Caspian Journal of Applied Mathematics, Ecology and Economics
V. 4, No 2, 2016, December
ISSN 1560-4055

On Strong Summability of Fourier-Gegenbauer Series

E.J. Ibrahimov, S.A. Jafarova

Abstract. In the paper [2] we consider the summability problems for Fourier-Gegenbauer series are studied via the Valle-Poussion's means. In this paper the strong summability of Fourier-Gegenbauer series we study.

Key Words and Phrases: Strong summable, Fourier-Gegenbauer series, Walle-Poussion mean, Chesaro mean, Lipschitz class.

2010 Mathematics Subject Classifications: Primary 26D07, 26D15,26D99

By definition $f \in L_{1,\lambda}[-1,1]$ is denote

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} |f(x)|\, dx < \infty.$$

The Fourier-Gegenbauer series of function $f \in L_{1,\lambda}[-1,1]$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n^{\lambda}(f) P_n^{\lambda}(x), \tag{1}$$

where

$$a_n^{\lambda}(f) = \frac{\Gamma(\lambda)\Gamma(2\lambda)(n+\lambda)\Gamma(n+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)} \int_{-1}^{1} (1-t^2)^{\lambda-\frac{1}{2}} f(t) p_n^{\lambda}(t) dt.$$

The Gengenbauer polynomials $p_n^{\lambda}(x)$ are the orthogonal system on segment [-1,1] with the weight $(1-x^2)^{\lambda-1/2}$, i.e.

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} P_{\nu}^{\lambda}(x) P_n^{\lambda}(x) dx = \begin{cases} 0, & \nu \neq n; \\ \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma(2\lambda)(n+\lambda)\Gamma(n+1)}, & \nu = n. \end{cases}$$

$S_n^{\lambda}(f;x)$ − denote the $n-th$ partial sum of series (1) and $\sigma_n^{(\lambda,\alpha)}(f:x)$− denote $(C,\delta)-$ means of series (1).

Let $\Lambda = \{\lambda_n\}$ is a non-decreasing sequence such that $\lambda_1 = 1,\ \lambda_{n+1} - \lambda_n \le 1$ and $p > 0$.

Generalized de la Valle-Passion strong mean of series (1) denote by

$$V_n^\lambda\left(f, \Lambda, p; x\right) = \left\{ \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n}^{n-1} \left| S_\nu^\lambda(f;x) - f(x) \right|^p \right\}^{\frac{1}{p}},$$

or $(V, \Lambda)-$ mean.

Here (see [2])

$$S_\nu^\lambda\left(f;x\right) = \frac{\Gamma\left(\lambda\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)} \int_0^\pi f_t(x) K_\nu^\lambda\left(\cos t\right) sh^{2\lambda} dt, \tag{2}$$

where

$$K_\nu^\lambda(\cos t) = \sum_{i=0}^{\nu} \left(i+\lambda\right) P_i^\lambda(\cos t), \tag{3}$$

and

$$f_t(x) = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma\left(\lambda\right)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi f\left(x\cos t + \sqrt{1-x^2}\sin t \cos\varphi\right) \left(\sin\varphi\right)^{2\lambda-1} d\varphi,$$

is a function of the generalized shift.

Neut target receipt nonperiodic analogues of some results of L. Leindler in [5]

Lemma 1 *Let $f \in L_{1,\lambda}[-1,1]$ and $f(x) = O(1)$. then for any $p > 0$*

$$\left\{ \frac{1}{n} \sum_{\nu=0}^{n} \left| S_\nu^\lambda\left(f;x\right) \right|^p \right\}^{\frac{1}{p}} = O(1).$$

Proof. From (3) we have

$$\sum = \left\{ \frac{1}{n} \sum_{\nu=0}^{n} \left| S_\nu^\lambda\left(t;x\right) \right|^p \right\}^{\frac{1}{p}} = O(1) \left\{ \frac{1}{n} \sum_{\nu=0}^{n} \left| \int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} + \int_{\pi-\frac{\pi}{n}}^{\pi} \right|^p \right\}^{\frac{1}{p}}$$

$$= O(1) \left\{ \left(\frac{1}{n} \sum_{\nu=0}^{n} \left| \int_0^{\frac{\pi}{n}} \right|^p \right)^{\frac{1}{p}} + \left(\frac{1}{n} \sum_{\nu=0}^{n} \left| \int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} \right|^p \right)^{\frac{1}{p}} + \left(\frac{1}{n} \sum_{\nu=0}^{n} \left| \int_{\pi-\frac{\pi}{n}}^{\pi} \right|^p \right)^{\frac{1}{p}} \right\}^{\frac{1}{p}}$$

$$= O(1) \left(\sum_1 + \sum_2 + \sum_3 \right). \tag{4}$$

Take into account the inequality

$$\sum_{\nu=0}^{n}\nu+\lambda)P_{\nu}^{\lambda}(x)=O\left(n^{2\lambda+1}\right),$$

which follows from the estimate (see [4], p.178)

$$P_{n}^{\lambda}(x)=O\left(n^{2\lambda-1}\right), \tag{5}$$

we obtain

$$\sum_{1}=O(1)\frac{1}{n}\sum_{\nu=0}^{n}\left(\nu^{2\lambda+1}\int_{0}^{\frac{\pi}{n}}|f_t(x)|\sin^{2\lambda}tdt\right)^{p},$$

since

$$|f_t(x)|\leq|f(x)|=O(1),$$

then

$$\sum_{1}=O(1)\frac{1}{n}\sum_{\nu=0}^{n}\left(\nu^{2\lambda+1}\int_{0}^{\frac{\pi}{n}}t^{2\lambda}dt\right)^{p}=O(1)\frac{1}{n}\sum_{\nu=0}^{n}\left(\frac{\nu}{n}\right)^{(2\lambda+1)p}=O(1). \tag{6}$$

