

On the Completeness of Double System of Exponents in the Weighted Lebesgue Space

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Abstract. This paper considers double system of exponentials with linear phase in the weighted space $L_{p,\rho}$ with power weight $\rho(\cdot)$ on the segment $[\pi, \pi]$. Under certain conditions on the weight function $\rho(\cdot)$ and on the perturbation parameters, the completeness of this system in $L_{p,\rho}$ is proved. An explicit expression for the biorthogonal system in the case of minimality is derived. The obtained results generalize all previously known results in this direction.

Key Words and Phrases: exponential system, completeness, weighted space.

2010 Mathematics Subject Classifications: 30B60, 42C15, 46A35

1. Introduction

Investigation of many partial differential equation by the application of Fourier method reduces to perturbed trigonometric system of sines (or cosines) of the form

$$\{\sin(nt + \alpha(t))\}_{n \in N}, \quad (1)$$

where $\alpha : [0, \pi] \rightarrow R$ is some function (N is a set of natural numbers). Similar problems were studied, for example, in the papers [4, 5, 6, 7, 8, 9, 11]. To justify the Fourier method it is necessary to study the basicity properties (completeness, minimality, basis property, etc.) of these systems in different functional spaces. Complex versions of these systems are perturbed system of exponents of the form

$$\left\{ e^{i(nt + \beta(t) \operatorname{sign} n)} \right\}_{n \in Z}, \quad (2)$$

where $\beta : [-\pi, \pi] \rightarrow R$ is some function (Z is the set of integers). Basis properties of the systems (1) and (2) in corresponding spaces are closely linked, in Lebesgue spaces L_p they are well studied by various mathematicians (see, for example [4, 5, 6, 12, 13, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 10]). The case $L_\infty = C[-\pi, \pi]$ is treated in [38]. In connection with application to solution of differential equations, the interest in Lebesgues spaces $L_{p(\cdot)}$ with

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variable summability power $p(\cdot)$ and in Morrey spaces $L^{p,\alpha}$ greatly increased in recent years. Problems of approximation in these spaces have also begun to be studied and basicity problems of the systems (1), (2) in $L_{p(\cdot)}$ are studied in [29, 30], but basicity of the classical system of exponents with linear phase in Morrey spaces are studied in [33, 34]. Note that study of basicity properties of the systems (1), (2) in weighted spaces $L_{p,\rho}$ is equivalent to the study of analogous properties of the systems (1), (2) with corresponding degenerate coefficients in the spaces L_p . For this reason, it can be assumed that the study of the basicity of trigonometric systems in weighted Lebesgue spaces takes its origin from the paper of K.Babenko [18]. Later this area was developed in the works [14, 15, 16, 28, 31, 32, 35, 36]. The problem of basicity of the exponential system in the weighted space $L_{p,\rho} \equiv L_{p,\rho}(-\pi, \pi)$, $1 < p < +\infty$, is solved in the paper [37]. Such a condition is a Muckenaupt condition with respect to the weight function $\rho(\cdot)$:

$$\sup_I \left(\frac{1}{|I|} \int_I \rho(t) dt \right) \left(\frac{1}{|I|} \int_I \rho^{-\frac{1}{p-1}} dt \right)^{p-1} < \infty, \quad (3)$$

where sup is taken over all intervals $I \subset [-\pi, \pi]$ and $|I|$ is the length of the interval I (see e.g. [3]).

In the papers [2, 15] the system (2) is considered in the case when $\beta(t) = \beta t$, where $\beta \in \mathbb{R}$ is some real parameter and its basicity in $L_{p,\rho}$, $1 < p < +\infty$, is studied when $\rho(\cdot)$ has the following form

$$\rho(t) = \prod_{k=-r}^r |t - t_k|^{\alpha_k},$$

where $-\pi = t_{-r} < t_{-r+1} < \dots < t_r = \pi$.

