & x \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

In [1], Alomari and Darus obtained inequalities for differentiable convex mappings and they used the following Lemma to prove them:

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° where $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)du = (a-b)\int_0^1 p(t)f'(ta + (1-t)b)dt, \tag{4}$$

for each $t \in [0,1]$, where

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{b-x}{b-a}\right] \\ t-1, & t \in \left(\frac{b-x}{b-a}, 1\right], \end{cases}$$

for all $x \in [a,b]$.

In [10], some inequalities of Hermite-Hadamard's type for differentiable convex mappings were presented as follows.

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left[\frac{|f'(a)| + |f'(b)|}{2}\right]. \tag{5}$$

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° where $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a,b]$, then we have:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \tag{6}$$

$$\leq \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left\{\left(\left|f'(a)\right|^{p/(p-1)}+3\left|f'(b)\right|^{p/(p-1)}\right)^{(p-1)/p}\right.$$
$$\left.+\left(3\left|f'(a)\right|^{p/(p-1)}+\left|f'(b)\right|^{p/(p-1)}\right)^{(p-1)/p}\right\}.$$

The aim of this paper is to establish new inequalities of weighted version of Hermite-Hadamard type inequality for functions whose derivatives absolute values are convex. The results presented here would provide extensions of those given in earlier works.

2. Main results

We will establish some new results connected with the left-hand side of (2) by using the following Lemma.

Lemma 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$. If $f', g \in L[a,b]$, then the following identity holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt = \int_a^b p(t)f'(t)dt, \tag{7}$$

for each $t \in [a,b]$, where

$$p(t) = \begin{cases} \int_a^t g(s)ds, & t \in \left[a, \frac{a+b}{2}\right) \\ -\int_t^b g(s)ds, & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_a^b p(t)f'(t)dt \\ &= \int_a^{\frac{a+b}{2}}\left(\int_a^t g(s)ds\right)f'(t)dt + \int_{\frac{a+b}{2}}^b\left(-\int_t^b g(s)ds\right)f'(t)dt. \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} I &= \left(\int_a^t g(s)ds\right)f(t)\bigg|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} g(t)f(t)dt \\ &\quad + \left(-\int_t^b g(s)ds\right)f(t)\bigg|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b g(t)f(t)dt \\ &= \left(\int_a^{\frac{a+b}{2}} g(s)ds\right)f\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} g(t)f(t)dt \end{aligned}$$

$$+\left(\int_{\frac{a+b}{2}}^{b} g(s)ds\right) f\left(\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^{b} g(t)f(t)dt$$
$$= f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b g(t)f(t)dt,$$

which completes the proof.

Remark 1. *If we choose $g(s) = 1$ in Lemma 3, then (7) reduces to (3).*

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$. If $f', g \in L[a, b]$ and $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt\right| \\
\leq\ & \frac{(b-a)^2}{24}\left\{\|g\|_{[a,\frac{a+b}{2}],\infty}\left(2\left|f'(a)\right| + \left|f'(b)\right|\right) + \|g\|_{[\frac{a+b}{2},b],\infty}\left(\left|f'(a)\right| + 2\left|f'(b)\right|\right)\right\} \\
\leq\ & \frac{(b-a)^2}{4}\|g\|_{[a,b],\infty}\left(\frac{|f'(a)| + |f'(b)|}{2}\right).
\end{aligned} \tag{8}$$

Proof. Using Lemma 3 and the convexity of $|f'|$, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt\right| \qquad (9)\\
\leq\ & \int_a^{\frac{a+b}{2}}\left|\int_a^t g(s)ds\right|\left|f'(t)\right|dt + \int_{\frac{a+b}{2}}^b\left|\int_t^b g(s)ds\right|\left|f'(t)\right|dt \\
\leq\ & \|g\|_{[a,\frac{a+b}{2}],\infty}\int_a^{\frac{a+b}{2}}(t-a)\left|f'(t)\right|dt + \|g\|_{[\frac{a+b}{2},b],\infty}\int_{\frac{a+b}{2}}^b(b-t)\left|f'(t)\right|dt
\end{aligned}$$

$$\begin{aligned}
=\ & \|g\|_{[a,\frac{a+b}{2}],\infty}\int_a^{\frac{a+b}{2}}(t-a)\left|f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right|dt \\
& + \|g\|_{[\frac{a+b}{2},b],\infty}\int_{\frac{a+b}{2}}^b(b-t)\left|f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right|dt \\
\leq\ & \|g\|_{[a,\frac{a+b}{2}],\infty}\int_a^{\frac{a+b}{2}}(t-a)\left[\left(\frac{b-t}{b-a}\right)\left|f'(a)\right| + \left(\frac{t-a}{b-a}\right)\left|f'(b)\right|\right]dt \\
& + \|g\|_{[\frac{a+b}{2},b],\infty}\int_{\frac{a+b}{2}}^b(b-t)\left[\left(\frac{b-t}{b-a}\right)\left|f'(a)\right| + \left(\frac{t-a}{b-a}\right)\left|f'(b)\right|\right]dt.
\end{aligned}$$