And take into account the inequality (see [1])

$$\sin^{\lambda}t\sin\frac{t}{2}\left|\sum_{\nu=0}^{n}(\nu+\lambda)P_{\nu}^{\lambda}(\cos t)\right|=O(n^{\lambda}),\quad t\in[0,\pi],$$

we will have

$$\sum_{3}=O(1)\frac{1}{n}\sum_{\nu=0}^{n}\left(\nu^{\lambda}\int_{\pi-\frac{\pi}{n}}^{\pi}\frac{\sin^{\lambda}t}{\sin\frac{t}{2}}dt\right)^{p}=O(1)\frac{1}{n}\sum_{\nu=0}^{n}\left(\nu^{\lambda}\int_{\pi-\frac{\pi}{n}}^{\pi}t^{\lambda-1}dt\right)^{p}$$

$$=O(1)\frac{1}{n}\sum_{\nu=0}^{n}\nu^{\lambda p}\left(\pi^{\lambda}-\left(\pi-\frac{\pi}{n}\right)^{\lambda}\right)^{p}=O(1)\frac{1}{n}\sum_{\nu=0}^{n}\nu^{\lambda p}\left(\pi-\pi+\frac{\pi}{n}\right)^{\lambda p}=O(1). \tag{7}$$

That estimate $\sum_2$ we transform the kernel $K_n^{\lambda}(x)$.

But us (see [6], p.95 and 93)

$$2(\nu+\lambda)P_{\nu}^{\lambda}(x)=\left(P_{\nu+1}^{\lambda}(x)-P_{\nu-1}^{\lambda}(x)\right)'=2\lambda\left(P_{\nu}^{\lambda+1}(x)-P_{\nu-2}^{\lambda+1}(x)\right),$$

then bu (3) we have

$$K_n^{\lambda}(\cos t)=\lambda\left(P_{n-1}^{\lambda+1}(\cos t)+P_{n}^{\lambda+1}(\cos t)\right). \tag{8}$$

Use of formula (see [4],p.204)

$$P_n^{\lambda+1}(\cos t)=\frac{2^{\lambda+1}(\pi n)^{-\frac{1}{2}}}{(\sin t)^{\lambda+1}}\cos\left[(n+\lambda+1)t-\frac{\pi}{2}(\lambda+1)\right]$$
$$+O\left(n^{-\frac{3}{2}}\right),\quad 0<t<\pi, \tag{9}$$

in (8) we obtain

$$\mathcal{K}_n^{\lambda}(\cos t)=O(1)\left(\nu^{-\frac{1}{2}}(\sin t)^{-\lambda-1}+\nu^{-\frac{3}{2}}\right),\quad 0<t<\pi.$$

From this we have

$$\sum_{2}=O(1)\frac{1}{n}\sum_{\nu=0}^{n}\left((\nu+1)^{-\frac{1}{2}}\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}(\sin t)^{\lambda-1}dt\right.$$
$$\left.+\nu^{-\frac{3}{2}}\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\sin^{2\lambda}tdt\right)^{p}=\sum_{2.1}+\sum_{2.2}. \tag{10}$$

Since

$$\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\frac{\sin^{\lambda}t\left(\cos^2 t+\sin^2 t\right)}{\sin t}dt=\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}(\sin t)^{\lambda-1}\cos^2 tdt$$
$$+\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}(\sin t)^{\lambda+1}dt\le\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}(\sin t)^{\lambda-1}\left|\cos t\right|dt+O(1)$$
$$=O(1)+\int_{\frac{\pi}{n}}^{\frac{\pi}{2}}(\sin t)^{\lambda-1}\cos tdt-\int_{\frac{\pi}{2}}^{\pi-\frac{\pi}{n}}(\sin t)^{\lambda-1}\cos tdt$$
$$=O(1)+\int_{\frac{\pi}{n}}^{\frac{\pi}{2}}(\sin t)^{\lambda-1}d\sin t-\int_{\frac{\pi}{2}}^{\pi-\frac{\pi}{n}}(\sin t)^{\lambda-1}d\sin t$$
$$=O(1)+\frac{2}{\lambda}\left(1-\sin\frac{\pi}{n}\right)=O(1). \tag{11}$$