The class of weights, satisfying the condition (3), is denoted by A_p . It is easy to see that

$$\rho \in A_p \Leftrightarrow -1 < \alpha_k < p - 1, \quad k = \overline{-r, r}.$$

It is additionally required in [2] that the condition $\alpha_{-r} = \alpha_r$ holds, which means that degeneration must be present at both ends of the segment $[-\pi, \pi]$. This effect does not take place in the paper [15].

In this paper the completeness of the exponential system

$$\left\{ e^{i(n + \frac{\beta}{2} \text{sign } n)t} \right\}_{n \in \mathbb{Z}}, \quad (4)$$

in the weighted space $L_{p,\rho}$, $1 < p < +\infty$, where $\beta \in \mathbb{C}$ is a complex parameter, is studied. Under certain conditions on the parameter β and the weight function $\rho(\cdot)$, the completeness of the system (4) is established in the space $L_{p,\rho}$.

2. Preliminaries. Main lemma

Consider the following double system of exponents

$$\left\{ e^{i[(n+\beta_1)t+\gamma]}, e^{-i[(k+\beta_2)t+\gamma_2]} \right\}_{n \in \mathbb{Z}_+; k \in \mathbb{N}}, \quad (5)$$

where $\beta_k = Re\beta_k + iIm\beta_k$, $\gamma_k = Re\gamma_k + iIm\gamma_k$, $k = 1, 2$, are complex parameters, $Z_+ = \{0\} \cup N$. We assume that the weight function $\rho(\cdot)$ is of the following power form

$$\rho(t) = \prod_{k=-r}^r |t - t_k|^{\alpha_k},$$

where $-\pi = t_{-r} < t_{-r+1} < \dots < t_0 = 0 < \dots < t_r = \pi$, $\{\alpha_k\}_{k=-r}^r \subset R$ are some numbers. We consider the weighted space $L_{p,\rho}$, $1 < p < +\infty$, with the norm $\|\cdot\|_{p,\rho}$:

$$\|f\|_{p,\rho} = \left(\int_{-\pi}^{\pi} |f(t)|^p \rho(t) dt \right)^{1/p}.$$

It is easy to see that basicity properties of the system (5) in $L_{p,\rho}$ are equivalent to basicity properties of the system

$$\left\{ e^{i(n+\beta_1)t}; e^{-i(k+\beta_2)t} \right\}_{n \in Z_+; k \in N}, \quad (6)$$

in $L_{p,\rho}$. We put $g(t) = e^{\frac{i}{2}(\beta_2 - \beta_1)t}$. It is evident that $\exists \delta > 0$:

$$0 < \delta \leq |g(t)| \leq \delta^{-1} < +\infty, \quad \forall t \in [-\pi, \pi].$$

Multiplying the system (6) to the function $g(t)$, we immediately obtain from here that the basicity properties of the system (6) on $L_{p,\rho}$ are equivalent to the basicity properties of the system (4) on $L_{p,\rho}$, $\beta = \beta_1 + \beta_2$. Thus, the study of basicity properties of the system (5) on $L_{p,\rho}$ is reduced to the investigation of corresponding properties with respect to the system (4) on $L_{p,\rho}$.

Let $\beta \in C$ be some complex number. We will assume throughout the paper that $(1+z)^\beta$ is some fixed branch of multivalued analytic function $(1+z)^\beta$ on the complex plane with the cut along the semiline $(-\infty, -1) \subset R$ on the real axis and take

$$(1+z)^{-\beta} = \frac{1}{(1+z)^\beta}.$$

Analogously, we define a branch z^β of a multivalued function z^β on C with the cut along $(-\infty, 0) \subset R$ and $z^{-\beta} = \frac{1}{z^\beta}$.

We will essentially use the following main lemma in the proof of main results.

