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On the Equivalence of Completeness of a System of Powers and Trivial Solvability of Homogeneous Riemann Problem

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Abstract. Double system of powers with degenerate coefficients is considered in this work. Some weighted Smirnov classes are introduced and conjugation problem for them is formulated. Equivalence of the completeness of a double system of powers in a weighted Lebesgue space and the trivial solvability of the corresponding homogeneous conjugation problem in weighted Smirnov classes is proved.

Key Words and Phrases: system of powers, completeness, weighted space, Smirnov classes.

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1. Introduction

Consider the following system of powers:

$$\left\{A^{+}(t)\,\omega^{+}(t)\,\varphi^{n}(t)\,;\;A^{-}(t)\,\omega^{-}(t)\,\bar{\varphi}^{n}(t)\right\}_{n\geq 0}, \tag{1}$$

where $A^{\pm}(t)\equiv\left|A^{\pm}(t)\right|e^{ia^{\pm}(t)}$ and $\varphi(t)$ are complex-valued functions on the interval $[a,b]$ with the degenerate coefficients $\omega^{\pm}(\cdot)$:

$$\omega^{\pm}(t)\equiv\prod_{i=1}^{l^{\pm}}\left|t-t_i^{\pm}\right|^{\beta_i^{\pm}},$$

where $\left\{t_i^{\pm}\right\}\subset(a,b)$, $\left\{\beta_i^{\pm}\right\}\subset R$ are some sets (R is the real axis).

Very special cases of the system (1) arise when considering spectral problems of the theory of differential operators. As a typical example, we can mention so-called Kostyuchenko system $\left\{e^{iant}\sin nt\right\}_{n\geq 1}$, where $a\in C$ is a complex parameter (C is the complex plane). Many researches have been dedicated to the basis properties of this system (see, e.g., [1-6]). Final results on the basis properties of this system (completeness, minimality, basicity)

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have been obtained in [5]. Theoretical foundations for the study of basis properties of the systems like (1) have been laid by J.L. Walsh [7]. [8] and [9] also treated the above mentioned problems. A special case of the system (1) with $\varphi(t) \equiv e^{it}$ was considered in [10,11], where basicity criteria for the exponential system with degenerate coefficients in L_p have been obtained.

In the present work, we study the completeness of the system (1) in the weighted space $L_{p,\rho} \equiv L_{p,\rho}(a,b)$, $1 < p < +\infty$, with the weight $\rho : [a,b] \to (0,+\infty)$.

2. Needful Information

Before stating our main result, we make the following assumptions.

1) $|A^{\pm}(t)|; |\varphi'(t)|$ are measurable on (a,b) and the following condition holds:

$$\sup_{(a,b)} vrai \left\{ \left|A^{+}(t)\right|^{\pm 1}; \left|A^{-}(t)\right|^{\pm 1}; \left|\varphi'(t)\right|^{\pm 1} \right\} < +\infty.$$

2) $\Gamma = \varphi\{[a,b]\}$ is a simple closed $(\varphi(a) = \varphi(b))$ rectifiable Jordan curve. Γ is either a Radon curve (i.e. the angle $\theta_0(\varphi(t))$ between the tangent line to Γ at the point $\varphi = \varphi(t)$ and the real axis is a function of bounded variation on $[a,b]$), or a piecewise Lyapunov curve.

For definiteness, we will assume that when the point $\varphi = \varphi(t)$ runs across the curve Γ as t increases, the internal domain $int\Gamma$ stays on the left side.

To state our theorem, we have to introduce weighted Smirnov classes of analytic functions.

Let $D \equiv int\Gamma$, and $E_1(D)$ be a usual Smirnov class of analytic functions in D. Let $\omega(\tau)$ be some weight function on Γ and $L_{p,\omega}(\Gamma)$ be a weighted Lebesgue class of p-summable functions on Γ:

$$L_{p,\omega}(\Gamma) \stackrel{def}{\equiv} \left\{ f : \int_{\Gamma} |f(\tau)|^p \omega(\tau) \, |d\tau| < +\infty \right\}.$$

By $f^{+}(\tau)$ we denote the non-tangential boundary values of the function $f(z) \in E_1(D)$. Introduce

$$E_{p,\omega}(D) \stackrel{def}{\equiv} \left\{ f \in E_1(D) : \; f^{+}(\tau) \in L_{p,\omega}(\Gamma) \right\}.$$