Take into account (11) in (10), we obtain

$$\sum_{2.1}=O(1)\frac{1}{n}\sum_{\nu=0}^{n}\nu^{-\frac{p}{2}}, \tag{12}$$

$$\sum_{2.2} = O(1)\frac{1}{n}\sum_{\nu=0}^{n}\nu^{-\frac{3}{2}p}. \qquad (13)$$

Now from (10) (12) and (13) we have

$$\sum_{2.1} = O(1)\frac{1}{n}\sum_{\nu=0}^{\nu}\nu^{-\frac{p}{2}}.$$

From this take into account the relation (see [1],p.355)

$$\frac{1}{\lambda_n}\sum_{\nu=0}^{n}\nu^{-\gamma} = O(1)\begin{cases} n^{-\gamma}, & 0<\gamma<1, \\ \frac{1}{n}\left(1+\log\frac{n}{n-\lambda_n+1}\right), & \gamma=1, \\ \frac{1}{n}\left(n-\lambda_n+1\right)^{1-\gamma}, & \gamma>1, \end{cases} \qquad (14)$$

for $\lambda_n = n$, we obtain

$$\sum_{2} = O(1)\begin{cases} n^{-\frac{p}{2}}, & 0<p<2, \\ \frac{1}{n}\log n, & p=2, \\ \frac{1}{n}, & p>2. \end{cases} \qquad (15)$$

Now taking into account (6), (7) and (15) in (4) we obtain assertion of the lemma.

Let H_n is the set of algebraic polynomials of degree $\leq n$, and $E_n(f)-$ is the best approximation of $f \in C[-1,1]$, i.e.

$$E_n(f) = \inf_{P_n \in H_n} \|f - P_n\|_C \, .$$

Theorem 1. *Let $f \in L_{1,\lambda}[-1,1]$ and $f(x) = O(1)$. Then for any $p > 0$*

$$V_n^{\lambda}(f,\Lambda,p;x) = O(1)\left(\frac{n}{\lambda_n}\right)^{\frac{1}{p}} E_{n-\lambda_n}.$$

Proof. Let $P_m(x)$ is algebraic polynomial *degree* $\leq m$. Since for $n-\lambda_n \geq m$

$$V_n^{\lambda}(f-p_m,\Lambda,p;x) = V_n^{\lambda}(f,\Lambda,p,x) - p_m(x),$$

then

$$\left\{\frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n}^{n-1}\left|S_\nu^{\lambda}(f:x)-f(x)\right|^p\right\}^{\frac{1}{p}}$$

$$\leq \left\{\frac{2^p}{\lambda_n}\left[\sum_{\nu=n-\lambda_n}^{n-1}\left|S_\nu^{\lambda}(f-P_{n-\lambda_n};x)\right| + \left|P_{n-\lambda_n}(x)-f(x)\right|^p\right]\right\}^{\frac{1}{p}}$$

$$\leq 2^{\frac{p+1}{p}}\left\{\left[\frac{n}{\lambda_n}\frac{1}{n}\sum_{\nu=n-\lambda_n}^{n-1}\left|S_\nu^{\lambda}(f-P_{n-\lambda_n};x)\right|^p + E_{n-\lambda_m}(f)\right]\right\}$$

$$= O(1)\left(\left(\frac{n}{\lambda_n}\right)^{\frac{1}{p}} E_{n-\lambda_m}(f)\right).$$

Theorem is proved.

We denote $\varphi_x(t) = f(x) - f_t(x)$.

Theorem 2. *If $f \in Lip\alpha(0 < \alpha < 1)$, then $\forall\delta > 0$ the uniformly estimate is true:*

$$f(x) - \sigma_n^{(\lambda,\delta)}(f;x) = O(1)\begin{cases} n^{-\alpha}, & \frac{1}{2} \le \lambda < 1; \\ n^{-\lambda-\frac{1}{2}}, & 0 < \lambda < \frac{1}{2}. \end{cases}$$

Proof. It is know, that

$$f(x) - \sigma_n^{(\lambda,\delta)}(f;x) = \frac{\Gamma(\lambda)}{2\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda+\frac{1}{2})}\int_0^{\pi} \varphi_x(t) K_n^{(\lambda,\delta)}(\cos t)\sin^{2\lambda} t dt, \tag{16}$$

where

$$K_n^{(\lambda,\delta)}(\cos t) = \frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta}(\nu+\lambda)\, p_\nu^{\lambda}(\cos t). \tag{17}$$

The integral (16) laying out by form

$$J = \int_0^{\pi} = \int_0^{\frac{\pi}{\nu+2}} + \int_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}} + \int_{\pi-\frac{\pi}{\nu+2}}^{\pi} = J_1 + J_2 + J_3. \tag{18}$$

If $f \in Lip\alpha$, then(see [2] Lemma 1)

$$|\varphi_x(t)| \le |x|^{\alpha}(1-\cos t)^2 + (1-x^2)^{\frac{\alpha}{2}}\sin^{\alpha} t.$$

We transform the kernel (4). About Abel transformation we have

$$K_n^{(\lambda,\delta)}(\cos t) = \frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1}\sum_{k=0}^{\nu}(k+\lambda)\, P_k^{\lambda}(\cos t). \tag{19}$$

Taking into account the inequalities (5) and (18), we will obtain

$$J_1 = O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1}(\nu+1)^{2\lambda+1}\int_0^{\frac{\pi}{\nu+2}} t^{\alpha+2\lambda} dt$$

$$= O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1}(\nu+1)^{-\alpha}.$$

From this, according by relations ([8], p.130, 131)

$$A_n^{\alpha+\gamma+1}=\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha}A_{\nu}^{\gamma}\text{ and }A_n^{\alpha}\sim\frac{n^{\alpha}}{\Gamma(\alpha+1)},\quad n\to\infty,$$

we have

$$J_1=O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}A_{\nu}^{-\alpha}=O(1)\frac{A_n^{\delta-\alpha}}{A_{\nu}^{\alpha}}=O(n^{-\alpha}). \tag{20}$$