Lemma 1. [38] *Let $Re\beta > -1$. Then the following Cauchy integral formulas hold*

$$J_m^-(z) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i(\beta+m)\theta} (1+e^{i\theta})^\beta}{e^{i\theta} - z} d\theta \equiv \begin{cases} 0, & |z| < 1, \\ -z^{-m-\beta-1} (1+z)^\beta, & |z| > 1, \end{cases}$$

$$J_m^+(z) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i(m+1)\theta} (1+e^{i\theta})^\beta}{e^{i\theta} - z} d\theta \equiv \begin{cases} 0, & |z| > 1, \\ z^m (1+z)^\beta, & |z| < 1, \end{cases}$$

$$\forall m \in Z_+.$$

Consider the following system of functions

$$\begin{aligned}\vartheta_n^+(t) &= \frac{e^{-i\frac{\beta}{2}t}}{2\pi} (1 + e^{it})^\beta \sum_{k=0}^n C_{-\beta}^{n-k} e^{-ikt}, \quad n \in Z_+; \\ \vartheta_m^-(t) &= -\frac{e^{-i\frac{\beta}{2}t}}{2\pi} (1 + e^{it})^\beta \sum_{k=1}^m C_{-\beta}^{m-k} e^{ikt}, \quad m \in N;\end{aligned}$$

where

$$C_{-\gamma}^k = \frac{\gamma(\gamma-1)\dots(\gamma-k+1)}{k!},$$

is a binomial coefficient. Accordingly, we denote

$$e_n^+(t) \equiv e^{i(n+\frac{\beta}{2})t}, \quad n \in Z_+; \quad e_k^-(t) \equiv e^{-i(n+\frac{\beta}{2})t}, \quad k \in N.$$

Assume that $Re\beta > -1$. The expansion in powers of z of the function $(1+z)^{-\beta} J_m^+(z)$ that is analytic on $|z| < 1$ is

$$(1+z)^{-\beta} J_m^+(z) = \sum_{n=0}^{\infty} a_{n;m}^+ z^n,$$

where

$$a_{n;m}^+ = \int_{-\pi}^{\pi} e^{i(m+\frac{\beta}{2})t} \vartheta_n^+(t) dt.$$

On the other hand, it follows from Lemma 1 that

$$(1+z)^{-\beta} J_m^+(z) \equiv z^m, \quad |z| < 1.$$

Comparing the corresponding coefficients, we arrive at the following equalities

$$\int_{-\pi}^{\pi} e_m^+(t) \vartheta_n^+(t) dt = \delta_{nm}, \quad \forall n, m \in Z_+.$$

Expanding the function $(1+z)^{-\beta} J_m^+(z)$ at infinity in powers of z^{-1} , we obtain

$$(1+z)^{-\beta} J_m^+(z) = \sum_{n=1}^{\infty} b_{n;m}^+ z^{-n}, \quad |z| > 1,$$

where

$$b_{n;m}^+ = \int_{-\pi}^{\pi} e^{i(m+\frac{\beta}{2})t} \vartheta_n^-(t) dt, \quad m \in Z_+, \quad n \in N.$$

It is easy to see that

$$\lim_{|z| \rightarrow \infty} (1+z)^{-\beta} J_m^+(z) = 0.$$

On the other hand, again, as follows from Lemma 1, we have

$$(1+z)^{-\beta} J_m^+(z) \equiv 0, \quad |z| > 1.$$

These two expansions imply

$$\int_{-\pi}^{\pi} e^{i(m+\frac{\beta}{2})t} \vartheta_n^-(t) dt = 0, \quad \forall m \in Z_+, \quad \forall n \in N.$$

The relations

$$\begin{aligned} \int_{-\pi}^{\pi} e_m^-(t) \vartheta_n^+(t) dt &= 0, \quad m \in N, \quad n \in Z_+; \\ \int_{-\pi}^{\pi} e_m^-(t) \vartheta_n^-(t) dt &= \delta_{nm}, \quad \forall n, m \in N \end{aligned}$$

can be proved analogously.

As a result, we obtain the validity of the following statement.

Proposition 1. *Let $Re\beta > -1$. Then for all admissible values of indices n and m the following relations*

$$\int_{-\pi}^{\pi} e_n^{\pm}(t) \vartheta_m^{\pm}(t) dt = \delta_{nm}, \quad \int_{-\pi}^{\pi} e_n^{\pm}(t) \vartheta_m^{\mp}(t) dt = 0,$$

hold.

