On Morrey type Spaces and Some Properties

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Abstract. Subspace $M_\rho^{p,\alpha}$ of the weighted Morrey -type space $L_\rho^{p,\alpha}$ is defined, it is proved that infinitely differentiable functions are dense in it. An approximation properties of the Poisson kernel is studied in $M_\rho^{p,\alpha}$. A sufficient condition for belonging of the product to the space $M_\rho^{p,\alpha}$ is obtained. It is proved that $M_\rho^{p,\alpha}$ is an invariant subspace of a singular integral operator.

Key Words and Phrases: Morrey-type classes, Minkowski inequality, Poisson kernel.

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1. Introduction

The concept of Morrey space was introduced by C. Morrey [1] in 1938 in the study of qualitative properties of the solutions of elliptic type equations with BMO (Bounded Mean Oscillations) coefficients (see also [2, 3]). This space provides a large class of weak solutions to the Navier-Stokes system [4]. In the context of fluid dynamics, Morrey-type spaces have been used to model the fluid flow in case where the vorticity is a singular measure supported on some sets in R^n [5]. There appeared lately a large number of research works which considered many problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc in Morrey-type spaces (for more details see [2-26]). It should be noted that the matter of approximation in Morrey-type spaces has only started to be studied recently (see, e.g., [11, 12, 16, 17]), and many problems in this field are still unsolved. This work is just dedicated to this field. Subspace $M_\rho^{p,\alpha}$ of the weighted Morrey -type space $L_\rho^{p,\alpha}$ is defined, it is proved that infinitely differentiable functions are dense in it. An approximation properties of the Poisson kernel is studied in $M_\rho^{p,\alpha}$. A sufficient condition for belonging of the product to the space $M_\rho^{p,\alpha}$ is obtained. It is proved that $M_\rho^{p,\alpha}$ is an invariant subspace of a singular integral operator .

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2. Needful Information

We will need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$. All the constants throughout this paper (can be different in different places) will be denoted by c.

The expression $f(x) \sim g(x)$, $x \in M$, means

$$\exists \delta > 0 : \delta \leq \left| \frac{f(x)}{g(x)} \right| \leq \delta^{-1}, \forall x \in M.$$

Similar meaning is intended by the expression $f(x) \sim g(x)$, $x \to a$.

By Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0 < \alpha \leq 1$, $p \geq 1$, we mean the normed space of all measurable functions $f(\cdot)$ on Γ with the finite norm

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_B \left(\left| B \bigcap \Gamma \right|_\Gamma^{\alpha-1} \int_{B \bigcap \Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty.$$

$L^{p,\alpha}(\Gamma)$ is a Banach space with $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$. Similarly we define the weighted Morrey-Lebesgue space $L_\mu^{p,\alpha}(\Gamma)$ with the weight function $\mu(\cdot)$ on Γ equipped with the norm

$$\|f\|_{L_\mu^{p,\alpha}(\Gamma)} = \|f\mu\|_{L^{p,\alpha}(\Gamma)}, f \in L_\mu^{p,\alpha}(\Gamma).$$

The inclusion $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 < \alpha_1 \leq \alpha_2 \leq 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall \alpha \in (0, 1]$, $\forall p \geq 1$. For $\Gamma = [-\pi, \pi]$ we will use the notation $L^{p,\alpha}(-\pi, \pi) = L^{p,\alpha}$.

More details on Morrey-type spaces can be found in [2-26].

In the sequel, we will need some auxiliary facts. Recall Minkowski's (integral) inequality.

Let $(X; A_x; \nu)$ and $(Y; A_y; \mu)$ be measurable spaces with $\sigma-$finite measures ν and μ, respectively. If $F(x; y)$ is $\nu \times \mu-$measurable, then we have

$$\left\| \int_X F(\cdot; y) \, d\nu(x) \right\|_{L^p(d\mu)} \leq \int_X \|F(x; \cdot)\|_{L^p(d\mu)} \, d\nu(x), 1 \leq p < +\infty,$$

where

$$\|g(y)\|_{L^p(d\mu)} = \left(\int_Y |g(y)|^p \, d\mu \right)^{1/p}.$$

Now let $Y \equiv R$ and $\mu(\cdot)$ be a Borel measure on R. We have

$$\left(\int_I \left| \int_X F(x; y) \, d\nu(x) \right|^p d\mu(y) \right)^{1/p} \leq$$

$$\le \int_X \left(\int_I |F(x;\,y)|^p\, d\mu(y) \right)^{1/p} d\nu(x).$$

Thus

$$\left(\frac{1}{|I|^{1-\alpha}} \int_I \left| \int_X F(x;\,y)\, d\nu(x) \right|^p d\mu(y) \right)^{1/p} \le$$

$$\le \int_X \left(\frac{1}{|I|^{1-\alpha}} \int_I |F(x;\,y)|^p\, d\mu(y) \right)^{1/p} d\nu(x) \le$$

$$\le \int_X \|F(x;\,y)\|_{L^{p,\,\alpha}(d\mu)}\, d\nu(x),\, \forall I \subset R.$$