Now we estimate J_3. Taking into account the Chiristoffel-Darboux formula, we obtain

$$J_3\leq\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}\int\limits_{\pi-\frac{\pi}{\nu+2}}^{\pi}\left|\sum_{k=0}^{\nu}(k+\lambda)P_k^{\lambda}(\cos t)\right|\left(|x|^{\alpha}\left(1-\cos t\right)^{\alpha}\right.$$

$$\left.+\left(1-x^2\right)^{\frac{\alpha}{2}}sh^{\alpha}t\right)\sin^{2\lambda}tdt=\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}\left(\int\limits_{\pi-\frac{\pi}{\nu+2}}^{\pi}\Big|(\nu+2\lambda)P_{\nu}^{\lambda}(\cos t)\right.$$

$$\left.-(\nu+1)P_{\nu+1}^{\lambda}(\cos t)\Big|\left(|x|^{\alpha}\left(1-\cos t\right)^{\alpha-1}+\left(1-x^2\right)^{\frac{\alpha}{2}}\frac{\sin^{\alpha}t}{1-\cos t}\right)\right)\sin^{2\lambda}tdt.$$

Here using the inequality ([6], p.179)

$$\sin^{\lambda}t|p_n^{\lambda}(\cos t)|<\frac{2^{1-\lambda}}{\Gamma(-1)}n^{\lambda-1},\ t\in[0,\pi],$$

we will have

$$J_3=O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}(\nu+1)^{\lambda}\left(\int\limits_{\pi-\frac{\pi}{\nu+2}}^{\pi}\frac{\sin^{\alpha}t}{\left(1-\cos t\right)^{1-\alpha}}dt\right.$$

$$\left.+\int\limits_{\pi-\frac{\pi}{\nu+2}}^{\pi}\frac{(\sin t)^{\alpha+\lambda}}{1-\cos t}dt\right)=\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}(\nu+1)^{\lambda}\left(\int\limits_{0}^{\frac{\pi}{\nu+2}}\frac{\sin^{\alpha}t}{\left(1+\cos t\right)^{1-\alpha}}dt\right.$$

$$\left.+\int\limits_{0}^{\frac{\pi}{\nu+2}}\frac{(\sin t)^{\alpha+\lambda}}{1+\cos t}dt\right)=O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}(\nu+1)^{\lambda}\left(\int\limits_{0}^{\frac{\pi}{\nu+2}}t^{\lambda}dt\right.$$

$$\left.+\int\limits_{0}^{\frac{\pi}{\nu+2}}t^{\alpha+\lambda}dt\right)=O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}\frac{1}{\nu+1}$$

$$=O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}(\nu+1)^{-\alpha}=O(n^{-\alpha}). \tag{21}$$

Taking into account (8) in (19), we obtain

$$K_n^{(\lambda,\delta)}(\cos t)=\frac{\lambda}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}\left(P_{\nu-1}^{\lambda+1}(\cos t)+P_\nu^{\lambda+1}(\cos t)\right)$$

$$=\frac{\lambda}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}P_{\nu-1}^{\lambda+1}(\cos t)+\frac{\lambda}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}P_{\nu-1}^{\lambda+1}(\cos t)$$

$$=K_{n.1}^{(\lambda,\delta)}(\cos t)+K_{n.2}^{(\lambda,\delta)}(\cos t).$$

According to (9) we have

$$K_{n,2}^{(\lambda,\delta)}(\cos t)=\frac{\lambda\cdot 2^{\lambda+1}}{\Gamma\left(\frac{1}{2}\right)}\cdot\frac{1}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}(\nu+1)^{-\frac{1}{2}}\cos\left[(\nu+\lambda+1)t\right.$$

$$\left.-\frac{\pi}{2}(\lambda+1)\right]+O(1)\frac{1}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}(\nu+1)^{-\frac{3}{2}}. \tag{22}$$

Taking into account (22) in (18), we obtain

$$J_2\left(K_{n,2}^{(\lambda,\delta)}(\cos t)\right)=\frac{\lambda\cdot 2^{\lambda+1}}{\Gamma\left(\frac{1}{2}\right)}\cdot\frac{1}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}(\nu+1)^{-\frac{1}{2}}$$

$$\times\int_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}}\varphi_x(t)\,(\sin t)^{\lambda-1}\cos\left[(\nu+\lambda+1)t-\frac{\pi}{2}(\lambda+1)\right]dt$$

$$+O(1)\frac{1}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}(\nu+1)^{-\frac{3}{2}}\int_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}}|\varphi_x(t)|\sin^{2\lambda}t\,dt=J_{2.1}+J_{2.2}. \tag{23}$$

First we estimate $J_{2.2}$.

$$J_{2.2}=O(1)\frac{1}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}(\nu+1)^{-\alpha}\int_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}}t^{\alpha+2\lambda}dt$$

$$=O(1)\frac{1}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta-1}A_\nu^{-\alpha}=O(n^{-\alpha}). \tag{24}$$

We suppose $\psi_x(t)=\varphi_x(t)(\sin t)^{\lambda-1}$ and consider the integral

$$A=\int_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}}\psi_x(t)\cos\left[(\nu+\lambda+1)t-\frac{\pi}{2}(\lambda+1)\right]dt.$$