Since

$$\int_a^{\frac{a+b}{2}} (t-a)\left(\frac{b-t}{b-a}\right) dt = \int_{\frac{a+b}{2}}^b (b-t)\left(\frac{t-a}{b-a}\right) dt = \frac{(b-a)^2}{12},$$

and

$$\int_a^{\frac{a+b}{2}} (t-a)\left(\frac{t-a}{b-a}\right) dt = \int_{\frac{a+b}{2}}^b (b-t)\left(\frac{b-t}{b-a}\right) dt = \frac{(b-a)^2}{24}.$$

Also

$$\|g\|_{\left[a,\frac{a+b}{2}\right],\infty} \leq \|g\|_{[a,b],\infty},$$

and

$$\|g\|_{\left[\frac{a+b}{2},b\right],\infty} \leq \|g\|_{[a,b],\infty}.$$

We obtain (8). This completes the proof.

Corollary 1. *Let $g : [a,b] \to \mathbb{R}_+$ be symmetric to $\frac{a+b}{2}$ in Theorem 4. Then we have the inequality*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\ &\leq \frac{(b-a)^2}{8} \|g\|_{[a,b],\infty} \left(\frac{|f'(a)| + |f'(b)|}{2}\right). \end{aligned} \tag{10}$$

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$. If $f', g \in L[a,b]$ and $|f'|^q$ is convex on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\ &\leq \frac{(b-a)^2}{4(p+1)^{1/p}} \left\{ \|g\|_{\left[a,\frac{a+b}{2}\right],\infty} \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} \right. \\ &\qquad \left. + \|g\|_{\left[\frac{a+b}{2},b\right],\infty} \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{11}$$

Proof. Using Lemma 3 and Hölder inequality, we obtain

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\ &\leq \int_a^{\frac{a+b}{2}} \left| \int_a^t g(s)ds \right| |f'(t)| \, dt + \int_{\frac{a+b}{2}}^b \left| \int_t^b g(s)ds \right| |f'(t)| \, dt \\ &\leq \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t g(s)ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q \, dt \right)^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&+\left(\int_{\frac{a+b}{2}}^{b}\left|\int_{t}^{b} g(s)ds\right|^{p} dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}\left|f'(t)\right|^{q} dt\right)^{\frac{1}{q}} \\
\leq\ & \|g\|_{[a,\frac{a+b}{2}],\infty}\left(\int_{a}^{\frac{a+b}{2}}|t-a|^{p} dt\right)^{\frac{1}{p}}\left(\int_{a}^{\frac{a+b}{2}}\left|f'(t)\right|^{q} dt\right)^{\frac{1}{q}} \\
&+\|g\|_{[\frac{a+b}{2},b],\infty}\left(\int_{\frac{a+b}{2}}^{b}|b-t|^{p} dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}\left|f'(t)\right|^{q} dt\right)^{\frac{1}{q}},
\end{aligned}
$$

for $\frac{1}{p}+\frac{1}{q}=1$. Since $|f'|^q$ is convex on $[a,b]$, we have

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt-\int_a^b g(t)f(t)dt\right| \\
\leq\ & \|g\|_{[a,\frac{a+b}{2}],\infty}\left[\frac{(b-a)^{p+1}}{(p+1)2^{p+1}}\right]^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}\left[\left(\frac{b-t}{b-a}\right)\left|f'(a)\right|^q+\left(\frac{t-a}{b-a}\right)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+\|g\|_{[\frac{a+b}{2},b],\infty}\left[\frac{(b-a)^{p+1}}{(p+1)2^{p+1}}\right]^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b\left[\left(\frac{b-t}{b-a}\right)\left|f'(a)\right|^q+\left(\frac{t-a}{b-a}\right)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left\{\|g\|_{[a,\frac{a+b}{2}],\infty}\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right. \\
&\left.+\|g\|_{[\frac{a+b}{2},b],\infty}\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

which is the inequality (11).