Taking sup over $I \subset R$, we get

$$\left\| \int_X F(x;\,y)\, d\nu(x) \right\|_{L^{p,\,\alpha}(d\mu)} \le \int_X \|F(x;\,y)\|_{L^{p,\,\alpha}(d\mu)}\, d\nu(x). \tag{1}$$

So the Minkowski inequality (1) holds in the Morrey-type space $L^{p,\,\alpha}(d\mu)$.

Thus, the following Minkowski's inequality regarding Morrey type spaces is true.

Statement 1. *Let $(X;\, A_x;\, \nu)$ be a measurable space with a $\sigma-$finite measure ν and $(R;\, B;;\, \mu)$ be a measurable space with a Borel measure μ on $\sigma-$algebra of Borel sets B of R. Then, for $\nu \times \mu-$ measurable function $F(x;\,y)$, the following analog of Minkowski's inequality is valid*

$$\left\| \int_X F(x;\,\cdot)\, d\nu(x) \right\|_{L^{p,\,\alpha}(d\mu)} \le \int_X \|F(x;\,\cdot)\|_{L^{p,\,\alpha}(d\mu)}\, d\nu(x).$$

By S_Γ we denote the following singular integral operator

$$(S_\Gamma f)(\tau) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)\, d\zeta}{\zeta - \tau}, \quad \tau \in \Gamma,$$

where $\Gamma \subset C$ is some rectifiable curve on complex plane C. Let $\omega = \{z \in C : |z| < 1\}$ be the unit disk on C and $\partial\omega = \gamma$ be its boundary. Define the Morrey-Hardy space $H_+^{p,\,\alpha}$ of analytic functions $f(z)$ inside ω equipped with the following norm

$$\|f\|_{H_+^{p,\,\alpha}} = \sup_{0 < r < 1} \left\| f(re^{it}) \right\|_{L^{p,\,\alpha}}.$$

In what follows, we assume that the function $f(\cdot)$ periodically continued on the whole axis R.

We will also use the following concepts. Let $\Gamma \subset C$ be some bounded rectifiable curve, $t = t(\sigma)$, $0 \le \sigma \le 1$, be its parametric representation with respect to the arc length σ, and l be the length of Γ. Let $d\mu(t) = d\sigma$, i.e. let $\mu(\cdot)$ be a linear measure on Γ. Let

$$\Gamma_t(r) = \{\tau \in \Gamma : |\tau - t| < r\}, \Gamma_{t(s)}(r) = \{\tau(\sigma) \in \Gamma : |\sigma - s| < r\}.$$

It is absolutely clear that $\Gamma_{t(s)}(r) \subset \Gamma_t(r)$.

Definition 1. *Curve Γ is said to be Carleson if $\exists c > 0$:*

$$\sup_{t \in \Gamma} \mu\left(\Gamma_t\left(r\right)\right) \leq cr, \forall r > 0.$$

Curve Γ is said to satisfy the chord-arc condition at the point $t_0 = t\left(s_0\right) \in \Gamma$ if there exists a constant $m > 0$ independent of t such that $|s - s_0| \leq m\left|t\left(s\right) - t\left(s_0\right)\right|$, $\forall t\left(s\right) \in \Gamma$. Γ satisfies a chord-arc condition uniformly on Γ if $\exists m > 0 : |s - \sigma| \leq m\left|t\left(s\right) - t\left(\sigma\right)\right|$, $\forall t\left(s\right), t\left(\sigma\right) \in \Gamma$.

Let's state the following lemma of [10] which is of independent interest.

Lemma 1. *[10] Let Γ be a bounded rectifiable curve. If the power function $|t - t_0|^\gamma$, $t_0 \in \Gamma$, belongs to the space $L^{p,\alpha}\left(\Gamma\right)$, $1 \leq p < \infty$, $0 < \alpha < 1$, then the inequality $\gamma \geq -\frac{\alpha}{p}$ holds. If Γ is a Carleson curve, then this condition is also sufficient.*

We will extensively use the following theorem of N.Samko [10].

