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On an Elementary Operator with Class $wA(s,t)$ Operator Entries

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Abstract. Given $\mathscr{H}$ a Hilbert space and $\mathscr{L}(\mathscr{H})$ the algebra of bounded linear operator in $\mathscr{H}$. For $s>0$ and $t>0$, a Hilbert space operator T belongs to class $wA(s,t)$, $T\in w\mathscr{A}(s,t)$, if $|\widetilde{T}(s,t)|^{\frac{2t}{s+t}} \geq |T|^{2t}$ and $|T|^{2s} \geq |\widetilde{T}^*|^{\frac{2s}{s+t}}$, where $\widetilde{T}(s,t) = |T|^s U |T|^t$ is the generalized Aluthge transformation of $T=U|T|$. Let $d_{AB}=\delta_{AB}$ or Δ_{AB}, where $\delta_{AB}\in\mathscr{L}(\mathscr{L}(\mathscr{H}))$ is the generalized derivation $\delta_{AB}(X)=AX-XB$ and $\Delta_{AB}\in\mathscr{L}(\mathscr{L}(\mathscr{H}))$ is the elementary operator $\Delta_{AB}(X)=AXB-X$. It is proved that if $A,B^*\in w\mathscr{A}(s,t)$ such that $s+t=1$, then, for all complex λ, $(d_{AB}-\lambda)^{-1}(0)\subseteq(d_{A^*B^*}-\bar{\lambda})^{-1}(0)$, the ascent of $\operatorname{asc}(d_{AB}-\lambda)\leq 1$. Furthermore, isolated points of $\sigma(d_{AB})$ are poles of the resolvent of d_{AB}. Also, it is proved that generalized Weyl's theorem holds for $f(d_{AB})$, generalized a-Weyl's theorem and property (gw) hold for $f(d^*_{AB})$ for every $f\in H(\sigma(d_{AB}))$ and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$, where $H(\sigma(d_{AB}))$ denotes the set of all analytic in a neighborhood of $\sigma(d_{AB})$.

Key Words and Phrases: Class $wA(s,t)$ operators, generalized derivation, elementary operators, SVEP, Weyl type theorems.

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1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded operators on a complex infinite dimensional Hilbert space $\mathscr{H}$. For $A,B\in\mathscr{L}(\mathscr{H})$, let $\delta_{AB}:\mathscr{L}(\mathscr{H})\to\mathscr{L}(\mathscr{H})$ and $\Delta_{AB}:\mathscr{L}(\mathscr{H})\to\mathscr{L}(\mathscr{H})$ denote the *generalized derivation* $\delta_{AB}=AX-XB$ and the *elementary operator* $\Delta_{AB}=AXB-X$. Let $d_{AB}=\delta_{AB}$ or Δ_{AB}. The following implications hold for a general bounded linear operator T on a normed linear space $\mathscr{X}$, in particular

$$\begin{aligned}(d_{AB})^{-1}(0)\perp\Re(d_{AB}) &\Rightarrow (d_{AB})^{-1}(0)\cap cl(\Re(d_{AB}))=\{0\}\\ &\Rightarrow (d_{AB})^{-1}(0)\cap\Re(d_{AB})=\{0\}\Leftrightarrow \operatorname{asc}(d_{AB})\leq 1,\end{aligned}$$

[11, Page 25]. Here $\operatorname{asc}(d_{AB})$ denotes the *ascent* of d_{AB}, $cl(\Re(d_{AB}))$ denote the closure of the range of d_{AB} and $(d_{AB})^{-1}(0)\perp\Re(d_{AB})$ denotes that the kernel of d_{AB} is orthogonal to the range of d_{AB} in the sense of G. Birkhoff. Recall that if $\mathscr{M},\mathscr{N}$ are linear subspaces

of a normed linear space $\mathscr{X}$, then $\mathscr{M} \perp \mathscr{N}$ in the sense of Birkhoff if $\|m\| \leq \|m+n\|$ for all $m \in \mathscr{M}$ and $n \in \mathscr{N}$. This concept of orthogonality is not symmetric, i.e., $\mathscr{M} \perp \mathscr{N}$ does not imply $\mathscr{N} \perp \mathscr{M}$, but the concept does agree with the usual concept of orthogonality in the case in which $\mathscr{X} = \mathscr{H}$. The range-kernel orthogonality of d_{AB} has been considered by a number of authors, see [5, 11, 14, 24, 36, 37]. A sufficient condition guaranteeing $(d_{AB})^{-1}(0) \perp \Re(d_{AB})$ is that $(d_{AB})^{-1}(0) \subseteq (d_{A^*B^*})^{-1}(0)$ [14]. The inclusion $(d_{AB})^{-1}(0) \subseteq (d_{A^*B^*})^{-1}(0)$, known in the literature as the Putnam-Fuglede commutativity theorem.
An operator $T \in \mathscr{L}(\mathscr{H})$ is p-hyponormal, $0 < p \leq 1$, if $|T|^{2p} \geq |T^*|^{2p}$ (a 1-hyponormal is hyponormal), and an invertible operator $T \in \mathscr{L}(\mathscr{H})$ is log-hyponormal if $\log|T|^2 \geq \log|T^*|^2$.

Definition 1.1. A pair (T, S) is said to have the Fuglede-Putnam property if $T^*X = XS^*$ whenever $TX = XS$ for every $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$.

Lemma 1.2. ([35]) Let $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$. Then the following assertions equivalent:

(a) The pair (T, S) satisfies Fuglede-Putnam theorem;

(b) if $TX = XS$ for some $X \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, then $\overline{\Re(X)}$ reduces T, $\ker(X)^{\perp}$ reduces S and $T|_{\overline{\Re(X)}}$ and $S|_{\ker(X)^{\perp}}$ are normal operators.

Lemma 1.3. ([23]) Let $T \in \mathscr{L}(\mathscr{H})$ and $S^* \in \mathscr{L}(\mathscr{H})$ be either log-hyponormal or p-hyponormal operators. Then the pair (T, S) has the Fuglede-Putnam property.