Corollary 2. *Let $g:[a,b]\to\mathbb{R}_+$ be symmetric to $\frac{a+b}{2}$ in Theorem 5. Then we have the inequality*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt-\int_a^b f(t)g(t)dt\right| \\
\leq\ & \frac{(b-a)^2}{8(p+1)^{1/p}}\|g\|_{[a,b],\infty}\left\{\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Now, we will give some new results connected with the inequality (2) by using the following Lemma:

Lemma 4. *Let $f:I\to\mathbb{R}$ be differentiable on I° and $a,b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$. If $f',g\in L[a,b]$, then the following identity holds:*

$$
\int_a^b f(u)g(u)du-f(x)\int_a^b g(u)du=(b-a)^2\int_0^1 k(t)f'(ta+(1-t)b)dt, \tag{12}
$$

for each $t\in[0,1]$ and $x,u\in[a,b]$, where

$$k(t)=\begin{cases}\int_0^t g(sa+(1-s)b)ds, & t\in\left[0,\frac{b-x}{b-a}\right)\\ -\int_t^1 g(sa+(1-s)b)ds, & t\in\left[\frac{b-x}{b-a},1\right].\end{cases}$$

Proof. It suffices to note that

$$\begin{aligned} J &= \int_0^1 k(t)f'(ta+(1-t)b)dt \\ &= \int_0^{\frac{b-x}{b-a}}\left(\int_0^t g(sa+(1-s)b)ds\right)f'(ta+(1-t)b)dt \\ &\quad +\int_{\frac{b-x}{b-a}}^1\left(-\int_t^1 g(sa+(1-s)b)ds\right)f'(ta+(1-t)b)dt \\ &= J_1+J_2. \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} J_1 &= \left(\int_0^t g(sa+(1-s)b)ds\right)\frac{f(ta+(1-t)b)}{a-b}\Bigg|_0^{\frac{b-x}{b-a}} \\ &\quad -\int_0^{\frac{b-x}{b-a}} g(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt \\ &= \frac{f(x)}{a-b}\left(\int_0^{\frac{b-x}{b-a}} g(sa+(1-s)b)ds\right) \\ &\quad -\int_0^{\frac{b-x}{b-a}} g(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt, \end{aligned}$$

and similarly

$$\begin{aligned} J_2 &= \frac{f(x)}{a-b}\left(\int_{\frac{b-x}{b-a}}^1 g(sa+(1-s)b)ds\right) \\ &\quad -\int_{\frac{b-x}{b-a}}^1 g(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt. \end{aligned}$$

Thus, we can write

$$\begin{aligned} J &= J_1+J_2 \\ &= \frac{f(x)}{a-b}\int_0^1 g(sa+(1-s)b)ds-\int_0^1 g(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt. \end{aligned}$$

Using the change of the variable $u=ta+(1-t)b$ for $t\in[0,1]$, and multiplying both sides by $(b-a)^2$, we have (12), which completes the proof.

Remark 2. *If we take $g(u) = 1$ in Lemma 4, then (12) reduces to (4).*

Theorem 6. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and $g : [a, b] \to [0, \infty)$ be differentiable mapping. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned} &\left| f(x)\int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \qquad (13) \\ \leq\ & \frac{1}{3(b-a)} \left\{ \|g\|_{[0,\frac{b-x}{b-a}],\infty} \left[(b-x)^3 \left|f'(a)\right| + \frac{(b-x)^2(b-3a+2x)}{2} \left|f'(b)\right| \right] \right. \\ & \left. + \|g\|_{[\frac{b-x}{b-a},1],\infty} \left[\frac{(x-a)^2(3b-a-2x)}{2} \left|f'(a)\right| + (x-a)^3 \left|f'(b)\right| \right] \right\}. \end{aligned}$$

Proof. Let $x \in [a, b]$. Using Lemma 4, we have

$$\begin{aligned} &\left| f(x)\int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \\ \leq\ & (b-a)^2 \left\{ \int_0^{\frac{b-x}{b-a}} \left| \int_0^t g(sa+(1-s)b)ds \right| \left| f'(ta+(1-t)b) \right| dt \right. \\ & \left. + \int_{\frac{b-x}{b-a}}^1 \left| \int_t^1 g(sa+(1-s)b)ds \right| \left| f'(ta+(1-t)b) \right| dt \right\} \\ \leq\ & (b-a)^2 \left\{ \|g\|_{[0,\frac{b-x}{b-a}],\infty} \int_0^{\frac{b-x}{b-a}} |t| \left| f'(ta+(1-t)b) \right| dt \right. \\ & \left. + \|g\|_{[\frac{b-x}{b-a},1],\infty} \int_{\frac{b-x}{b-a}}^1 |1-t| \left| f'(ta+(1-t)b) \right| dt \right\}. \end{aligned}$$