Theorem 1. *[10] Let the curve Γ satisfy the chord-arc condition and the weight $\rho\left(\cdot\right)$ be defined by*

$$\rho\left(t\right) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k} ; \ \{t_k\}_1^m \subset \Gamma, t_i \neq t_j, i \neq j. \tag{2}$$

A singular operator S_Γ is bounded in the weighted space $L_\rho^{p,\alpha}\left(\Gamma\right)$, $1 < p < +\infty$, $0 < \alpha \leq 1$, if the following inequalities are satisfied

$$-\frac{\alpha}{p} < \alpha_k < -\frac{\alpha}{p} + 1, k = \overline{1, \, m}. \tag{3}$$

Moreover, if Γ is smooth in some neighborhoods of the points t_k, $k = \overline{1, \, m}$, then the validity of inequalities (3) is necessary for the boundedness of the operator S_Γ in $L_\rho^{p,\alpha}\left(\Gamma\right)$.

In what follows, as Γ we will consider a boundary of unit disk: $\gamma = \partial\omega$. Consider the weighted space $L_\rho^{p,\alpha}\left(\gamma\right) =: L_\rho^{p,\alpha}$ with the weight $\rho\left(\cdot\right)$. In an absolutely similar way to the non-weighted case, we define the space $M_\rho^{p,\alpha}$ with the weight $\rho\left(\cdot\right)$. Denote by $\tilde{M}_\rho^{p,\alpha}$ the set of functions whose shifts are continuous in $L_\rho^{p,\alpha}$, i.e.

$$\|S_\delta f - f\|_{p,\alpha;\rho} = \|f\left(\cdot + \delta\right) - f\left(\cdot\right)\|_{p,\alpha;\rho} \to 0, \delta \to 0,$$

where S_δ is a shift operator: $\left(S_\delta f\right)\left(x\right) = f\left(x + \delta\right)$ and we will consider that the function $f\left(\cdot\right)$ (in sequel also) periodically continued to the whole real axis R. It is not difficult to see that $\tilde{M}_\rho^{p,\alpha}$ is a linear subspace of $L_\rho^{p,\alpha}$. Denote the closure of $\tilde{M}_\rho^{p,\alpha}$ in $L_\rho^{p,\alpha}$ by $M_\rho^{p,\alpha}$.

Consider the following class

$$\tilde{L}_\rho^{p,\alpha} =: \left\{f \in L_\rho^{p,\alpha} : \|T_\delta f - f\|_{p,\alpha;\rho} \to 0, \, \delta \to 0\right\}.$$

It is evident that the class $\tilde{L}_\rho^{p,\alpha}$ is a linear subspace of $L_\rho^{p,\alpha}$. Let us denote by $M_\rho^{p,\alpha}$ the closure of $\tilde{L}_\rho^{p,\alpha}$ in $L_\rho^{p,\alpha}$.

Let us remember the following properties of Poisson kernel $P_r(\varphi)$:

$$P_r(\varphi) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\varphi + r^2} , \quad 0 < r < 1.$$

(a) $\sup\limits_{|t|<\delta} P_r(t) \to 0$ as $|r| \to 1$;

(b) $\int_{|t|>\delta} P_r(t)\,dt \to 0$ as $|r| \to 1$ for $\forall \delta > 0$.

These properties directly follows from the expression for $P_r(\varphi)$. We have

$$\|(P_r * f)(\cdot) - f(\cdot)\|_{p,\alpha;\rho} = \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(t-s)\,dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(s)\,dt \right\|_{p,\alpha;\rho} \leq$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt =$$

$$= \frac{1}{2\pi} \left[\int_{|t|>\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt + \int_{|t|<\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt \right].$$

Regarding the second integral in the right-hand side, we have

$$\frac{1}{2\pi} \int_{|t|<\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt \leq$$

$$\leq \sup_{|t|<\delta} \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}, \quad \text{as} \quad \delta \to 0.$$

To estimate the first integral, consider

$$\|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho} \leq \|f\|_{p,\alpha;\rho} + \|f(t-\cdot)\|_{p,\alpha;\rho} = 2\|f\|_{p,\alpha;\rho}.$$

We have

$$\|f(t-\cdot)\|_{p,\alpha;\rho}^p = \sup_B \frac{1}{|B \bigcap \gamma|^{1-\alpha}} \int_{B \bigcap \gamma} |f(t-s)|^p\,ds =$$

$$= \sup_B \frac{1}{|B \bigcap \gamma|^{1-\alpha}} \int_{(B \bigcap \gamma)_t} |f(s)|^p\,ds,$$

where $(B \bigcap \gamma)_t \equiv \{s : t - s \in B \bigcap \gamma\}$. It is clear that

$$\left| B \bigcap \gamma \right| = \left| \left(B \bigcap \gamma \right)_t \right|,$$

holds. Therefore

$$\|f(t-\cdot)\|_{p,\alpha;\rho} = \|f\|_{p,\alpha;\rho}.$$

As a result

$$\int_{|t|>\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt \leq$$

$$\leq 2\|f\|_{p,\alpha;\rho} \int_{|t|>\delta} P_r(t)\,dt \to 0 \text{ as } |r| \to 1.$$

So we have proved the following

Theorem 2. *If $f \in M_\rho^{p,\alpha}$, $1 \leq p < +\infty \wedge 0 \leq \alpha \leq 1$, then $\|P_r * f - f\|_{p,\alpha;\rho} \to 0$ as $|r| \to 1$.*

From Theorem 2 we immediately get the validity of the following

Theorem 3. *Let $f \in M_\rho^{p,\alpha}$, $0 < \alpha \leq 1$, $1 < p < +\infty$. Then it holds*

$$\left\|(Kf)(r\xi) - f^+(\xi)\right\|_{p,\alpha;\rho} \to 0, \ r \to 1 - 0.$$

Similar assertion is true in case of $f^-(\xi)$ when $r \to 1 + 0$.