Doing replacement of variables $t = u + \frac{\pi}{\nu+\lambda+1}$ we obtain

$$A = -\int_{\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \psi_x\left(t+\frac{\pi}{\nu+\lambda+1}\right)\cos\left[(\nu+\lambda+1)t-\frac{\pi}{2}(\lambda+1)\right]dt.$$

Putting next expression for A, we will have

$$A = \frac{1}{2}\int_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \left[\psi_\varkappa(t)-\psi_\varkappa\left(t+\frac{\pi}{\nu+\lambda+1}\right)\right]$$

$$\times\cos\left[(\nu+\lambda+1)t-\frac{\pi}{2}(\lambda+1)\right]dt$$

$$+\frac{1}{2}\int_{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\pi-\frac{\pi}{\nu+2}} \psi_x(t)\cos\left[(\nu+\lambda+1)t-\frac{\pi}{2}(\lambda+1)\right]dt$$

$$-\frac{1}{2}\int_{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\pi-\frac{\pi}{\nu+2}} \psi_x\left(t+\frac{\pi}{\nu+\lambda+1}\right)$$

$$\times\cos\left[(\nu+\lambda+1)t-\frac{\pi}{2}(\lambda+1)\right]dt = A_1+A_2+A_3. \tag{25}$$

Further

$$\psi_x(t)-\psi_x\left(t+\frac{\pi}{\nu+\lambda+1}\right) = \left[\varphi_x(t)-\varphi_x\left(t+\frac{\pi}{\nu+\lambda+1}\right)\right](\sin t)^{\lambda-1}$$

$$+\left[(\sin t)^{\lambda-1}-\sin^{\lambda-1}\left(t+\frac{\pi}{\nu+\lambda+1}\right)\right]\psi_x\left(t+\frac{\pi}{\nu+\lambda+1}\right). \tag{26}$$

On the other hand

$$\left|\varphi_x(t)-\varphi_x\left(t+\frac{\pi}{\nu+\lambda+1}\right)\right| = \left|f_t(x)-f_{t+\frac{\pi}{\nu+\lambda+1}}\right|$$

$$\le |x|^\alpha\left|\cos\left(t+\frac{\pi}{\nu+\lambda+1}\right)-\cos t\right|^\alpha+(1-x^2)^{\frac{\alpha}{2}}\left|\sin\left(t+\frac{\pi}{\nu+\lambda+1}\right)\right|^\alpha$$

$$= |x|^\alpha\left|2\sin\left(t+\frac{\pi}{2\nu+2\lambda+2}\right)\sin\frac{\pi}{2\nu+2\lambda+2}\right|^\alpha$$

$$+(1-x^2)^{\frac{\alpha}{2}\alpha}\left|2\cos\left(t+\frac{\pi}{2\nu+2\lambda+2}\right)\sin\frac{\pi}{2\nu+2\lambda+2}\right|^\alpha = O(\nu)^{-\alpha}. \tag{27}$$

And

$$\left|\sin^{\lambda-1} t - \sin^{\lambda-1}\left(t + \frac{\pi}{\nu+\lambda+1}\right)\right|$$

$$\leq \frac{\left|2\cos\left(t + \frac{\pi}{2\nu+2\lambda+2}\right)\sin\frac{\pi}{2\nu+2\lambda+2}\right|^{1-\lambda}}{\sin^{1-\lambda} t \sin^{1-\lambda}\left(t + \frac{\pi}{\nu+\lambda+1}\right)}$$

$$= O(1)\frac{|\cos t|\,(\nu+1)^{\lambda-2} + (\nu+1)^{\lambda-1}\sin t}{\sin^{1-\lambda} t \sin^{1-\lambda}\left(t + \frac{\pi}{\nu+\lambda+1}\right)}. \tag{28}$$

At last

$$\left|\psi_x\left(t + \frac{\pi}{\nu+\lambda+1}\right)\right| = \left|f(x) - f_{t+\frac{\pi}{\nu+\lambda+1}}\right|$$

$$\leq |x|^\alpha \left[1 - \cos\left(t + \frac{\pi}{\nu+\lambda+1}\right)\right]^\alpha + (1-x^2)^{\frac{\alpha}{2}}\sin^\alpha\left(t + \frac{\pi}{\nu+\lambda+1}\right). \tag{29}$$

Taking into account (27) -(29) in (26) we obtain

$$\left|\psi_x(t) - \psi_x\left(t + \frac{\pi}{\nu+\lambda+1}\right)\right| = O(1)\,(\nu+1)^{-\alpha}\sin^{\lambda-1} t$$

$$+O(1)\frac{\left[(\nu+1)^{\lambda-2}\,|\cos t| + (\nu+1)^{\lambda-1}\sin t\right]}{\sin^{1-\lambda} t \sin^{1-\lambda}\left(t + \frac{\pi}{\nu+\lambda+1}\right)}\left[\sin^{2\alpha}\left(\frac{t}{2} + \frac{\pi}{2\nu+2\lambda+2}\right)\right.$$

$$\left. + \sin^\alpha\left(t + \frac{\pi}{\nu+\lambda+1}\right)\right] = O(1)\,(\nu+1)^{-\alpha}\sin^{\lambda-1} t$$

