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A Remark on the Levelling Algorithm for the Approximation by Sums of Two Compositions

A.Kh. Asgarova, V.E. Ismailov*

Abstract. Let X be compact subset of the d-dimensional Euclidean space and $C(X)$ be the space of continuous functions on X. In [6], the second author, under suitable conditions, showed that the Diliberto-Straus levelling algorithm holds for a subspace of $C(X)$ consisting of sums of two compositions. In the proof, he substantially used the theory of bolts and bolt functionals. In the current paper, we prove the result differently, by implementing Golomb's and also Light and Cheney's ideas.

Key Words and Phrases: uniform approximation, levelling algorithm, best approximation operator, central proximity map.
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1. Introduction

Assume E is a Banach space and X and Y are closed subspaces of E. In addition, assume A and B are best approximation operators acting from E onto X and Y respectively. There are many papers devoted to methods of computing the distance to a given element $z \in E$ from $X+Y$. In this paper, we consider a method called the *levelling algorithm*. This method can be described as follows: Starting with $z_1 = z$ compute $z_2 = z_1 - Az_1$, $z_3 = z_2 - Bz_2$, $z_4 = z_3 - Az_3$, and so forth. Obviously, $z - z_n \in X + Y$ and the sequence $\{\|z_n\|\}_{n=1}^{\infty}$ is nonincreasing. J. von Neumann [17] was the first to prove that in Hilbert space setting the sequence $\{\|z_n\|\}_{n=1}^{\infty}$ converges to the error of approximation from $X+Y$. But for other Banach spaces, the convergence of the algorithm depends on certain additional conditions. The general result of Golomb [5] (see also Light and Cheney [13, p.57]) states that in the above Banach space setting the sequence $\{\|z_n\|\}_{n=1}^{\infty}$ converges in norm to the error of approximation from $X+Y$ provided that the sum $X+Y$ is closed and the equalities

$$\|z - Az + x\| = \|z - Az - x\|, \ \ \|z - Bz + y\| = \|z - Bz - y\|, \tag{1}$$

hold for all $z \in E$, $x \in X$ and $y \in Y$. Note that best approximation operators with the property (1) are called central proximity maps (see [13]).

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*Corresponding author.

In 1951, Diliberto and Straus [3] considered the levelling algorithm in the space of continuous functions. They proved that for the problem of uniform approximation of a bivariate function defined on a unit square by sums of univariate functions, the sequence produced by the levelling algorithm converges to the desired quantity. In this paper, we generalize Diliberto-Straus's result to linear superpositions consisting of two summands. More precisely, we consider the levelling algorithm in the problem of approximating from the set of sums of superpositions, which contains functions of the form $f(s(x))+g(p(x))$,where $s(x)$ and $p(x)$ are fixed continuous mappings and f and g are variable univariate continuous functions on the images of s and p, respectively. Under suitable assumptions, we prove that the sequence produced by the levelling algorithm converges to the error of approximation. It should be noted that using the idea of *bolts* (for this terminology see [2, 4, 7, 9, 10, 11, 12]) and methods of Functional Analysis, the second author [6] proved the convergence of the algorithm in the setting considered in this paper. The method of the proof presented here is different and quite short. It is mainly based on the above result of Golomb [5] and ideas of Light and Cheney [13].

2. Levelling algorithm for the sum of two compositions

Let Q be a compact subset of the space $\mathbb{R}^d$. Fix two continuous maps $s : Q \longrightarrow \mathbb{R}$, $p : Q \longrightarrow \mathbb{R}$ and consider the following spaces

$$\begin{aligned} D_1 &= \{f(s(x)) : \ f \in C(\mathbb{R})\}, \\ D_2 &= \{g(p(x)) : \ g \in C(\mathbb{R})\}, \\ D &= D_1 + D_2. \end{aligned}$$