3. Subspace $M_\rho^{p,\alpha}$

Let $\rho : [-\pi, \pi] \to (0, +\infty)$ be some weight function and consider the space $M_\rho^{p,\alpha}$. It is easy to see that if $\rho \in L^{p,\alpha}$, then $C[-\pi, \pi] \subset M_\rho^{p,\alpha}$ is true. Indeed, let $f \in C[-\pi, \pi]$. Without loss of generality, we assume that the function f periodically continued on the whole axis. We have

$$|f(x + \delta) - f(x)| \leq \|f(\cdot + \delta) - f(\cdot)\|_\infty \to 0, \delta \to 0.$$

Consequently

$$\|f(\cdot + \delta) - f(\cdot)\|_{p,\alpha;\rho} = \|(f(\cdot + \delta) - f(\cdot))\rho(\cdot)\|_{p,\alpha} \leq$$

$$\leq \|f(\cdot + \delta) - f(\cdot)\|_\infty \|\rho(\cdot)\|_{p,\alpha} \to 0, \delta \to 0.$$

Hence, we have $f \in M_\rho^{p,\alpha}$.

Let us show that the set of infinitely differentiable functions is dense in $M_\rho^{p,\alpha}$. Consider the following averaged function

$$\omega_\varepsilon(t) = \begin{cases} c_\varepsilon \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |t|^2}\right), & |t| < \varepsilon, \\ 0, & |t| \geq \varepsilon, \end{cases}$$

where

$$c_\varepsilon \int_{-\infty}^{+\infty} \omega_\varepsilon(t)\, dt = 1.$$

Take $\forall f \in M_\rho^{p,\alpha}$ and consider the convolution $f * g$:

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(t - s)\, g(s)\, ds,$$

and let

$$f_\varepsilon(t) = (\omega_\varepsilon * f)(t) = (f * \omega_\varepsilon)(t).$$

It is clear that f_ε is infinitely differentiable on $[-\pi, \pi]$. We have

$$\|f_\varepsilon - f\|_{p,\alpha;\rho} = \left\| \int_{-\infty}^{+\infty} \omega_\varepsilon(s) f(\cdot - s) ds - f(\cdot) \right\|_{p,\alpha;\rho} =$$

$$= \left\| \int_{-\infty}^{+\infty} \omega_\varepsilon(s) [f(\cdot - s) - f(\cdot)] ds \right\|_{p,\alpha;\rho}.$$

Applying Minkowski inequality (1) to this expression, we obtain

$$\|f_\varepsilon - f\|_{p,\alpha;\rho} \leq \int_{-\infty}^{+\infty} \omega_\varepsilon(s) \|f(\cdot - s) - f(\cdot)\|_{p,\alpha;\rho} ds =$$

$$= \int_{-\varepsilon}^{\varepsilon} \omega_\varepsilon(s) \|f(\cdot - s) - f(\cdot)\|_{p,\alpha;\rho} ds \leq$$

$$= \sup_{|s| \leq \varepsilon} \|f(\cdot - s) - f(\cdot)\|_{p,\alpha;\rho} \to 0, \varepsilon \to 0.$$

Thus, the following theorem is true.

Theorem 4. *Let $\rho \in L^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$. Then infinitely differentiable functions are dense in $M_\rho^{p,\alpha}$.*

Consider the singular operator $S(\cdot)$:

$$Sf(t) = \frac{1}{\pi} \int_\gamma \frac{f(\tau) d\tau}{\tau - t}, t \in \gamma.$$

Applying Theorem 1 [10] to the operator S we obtain the following result.