If $A, B^* \in \mathscr{L}(\mathscr{H})$ are hyponormal operators, then d_{AB} satisfies the asymmetric Putnam-Fuglede commutativity property $d_{AB}^{-1}(0) \subseteq d_{A^*B^*}^{-1}(0)$. Hence $d_{A^*B^*}^{-1}(0) \perp \Re(d_{AB})$ [15, Lemma 4] and $\operatorname{asc}(d_{AB}) \leq 1$ [16, Proposition 2.3], where $\operatorname{asc}(d_{AB})$ denotes the ascent of d_{AB}. The class of hyponormal operators is closed under translation and multiplication by scalars; hence, since $\delta_{AB} - \lambda = \delta_{(A-\lambda)B}$ and $\Delta_{AB} - \lambda = (1+\lambda)\Delta_{(\frac{1}{1+\lambda}A)B}$ for all $-1 \neq \lambda \in \mathbb{C}$ (=the set of complex numbers), $(\delta_{AB} - \lambda)^{-1}(0) \subseteq (\delta_{A^*B^*} - \bar{\lambda})^{-1}(0)$ for all $\lambda \in \mathbb{C}$ and $(\Delta_{AB} - \lambda)^{-1}(0) \subseteq (\Delta_{A^*B^*} - \bar{\lambda})^{-1}(0)$ for all $-1 \neq \lambda \in \mathbb{C}$. If we let L_A and R_A denote the operators of left multiplication and right multiplication by A, respectively, then for $\lambda = -1$, $(\Delta_{AB} - \lambda)^{-1}(0) = (L_A R_B)^{-1}(0) \subseteq (L_A^* R_B^*)^{-1}(0)$ for hyponormal A, B^*; hence $(d_{AB} - \lambda)^{-1}(0) \subseteq (d_{A^*B^*} - \lambda)^{-1}(0)$ for all $\lambda \in \mathbb{C}$. This paper considers the operator d_{AB} with entries A and B^* are $wA(s,t)$ operators. Since the class $wA(s,t)$ operators is not closed under translation by scalars, it is of interest to find out if $d_{AB} - \lambda$ has properties, in particular those related to kernel-range orthogonality, in common with the case in which the entries A and B^* are hyponormal. It is proved for $w\mathscr{A}(s,t)$ entries A and B^* that $(\delta_{AB} - \lambda)^{-1}(0) \subseteq (\delta_{A^*B^*} - \bar{\lambda})^{-1}(0)$, $\operatorname{asc}(d_{AB} - \lambda) \leq 1$ and $(\delta_{AB} - \lambda)^{-1}(0) \perp \Re(d_{AB} - \lambda)$ for all $\lambda \in \mathbb{C}$. Furthermore, if λ is isolated in the spectrum of d_{AB}, $\lambda \in \operatorname{iso}\sigma(d_{AB})$, then the quasinilpotent part $H_0(d_{AB} - \lambda)$ of $d_{AB} - \lambda$ coincides with $(d_{AB} - \lambda)^{-1}(0)$; consequently, λ is a simple pole of the resolvent of d_{AB}. Also, we prove that the operator $f(d_{AB})$, for $A, B^* \in w\mathscr{A}(s,t)$, satisfies generalized Weyl's theorem, generalized a-Weyl's theorem and property (gw) hold for $f(d^*_{AB})$ for every $f \in H(\sigma(d_{AB}))$ and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$, where $H(\sigma(d_{AB}))$ denotes the set of all analytic in a neighborhood of $\sigma(d_{AB})$.

2. Complementary results

Let $A \in \mathscr{L}(\mathscr{H})$ have the polar decomposition $A = U|A|$. If A belongs to class $wA(s,t)$ for $s,t > 0$, then A belongs to class $wA(r,r)$, where $r = \max\{s,t\}$. Hence, the first generalized Aluthge transform $\widetilde{A} = |A|^r U|A|^r$ of A is semi-hyponormal, and if $\widetilde{A}(r,r)$ has the polar decomposition $\widetilde{A}(r,r) = V|\widetilde{A}(r,r)|$, then the second generalized Aluthge $\widetilde{\widetilde{A}}(r,r) = |\widetilde{A}(r,r)|^r V|\widetilde{A}(r,r)|^r$ of A is hyponormal [33]. It is known that $A, \widetilde{A}(s,t)$ and $\widetilde{\widetilde{A}}(s,t)$ have the same point spectrum, the same approximate point spectrum and the same spectrum. Furthermore, $\widetilde{\widetilde{\widetilde{A}}}(s,t)$ has a normal part if and only if A has a normal part. Hyponormal operators are closed under translations by (λI); class $wA(s,t)$ operators are not closed under translation.

Definition 2.1. Let $s > 0$ and $t > 0$ and $T = U|T|$ be the polar decomposition of T.

(i) T belongs to class $A(s,t) \Leftrightarrow (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{t+s}} \geq |T^*|^{2t}$ [19].

(ii) T belongs to class $wA(s,t)$ (in symbol, $T \in w\mathscr{A}(s,t)$)

$$\Leftrightarrow (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{t+s}} \geq |T^*|^{2t} \text{ and } |T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}.$$
$$\Leftrightarrow |\widetilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t} \text{ and } |T|^{2s} \geq |\widetilde{T}^*_{s,t}|,$$

where $\widetilde{T}_{s,t} = |T|^s U|T|^t$ is the generalized Aluthge transformation [20].

Lemma 2.2. *Let $T \in \mathscr{L}(\mathscr{H})$. If T is an invertible class $wA(s,t)$, then so is T^{-1}.*

Proof. Let $r = \max\{s,t\}$. Then $T \in w\mathscr{A}(r,r)$ [20]. Since $|T^{-1}| = |T^*|^{-1}$, $|T^{-1^*}| = |T|^{-1}$ and $T \geq I \Longleftrightarrow T^{-1} \leq I$. Applying (ii) of Definition 2.1, we have

$$(|T^*|^r|T|^{2r}|T^*|^r)^{\frac{1}{2}} \geq |T^*|^{2r}$$
$$\Longleftrightarrow |T^*|^{-r}(|T^*|^r|T|^{2r}|T^*|^r)^{\frac{1}{2}}|T^*|^{-r} \geq I$$
$$\Longleftrightarrow \left(|T^*|^{-r}(|T^*|^r|T|^{2r}|T^*|^r)^{\frac{1}{2}}|T^*|^{-r}\right)^{-1} \leq I$$
$$\Longleftrightarrow |T^*|^r(|T^*|^r|T|^{2r}|T^*|^r)^{-\frac{1}{2}}|T^*|^r \leq I \Longleftrightarrow (|T^*|^{-r}|T|^{-2r}|T^*|^{-r})^{\frac{1}{2}} \leq |T^*|^{-2r}$$
$$\Longleftrightarrow (|T^{-1}|^r|T^{*^{-1}}|^{2r}|T^{-1}|^r)^{\frac{1}{2}} \leq |T^{-1}|^{2r}.$$

Similarly

$$|T|^{2r} \geq (|T|^r|T^*|^{2r}|T|^r)^{\frac{1}{2}} \Longleftrightarrow$$

$$(|T|^r|T^*|^{2r}|T|^r)^{-\frac{1}{4}}|T|^{2r}(|T|^{\frac{1}{2}}|T^*|^{2r}|T|^r)^{-\frac{1}{4}} \geq I$$
$$\Longleftrightarrow (|T|^r|T^*||T|^r)^{\frac{-1}{4}}|T|^{-2r}(|T|^r|T^*|^{2r}|T|^r)^{\frac{-1}{4}} \leq I \Longleftrightarrow |T^{*^{-1}}|^{2r} \leq (|T|^r|T^*||T|^r)^{-\frac{1}{2}}$$
$$\Longleftrightarrow |T^{*^{-1}}|^{2r} \leq (|T|^{-r}|T^*|^{-2r}|T|^{-r})^{\frac{1}{2}}$$
$$\Longleftrightarrow |T^{*^{-1}}|^{2r} \leq (|T^{*^{-1}}|^r|T^{-1}|^{2r}|T^{*^{-1}}|^r)^{\frac{1}{2}}.$$

That is, T^{-1} is $w\mathscr{A}(r,r)$ operator.