Note that the space D, in particular cases, turn into sums of univariate functions, sums of two ridge functions, sums of two radial functions, etc. The literature abounds with the use of ridge functions (see, e.g., [2, 7, 8, 15, 18, 20]) and radial functions (see, e.g., [4, 14, 16] and a great deal of references therein). Ridge functions and radial functions are defined as multivariate functions of the form $g(\mathbf{a} \cdot \mathbf{x})$ and $g(\|\mathbf{x} - \mathbf{a}\|)$ respectively, where $\mathbf{a} \in \mathbb{R}^d$ is a fixed vector, $\mathbf{x} \in \mathbb{R}^d$ is the variable, $\mathbf{a} \cdot \mathbf{x}$ is the usual inner product, $\|\cdot\|$ is the norm induced by this inner product and g is a univariate function.

We are going to deal with the problem of approximating a continuous function $h : Q \to \mathbb{R}$ using functions from the space D. By $s(Q)$ and $p(Q)$ we will denote the images of Q under the mappings s and p respectively. Define the following operators

$$F : C(Q) \to D_1, \quad (Fh)(a) = \frac{1}{2}\left(\max_{\substack{x\in Q \\ s(x)=a}} h(x) + \min_{\substack{x\in Q \\ s(x)=a}} h(x) \right), \quad \text{for all } a \in s(Q),$$

and

$$G : C(Q) \to D_2, \quad (Gh)(b) = \frac{1}{2}\left(\max_{\substack{x\in Q \\ p(x)=b}} h(x) + \min_{\substack{x\in Q \\ p(x)=b}} h(x) \right), \quad \text{for all } b \in p(Q).$$

In the sequel, we need that the above max and min functions be continuous. For this reason, we will chose the functions $s(x)$ and $p(x)$ from the certain class of functions defined below.

Definition 1. *We say that a function $f \in C(Q)$ belongs to the class $\mathcal{M}(Q)$, if for any two points x and y with $f(x) = f(y)$ and any sequence $\{x_n\}_{n=1}^{\infty}$ tending to x, there exist a sequence $\{y_k\}_{k=1}^{\infty}$ tending to y and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $f(y_k) = f(x_{n_k})$, for all $k = 1, 2, ...$*

Note that the class $\mathcal{M}(Q)$ strictly depends on the considered set Q. That is, a continuous function $f : Q \to \mathbb{R}$ may be in $\mathcal{M}(Q)$, but for many subsets $P \subset Q$, it may happen that the restriction of f to P is not in $\mathcal{M}(P)$. For example, let K be the unit square in $\mathbb{R}^2$ and $K_1 = [0, 1] \times [0, \frac{1}{2}] \cup [0, \frac{1}{2}] \times [0, 1]$. Clearly, the coordinate function $f(x, y) = x$ is in $\mathcal{M}(K)$, but not in $\mathcal{M}(K_1)$. Indeed, for the sequence $\{(\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2})\}_{n=1}^{\infty} \subset K_1$, which tends to $(\frac{1}{2}, \frac{1}{2})$, we cannot find a sequence $\{(x_k, y_k)\}_{k=1}^{\infty} \subset K_1$ tending to $(\frac{1}{2}, 1)$ such that $\{x_k\}_{k=1}^{\infty}$ is a subsequence of $\{\frac{1}{2} + \frac{1}{n+1}\}_{n=1}^{\infty}$.

Let f be a fixed continuous function on a compact set $Q \subset \mathbb{R}^d$. For each continuous function $h : Q \to \mathbb{R}$ consider the following max and min functions

$$r(a) = \max_{\substack{x\in Q \\ f(x)=a}} h(x) \text{ and } u(a) = \min_{\substack{x\in Q \\ f(x)=a}} h(x),\ a \in f(Q). \tag{2}$$

When these functions inherit continuity properties of the given f? It turns out that if $f \in \mathcal{M}(Q)$, then the functions in (2) are continuous for all $h \in C(Q)$.