Let's consider the following conjugation problem in the classes $E_{p^{\pm},\rho^{\pm}}(D)$:

$$F_1^{+}(\tau) + G(\tau)\overline{F_2^{+}(\tau)} = g(\tau), \; \tau \in \Gamma, \qquad (2)$$

where $\overline{F_2^{+}(\tau)}$ is a complex conjugation, $g(\tau) \in L_{p,\omega}(\Gamma)$ is some function, and $G(\tau)$ is a given function. $g(\tau)$ and $G(\tau)$ are called the free term and the coefficient of the problem (2), respectively. By the solution of the problem (2) we mean a pair of analytic functions $F_1(z)$ and $F_2(z)$ in D, which belong to the classes $E_{p^{+},\rho^{+}}(D)$ and $E_{p^{-},\rho^{-}}(D)$, respectively, and whose boundary values $F_1^{+}(\tau)$ and $F_2^{+}(\tau)$ satisfy the equality (2) almost everywhere on Γ.

Further, denote by $t = \psi(\varphi)$ the inverse of the function $\varphi = \varphi(t)$ defined on $\Gamma \backslash \{\varphi(a) = \varphi(b)\}$. The point $\varphi_0 = \varphi(a) = \varphi(b)$ is considered as two different "stuck-together" endpoints $\varphi_0^+ = \varphi(a)$ and $\varphi_0^- = \varphi(b)$. Then, it is quite natural to assume that $\psi(\varphi_0^+) = a$ and $\psi(\varphi_0^-) = b$.

3. t-Besselian systems

Consider the following homogeneous conjugation problem:

$$F_1^+(\tau) - G(\tau)\overline{F_1^+(\tau)} = 0 \quad \text{a.e. on } \Gamma, \tag{3}$$

where the coefficient $G(\tau)$ is defined by the formula

$$G(\tau) = \frac{A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\,\bar{\varphi}'(\psi(\varphi))}{A^-(\psi(\varphi))\,\omega^-(\psi(\varphi))\,\varphi'(\psi(\varphi))}.$$

The following theorem is true.

Theorem 1. *Let $\rho : [a,b] \to (0,+\infty)$ be some weight function, the coefficients $A^\pm(t)$ satisfy the conditions 1), 2,) and $\omega^\pm \in L_{p,\rho}(a,b)$, where $p \in (1,+\infty)$ is some number. Then the system* (1) *is complete in $L_{p,\rho}(a,b)$ only when the homogeneous conjugation problem* (3) *has only the trivial solution in the classes $E_{q,\rho^\pm}(D)$, $\frac{1}{p}+\frac{1}{q}=1$, where*

$$\rho^\pm(\varphi) = \left|\omega^\pm(\psi(\varphi))\right|^{-q} \rho^{1-q}(\psi(\varphi)), \quad \varphi \in .$$

Proof. The completeness of the system (1) in $L_{p,\rho}(a,b)$ is equivalent to saying that every function $f(t) \in L_{q,\rho}(a,b)$, $\frac{1}{p}+\frac{1}{q}=1$, is equal to zero almost everywhere with

$$\left.\begin{array}{l} \int_a^b A^+(t)\,\omega^+(t)\,\varphi^n(t)\,\bar{f}(t)\,\rho(t)\,dt = 0\,, \\ \int_a^b A^-(t)\,\omega^-(t)\,\bar{\varphi}^n(t)\,\bar{f}(t)\,\rho(t)\,dt = 0\,,\ n \geq 0. \end{array}\right\} \tag{4}$$

From the first of (4) we have

$$\int_a^b A^+(t)\,\omega^+(t)\,\bar{f}(t)\,\varphi^n(t)\,\rho(t)\,dt = \int_\Gamma A^+(\psi(\varphi))\,\omega^+(\psi(\varphi)) \times$$

$$\times \bar{f}(\psi(\varphi))\left[\varphi'(\psi(\varphi))\right]^{-1}\rho(\psi(\varphi))\,\varphi^n d\varphi = \int_\Gamma F_1(\varphi)\,\varphi^n d\varphi = 0, \tag{5}$$

where

$$F_1(\varphi) = A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\left[\varphi'(\psi(\varphi))\right]^{-1}\bar{f}(\psi(\varphi))\,\rho(\psi(\varphi))\,.$$

It is not difficult to conclude from the conditions of the theorem that $F_1(\varphi) \in L_1(\Gamma)$. Then, due to the results of [12], the equalities (5) are equivalent to the existence of the function $F_1 \in E_1(D)$ such that $F_1^+(\varphi) = F_1(\varphi)$ a.e. on Γ.