Lemma 2.3. ([12]) *If $[A,B]=[A^*,B]=0$ for some operators $A,B \in \mathscr{L}(\mathscr{H})$, then*

(i) $[|A|,B] = [A,|B|] = [|A^*|,B] = [A,|B^*|] = [|A|,|B|] = [|A^*|,|B|] = [|A|,|B^*|] = [|A^*|,|B^*|] = 0$;

(ii) $[||A^*|^{\frac{1}{2}}|A|^{\frac{1}{2}}|, ||B^*|^{\frac{1}{2}}|B|^{\frac{1}{2}}|] = [||A|^{\frac{1}{2}}|A^*|^{\frac{1}{2}}|, ||B|^{\frac{1}{2}}|B^*|^{\frac{1}{2}}|] = 0$.

Lemma 2.4. *If $A,B \in w\mathscr{A}(s,t)$ are such that $s+t=1$ and $[A,B]=[A^*,B]=0$, then $AB \in w\mathscr{A}(s,t)$.*

Proof. Let $r=\max\{s,t\}$. Then $T \in w\mathscr{A}(r,r)$ [20]. Since $|AB|=|A||B|=|B||A|$ (etc.) we have:

$$\begin{aligned}(|AB|^r|(AB)^*|^{2r}|AB|^r)^{\frac{1}{2}} &= \left(|A|^r|B|^r|B^*|^{2r}|A^*|^{2r}|B|^r|A|^r\right)^{\frac{1}{2}}\\ &= \left((|B|^r|B^*|^{2r}|B|^r)(|A|^r|A^*|^{2r}|A|^r)\right)^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}&= \left(||B^*|^r|B|^r|^2\,||A^*|^r|A|^r|^2\right)^{\frac{1}{2}}\\ &= \left(||A^*|^r|A|^r|^2 \,.\, ||B^*|^r|B|^r|^2\right)^{\frac{1}{2}}\\ &= ||A^*|^r|A|^r|\,||B^*|^r|B|^r|\\ &= ||B^*|^r|B|^r|^{\frac{1}{2}}\left(|A|^r|A^*|^{2r}|A|^r\right)^{\frac{1}{2}}||B^*|^r|B|^r|^{\frac{1}{2}}\\ &\leq ||B^*|^r|B|^r|^{\frac{1}{2}}\,|A|^{2r}\,||B^*|^r|B|^r|^{\frac{1}{2}}\\ &= |A|^r\left(|B|^r|B^*|^{2r}|B|^r\right)^{\frac{1}{2}}|A|^r\\ &\leq |A|^r|B|^{2r}|A|^r = |A|^{2r}|B|^{2r} = |AB|^{2r}\end{aligned}$$

and by a similar argument

$$\begin{aligned}(|(AB)^*|^r|AB|^{2r}|(AB)^*|^r)^{\frac{1}{2}} &= ||A|^r|A^*|^r|\,||B|^r|B^*|^r|\\ &\geq |A^*|^r\left(|B^*|^r|B|^{2r}|B^*|^r\right)^{\frac{1}{2}}|A^*|^r\\ &\geq |A^*|^{2r}|B^*|^{2r} = |(AB)^*|^{2r}.\end{aligned}$$

The proof of the lemma is achieved.

Lemma 2.5. *If $A, B^* \in w\mathscr{A}(s,t)$ are such that $s+t=1$, $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ and $B^{-1}(0) \subseteq {B^*}^{-1}(0)$, then $d_{AB}(X)=0$ implies $d_{A^*B^*}(X)=0$.*

Proof. We consider only the case $d_{AB}=\delta_{AB}$ (the proof of the other case is similar). Let $\overline{\Re(X)}$ and $X^{-1}(0)^{\perp}$ be the closure of $\Re(X)$ and the orthogonal complement of $X^{-1}(0)$, respectively. Let $A_1 = A|_{\overline{\Re(X)}}, B_1^* = B^*|_{X^{-1}(0)}$, and define the quasiaffinity $X_1 : X^{-1}(0)^{\perp} \to \overline{\Re(X)}$ by setting $X_1x = Xx$ for all $x \in X^{-1}(0)^{\perp}$. Then $\delta_{A_1B_1}(X_1)=0$, where A_1 and B_1^* are $w\mathscr{A}(s,t)$ operators. Since X_1 is a quasiaffinity, A_1 and B_1^* have the polar decompositions $A_1 = U_1|A_1|$ and $B_1^* = V_1|B_1^*|$. Since X_1 is a quasiaffinity, A_1 has dense range and B_1 is injective. The hypotheses $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ and $B^{-1}(0) \subseteq {B^*}^{-1}(0)$ imply that A_1 and B_1 are quasiaffinities. (Indeed, if $Ax=0$ for some non-trivial x, then $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ implies $A^*(x \oplus 0)=0$, which is a contradiction since $A_1^*x=0$ implies x=0.) Hence both $|A_1|$ and $|B_1^*|$ are quasiaffinities (and U_1 and V_1 are unitaries). Set $Y_1 = |A_1|^s X_1 |B_1^*|^t$; then Y_1 is a quasiaffinity. The first generalized Aluthge transforms $\widetilde{A}_1(s,t) = |A_1|^s U_1 |A_1|^t$ and $\widetilde{B}^*(s,t) = |B_1^*|^s U_1 |B_1^*|^t$ of A_1 and B_1^* are $\frac{\min\{s,t\}}{s+t}$-hyponormal ([20]) which satisfy $\delta_{\widetilde{A}_1^*(s,t)\widetilde{B}_1^*(s,t)}(Y_1)=0$. Let $\widetilde{A}_1(s,t)$ and $\widetilde{B}_1^*(s,t)$ have the polar decompositions $\widetilde{A}_1(s,t) = U_2|\widetilde{A}_1(s,t)|$ and $\widetilde{B}_1^*(s,t) = V_2|\widetilde{B}_1^*(s,t)|$, and let $C = \widetilde{\widetilde{A}}_1 = |\widetilde{A}_1(s,t)|^s U_2 |\widetilde{A}_1(s,t)|^t$ and $D^* = \widetilde{\widetilde{B}}_1^* = |\widetilde{B}_1^*(s,t)|^s U_2 |\widetilde{B}_1^*(s,t)|^t$ denote the second generalized Aluthge transforms of A_1 and B_1^*, respectively. Then C and D^* are hyponormal ([33]) which satisfy $\delta_{CD}(Y)=0$, where Y is the quasiaffinity defined by $Y = |\widetilde{A}_1(s,t)|^s Y_1 |\widetilde{B}_1^*(s,t)|^t$. Apparently, ${D^*}^{-1}$ exists as a closed densely defined (possibly unbounded) hyponormal operator (with non-empty resolvent). Then $\delta_{CD}(Y)=0$ implies $YD^{-1} \subset CY$ and Y has dense range, $\sigma(C) \subset \sigma(D^{-1})$ [28, Theorem 3.3]. Hence, since the resolvent set of D is not empty, there is a λ in the resolvent sets C and D^{-1} such that $C-\lambda$ and $D^{-1}-\lambda$ are bounded invertible and satisfy $(C-\lambda)^{-1}Y = Y(D^{-1}-\lambda)^{-1}$ [22, Lemma 1]. Applying the asymmetric Putnam-Fuglede theorem for bounded hyponormal operators it follows that ${(C-\lambda)^*}^{-1}Y = Y{(D^{-1}-\lambda)^*}^{-1}$, and hence that ${(C-\lambda)^*}^{-1}$ and ${(D^{-1}-\lambda)^*}^{-1}$ are unitarily equivalent normal operators. Consequently, C and D are normal operators, and this by [33, Lemma 2.7] implies that A_1 and B_1 are normal operators. From this we conclude that if $E, F^* \in w\mathscr{A}(s,t)$, $E^{-1}(0) \subseteq {E^*}^{-1}$ and ${F^*}^{-1} \subseteq F^{-1}(0)$, and E or F^* is pure (i.e., completely non-normal), then $\delta_{EF}(X)=0$ implies $X=0$.