Lemma 1. *Let Q be a compact set in $\mathbb{R}^d$ and $f \in \mathcal{M}(Q)$. Then the functions $r(a)$ and $u(a)$ are continuous for each function $h \in C(Q)$.*

Proof. Suppose the contrary. Suppose that $f \in \mathcal{M}(Q)$, but one of the functions $r(a)$ and $u(a)$ is not continuous. Without loss of generality we may assume that $r(a)$ is not continuous on the image of f. Let $r(a)$ be discontinuous at a point $a_0 \in f(Q)$. Then there exists a number $\varepsilon > 0$ and a sequence $\{a_n\}_{n=1}^{\infty} \subset f(Q)$ tending to a_0, such that

$$|r(a_n) - r(a_0)| > \varepsilon, \tag{3}$$

for all $n = 1, 2, ...$ Since the function h is continuous on Q, there exist points $x_k \in Q$, $k = 0, 1, 2, ...$, such that $h(x_k) = r(a_k)$, $f(x_k) = a_k$, for $k = 0, 1, 2, ...$ Thus the inequality (3) can be written as

$$|h(x_n) - h(x_0)| > \varepsilon, \tag{4}$$

for all $n = 1, 2, ...$ Since Q is compact, the sequence $\{x_n\}_{n=1}^{\infty}$ has a converging subsequence. Without loss of generality assume that $\{x_n\}_{n=1}^{\infty}$ itself converges to a point $y_0 \in Q$. Then

$f(x_n) \to f(y_0)$, as $n \to \infty$. But by the assumption, we also have $f(x_n) \to f(x_0)$, as $n \to \infty$. Therefore, $f(y_0) = f(x_0) = a_0$. Note that x_0 and y_0 cannot be the same point, for the equality $x_0 = y_0$ violates the condition (4). By the definition of the class $\mathcal{M}(Q)$, we must have a subsequence $x_{n_k} \to y_0$ and a sequence $z_k \to x_0$ such that

$$f(x_{n_k}) = f(z_k),$$

for all $k = 1, 2, ...$ Since $f(x_{n_k}) = a_{n_k}$, $k = 1, 2, ...$, and on each level set $\{x \in Q : f(x) = a_{n_k}\}$, the function h takes its maximum value at x_{n_k}, we obtain that

$$h(z_k) \leq h(x_{n_k}),\ k = 1, 2, ...$$

Taking the limit in the last inequality as $k \to \infty$, gives us the new inequality

$$h(x_0) \leq h(y_0). \tag{5}$$

Recall that on the level set $\{x \in Q : f(x) = a_0\}$, the function h takes its maximum at x_0. Thus from (5) we conclude that $h(x_0) = h(y_0)$. This last equality contradicts the choice of the positive ε in (4), since $h(x_n) \to h(y_0)$, as $n \to \infty$. Hence the function r is continuous on $f(Q)$. By the same way one can prove that u is continuous on $f(Q)$. ◀

The following theorem plays a key role in the proof of our main result (Theorem 2).

Theorem 1. *Let the continuous mappings $s : Q \longrightarrow \mathbb{R}$, $p : Q \longrightarrow \mathbb{R}$ be in the class $\mathcal{M}(Q)$. Then the operators F and G are best approximation operators onto the spaces D_1 and D_2 respectively, both enjoying the properties of centrality and non-expansiveness.*

Proof. We prove this theorem for the operator F. A proof for G can be carried out by the same way.

Clearly, on the level set $s(x) = a$, the constant $(Fh)(a)$ is a best approximation to h, among all constants. Varying over $a \in s(Q)$, we obtain a best approximating function $Fh : s(Q) \to \mathbb{R}$, which is, due to Lemma 1, in the space D_1.