Define the following system of functions

$$h_n^{\pm}(t) = \rho^{-1}(t) \overline{\vartheta_n^{\pm}(t)}.$$

3. Completeness in $L_{p,\rho}$

The following lemma on the uniform convergence plays an important role in the study of the completeness of the exponential system (4) in $L_{p,\rho}$.

Lemma 2. *Let $-1 < Re\beta < 0$, or $\beta = 0$. If $\psi(\cdot)$ is an arbitrary Hlder function on $[-\pi, \pi] : e^{i\beta\pi}\psi(-\pi) = \psi(\pi) = 0$, then the series*

$$\sum_{n=0}^{\infty} a_n^+ e_n^+(t) + \sum_{n=1}^{\infty} a_n^- e_n^-(t),$$

uniformly converges to $\psi(\cdot)$ on $[-\pi, \pi]$, where $a_n^{\pm} = \int_{-\pi}^{\pi} \psi(t) h_n^{\pm}(t) \rho(t) dt$.

Proof. Consider the following conjugate problem: find a piecewise analytic function $F(z)$ inside and outside of the unit circle, which the boundary values on the unit circle satisfy the following condition

$$F^+(e^{it}) + e^{-i\beta t} F^-(e^{it}) = e^{-i\frac{\beta}{2}t} \psi(t), \quad t \in (-\pi, \pi]. \quad (7)$$

We will solve this problem by the method developed in the monograph F.D. Gakhov [1] (see page 427). Consider the following multi-valued analytic function in the complex plane

$$\omega(z) = (z+1)^\gamma.$$

We carry out cut on the plane z from zero to infinity ($-\infty$) along the negative real axis. On the plane that have cut in this way, this function will be unique and the incision for it will be line of the rupture . Denote this branch by

$$\omega_{-1}(z) = (z+1)_{-1}^{\gamma}.$$

Let us define

$$\gamma = \frac{1}{2\pi i} \ln e^{-i2\beta\pi} \Rightarrow Re\gamma = -Re\beta.$$

A solution of problem (7) is the following Cauchy type integral

$$\begin{aligned} F^+(z) &= (z+1)_{-1}^{\gamma} X_1^+(z) \Psi^+(z), \\ F^-(z) &= \left(\frac{z+1}{z}\right)_{-1}^{\gamma} X_1^-(z) \Psi^-(z), \end{aligned}$$

where

$$\begin{aligned} X_1(z) &= \exp[\Gamma(z)], \\ \Gamma(z) &= \frac{1}{2\pi i} \int_L \frac{\ln[\tau^{-\gamma} G(\arg \tau)]}{\tau - z} d\tau, \\ \Psi(z) &= \frac{1}{2\pi i} \int_L \frac{(\tau+1)_{-1}^{-\gamma} \varphi(\arg \tau)}{X_1^+(\tau)(\tau - z)} d\tau, \\ G(t) &= e^{-i\beta t}; \varphi(t) = e^{-i\frac{\beta}{2}t} \psi(t), \end{aligned}$$

$L-$ is a unit circle, which goes around from the point $e^{-i\pi}$ to the point $e^{i\pi}$ in the positive direction.

The fact that $F(z)$ satisfies the boundary condition (7), follows directly from the Sokhotskii-Plemelj formulas. Let $0 < Re\gamma < 1$. It is clear that the function $G(t)$ satisfies the Holder condition on the interval $[-\pi, \pi]$. Moreover, it is easy to verify that the function $\tau^{-\gamma} G(\arg \tau)$ is continuous at a point $\tau = -1$, and as a result it satisfies a certain Holder condition on the unit circle. Then according to the results of the monograph F.D.Gakhov [1] (see page 55) the function $X_1^{\pm}(\tau)$ satisfies the Holder condition on L . Denote

$$L_{-\pi} = \left\{ z = e^{it} : t \in \left[-\pi, -\frac{\pi}{2} \right] \right\}.$$

Assume

$$\varphi^*(\tau) = \frac{\psi(\arg \tau)}{X^+(\tau) \tau^{\frac{\beta}{2}}}, \quad \tau \in L.$$