Now decompose A and B^* into their normal and pure parts by $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$ and let X have the corresponding representation $X = [X_{ij}]_{i,j=1}^2$. Then

$$\delta_{AB}(X) = \begin{pmatrix} \delta_{A_1B_1}(X_{11} & \delta_{A_1B_2}(X_{12} \\ \delta_{A_2B_1}(X_{21} & \delta_{A_2B_2}(X_{22} \end{pmatrix} = 0.$$

In the view of the observation above, we have $X_{ij}=0$, except X_{11}. Recall from [34, Theorem 5] that if Δ_n and $\Delta_n^* \in \mathscr{L}(\mathscr{L}(\mathscr{H}))$ are the operators $\Delta_n(X) = \sum_i^n A_iXB_i$ and $\Delta_n^* = \sum_{i=1}^n A_i^*XB_i^*$, where $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are commuting families of normal operators in $\mathscr{L}(\mathscr{H})$, then $\Delta_n^{-(n-1)}(0) = \Delta_n^{*-(n-1)}(0)$. Choosing $n=2, B_1=I, A_2=-I$ and $B_2=B_1$, it follows that $A_1X - XB_1 = 0$ implies $A_1^*X - XB_1^* = 0$. Hence $\delta_{AB}(X)=0$ implies $\delta_{A^*B^*}(X)=0$.

Lemma 2.6. *Let $A, B \in \mathscr{L}(\mathscr{H})$. If $A, B^* \in w\mathscr{A}$ are such that $s+t=1$, $A^{-1}(0) \subseteq {A^*}^{-1}(0)$ and $B^{-1}(0) \subseteq {B^*}^{-1}(0)$, all combinations are allowed, then $(\delta_{AB}-\lambda)^{-1}(0) \subseteq (\delta_{A^*B^*}-\overline{\lambda})^{-1}(0)$ for all $\lambda \in \mathbb{C}$, where $\overline{\lambda}$ denote the complex conjugate of λ.*

Proof. We consider the cases $d_{AB}=\delta_{AB}$ and $d_{AB}=\Delta_{AB}$ separately.
Case I. $d_{AB}=\delta_{AB}$. Decompose A and B into their normal and pure (=completely non-normal) parts, with respect to some decompositions $\mathscr{H}=\mathscr{H}_0 \oplus (\mathscr{H} \ominus \mathscr{H}_0)$ and $\mathscr{H}=\mathscr{H}_1 \oplus (\mathscr{H} \ominus \mathscr{H}_1)$, by $A=A_n \oplus A_p$ and $B=B_n \oplus B_p$, let $X \in (\delta_{AB}-\lambda)^{-1}(0)$, $X: \mathscr{H}_1 \oplus (\mathscr{H} \ominus \mathscr{H}_1) \longrightarrow \mathscr{H}_0 \oplus (\mathscr{H} \ominus \mathscr{H}_0)$ have the corresponding matrix representation $X=[X_{ij}]_{i,j=1}^2$. Then

$$(\delta_{AB}-\lambda)^{-1}(0)=\begin{pmatrix} (\delta_{A_nB_n}-\lambda)X_{11} & (\delta_{A_nB_p}-\lambda)X_{12} \\ (\delta_{A_pB_n}-\lambda)X_{21} & (\delta_{A_pB_p}-\lambda)X_{22} \end{pmatrix}=0.$$

Since the operator $A_n-\lambda$ (resp., $B_n-\lambda$) is normal and the pure parts $B_p^* \in w\mathscr{A}(s,t)$ (resp., the pure operator $A_p \in w\mathscr{A}(s,t)$), it follows from application of the Putnam-Fuglede property to $(\delta_{A_nB_p}-\lambda)X_{12}=(\delta_{A_pB_n}-\lambda)X_{21}=0$ that $X_{12}=X_{21}=0$. Define the second generalized Aluthge transforms $\widetilde{\widetilde{A}}(s,t)$ and $\widetilde{\widetilde{B}}^*(s,t)$ as above. Then

$$(\delta_{A_pB_p}-\lambda)X_{22}=0 \Longleftrightarrow (\delta_{A_pT_p}-\lambda)Y=0,$$