$$+O(1)\,(\nu+1)^{\lambda-2}\,\frac{|\cos t|\sin^{2\alpha}\left(\frac{t}{2} + \frac{\pi}{2\nu+2\lambda+2}\right)}{\sin^{1-\lambda} t \sin^{1-\lambda}\left(t + \frac{\pi}{\nu+\lambda+1}\right)}$$

$$O(1)\,(\nu+1)^{\lambda-2}\,\frac{|\cos t|\sin^{\alpha}\left(t + \frac{\pi}{\nu+\lambda+1}\right)}{\sin^{1-\lambda} t \sin^{1-\lambda}\left(t + \frac{\pi}{\nu+\lambda+1}\right)}$$

$$+O(1)\,(\nu+1)^{\lambda-1}\sin^\lambda t \sin^{2\alpha}\left(\frac{t}{2} + \frac{\pi}{2\nu+2\lambda+2}\right)\sin^{\lambda-1}\left(t + \frac{\pi}{\nu+\lambda+2}\right)$$

$$+O(1)\,(\nu+1)^{\lambda-1}\sin^\lambda t \sin^{\alpha}\left(t + \frac{\pi}{\nu+\lambda+1}\right)\sin^{1-\lambda}\left(t + \frac{\pi}{\nu+\lambda+2}\right). \tag{30}$$

From (30) and (25), we have

$$A_1 = O(1)\,(\nu+1)^{-\alpha}\int\limits_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}\sin^{\lambda-1} t dt + O(1)\,(\nu+1)^{\lambda-2}$$

$$\times \int\limits_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \frac{|\cos t| \sin^{2\alpha}\left(\frac{t}{2}+\frac{\pi}{2\nu+2\lambda+2}\right)}{\sin^{1-\lambda} t \sin^{1-\lambda}\left(\frac{t}{2}+\frac{\pi}{2\nu+2\lambda+2}\right) \cos^{1-\lambda}\left(\frac{t}{2}+\frac{\pi}{2\nu+2\lambda+2}\right)} dt$$

$$+O(1)\,(\nu+1)^{\lambda-2} \int\limits_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \frac{|\cos t| \sin^{\alpha}\left(t+\frac{\pi}{\nu+\lambda+1}\right)}{\sin^{1-\lambda} t \sin^{1-\lambda}\left(t+\frac{\pi}{\nu+\lambda+1}\right)} dt$$

$$+O(1)\,(\nu+1)^{\lambda-1} \int\limits_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \frac{\sin^{\lambda} t \sin^{2\alpha}\left(\frac{t}{2}+\frac{\pi}{2\nu+2\lambda+2}\right)}{\sin^{1-\lambda}\left(\frac{t}{2}+\frac{\pi}{2\nu+2\lambda+2}\right) \cos^{1-\lambda}\left(\frac{t}{2}+\frac{\pi}{2\nu+2\lambda+2}\right)} dt$$

$$+O(1)\,(\nu+1)^{\lambda-1} \int\limits_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \frac{\sin^{\lambda} t \sin^{\alpha}\left(t+\frac{\pi}{\nu+\lambda+1}\right)}{\sin^{1-\lambda}\left(t+\frac{\pi}{\nu+\lambda+1}\right)} dt$$

$$= A_{1.1} + A_{1.2} + A_{1.3} + A_{1.4} + A_{1.5}. \tag{31}$$

We estimate $A_{1.1}$.

$$A_{1.1} = O(1)\,(\nu+1)^{-\alpha} \left(\int\limits_{\frac{\pi}{\nu+2}}^{\frac{\pi}{2}} + \int\limits_{\frac{\pi}{2}}^{\pi} \right) \sin^{\lambda-1} t dt$$

$$= O(1)\,(\nu+1)^{-\alpha} \left(\int\limits_{\frac{\pi}{\nu+2}}^{\frac{\pi}{2}} t^{\lambda-1} dt + \int\limits_{0}^{\frac{\pi}{2}} t^{\lambda-1} dt \right) = O(1)\,(\nu+1)^{-\alpha}. \tag{32}$$

Further, taking into account the inequalities $\sin t \le t$ and $\sin t \ge \frac{\pi}{2}$ for $0 \le t \ge \frac{\pi}{2}$, we obtain

$$A_{1.2} = O(1)\,(\nu+1)^{\lambda-2} \int\limits_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \frac{\left((\nu+1)^{-2\alpha} + t^{\alpha}\right) |\cos t|\, dt}{t^{1-\lambda-\alpha} \sin^{1-\lambda} t \cos^{1-\lambda}\left(\frac{t}{2}+\frac{\pi}{2\nu+2\lambda+2}\right)}$$

$$= O(1) \frac{1}{\nu+1} \int\limits_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \frac{\left((\nu+1)^{-2\alpha} + t^{\alpha}\right) |\cos t|\, dt}{t^{1-\lambda-\alpha} \sin^{1-\lambda} t}$$

$$= O(1)\,(\nu+1)^{\lambda-3\alpha} \left(\int\limits_{\frac{\pi}{\nu+2}}^{\frac{\pi}{2}} \frac{\cos t dt}{\sin^{1-\lambda} t} - \int\limits_{\frac{\pi}{2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \frac{\cos t dt}{\sin^{1-\lambda} t} \right)$$

$$+O(1)\,(\nu+1)^{-\lambda-\alpha}\left(\int\limits_{\frac{\pi}{\nu+2}}^{\frac{\pi}{2}}\frac{t^{\alpha}\cos t}{\sin^{1-\lambda}t}dt-\int\limits_{\frac{\pi}{2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}\frac{t^{\alpha}\cos tdt}{\sin^{1-\lambda}t}\right)$$

$$=O(1)\,(\nu+1)^{-\lambda-3\alpha}+O(1)\,(\nu+1)^{-\lambda-\alpha}=O(1)\,(\nu+1)^{-\alpha-\lambda}. \tag{33}$$

Some estimate for $A_{1.3}$ is true.