Let $[(z+1)^{-\gamma}]^*$ be a branch, holomorphic function $(z+1)^{-\gamma}$ in the cut along a plane $L_{-\pi}$ that takes values $(t+1)_{-1}^{-\gamma}$ on the left side $L_{-\pi}$. So

$[(t+1)^{-\gamma}]^* = (t+1)_{-1}^{-\gamma}$ on $L_{\pi} = \left\{ z = e^{it} : t \in \left[\frac{\pi}{2}, \pi \right] \right\}$, then using the results from the monograph of [1] (see page 74), the function $\Psi(z)$ near the point $z = -1$ on the contour L can be represented as

$$\Psi(t) = \left[\frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1+0) - \frac{ctg \gamma\pi}{2i} \varphi^*(-1-0) \right] \frac{1}{[(t+1)^\gamma]^*} + \Phi(t), \quad \text{for } t \in L_\pi; \quad (8)$$

$$\Psi(t) = \left[\frac{ctg \gamma\pi}{2i} \varphi^*(-1+0) - \frac{e^{-i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1-0) \right] \frac{1}{[(t+1)^\gamma]^*} + \Phi(t), \quad \text{for } t \in L_{-\pi}, \quad (9)$$

where under $[(t+1)^{-\gamma}]^*$ we mean the limit of the function $[(z+1)^{-\gamma}]^*$, when z tends to t on the left of $L_{-\pi} \cup L_\pi$, and moreover

$$\Phi(t) = \frac{\Phi^*(t)}{|t+1|^{\gamma_0}}, \quad \gamma_0 < Re \gamma, \quad (10)$$

and the function $\Phi^*(t)$ belongs to the Holder class at the neighborhood of the point $z = -1$.

Applying the Sokhotskii-Plemelj formula from these representations near the point $z = -1$ we have

$$\begin{aligned} F^+(t) &= (t+1)_{-1}^\gamma X_1^+(t) \left[\frac{1}{2} (t+1)_{-1}^{-\gamma} \varphi^*(t) + \frac{1}{2\pi i} \int_L \frac{\varphi^*(\tau)}{(\tau+1)_{-1}^\gamma (\tau-t)} d\tau \right] = X_1^+(t) \left[\frac{1}{2} \varphi^*(t) + \right. \\ &\quad \left. + \frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1+0) - \frac{ctg \gamma\pi}{2i} \varphi^*(-1-0) \right] + (t+1)_{-1}^\gamma X_1^+(t) \Phi(t). \end{aligned}$$

Passing to the limit as $t \rightarrow -1-0$, and taking into account the relation (10), we obtain

$$\begin{aligned} F^+(-1-0) &= X_1^+(-1) \left[\frac{1}{2} \varphi^*(-1-0) + \frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1+0) - \frac{ctg \gamma\pi}{2i} \varphi^*(-1-0) \right] = \\ &= X_1^+(-1) \left[\frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1+0) - \frac{e^{-i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1-0) \right]. \end{aligned}$$

Similarly, from the expressions (8) and (9) we obtain

$$\begin{aligned} F^+(-1+0) &= X_1^+(-1) \left[\frac{1}{2} \varphi^*(-1+0) + \frac{ctg \gamma\pi}{2i} \varphi^*(-1+0) - \frac{e^{-i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1-0) \right] = \\ &= X_1^+(-1) \left[\frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1+0) - \frac{e^{-i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1-0) \right]. \end{aligned}$$

Thus, $F^+(-1-0) = F^+(-1+0)$, i.e. $F^+(t)$ is continuous at the point $z = -1$, and as a result, it satisfies a certain Holder condition on L . Expanding $F^+(z)$ on z at zero, we obtain

$$F^+(z) = \sum_{n=0}^{\infty} a_n^+ z^n,$$

where

$$a_n^+ = \int_{-\pi}^{\pi} \psi(t) h_n^+(t) \rho(t) dt, \quad n \in Z_+.$$

We have

$$\frac{1}{2\pi i} \int_{|z|=r<1} F^+(z) z^{-n-1} dz = \begin{cases} a_n^+, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Passing to the limit as $r \rightarrow 1 - 0$, hence we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F^+(e^{it}) e^{-int} dt = \begin{cases} a_n^+, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