where we have set $(\widetilde{\widetilde{B}}_p^*)^*=T_p$ and $Y=|\widetilde{A}_p(s,t)|^s|A_p|^sX_{22}|B_p^*|^t|\widetilde{B}_p^*(s,t)|^t$. The operators $\widetilde{\widetilde{A}}(s,t)$ and T_p^* being pure hyponormal operators, the Putnam-Fuglede theorem for hyponormal operators implies that $Y=0$. Recall that the eigenvalues of operators in $w\mathscr{A}(s,t)$ are normal ([38]) (i.e., the eigenspaces are reducing); in particular, the pure part of an operator in $w\mathscr{A}(s,t)$ is injective. Hence $|A_p|^s$, $|\widetilde{A}|^s$, $|B_p^*|^t$ and $|\widetilde{B}_p^*|^t$ are quasiaffinities, which implies that $X_{22}=0$ and $X=X_{11}\oplus 0$. Since $(\delta A_nB_n-\lambda)^{-1}(0) \subseteq (\delta_{A_n^*B_n^*}-\lambda)^{-1}(0)$ we have $(\delta_{AB}-\lambda)^{-1}(0) \subseteq (\delta_{A_n^*B_n^*}-\lambda)^{-1}(0)$.
Case II. $d_{AB}=\Delta_{AB}$. Here we divide the proof into the cases $\lambda=-1$ and $\lambda \neq -1$. If $\lambda=-1$, then $(\Delta_{AB}-\lambda)X=0$ if and only if $AXB=0$. If $A, B^* \in w\mathscr{A}(s,t)$, then $AXB=0$ if and only if $X=0$: trivially, $A^*XB^*=(\Delta_{A^*B^*}-\overline{\lambda})X=0$. If $A, B^* \in w\mathscr{A}(s,t)$, then $AXB=0 \Leftrightarrow XB=0 \Rightarrow XB^*=0 \Rightarrow A^*XB^*=0 \Rightarrow (\Delta_{A^*B^*}-\overline{\lambda})X=0$. Decompose A and B into their normal and pure parts and letting X have the matrix representation $X=[X_{ij}]_{i,j=1}^2$ as in the case I, it is seen that

$$0=A_nX_{11}B_n=X_{11}B_n.0 \Rightarrow A_n^*X_{11}B_n=0=A_n^*X_{11}.0 \Rightarrow A_n^*X_{11}B_n^*=0;$$

$$A_nX_{12}B_p=0 \Rightarrow A_n^*X_{12}B_p^*=0; A_pX_{12}B_n=0 \Rightarrow A_p^*X_{21}B_n^*=0$$

and since A_p is injective and B_p has dense range

$$A_pX_{22}B_p=0 \Leftrightarrow X_{22}=0.$$

Therefore, $(\Delta_{A^*B^*}-\overline{\lambda})X=0$. Now let $\lambda \neq -1$. Then $(\Delta_{AB}-\lambda)X=0 \Leftrightarrow \Delta_{(\frac{1}{1+\lambda}A)B}X=0$. Since $\frac{1}{1+\lambda}A \in pw-\mathscr{H}$ and since $\Delta_{(\frac{1}{1+\overline{\lambda}}A^*)B^*}X=0 \Leftrightarrow (\Delta_{A^*B^*-\bar{\lambda}})X=0$, it would suffice to prove that $\Delta_{AB}X = 0 \Rightarrow \Delta_{A^*B^*}X = 0$. If $A, B^* \in w\mathscr{A}(s,t)$ and $A^{-1}(0) \subseteq A^{*^{-1}}(0)$ and $B^{-1}(0) \subseteq B^{*^{-1}}(0)$, then the implication follows from an application of [33, Theorem 3.3] followed by an application of [15, Theorem 2].

The ascent of a Banach space operator $T \in \mathscr{L}(\mathscr{X})$, asc (T), is the least non-negative integer k such that $T^{-k}(0) = T^{-(k+1)}(0)$. The following corollary is a consequence of Lemma 2.6 and [16, Proposition 2.3].

Corollary 2.7. *Let $A, B \in \mathscr{L}(\mathscr{H})$. If $A, B^* \in w\mathscr{A}(s,t)$ are such that $s+t = 1$, $A^{-1}(0) \subseteq A^{*^{-1}}(0)$ and $B^{-1}(0) \subseteq B^{*^{-1}}(0)$, all combinations are allowed, then asc $(d_{AB} - \lambda) \leq 1$ for all $\lambda \in \mathbb{C}$.*

If $A, B \in w\mathscr{A}(s,t)$ are such that $[A,B] = [A^*, B] = 0$, B is invertible and $A^{-1}(0) \subseteq A^{*^{-1}}$, then $AB^{-1} \in w\mathscr{A}(s,t)$ and $(AB^{-1})^{-1}(0) \subseteq (B^{*^{-1}}A^*)^{-1}(0)$. To see this, we recall from Lemma 2.4 and Lemma 2.5 that $AB^{-1} \in w\mathscr{A}(s,t)$. Also, if $x \in (AB^{-1})^{-1}(0)$, then $B^{-1}x \in A^{-1}(0)$ implies $B^{-1}x \in A^{*^{-1}}(0)$, i.e., $A^*B^{-1}x = 0$ and $A^*x = 0$, which implies that $B^{*^{-1}}A^*x = 0$.

Corollary 2.8. *Let $\Psi \in \mathscr{L}(\mathscr{L}(\mathscr{H}))$ be the elementary operator $\Psi(X) = AXB - CXD$. Suppose that A and C, and B^* and D^* are doubly commuting $w\mathscr{A}(s,t)$ operators. If either*

(i) B and C are invertible, $A^{-1}(0) \subseteq A^{*^{-1}}$ and $D^{*^{-1}} \subseteq D^{-1}(0)$ or

(ii) C and D are invertible, $A^{-1}(0) \subseteq A^{*^{-1}}$ and $B^{*^{-1}} \subseteq B^{-1}(0)$,

then $\Psi(X) = 0$ implies $A^*XB^* - C^*XD^* = 0$.

Proof. Apply Lemma 2.5 to $\delta_{(C^{-1}A)(D^{-1}B)}(X) = 0$.

3. The operator d_{AB} and Weyl's theorem

Let $\mathscr{X}$ be a complex Banach space. A Banach space operator $T \in \mathscr{L}(\mathscr{X})$ has the single-valued extension property, or SVEP, at a point $\lambda \in \sigma(T)$ if for every open disc $\mathcal{D}$ centered at λ the only analytic function $f : \mathcal{D} \longrightarrow \mathscr{X}$ satisfying $(T - \mu)f(\mu) = 0$ is the function $f \equiv 0$; T has SVEP if it has SVEP at every $\lambda \in \sigma(T)$.
Let us denote by $\alpha(T)$ the dimension of the kernel and by $\beta(T)$ the codimension of the range. Recall that the operator $T \in \mathscr{L}(\mathscr{X})$ is said to be *upper semi-Fredholm*, $T \in SF_+(\mathscr{X})$, if the range of $T \in \mathscr{L}(\mathscr{X})$ is closed and $\alpha(T) < \infty$, while $T \in \mathscr{L}(\mathscr{X})$ is said to be *lower semi-Fredholm*, $T \in SF_-(\mathscr{X})$, if $\beta(T) < \infty$. An operator $T \in B(\mathscr{X})$ is said to be *semi-Fredholm* if $T \in SF_+(\mathscr{X}) \cup SF_-(\mathscr{X})$ and Fredholm if $T \in SF_+(\mathscr{X}) \cap SF_-(\mathscr{X})$. If T is semi-Fredholm then the *index* of T is defined by ind $(T) = \alpha(T) - \beta(T)$. A bounded linear operator T acting on a Banach space $\mathscr{X}$ is *Weyl* if it is Fredholm of index zero and Browder if T is Fredholm of finite ascent and descent. The *Weyl spectrum* $\sigma_w(T)$ and *Browder spectrum* $\sigma_b(T)$ of T are defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$ and $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}$. Let $E^0(T) = \{\lambda \in \text{iso}\,\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$ and let $\pi_0(T) := \sigma(T) \setminus \sigma_b(T)$ all *Riesz points* of T. According to Coburn [13], *Weyl's*