We estimate $A_{1.4}$. From (35) we have

$$A_{1.4}=O(1)\,(\nu+1)^{\lambda-1}\int\limits_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}\frac{t^{\alpha+\lambda}dt}{t^{1-\lambda-\alpha}\cos^{1-\lambda}\left(\frac{t}{2}+\frac{\pi}{2\nu+2\lambda+2}\right)}.$$

From this for $\alpha+\lambda\leq 1$ we obtain

$$A_{1.4}=O(1)\,(\nu+1)^{-\alpha}\int\limits_{\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}\frac{dt}{\cos^{1-\lambda}\left(\frac{t}{2}+\frac{\pi}{2\nu+2\lambda+2}\right)}$$

$$=O(1)\,(\nu+1)^{-\alpha}\int\limits_{\frac{\pi}{2\nu+4}+\frac{\pi}{2\nu+2\lambda+2}}^{\frac{\pi}{2}-\frac{\pi}{2\nu+4}}\left(\cos^2 t+\sin^2 t\right)\cos^{\lambda-1}tdt$$

$$=O(1)\,(\nu+1)^{-\alpha}+O(1)\,(\nu+1)^{-\alpha}\int\limits_{\frac{\pi}{2}}^{0}\cos^{\lambda-1}td\cos t=O(1)\,(\nu+1)^{-\alpha}. \tag{34}$$

The same estimate is true for $A_{1.4}$.

Taking into account (32)-(34) in (31), we obtain that,

$$A_1=O(1)\,(\nu+1)^{-\alpha}. \tag{35}$$

Consider the integral A_2.

$$A_2=O(1)\int\limits_{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\pi-\frac{\pi}{\nu+2}}\frac{(1-\cos t)^{\alpha}+\sin^{\alpha}t}{\sin^{1-\lambda}t}dt$$

$$=O(1)\int\limits_{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\pi-\frac{\pi}{\nu+2}}\frac{\sin^{2\alpha}\frac{t}{2}+\sin^{\alpha}\frac{t}{2}\cos^{\alpha}\frac{t}{2}}{\left(\sin\frac{t}{2}\right)^{1-\lambda}\left(\cos\frac{t}{2}\right)^{1-\lambda}}dt$$

$$=O(1)\int\limits_{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\pi-\frac{\pi}{\nu+2}}\frac{t^{2\alpha+\lambda-1}\left(\cos^2\frac{t}{2}+\sin^2\frac{t}{2}\right)}{\left(\cos\frac{t}{2}\right)^{1-\lambda}}dt$$

$$= O(1) \int\limits_{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\pi-\frac{\pi}{\nu+2}} \frac{t^{\alpha+\lambda-1}\left(\cos^2\frac{t}{2}+\sin^2\frac{t}{2}\right)}{\left(\cos\frac{t}{2}\right)^{1-\lambda-\alpha}} dt$$

$$+O(1) \int\limits_{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\pi-\frac{\pi}{\nu+2}} \left(\cos\frac{t}{2}\right)^{1+\lambda} dt + O(1) \int\limits_{\pi-\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \left(\cos\frac{t}{2}\right)^{\alpha+\lambda-1} d\cos\frac{t}{2}$$

$$+O(1) \int\limits_{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\pi-\frac{\pi}{\nu+2}} \left(\cos\frac{t}{2}\right)^{\alpha+\lambda-1} dt + O(1) \int\limits_{\pi-\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} \left(\cos\frac{t}{2}\right)^{\alpha+\lambda-1} d\cos\frac{t}{2}$$

$$= O(1)\frac{1}{\nu+1} + O(1)\cos^{\lambda}\frac{t}{2}\Bigg|_{\pi-\frac{\pi}{\nu+2}}^{\pi-\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}} =$$

$$= O(1)\frac{1}{\nu+1} + O(1)\left(\sin^{\lambda}\left(\frac{\pi}{2\nu+4}-\frac{\pi}{2\nu+2\lambda+2}\right) - \sin^{\lambda}\frac{\pi}{2\nu+4}\right)$$

$$= O(1)\frac{1}{\nu+1} + O(1)\sin^{\lambda}\frac{\pi}{4(\nu+\lambda+1)} = O(1)(\nu+1)^{-\lambda}. \tag{36}$$

From (36) and (23), we obtain

$$J_{2.1}(A_2) = O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} A_{\nu}^{-\lambda-\frac{1}{2}} = O(1) n^{-\lambda-\frac{1}{2}}. \tag{37}$$

And for $\lambda \geq \frac{1}{2}$ we have

$$J_{2.1}(A_2) = O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1}(\nu+1)^{-1}$$

$$= O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1}(\nu+1)^{-\alpha}$$

$$= O(1)\frac{1}{A_n^{\delta}}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} A_{\nu}^{-\alpha} = O(1)(n+1)^{-\alpha}. \tag{38}$$