Theorem 5. *Let the weight $\rho(\cdot)$ be defined by the expression (2), where $\Gamma = \gamma$. Then the operator S is bounded in $L_\rho^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$, i.e. the following inequality holds*

$$\|Sf\|_{p,\alpha;\rho} \leq c \|f\|_{p,\alpha;\rho}, \forall f \in L_\rho^{p,\alpha},$$

if and only if the following inequalities are fulfilled

$$-\frac{\alpha}{p} < \alpha_k < -\frac{\alpha}{p} + 1, k = \overline{1, m}. \tag{4}$$

Let us show that the subspace $M_\rho^{p,\alpha}$ is an invariant with respect to the operator S. It is sufficient to prove that the shift operator S is continuous in $M_\rho^{p,\alpha}$. So, let $f \in M_\rho^{p,\alpha}$ and $\delta \in R$. Consider the shift operator S:

$$(Sf)\left(te^{i\delta}\right) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau) d\tau}{\tau - te^{i\delta}}, t \in \gamma.$$

We have

$$(Sf)\left(te^{i\delta}\right) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau) d\tau e^{-i\delta}}{\tau e^{-i\delta} - t} =$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f\left(\tau e^{-i\delta} e^{i\delta}\right) d\tau e^{-i\delta}}{\tau e^{-i\delta} - t} = \frac{1}{2\pi i} \int_\gamma \frac{f\left(\xi e^{i\delta}\right) d\xi}{\xi - t}.$$

Consequently

$$(Sf)\left(te^{i\delta}\right) - (Sf)(t) = \frac{1}{2\pi i} \int_\gamma \frac{f\left(\xi e^{i\delta}\right) - f\left(\xi\right)}{\xi - t} d\xi =$$

$$= \left(S\left(f\left(\cdot e^{i\delta}\right) - f\left(\cdot\right)\right)\right)(t).$$

Paying attention to Theorem 5, hence we immediately obtain

$$\left\|(Sf)\left(te^{i\delta}\right) - (Sf)(t)\right\|_{p,\alpha;\rho} \le$$

$$\le c \left\|f\left(e^{i\delta}\right) - f\left(\cdot\right)\right\|_{p,\alpha;\rho} \to 0, \delta \to 0,$$

as $f \in M_\rho^{p,\alpha}$. Thus, the following theorem is true.

Theorem 6. *Let the weight ρ be defined by the expression*

$$\rho(t) = \prod_{k=1}^m |t - t_k|^{\alpha_k}, t \in \gamma, \tag{5}$$

where $\{t_k\}_{k=\overline{1,m}} \subset \gamma-$ are different points. If the inequalities (4) hold, then the operator S boundedly acts in $M_\rho^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \le 1$.

Remark 1. *In previous statements and in their proofs the spaces $L_\rho^{p,\alpha}$, $M_\rho^{p,\alpha}(-\pi, \pi)$ and $L_\rho^{p,\alpha}$, $M_\rho^{p,\alpha}$, are naturally identified, respectively, i.e. $L_\rho^{p,\alpha} = L_\rho^{p,\alpha}(-\pi, \pi) = L_\rho^{p,\alpha}(\gamma)$ & $M_\rho^{p,\alpha} = M_\rho^{p,\alpha}(-\pi, \pi) = M_\rho^{p,\alpha}(\gamma)$.*

Consider the following Cauchy integral

$$(Kf)(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi) d\xi}{\xi - z}, z \notin \gamma.$$

Let $f \in L_\rho^{p,\alpha}$, where the weight $\rho(\cdot)$ is defined by the expression (5). Applying Holder's inequality we obtain

$$\|f\|_{L_1} = \left\|f\rho\rho^{-1}\right\|_{L_1} \le c \|f\rho\|_{p,\alpha} \left\|\rho^{-1}\right\|_{q,\alpha} =$$

$$= c \|f\|_{p,\alpha;\rho} \left\|\rho^{-1}\right\|_{q;\alpha}. \tag{6}$$

Suppose that the following inequalities are fulfilled

$$-\frac{\alpha}{p} < \alpha_k < \frac{\alpha}{q}, k = \overline{1,m}, \frac{1}{p} + \frac{1}{q} = 1. \tag{7}$$

Then from (6) it follows that $f \in L_1$. As a result, according to the classical facts, the following Sokhotskii-Plemelj is true

$$f^{\pm}(\xi) = \pm \frac{1}{2} f(\xi) + (Sf)(\xi), \xi \in \gamma, \tag{8}$$

where $f^+(\xi)$ (respectively, $f^-(\xi)$) boundary values of a Cauchy integral $(Kf)(z)$ on γ inside ω (outside ω). Paying attention to Theorem 6, form (8) we obtain that if $f \in M_{\rho}^{p,\alpha}$, then $f^{\pm} \in M_{\rho}^{p,\alpha}$ and the following inequality holds.

$$\|f^{\pm}\|_{p,\alpha;\rho} \le c \|f\|_{p,\alpha;\rho}, \forall f \in M_{\rho}^{p,\alpha}. \tag{9}$$