theorem holds for T if $\sigma(T) \setminus \sigma_w(T) = E^0(T)$, and that *Browder's theorem* holds for T if $\sigma(T)\setminus\sigma_w(T) = \pi^0(T)$. Let $SF_+^-(\mathscr{X}) = \{T \in SF_+ : \text{ind}\,(T) \leq 0\}$. The *upper semi Weyl spectrum* is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathscr{X})\}$. According to Rakočević [30], an operator $T \in B(\mathscr{X})$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T)\setminus\sigma_{SF_+^-}(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \text{iso}\,\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}$. It is known [30] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

In the following we prove that if $A, B \in pw - \mathscr{H}$, then d_{AB} has SVEP and satisfies the property that its quasinilpotent part $H_0(d_{AB} - \lambda)$,

$$H_0(d_{AB} - \lambda) = \{X \in \mathscr{L}(\mathscr{H}) : \lim_{n\to\infty} \|(d_{AB} - \lambda)^n X\| = 0\}$$

equals $(d_{AB} - \lambda)^{-1}(0)$ for all $\lambda \in \text{iso}\,\sigma(d_{AB})$. This implies that d_{AB} satisfies Weyl's theorem, d_{AB}^* satisfies a-Weyl's theorem, and that if Ψ is the operator of Corollary 2.8 with $0 \in \text{iso}\,\sigma(\Psi)$, then 0 is a pole of the resolvent of Ψ.

Bishop's property (β), implies (Dunford's) condition (C); also T satisfies property (β) if and only if $T^* \in \mathscr{L}(\mathscr{X}^*)$ satisfies (the decomposition) property (δ) [25, Theorem 2.5.5]. Since we have no more than a passing interest in properties (β), (δ) and condition (C), we refer the interested reader to pages 11, 22 and 32 of [25] for the definitions of these properties.

Let L_T and R_T , $T \in \mathscr{L}(\mathscr{X})$, denote the operators of left and right multiplication (respectively) by T.

Lemma 3.1. *If $A, B^* \in w\mathscr{A}(s,t)$ are such that $s + t = 1$, then d_{AB} has SVEP.*

Proof. Since $A, B^* \in w\mathscr{A}(s,t)$, A satisfies property (β) [32, Corollary 2.13] and B^* satisfies property (δ). Hence both L_A and R_B satisfy condition (C) [25, Corollary 3.6.11]. Apparently, L_A and R_B commute. By Theorem 3.6.3 and Note 3.6.19 on page 283 of [25], $L_A - R_B$ and $L_A R_B$ have SVEP, which implies that d_{AB} has SVEP.

Remark 3.2. Recall from [18] that $\sigma(\delta_{AB}) = \{\lambda \in \sigma(A) - \sigma(B) : \lambda = \alpha - \beta, \alpha \in \sigma(A) \text{ and } \beta \in \sigma(B)\}$ and $\sigma(\Delta_{AB}) = \{\lambda \in \sigma(A)\sigma(B) - 1 : \lambda = \alpha\beta - 1, \alpha \in \sigma(A), \beta \in \sigma(B)\}$. If $\lambda \in \text{iso}\,\sigma(d_{AB})$, then we have one of the following two cases:

(i) $\lambda \neq -1$ if $d_{AB} = \Delta_{AB}$. Then there exist finite sequence $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^m$ of isolated points in $\sigma(A)$ and $\sigma(B)$, respectively, such that $\lambda = \alpha_i - \beta_i$ if $\lambda \in \text{iso}\,\sigma(\delta_{AB})$ and $\lambda = \alpha_i\beta_i - 1$ if $\lambda \in \text{iso}\,\sigma(\Delta_{AB})$, for all $1 \leq i \leq m$.

(ii) $d_{AB} = \Delta_{AB}$ and $\lambda = -1$. Then either $0 \in \text{iso}\,\sigma(A)$ and $0 \in \text{iso}\,\sigma(B)$, or, $0 \in \text{iso}\,\sigma(A)$ and $0 \notin \sigma(B)$, or, $0 \in \text{iso}\,\sigma(B)$ and $0 \notin \sigma(A)$.

Remark 3.3. Let $T \in w\mathscr{A}(s,t)$ be such that $s + t = 1$. If an $\alpha \in \text{iso}\,\sigma(T)$, then $\mathscr{H} = (T-\alpha)^{-1}(0) \oplus (T-\alpha)\mathscr{H}, \sigma(T_{11}) = \sigma(T|_{(T-\alpha)^{-1}(0)}) = \{\alpha\}, T_{11} - \alpha = T_{11} - \alpha I|_{(T-\alpha)^{-1}(0)} = 0$ and $\sigma(T|_{(T-\alpha)\mathscr{H}}) = \sigma(T) \setminus \{\alpha\}$, see [32, 33].

Lemma 3.4. *If $A, B^* \in w\mathscr{A}(s,t)$ are such that $s+t=1$, then $H_0(d_{AB}-\lambda) = (d_{AB}-\lambda)^{-1}(0)$ for all $\lambda \in iso\,\sigma(d_{AB})$.*

Proof. We start by considering Case (i) above. Evidently, the non-zero points α_i (resp.,$\overline{\beta}$), $1 \le i \le m$, are normal eigenvalues of A (resp., B^*). Let $M_{1i} = (A-\alpha_i)^{-1}(0)$, $N_{1i} = (B-\mathscr{B}_i)^{-1}(0)$ $(= (B-\beta_i)^{*^{-1}})$, $M_1 = \bigoplus_{i=1}^m M_{1i}, N_1 = \bigoplus_{i=1}^m, M_2 = M_1^\perp$ and $N_2 = N_1^\perp$; let $A = A_1 \oplus A_2 \in \mathscr{L}(M_1 \oplus M_2)$ and $B = B_1 \oplus B_2 \in \mathscr{L}(N_1 \oplus N_2)$. Then $\sigma(A_2) = \sigma(A) \setminus \{\alpha_1, \cdots, \alpha_m\}$ and $\sigma(B_2) = \sigma(B) \setminus \{\beta_1, \cdots, \beta_m\}$ $\lambda \notin \sigma(d_{A_kB_t})$ for all $1 \le k, t \le 2$ other than $k=t=1$.