As, the function $F^+(e^{it})$ satisfies Holder condition on L , then its Fourier series on classical system of exponents $\{e^{int}\}_{n \in Z}$ uniformly converges to it on $[-\pi, \pi]$, and consequently

$$F^+(e^{it}) = \sum_{n=0}^{\infty} a_n^+ e^{int}, \quad t \in [-\pi, \pi].$$

Now, we investigate the boundary properties of the function $F^-(z)$. Similarly to the case $F^+(z)$, using the representation (8) - (10), and the Sohotskogo- Plemelj formula, we obtain

$$\begin{aligned} F^-(t) &= t_{-1}^{-\gamma} (1+t)_{-1}^{\gamma} X_1^-(t) \Psi^-(t) = t_{-1}^{-\gamma} (1+t)_{-1}^{\gamma} X_1^-(t) \\ &\left\{ -\frac{1}{2} \varphi^*(t) (1+t)_{-1}^{-\gamma} + \left[\frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1+0) - \frac{ctg\gamma\pi}{2i} \varphi^*(-1-0) \right] (1+t)_{-1}^{-\gamma} + \Phi(t) \right\} = \\ &= t_{-1}^{-\gamma} X_1^-(t) \left[-\frac{\varphi^*(t)}{2} + \frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1+0) - \frac{ctg\gamma\pi}{2i} \varphi^*(-1-0) + (1+t)_{-1}^{\gamma} \Phi(t) \right]. \end{aligned}$$

Passing to the limit as $t \rightarrow 1 - 0$, we have

$$\begin{aligned} F^-(-1-0) &= e^{-i\gamma\pi} X_1^-(-1) \left[-\frac{\varphi^*(-1-0)}{2} + \frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1+0) - \frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1-0) \right] = \\ &= e^{-i\gamma\pi} X_1^-(-1) \frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} [\varphi^*(-1+0) - \varphi^*(-1-0)] = 0. \end{aligned}$$

$$\begin{aligned} F^-(-1+0) &= e^{i\gamma\pi} X_1^-(-1) \left[-\frac{\varphi^*(-1+0)}{2} + \frac{e^{i\gamma\pi}}{2i \sin \gamma\pi} \varphi^*(-1+0) - \frac{ctg\gamma\pi}{2i} \varphi^*(-1-0) \right] = \\ &= e^{i\gamma\pi} X_1^-(-1) \frac{ctg\gamma\pi}{2i} [\varphi^*(-1+0) - \varphi^*(-1-0)] = 0. \end{aligned}$$

Thus, $F^-(-1-0) = F^-(-1+0) = 0$, and as a result, $F^-(t)$ satisfies Holder condition on L . Similarly to the case $F^+(t)$, it is proved that the series

$$\sum_{n=1}^{\infty} a_n^- e^{-int},$$

uniformly converges to $F^-(e^{it})$ on $[-\pi, \pi]$. Then from the boundary condition (7) it follows that the biorthogonal series

$$\sum_{n=0}^{\infty} a_n^+ e_n^+(t) + \sum_{n=1}^{\infty} a_n^- e_n^-(t),$$

uniformly converges to $\psi(t)$ on $[-\pi, \pi]$. ◀

Using the representation (8), (9) and the expression of the functions $F^\pm(z)$ we establish the validity of the following lemma.