Since $|f'|$ is convex on $[a, b]$, we obtain

$$\begin{aligned} &\left| f(x)\int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \\ \leq\ & (b-a)^2 \left\{ \|g\|_{[0,\frac{b-x}{b-a}],\infty} \int_0^{\frac{b-x}{b-a}} t \left[t\left|f'(a)\right| + (1-t)\left|f'(b)\right| \right] dt \right. \\ & \left. + \|g\|_{[\frac{b-x}{b-a},1],\infty} \int_{\frac{b-x}{b-a}}^1 (1-t) \left[t\left|f'(a)\right| + (1-t)\left|f'(b)\right| \right] dt \right\} \\ =\ & \frac{1}{3(b-a)} \left\{ \|g\|_{[0,\frac{b-x}{b-a}],\infty} \left[(b-x)^3 \left|f'(a)\right| + \frac{(b-x)^2(b-3a+2x)}{2} \left|f'(b)\right| \right] \right. \\ & \left. + \|g\|_{[\frac{b-x}{b-a},1],\infty} \left[\frac{(x-a)^2(3b-a-2x)}{2} \left|f'(a)\right| + (x-a)^3 \left|f'(b)\right| \right] \right\}. \end{aligned}$$

This completes the proof.

Corollary 3. *In Theorem 6, if we choose* $x=\frac{a+b}{2}$*, then we have the following Fejér type inequality:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)\int_a^b g(u)du-\int_a^b g(u)f(u)du\right| \qquad (14)\\ \le\ & \frac{(b-a)^2}{24}\left\{\|g\|_{[0,\frac{1}{2}],\infty}\left[\left|f'(a)\right|+2\left|f'(b)\right|\right]+\|g\|_{[\frac{1}{2},1],\infty}\left[2\left|f'(a)\right|+\left|f'(b)\right|\right]\right\}\\ \le\ & \frac{(b-a)^2}{4}\|g\|_{[0,1],\infty}\left[\frac{|f'(a)|+|f'(b)|}{2}\right].\end{aligned}$$

Remark 3. *If we take* $g(u)=1$ *in Corollary 3, then the inequality (14) reduces to (5).*

Theorem 7. *Let* $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ *be differentiable mapping on* I° *and* $a,b\in I^\circ$ *with* $a<b$ *and let* $g:[a,b]\to[0,\infty)$ *be differentiable mapping. If* $|f'|^q$ *is convex on* $[a,b]$*,* $q>1$*, then the following inequality holds:*

$$\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right| \qquad (15)$$

$$\begin{aligned}\le\ & \frac{1}{(b-a)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left\{\|g\|_{[0,\frac{b-x}{b-a}],\infty}\left[\frac{(b-x)^{2q+1}|f'(a)|^q+(b-x)^{2q}(b-2a+x)|f'(b)|^q}{2}\right]^{\frac{1}{q}}\right.\\ &\left.+\|g\|_{[\frac{b-x}{b-a},1],\infty}\left[\frac{(x-a)^{2q}(2b-a-x)|f'(a)|^q+(x-a)^{2q+1}|f'(b)|^q}{2}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof. Using Lemma 4, Hölder's inequality and the convexity of $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, it follows that

$$\begin{aligned}&\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right|\\ \le\ &(b-a)^2\\ &\times\left\{\left(\int_0^{\frac{b-x}{b-a}}\left|\int_0^t g(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\frac{b-x}{b-a}}^1\left|\int_t^1 g(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}\\ \le\ &(b-a)^2\left\{\|g\|_{[0,\frac{b-x}{b-a}],\infty}\left(\int_0^{\frac{b-x}{b-a}}t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\\ &\left.+\|g\|_{[\frac{b-x}{b-a},1],\infty}\left(\int_{\frac{b-x}{b-a}}^1(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}\end{aligned}$$

$$= \frac{1}{(b-a)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}} \left\{ \|g\|_{[0,\frac{b-x}{b-a}],\infty} \left[\frac{(b-x)^{2q+1}|f'(a)|^q + (b-x)^{2q}(b-2a+x)|f'(b)|^q}{2} \right]^{\frac{1}{q}} \right.$$

$$\left. + \|g\|_{[\frac{b-x}{b-a},1],\infty} \left[\frac{(x-a)^{2q}(2b-a-x)|f'(a)|^q + (x-a)^{2q+1}|f'(b)|^q}{2} \right]^{\frac{1}{q}} \right\}$$

which is the inequality (15).

Using Theorem 7, we have the following Corollary which is connected with the left-hand side of Fejér inequality.

Corollary 4. *In Theorem 7, if we choose $x = \frac{a+b}{2}$, then we have the following inequality:*

$$\left| f\left(\frac{a+b}{2}\right) \int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \tag{16}$$

$$\leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}} \|g\|_{[0,1],\infty} \left\{ \left[\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right]^{\frac{1}{q}} + \left[\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right]^{\frac{1}{q}} \right\}.$$

Remark 4. *If we take $g(u) = 1$ in Corollary 4, then the inequality (16) reduces to (6).*

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