Now let us prove that the operator F is central. In other words, we must prove that for any functions $h(x) \in C(Q)$ and $f(s(x)) \in D_1$,

$$\|h - Fh - f\| = \|h - Fh + f\|. \tag{6}$$

Put $u = h - Fh$. There exists a point $x_0 \in Q$ such that

$$\|u + f\| = |u(x_0) + f(s(x_0))|.$$

First assume that $|u(x_0) + f(s(x_0))| = u(x_0) + f(s(x_0))$. Note that $Fu = 0$. This means that

$$\max_{\substack{x \in Q \\ s(x)=a}} u(x) = - \min_{\substack{x \in Q \\ s(x)=a}} u(x),\ \text{for all } a \in s(Q). \tag{7}$$

Let

$$\min_{\substack{x\in Q \\ s(x)=s(x_0)}} u(x) = u(x_1). \tag{8}$$

From (7) and (8) it follows that

$$-u(x_1) \geq u(x_0).$$

Taking the last inequality and the equality $s(x_1) = s(x_0)$ into account we may write

$$\|u - f\| \geq f(s(x_1)) - u(x_1) \geq f(s(x_0)) + u(x_0) = \|u + f\|. \tag{9}$$

Changing in (9) the function f to $-f$ gives the reverse inequality $\|u + f\| \geq \|u - f\|$. Thus (6) holds.

Note that if $|u(x_0) + f(s(x_0))| = -(u(x_0) + f(s(x_0)))$, then by replacing Eq (8) by

$$\max_{\substack{x\in Q \\ s(x)=s(x_0)}} u(x) = u(x_1). \tag{10}$$

we will derive from (7) and (10) that $u(x_1) \geq -u(x_0)$. This inequality is then used to obtain the estimation

$$\|u - f\| \geq -(f(s(x_1) - u(x_1)) \geq -(f(s(x_0)) + u(x_0)) = \|u + f\|,$$

which in turn yields (6). The centrality has been proven.

Now we prove that the operator F is non-expansive. First note that it is nondecreasing. That is, if $h_1(x) \leq h_2(x)$, then $Fh_1(s(x)) \leq Fh_2(s(x))$ for all $x \in Q$. Besides, $F(h + c) = Fh + c$, for any real number c. Put $c = \|h_1 - h_2\|$. Then for any $x \in Q$, we can write

$$h_2(x) - c \leq h_1(x) \leq h_2(x) + c$$

and further

$$Fh_2(s(x)) - c \leq Fh_1(s(x)) \leq Fh_2(s(x)) + c.$$

From the last inequality we obtain that

$$\|Fh_1(s(x)) - Fh_2(s(x))\| \leq c = \|h_1 - h_2\|.$$

Thus we see that F is non-expansive. ◄

Consider the iteration

$$h_1(x) = h(x), h_{2n} = h_{2n-1} - Fh_{2n-1}, h_{2n+1} = h_{2n} - Gh_{2n}, n = 1, 2, ...$$

Our main result is the following theorem.

Theorem 2. *Assume $s, p \in \mathcal{M}(Q)$ and D is closed in $C(Q)$. Then $\|h_n\|$ converges to the error of approximation $E(h)$.*

Proof of Theorem 2 easily follows from Theorem 1 and the result of Golomb [5]: Let E be a Banach space and X and Y be closed subspaces of E. In addition, let the sum $X+Y$ be closed in E. If A and B are central proximity maps (see Introduction), then for an element $z \in E$ the sequence produced by the levelling algorithm $z_1 = z$, $z_2 = z_1 - Az_1$, $z_3 = z_2 - Bz_2$, $z_4 = z_3 - Az_3, ...$, converges in norm to the distance $dist(z, X+Y)$.

Remark 1. Theorem 2 in a more general form involving any compact Hausdorff space X and closed subalgebras of $C(X)$ will appear in [1].

Remark 2. A version of Theorem 2 was proved by the second author differently in [6]. He did not consider the classes $\mathcal{M}(Q)$ and assumed directly that the functions $r(a)$ and $u(a)$ are continuous for each function $h \in C(Q)$.

Remark 3. We do not yet know if the Diliberto and Straus algorithm converges without the closedness assumption on the subspace D. Note that this problem was posed in various settings in several works (see, e.g., [18, 19, 20]).

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Aida Kh. Asgarova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az1141, Baku, Azerbaijan
E-mail: aidaasgarova@gmail.com

Vugar E. Ismailov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az1141, Baku, Azerbaijan
E-mail: vugaris@mail.ru

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