Lemma 3. *Let $0 < \operatorname{Re}\beta < 1$ and $\psi(\cdot)$ be an arbitrary Holder function on $[-\pi, \pi]$. Then the series*

$$\sum_{n=0}^{\infty} a_n^+ e_n^+(t) + \sum_{n=1}^{\infty} a_n^- e_n^-(t),$$

where

$$a_n^\pm = \int_{-\pi}^{\pi} \psi(t) h_n^\pm(t) \rho(t) dt,$$

uniformly converges to $\psi(\cdot)$ on every compact $G \subset (-\pi, \pi)$, and if $|1 + e^{it}|^{-\operatorname{Re}\beta} \in L_{p,\rho}$ converges to it in $L_{p,\rho}$, and the following inequality holds

$$-1 < \alpha_k < \frac{p}{q}, \quad k = -\bar{r}, \bar{r}. \quad (11)$$

Indeed, the first part follows from the fact that in this case the functions $F^\pm(e^{it})$ are Holder functions on each compact $G \subset (-\pi, \pi)$. And under fulfilling the inequality (11) the system of exponents $\{e^{int}\}_{n \in \mathbb{Z}}$ forms a basis for $L_{p,\rho}$ and from the inclusion $|1 + e^{it}|^{-\operatorname{Re}\beta} \in L_{p,\rho}$ follows that $F^\pm(e^{it}) \in L_{p,\rho}$.

The following theorem follows directly from this lemma.

Theorem 1. *Let $\rho \in L_1$ and the parameter β satisfy one of the following conditions:*

- i) $-\operatorname{Re}\beta \in \bigcup_{k=0}^{\infty} (k, k+1)$;*
- ii) $-\beta \in \mathbb{Z}_+$;*
- iii) $|1 + e^{it}|^{-\operatorname{Re}\beta} \in L_{p,\rho}$ and the following inequalities hold*

$$-1 < \alpha_k < \frac{p}{q}, \quad k = -\bar{r}, \bar{r}.$$

Then the system (4) is complete in $L_{p,\rho}$, for $\forall p \geq 1$, if $\rho \in L_1$.

Indeed, let us consider the case *i)*, the case *ii)* proves similarly. Let, for example, $-\operatorname{Re}\beta \in (1, 2)$, i.e. $-2 < \operatorname{Re}\beta < -1 \Rightarrow \operatorname{Re}\tilde{\beta} < 0$, where $\tilde{\beta} = \beta + 1$. Consider the system

$$\left\{ e^{i(n+\frac{\beta}{2})t}; e^{-i(n+\frac{\beta}{2})t} \right\}_{n \in \mathbb{N}}. \quad (12)$$

Presenting this system in the form of

$$\left\{ e^{i(n-1+1+\frac{\beta}{2})t}; e^{-i(n+\frac{\beta}{2})t} \right\}_{n \in N},$$

and multiplying it by $e^{-i\frac{\beta}{2}t}$, we get the following system

$$\left\{ e^{i(n+\frac{\tilde{\beta}}{2})t} \right\}_{n \in Z}, \quad (13)$$

where $\tilde{\beta} = \beta + 1$, as a result, as it follows from Lemma 2, the corresponding biorthogonal series of an arbitrary Holder function $f \in C^\alpha[-\alpha, \pi] : f(-\pi) = f(\pi) = 0$ uniformly converges to it on $[-\pi, \pi]$. Denote the partial sums of this series by $S_m(f)$, $m \in N$. Consequently

$$\begin{aligned} \|f - S_m(f)\|_{p,\rho}^p &= \int_{-\pi}^{\pi} |f(t) - S_m(f)(t)|^p \rho(t) dt \leq \\ &\leq \int_{-\pi}^{\pi} \rho(t) dt \max_{[-\pi,\pi]} |f(t) - S_m(f)(t)|^p \rightarrow 0, \quad m \rightarrow \infty. \end{aligned}$$

Since the set of such functions is dense in $L_{p,\rho}$, hence, we obtain the completeness of the system (13), and at the same time of the system (12) in $L_{p,\rho}$. From the completeness of the system (12) follows the completeness of the system (4) in $L_{p,\rho}$. The remaining cases are proved by mathematical induction. Case *iii*) directly follows from the Lemma 3.

Acknowledgements

The authors would like to express their profound gratitude to Corres. Member of ANAS, Prof. Bilal Bilalov, for his valuable guidance to this article.

This work was supported by the Research Program Competition launched by the National Academy of Sciences of Azerbaijan (Program: Frame theory Applications of Wavelet Analysis to Signal Processing in Seismology and Other Fields)

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Received 01 January 2016

Accepted 20 March 2016