Let $X \in H_0(d_{AB}-\lambda)$, and let $X \in \mathscr{L}(N_1 \oplus N_2, M_1 \oplus M_2)$ have the representation $X = [X_{ij}]_{i,j=1}^2$. Then

$$(d_{AB}-\lambda)^n X = \begin{pmatrix} * & * \\ * & (d_{A_2B_2}-\lambda)^n X_{22} \end{pmatrix}$$

(for some, as yet, non specified entries $*$). Since $\lim\limits_{n\to\infty} \|(d_{AB}-\lambda)^n X\|^{\frac{1}{n}} = 0$ implies $\lim\limits_{n\to\infty} \|(d_{A_2B_2}-\lambda)^n X_{22}\|^{\frac{1}{n}} = 0$, and since $d_{A_2B_2}-\lambda$ is invertible, we have $X_{22}=0$, and then

$$(d_{AB}-\lambda)^n X = \begin{pmatrix} * & (d_{A_1B_2}-\lambda)^n X_{12} \\ (d_{A_2B_1}-\lambda)^n X_{21} & 0 \end{pmatrix}$$

(for some, as yet, non specified entry $*$). Again, $\lim\limits_{n\to\infty} \|(d_{AB}-\lambda)^n X\|^{\frac{1}{n}} = 0$ implies $\lim\limits_{n\to\infty} \|(d_{A_1B_2}-\lambda)^n X_{12}\|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \|(d_{A_2B_1}-\lambda)^n X_{21}\|^{\frac{1}{n}} = 0$, and since $d_{A_1B_2}-\lambda$ and $d_{A_2B_1}-\lambda$ are invertible, we have $X_{12} = 0 = X_{21}$. Hence, $(d_{AB}-\lambda)^n X = (d_{A_1B_1}-\lambda)^n X_{11}$. Let $X_{11} = [Y_{ij}]_{1\le i,j\le m} \in \mathscr{L}(\bigoplus_{i=1}^m N_{1i}, \bigoplus_{i=1}^m M_{1i})$. Then, for $1 \le i,j \le m$,

$$\begin{aligned}(\delta_{A_1B_1}-\lambda)^n(X_{11}) &= ((L_{A_1-\alpha_i} - R_{B_1-\beta_j}) + (\alpha_i - \beta_j - \lambda))^n [Y_{ij}]_{1\le i,j\le m} \\ &= \left(\sum_{k=0}^n \binom{n}{k} (L_{A_1-\alpha_i} - R_{B_1-\beta_j})^k (\alpha_i-\beta_j-\lambda)^{n-k}\right) [Y_{ij}]_{1\le i,j\le m}\end{aligned}$$

and

$$\begin{aligned}(\Delta_{A_1B_1}-\lambda)^n(X_{11}) &= (T + \alpha_i\beta_j - 1 - \lambda)^n [Y_{ij}]_{1\le i,j\le m} \\ &= \left(\sum_{k=0}^n \binom{n}{k} T^k (\alpha_i\beta_j - 1 - \lambda)^{n-k}\right) [Y_{ij}]_{1\le i,j\le m},\end{aligned}$$

where we have set $L_{A_1-\alpha_i}R_{B_1} + \alpha_i R_{B_1-\beta_j} = T$. Since $(A-\alpha_i)|_{M_{1i}} = 0 = (B_1-\beta_i)|_{N_{1i}}$, it follows that

$$(\delta_{A_1B_1}-\lambda)^n(X_{11}) = (\alpha_i - \beta_j - \lambda))^n [Y_{ij}]_{1\le i,j\le m}$$

and

$$(\Delta_{A_1B_1}-\lambda)^n(X_{11}) = (\alpha_i\beta_j - 1 - \lambda)^n [Y_{ij}]_{1\le i,j\le m}.$$

Recall, $\lim\limits_{n\to\infty}\|(d_{A_1B_1}-\lambda)^nX_{11}\|^{\frac{1}{n}}=0$; hence $\lim\limits_{n\to\infty}|\alpha_i-\beta_j-\lambda|\,\|Y_{ij}\|^{\frac{1}{n}}=0$ in the case in which $d=\delta$ and $\lim\limits_{n\to\infty}|\alpha_i\beta_j-1-\lambda|\,\|Y_{ij}\|^{\frac{1}{n}}=0$ in the case in which $d=\Delta$. Thus $Y_{ij}=0$ for all i,j such that $i\neq j$. This implies that $X=X_{11}=\bigoplus_{i=1}^m Y_{ij}\in(d_{AB}-\lambda)^{-1}(0)$. Hence $H_0(d_{AB}-\lambda)\subset(d_{AB}-\lambda)^{-1}(0)$. Since the reverse inclusion holds for every operator, we must have $H_0(d_{AB}-\lambda)=(d_{AB}-\lambda)^{-1}(0)$.

To complete the proof, we now consider Case (ii). If 0 is both in $\operatorname{iso}\sigma(A)$ and $\operatorname{iso}\sigma(B)$, then, upon letting $M_1=A^{-1}(0)$, $N_1={B^*}^{-1}(0)$, $M_2=\mathscr{H}\ominus M_1$ and $N_2=\mathscr{H}\ominus N_1$, it is seen that $A=\begin{pmatrix}0 & C_1\\ 0 & A_2\end{pmatrix}\in\mathscr{L}(M_1\oplus M_2)$ and $B=\begin{pmatrix}0 & 0\\ C_2 & B_2\end{pmatrix}\in\mathscr{L}(N_1\oplus N_2)$ for some operators C_1 and C_2. Here both A_2 and B_2 are invertible (which implies that $\Delta_{A_2B_2}-\lambda=L_{A_2}R_{B_2}$ is invertible). Letting $X=[X_{kl}]_{k,l=1}^2$ as above, it follows that $X_{22}=0$. Hence $L_AR_B(X)=0$ for every $X\in H_0(L_AR_B)=H_0(\Delta_{AB}-\lambda)$. Consequently, $H_0(\Delta_{AB}-\lambda)=(\Delta_{AB}-\lambda)^{-1}(0)$. The proof of the other two (remaining cases) is similar: we consider $0\in\operatorname{iso}\sigma(A)$ and $0\notin\sigma(B)$. If $0\notin\sigma(B)$ and $X\in H_0(L_AR_B)$, then $\lim\limits_{n\to\infty}\|L_A^nX\|^{\frac{1}{n}}\leq\|B^{-1}\|\lim\limits_{n\to\infty}\|(L_AR_B)^nX\|^{\frac{1}{n}}=0$. Again, if $L_A\in H_0(L_A)$, then $\lim\limits_{n\to\infty}\|(L_AR_B)^nX\|^{\frac{1}{n}}\leq\|B\|\lim\limits_{n\to\infty}\|L_A^nX\|^{\frac{1}{n}}=0$. Hence $H_0(\Delta_{AB}-\lambda)=H_0(L_AR_B)=(L_A)^{-1}(0)=(\Delta_{AB}-\lambda)^{-1}(0)=0$.