Assume
$K_z(s) = \frac{e^{is}}{e^{is}-z} -$ Cauchy kernel; $P_z(s) = Re \frac{e^{is}+z}{e^{is}-z} -$ Poisson kernel;
$Q_z(s) = Im \frac{e^{is}+z}{e^{is}-z} -$ is the conjugate Poisson kernel,
($Re-$ is a real part, $Im-$ is an imaginary part). We have

$$K_z(s) = \frac{1}{2} + \frac{1}{2}(P_z(s) + iQ_z(s)) = \frac{1}{2} + \frac{1}{2}\frac{e^{is}+z}{e^{is}-z}, z \in \omega. \tag{10}$$

Let $F(z) = u(z) + i\vartheta(z)$ be an analytic function in ω. It is clear that $F \in H_{\rho}^{p,\alpha}$ if and only if $u; \vartheta \in h_{\rho}^{p,\alpha}$. Paying attention to the relation (10) we arrive at the conclusion that many of the properties of functions from $h_{\rho}^{p,\alpha}$ transferred to the function from $H_{\rho}^{p,\alpha}$. For example, $\forall F \in H_{\rho}^{p,\alpha}$ has a.e. on γ the nontangential boundary values F^+, since, $\lim_{r \to 1-0} F(re^{it}) = F^+(e^{it})$, a.e. $t \in [-\pi, \pi]$. We have

$$F^+(\tau) = u^+(\tau) + i\vartheta^+(\tau), \tau \in \gamma.$$

Let all the conditions of Theorem 2 be fulfilled. Then the following representation is true.

$$u\left(re^{i\theta}\right) = (P_r * u^+)(\theta), \vartheta\left(re^{i\theta}\right) = (P_r * u^+)(\theta),$$

and, as a result

$$F\left(re^{i\theta}\right) = (P_r * F^+)(\theta).$$

Paying attention to Theorem 2, we obtain the following result.

Theorem 7. *Let the weight $\rho(\cdot)$ is defined by the expression (5) and the inequality (7) holds. Then the Sokhotskii-Plemelj formula (8) is valid and for the boundary values the inequality (9) holds .*

Let us prove the following

Theorem 8. *Let $f(\cdot) \in L_{\infty} \bigcap M_{\rho}^{p,1} \wedge g(\cdot) \in M_{\rho}^{p,\alpha}$ and let the weight function ρ satisfies the following condition*

$$\exists \delta_0 > 0 : \frac{1}{|I|^{1-\alpha}} \int_I \rho^p(t) dt \le c |I|^{\delta_0}, \forall I \in [-\pi, \pi].$$

Then $f(\cdot) g(\cdot) \in M_{\rho}^{p,\alpha}$ when $0 < \alpha \le 1$ and $p \ge 1$.

Proof. Let $f\left(\cdot\right)\in L_{\infty}\bigcap M_{\rho}^{p,\,1}$ and $g\left(\cdot\right)\in M_{\rho}^{p,\,\alpha}$, $0<\alpha\leq 1$, $p\geq 1$. For $\alpha=1$ it is evident that $M_{\rho}^{p,\,1}=L_{p,\,\rho}$ and the following estimation is true

$$\int_{-\pi}^{\pi}\left|f\left(t\right)g\left(t\right)\right|^{p}\rho^{p}\left(t\right)dt\leq c\int_{-\pi}^{\pi}\left|g\left(t\right)\right|^{p}\rho^{p}\left(t\right)dt=c\left\|g\right\|_{p,\,\rho}^{p}<+\infty.$$

So, $f\left(\cdot\right)g\left(\cdot\right)\in M_{\rho}^{p,\,\alpha}$ when $\alpha=1$.

Consider the case $0<\alpha<1$. We have

$$\sup_{I}\left(\frac{1}{\left|I\right|^{1-\alpha}}\int_{I}\left|f\left(t\right)g\left(t\right)\right|^{p}\rho^{p}\left(t\right)dt\right)^{\frac{1}{p}}\leq$$

$$\leq c\sup_{I}\left(\frac{1}{\left|I\right|^{1-\alpha}}\int_{I}\left|g\left(t\right)\right|^{p}\rho^{p}\left(t\right)dt\right)^{\frac{1}{p}}=c\left\|g\right\|_{p,\,\alpha;\,\rho}<+\infty.$$

Let us consider

$$\Delta_{\delta}=\left\|f\left(\cdot+\delta\right)g\left(\cdot+\delta\right)-f\left(\cdot\right)g\left(\cdot\right)\right\|_{p,\,\alpha;\,\rho}.$$