From (37) and (38) we obtain that

$$J_{2.1}(A_2) = O(1)\begin{cases} n^{-\lambda-\frac{1}{2}}, & 0 < \lambda < \frac{1}{2}; \\ n^{-\alpha}, & \frac{1}{2} \leq \lambda < 1. \end{cases} \tag{39}$$

And taking into account (35) in (23), we have

$$J_{2.1}(A_2) = O(1)(n+1)^{-\alpha}. \tag{40}$$

From (29) we have

$$A_3 = O(1) \int\limits_{\frac{\pi}{\nu+2}-\frac{\pi}{\nu+\lambda+1}}^{\frac{\pi}{\nu+2}} \left(t + \frac{\pi}{\nu+\lambda+1}\right)^{\alpha} dt = O(1)(\nu+1)^{-\alpha}. \tag{41}$$

Taking into account (41) in (23), we obtain

$$J_{2.1}(A_2) = O(n^{-\alpha}). \tag{42}$$

From (39), (40) and (42), we obtain

$$J_{2.1} = O(1) \begin{cases} n^{-\lambda-\frac{1}{2}}, & 0 < \lambda < \frac{1}{2}, \\ n^{-\alpha}, & \frac{1}{2} \le \lambda < 1. \end{cases} \tag{43}$$

And taking into account (24) and (43) in (23), we obtain that for $0 < \alpha < 1$

$$J_2\left(K_{n.2}^{(\lambda,\delta)}(\cos t)\right) = O(1) \begin{cases} n^{-\lambda-\frac{1}{2}}, & 0 < \lambda < \frac{1}{2}, \\ n^{-\alpha}, & \frac{1}{2} \le \lambda < 1. \end{cases} \tag{44}$$

The same estimate and for $J_2\left(K_{n.2}^{(\lambda,\delta)}(\cos t)\right)$ is just. Now taking into account (23), (24) and (44) in (18), we obtain the approval of the theorem.

Theorem 3. *Let* $f \in Lip\alpha (0 < \alpha < 1), \delta > 0$ *and* $0 < \lambda < 1$. *Then for* $0 < \lambda < \frac{1}{2}$

$$\left(\frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n}^{n} \left|\sigma_n^{(\lambda,\delta)}(f;x) - f(x)\right|^p\right)^{\frac{1}{p}}$$

$$= O(1) \begin{cases} n^{-\lambda-\frac{1}{2}}, & 0 < \left(\lambda + \frac{1}{2}\right)p < 1; \\ \left(\frac{1}{n}\left(1 + \log\frac{n}{n-\lambda n}\right)\right)^{\frac{1}{p}}, & \left(\lambda+\frac{1}{2}\right)p = 1; \\ n^{-\frac{1}{p}}\left(n - \lambda n\right)^{1-\left(\lambda+\frac{1}{2}\right)p}, & \left(\lambda+\frac{1}{2}\right)p > 1; \end{cases}$$

And for $\frac{1}{2} \le \lambda < 1$

$$\left(\frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n}^{n} \left|\sigma_n^{(\lambda,\delta)}(f;x) - f(x)\right|^p\right)^{\frac{1}{p}}$$

$$= O(1) \begin{cases} n^{-\alpha}, & 0 < \alpha p < 1; \\ n^{-\alpha}\left(1 + \log\frac{n}{n-\lambda n}\right)^{\alpha}, & \alpha p = 1; \\ n^{-\frac{1}{p}}\left(n - \lambda n\right)^{1-\alpha p}, & \alpha p > 1. \end{cases}$$

The assertion of the theorem follows from Theorem 2 and (14).

References

[1] V.G. Dolgolaptov, *Strong summability of Riesz means*, Mat. Zametki, **40(3)**, 1986, 353-363 (in Russian).

[2] E.I. Ibrahimov, *On summability of Fourier-Gegenbauer series*, Proc. of A.Razmadze, Math. Institute, **466**, 2014, 69-93.

[3] S.O. Kallayev, *On the Valle-Poussion mean of Fourier-Gegenbauer series*, Math. Zam. **7**, 1970, 19-30 (in Russian).

[4] E. Kogbetliantz, *Sur la sommability de le serie ultraspherique a l'interieur de l'intervalle(-1,1) par la methode des moyennes arthmetiques*, Bull. Soc.Math. de France, **1**, 1923, 244-295.

[5] L. Leindler, *On the strong approximation of Fourier series*, Acta Sci. Math., **38**, 1976, 317-324.

[6] G. Szegö, *Orthogonal polynomials*, Izd. Phyz.-Mat.Lit., Moscow, (1962) (Translation into Russian).

[7] A.F. Timan, *Theory of approximation of functions of a real variable*, Fizmatlit, Moscow, 1960 (in Russian).

[8] A. Zugmund, *Trigonometric series*, Izd. "Mir", Moscow, **1**, 1965.

E.I. Ibrahimov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az1141, Baku, Azerbaijan
E-mail:elmanibrahimov@yahoo.com

S.A. Jafarova
Azerbaijan State Economic University 6, Istiqlaliyyat str., Baku AZ 1001, Baku, Azerbaijan
E-mail: sada-jafarova@rambler.ru

Received 22 October 2016
Accepted 15 December 2016