For an operator $T\in\mathscr{L}(\mathscr{X})$, the *analytic core* $\mathcal{K}(T-\lambda)$ of $T-\lambda$ is defined by

$$\mathcal{K}(T-\lambda)=\{x\in\mathscr{X}:\text{there exists a sequence }\{x_n\}\subset\mathscr{X}\text{ and }\delta>0\text{ for which }x=x_0,(T-\lambda)x_{n+1}=x_n\text{ and }\|x_n\|\leq\delta^n\|x\|\text{ for all }n=1,2,\cdots\}.$$

We note that $H_0(T-\lambda)$ and $\mathcal{K}(T-\lambda)$ are generally non-closed hyperinvariant subspaces of $T-\lambda$ such that $(T-\lambda)^{-q}(0)\subseteq H_0(T-\lambda)$ for all $q=0,1,2,\cdots$ and $(T-\lambda)\mathcal{K}(T-\lambda)=\mathcal{K}(T-\lambda)$ [26]. Recall from [26] that if $0\in\operatorname{iso}\sigma(T)$, then $H_0(T)$ and $\mathcal{K}(T)$ are closed and $\mathscr{X}=H_0(T)\oplus\mathcal{K}(T)$. An operator $T\in\mathscr{L}(\mathscr{X})$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid then it is isoloid.

Lemma 3.5. *If $A,B^*\in w\mathscr{A}(s,t)$ are such that $s+t=1$, then d_{AB} is polaroid, in particular, d_{AB} is isoloid.*

Proof. Let $\lambda\in\operatorname{iso}\sigma(d_{AB})$. Then by Lemma 3.4, $H_0(d_{AB}-\lambda)=(d_{AB}-\lambda)^{-1}(0)$ implies

$$\mathscr{L}(\mathscr{H})=H_0(d_{AB}-\lambda)\oplus\mathcal{K}(d_{AB}-\lambda)=(d_{AB}-\lambda)^{-1}(0)\oplus\mathcal{K}(d_{AB}-\lambda).$$

Hence

$$\Re(d_{AB}-\lambda)=0\oplus(d_{AB}-\lambda)(\mathcal{K}(d_{AB}-\lambda))=\mathcal{K}(d_{AB}-\lambda),$$

and

$$\mathscr{L}(\mathscr{H})=(d_{AB}-\lambda)^{-1}(0)\oplus\Re(d_{AB}-\lambda).$$

Thus, isolated points of $\sigma(d_{AB})$ are simple poles of the resolvent of d_{AB}. Hence, d_{AB} is polaroid. Since polaroid operators are always isoloid, we have that d_{AB} is isoloid.

For $T \in \mathscr{L}(\mathscr{X})$ and a non negative integer n define $T_{[n]}$ to be the restriction T to $\Re(T^n)$ viewed as a map from $\Re(T^n)$ to $\Re(T^n)$(in particular $T_{[0]} = T$). If for some integer n the range space $\Re(T^n)$ is closed and $T_{[n]}$ is an upper (resp., lower) semi-Fredholm operator, then T is called *upper* (resp., *lower*) *semi-B-Fredholm* operator. In this case index of T is defined as the index of semi-B-Fredholm operator $T_{[n]}$. A *semi-B-Fredholm operator* is an upper or lower semi-Fredholm operator [10]. Moreover, if $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm* operator [6]. An operator T is called a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl operator } \}$ [7]. Let $E(T)$ be the set of all eigenvalues of T which are isolated in $\sigma(T)$. According to [8], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to satisfy *generalized Weyl's theorem*, if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$. In general, generalized Weyl's theorem implies Weyl's theorem but the converse is not true [9].

Let $SBF_+^-(\mathscr{X})$ denote the class of all is *upper B-Fredholm* operators such that ind (T) $\leq$ 0. The *upper B-Weyl spectrum* $\sigma_{SBF_+^-}(T)$ of T is defined by

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathscr{X})\}.$$

Following [9], we say that *generalized a-Weyl's theorem* holds for $T \in \mathscr{L}(\mathscr{X})$ if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in iso\sigma_a(T) : \alpha(T - \lambda) > 0\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$. It is known from [9, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem.

Let $H(\sigma(T))$ denote the set of all analytic functions defined on an open neighborhood of $\sigma(T)$ define, by the classical functional calculus, $f(T)$ for every $f \in H(\sigma(T))$.

Theorem 3.6. *Let $A, B^* \in w\mathscr{A}(s,t)$ be such that $s + t = 1$. Then generalized Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. By Lemma 3.1 and Lemma 3.5, we have that d_{AB} has SVEP and dab is polaroid. So, we have that generalized Weyl's theorem holds for d_{AB} by Theorem 3.10 (ii) of [3]. Since d_{AB} has SVEP and d_{AB} is isoloid, we have that generalized Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB}))$ by [40, Theorem 2.2].

Corollary 3.7. *Let $A, B^* \in w\mathscr{A}(s,t)$ be such that $s + t = 1$. Then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

A bounded operator $T \in \mathscr{L}(\mathscr{X})$ is called a-polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T and that $T \in \mathscr{L}(\mathscr{X})$ is a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. In general, if T is a-polaroid, then it is a-isoloid.

Lemma 3.8. *If $A, B^* \in w\mathscr{A}(s,t)$ are such that $s + t = 1$, then d^*_{AB} is a-polaroid, in particular, d^*_{AB} is a-isoloid.*

Proof. Let $\lambda \in \sigma_a(d^*_{AB})$. By Lemma 3.1 and Lemma 3.5, we have that d_{AB} has SVEP and d^*_{AB} is polaroid (A Banach space operator T is polaroid if and only if T^* is polaroid). Hence $\sigma_a(d^*_{AB}) = \sigma(d^*_{AB})$ by [17, Corollary 7]. We have that λ is an isolated point of $\sigma(d^*_{AB})$. Since d^*_{AB} is isoloid, we have that λ is a pole of the resolvent of d^*_{AB}.

Hence d^*_{AB} is a-polaroid. Since a-polaroid operators are always a-isoloid, we have that d^*_{AB} is a-isoloid.

Following [29], we say that $T \in B(X)$ possesses *property* (w) if $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. The property (w) has been studied in [1, 2, 29]. In Theorem 2.8 of [2], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in B(X)$ possesses *property* (gw) if $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Property (gw) has been introduced and studied in [4]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [4] that an operator possessing property (gw) possesses property (w) but the converse is not true in general.

Theorem 3.9. *Let $A, B^* \in w\mathscr{A}(s,t)$ be such that $s+t=1$. Suppose that $f \in H(\sigma(T))$ is not constant on each of the components of its domain. Then a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem hold for $f(d^*_{AB})$.*

Proof. By Lemma 3.1 and Lemma 3.5, we have that d_{AB} has SVEP and d^*_{AB} is polaroid. The result follows now by Theorem 3.12 of [3].

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