For any $\varepsilon>0$ and $m>0$ there is a $\varphi\left(\cdot\right)\in C\left[-\pi;\,\pi\right]$ such that $\left\|g\left(\cdot\right)-\varphi\left(\cdot\right)\right\|_{p,\,\alpha;\,\rho}<\frac{\varepsilon}{m}$, as $g\in M_{\rho}^{p,\,\alpha}$. We have

$$\Delta_{\delta}=\left\|f\left(\cdot+\delta\right)\left[g\left(\cdot+\delta\right)-\varphi\left(\cdot+\delta\right)+\varphi\left(\cdot+\delta\right)\right]-f\left(\cdot\right)\left[g\left(\cdot\right)-\varphi\left(\cdot\right)+\varphi\left(\cdot\right)\right]\right\|_{p,\,\alpha;\,\rho}\leq$$

$$\leq c_{f}\left\|g\left(\cdot+\delta\right)-\varphi\left(\cdot+\delta\right)\right\|_{p,\,\alpha;\,\rho}+\left\|f\left(\cdot+\delta\right)\varphi\left(\cdot+\delta\right)-f\left(\cdot\right)\varphi\left(\cdot\right)\right\|_{p,\,\alpha;\,\rho}+c_{f}\left\|g\left(\cdot\right)-\varphi\left(\cdot\right)\right\|_{p,\,\alpha;\,\rho},$$

where $c_{f}=\left\|f\left(\cdot\right)\right\|_{L_{\infty}}$. From $\left\|g\left(\cdot\right)-\varphi\left(\cdot\right)\right\|_{p,\,\alpha;\,\rho}<\frac{\varepsilon}{m}$ it follows

$$\left\|g\left(\cdot+\delta\right)-\varphi\left(\cdot+\delta\right)\right\|_{p,\,\alpha;\,\rho}\leq\left\|g\left(\cdot+\delta\right)-g\left(\cdot\right)\right\|_{p,\,\alpha;\,\rho}+$$

$$+\left\|g\left(\cdot\right)-\varphi\left(\cdot\right)\right\|_{p,\,\alpha;\,\rho}+\left\|\varphi\left(\cdot\right)-\varphi\left(\cdot+\delta\right)\right\|_{p,\,\alpha;\,\rho}.$$

It is obvious that $\left\|g\left(\cdot+\delta\right)-g\left(\cdot\right)\right\|_{p,\,\alpha;\,\rho}\to 0$, $\delta\to 0$.

Let the weight function ρ satisfies the following condition

$$\exists\delta_{0}>0:\frac{1}{\left|I\right|^{1-\alpha}}\int_{I}\rho^{p}\left(t\right)dt\leq c\left|I\right|^{\delta_{0}},\forall I\in\left[-\pi;\,\pi\right],$$

where $c>0$ is some constant.

It follows from uniformly continuity that for $\forall m>0$, $\varepsilon>0$ there exists $\delta_{1}>0:\forall\delta\in\left(-\delta_{1},\,\delta_{1}\right)$:

$$\left\|\varphi\left(\cdot\right)-\varphi\left(\cdot+\delta\right)\right\|_{p,\,\alpha;\,\rho}=\sup_{I}\left(\frac{1}{\left|I\right|^{1-\alpha}}\int_{I}\left|\varphi\left(t\right)-\varphi\left(t+\delta\right)\right|^{p}\rho^{p}\left(t\right)\right)^{\frac{1}{p}}<$$

$$<\frac{\varepsilon}{m}\sup_{I}\left(\frac{1}{\left|I\right|^{1-\alpha}}\int_{I}\rho^{p}\left(t\right)\right)^{\frac{1}{p}}<\frac{\varepsilon}{m}\sup_{I}\left(c\left|I\right|^{\delta_{0}}\right)^{\frac{1}{p}}=c\frac{\varepsilon}{m}\left(2\pi\right)^{\frac{\delta_{0}}{p}}.$$

Then the previous inequality implies

$$\Delta_\delta \leq c_f \frac{\varepsilon}{m} \left(2 + c \left(2\pi \right)^{\frac{\delta_0}{p}} \right) + \left\| f \left(\cdot + \delta \right) \varphi \left(\cdot + \delta \right) - f \left(\cdot \right) \varphi \left(\cdot \right) \right\|_{p, \alpha; \rho}.$$

Thus, it is suffices to prove that for $\varphi \left(\cdot \right) \in C \left[-\pi; \pi \right]$ it is true

$$\lim_{\delta \to 0} \left\| f \left(\cdot + \delta \right) \varphi \left(\cdot + \delta \right) - f \left(\cdot \right) \varphi \left(\cdot \right) \right\|_{p, \alpha; \rho} = 0.$$