It is not difficult to see that $F_1(\varphi) \in L_{q,\rho^+}(\Gamma)$, where $\rho^+ \equiv |\omega^+|^{-q}\rho^{1-q}$. Consequently, by definition, the function $F_1(z)$ belongs to the class $E_{q,\rho^+}(D)$.

Similarly, from the second of (4) we have

$$\int_a^b \overline{A^-(t)}\omega^-(t)\, f(t)\,\varphi^n(t)\,\rho(t)\,dt = \int_\Gamma \overline{A^-(\psi(\varphi))}\omega^-(\psi(\varphi)) \times$$

$$\times f(\psi(\varphi))\left[\varphi'(\psi(\varphi))\right]^{-1}\rho(\psi(\varphi))\,\varphi^n d\varphi = \int_\Gamma F_2(\varphi)\,\varphi^n d\varphi = 0\,,\ n\geq 0,$$

where

$$F_2(\varphi) = \overline{A^-(\psi(\varphi))}\omega^-(\psi(\varphi))\left[\varphi'(\psi(\varphi))\right]^{-1} f(\psi(\varphi))\,\rho(\psi(\varphi))\,.$$

Proceeding as above, we arrive at the conclusion that there exists the function $F_2(z)\in E_1(D)$ such that $F_2^+(\varphi) = F_2(\varphi)$ a.e. on Γ, where $F_2^+(\varphi)$ is a non-tangential boundary value of $F_2(z)$ on Γ. From $F_2(\varphi)\in L_{q,\rho^-}(\Gamma)$, $\rho^-\equiv|\omega^-|^{-q}\rho^{1-q}$, it follows that the function $F_2(z)$ belongs to the class $E_{q,\rho^-}(D)$. Expressing the function $f(t)$ in terms of $F_1(\varphi)$ and $F_2(\varphi)$, we have

$$F_1^+(\varphi) = G(\varphi)\,\overline{F_2^+(\varphi)}\,,\varphi\in\Gamma,$$

where

$$G(\varphi) = \frac{A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\,\bar{\varphi}'(\psi(\varphi))}{A^-(\psi(\varphi))\,\omega^-(\psi(\varphi))\,\varphi'(\psi(\varphi))}.$$

Thus, if the system (1) is not complete in $L_{p,\rho}(a,b)$, then the homogeneous conjugation problem (3) is non-trivially solvable in the classes $E_{p,\rho^\pm}(D)$.

Now suppose to the contrary that the problem (3) is non-trivially solvable in the classes $E_{p,\rho^\pm}(D)$. From the definition of the classes $E_{p,\rho^\pm}(D)$ and from $F_i(z)\in E_1(D)\,,\ i=\overline{1,2}$, it follows that

$$\int_\Gamma F_i^+(\varphi)\,\varphi^n d\varphi = 0, n\geq 0.$$

Taking into account the expression for the function $G(\tau)$, we have

$$\frac{F_1^+(\varphi)\,\varphi'(\psi(\varphi))}{A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\,\rho(\psi(\varphi))} = \frac{F_2^+(\varphi)\,\varphi'(\psi(\varphi))}{A^-(\psi(\varphi))\,\omega^-(\psi(\varphi))\,\rho(\psi(\varphi))}.$$

Denoting the last expression by $\overline{f(\varphi)}$, we obtain

$$\int_\Gamma A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\,\frac{\overline{f(\varphi)}}{\varphi'(\psi(\varphi))}\varphi^n\rho(\psi(\varphi))\,d\varphi =$$

$$= \int_a^b A^+(t)\,\omega^+(t)\,\overline{f(\varphi(t))}\varphi^n(t)\,\rho(t)\,dt = 0\,,\ n\geq 0.$$

Similarly we have

$$\int_a^b A^-(t)\,\omega^-(t)\,\overline{f(\varphi(t))}\bar{\varphi}^n(t)\,\rho(t)\,dt = 0\,,\ n\geq 0.$$

From the conditions of the theorem, by the definition of the classes $E_{p,\rho^{\pm}}(D)$ it follows that the function $f(\varphi(t))$ belongs to the space $L_{q,\rho}(a,b)$. It is absolutely clear that this function is different from zero. Then, the previous relations imply that the system (1) is not complete in $L_{p,\rho}(a,b)$. ◄

Remark 1. *One of the results obtained by Smirnov implies that if the domain D belongs to the Smirnov class and* $\rho^{\pm} \equiv 1$, $p \geq 1$, *then the definition of the classes* $E_{p,\rho^{\pm}}$ *is equivalent to the classical definition for the classes* E_p.

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