We have

$$\left\| f \left(\cdot + \delta \right) \varphi \left(\cdot + \delta \right) - f \left(\cdot \right) \varphi \left(\cdot \right) \right\|_{p, \alpha; \rho} \leq \left\| f \left(\cdot + \delta \right) \left[\varphi \left(\cdot + \delta \right) - \varphi \left(\cdot \right) \right] \right\|_{p, \alpha; \rho} +$$

$$+ \left\| \left[f \left(\cdot + \delta \right) - f \left(\cdot \right) \right] \varphi \left(\cdot \right) \right\|_{p, \alpha; \rho} \leq c_f \left\| \varphi \left(\cdot + \delta \right) - \varphi \left(\cdot \right) \right\|_{p, \alpha; \rho} + c_\varphi \left\| f \left(\cdot + \delta \right) - f \left(\cdot \right) \right\|_{p, \alpha; \rho}.$$

where $c_\varphi = \left\| \varphi \left(\cdot \right) \right\|_{L_\infty}$. Let us take

$$\Delta_\delta \left(f \right) = \left\| f \left(\cdot + \delta \right) - f \left(\cdot \right) \right\|_{p, \alpha; \rho}.$$

Let $\vartheta > 0$ be an arbitrary number. We have

$$\Delta_\delta \left(f \right) = \max \left\{ \Delta_\delta^{(1)} \left(f \right), \Delta_\delta^{(1)} \left(f \right) \right\},$$

where

$$\Delta_\delta^{(1)} \left(f \right) = \sup_{I : |I| \leq \vartheta} \left(\frac{1}{|I|^{1-\alpha}} \int_I |f \left(t + \delta \right) - f \left(t \right)|^p \rho^p \left(t \right) dt \right)^{\frac{1}{p}},$$

$$\Delta_\delta^{(21)} \left(f \right) = \sup_{I : |I| \leq \vartheta} \left(\frac{1}{|I|^{1-\alpha}} \int_I |f \left(t + \delta \right) - f \left(t \right)|^p \rho^p \left(t \right) dt \right)^{\frac{1}{p}}.$$

Regarding $\Delta_\delta^{(1)} \left(f \right)$, we have

$$\Delta_\delta^{(1)} \left(f \right) \leq 2 c_f \sup_{|I| \leq \vartheta} \left(\frac{1}{|I|^{1-\alpha}} \int_I \rho^p \left(t \right) dt \right)^{\frac{1}{p}} \leq$$

$$\leq 2 c_f \sup_{|I| \leq \vartheta} \left(|I| \right)^{\frac{\delta_0}{p}} = \tilde{c} \vartheta^{\frac{\delta_0}{p}}.$$

Regarding $\Delta_\delta^{(2)} \left(f \right)$, we have

$$\Delta_\delta^{(2)} \left(f \right) \leq \vartheta_\varepsilon^{\frac{\alpha-1}{p}} \sup_{|I| \geq \vartheta} \left(\int_I |f \left(t + \delta \right) - f \left(t \right)|^p \rho^p \left(t \right) dt \right)^{\frac{1}{p}} \leq$$

$$\leq \vartheta_\varepsilon^{\frac{\alpha-1}{p}} \left(\int_{-\pi}^{\pi} |f \left(t + \delta \right) - f \left(t \right)|^p \rho^p \left(t \right) dt \right)^{\frac{1}{p}} \leq$$

$$\vartheta_\varepsilon^{\frac{\alpha-1}{p}} \|f(\cdot+\delta) - f(\cdot)\|_{p,\rho}.$$

where $\vartheta = \left(\frac{\varepsilon}{c}\right)^{\frac{p}{\delta_0}} := \vartheta_\varepsilon$. It is clear that $\exists \delta_2 > 0$:

$$\|f(\cdot+\delta) - f(\cdot)\|_{p,\rho} < \vartheta_\varepsilon^{\frac{1}{p}}, \forall \delta \in (-\delta_2,\,\delta_2),$$

where we can choose $\varepsilon_1 = \vartheta_\varepsilon^{\frac{1}{p}}$ for any $\varepsilon_1 > 0$. Hence we get $\Delta_\delta^{(2)}(f) \le \left(\frac{\varepsilon}{c}\right)^{\frac{\alpha}{\delta_0}}$.
Now let us take $\Delta_\delta(f) \le \max\left\{\varepsilon_2, \left(\frac{\varepsilon_2}{c}\right)^{\frac{\alpha}{\delta_0}}\right\}$, where $\varepsilon_2 = \frac{\varepsilon}{m}$ for any $m > 0$.
Consequently

$$\Delta_\delta \le \frac{\varepsilon}{m}\left(c_f\left(2 + 2c\left(2\pi\right)\right)^{\frac{\delta_0}{p}} + c_\varphi\right), \forall \delta \in (-\delta_3,\,\delta_3),$$

where $\delta_3 = \min\{\delta_1,\,\delta_2\}$. By taking $m = \left(c_f\left(2 + 2c\left(2\pi\right)\right)^{\frac{\delta_0}{p}} + c_\varphi\right)$, we get $\Delta_\delta \le \varepsilon$, $\forall \delta \in (-\delta_3, \delta_3)$. It follows that $\Delta_\delta \to 0$ as $\delta \to 0$. ◄

Acknowledgements

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