Parabolic Fractional Integral Operators with Rough Kernels in Parabolic Local Generalized Morrey Spaces

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Abstract. Let P be a real $n \times n$ matrix, whose all the eigenvalues have positive real part, $A_t = t^P$, $t > 0$, $\gamma = \mathrm{tr}P$ is the homogeneous dimension on $\mathbb{R}^n$ and Ω is an A_t-homogeneous of degree zero function, integrable to a power $s > 1$ on the unit sphere generated by the corresponding parabolic metric. We study the parabolic fractional integral operator $I_{\Omega,\alpha}^P$, $0 < \alpha < \gamma$ with rough kernels in the parabolic local generalized Morrey space $LM_{p,\varphi,P}^{\{x_0\}}(\mathbb{R}^n)$. We find conditions on the pair (φ_1, φ_2) for the boundedness of the operator $I_{\Omega,\alpha}^P$ from the space $LM_{p,\varphi_1,P}^{\{x_0\}}(\mathbb{R}^n)$ to another one $LM_{q,\varphi_2,P}^{\{x_0\}}(\mathbb{R}^n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $LM_{1,\varphi_1,P}^{\{x_0\}}(\mathbb{R}^n)$ to the weak space $WLM_{q,\varphi_2,P}^{\{x_0\}}(\mathbb{R}^n)$, $1 \le q < \infty$, $1 - 1/q = \alpha/\gamma$.

Key Words and Phrases: parabolic fractional integral, parabolic local generalized Morrey space.

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1. Introduction

The boundedness of classical operators of the real analysis, such as the fractional integral operators, from one weighted Lebesgue space to another one is well studied by now, and there are well known various applications of such results in partial differential equations. Besides Lebesgue spaces, Morrey spaces, both the classical ones (the idea od their definition having appeared in [13]) and generalized ones, also play an important role in the theory of partial differential equations.

In this paper, we find conditions for the boundedness of the parabolic fractional integral operators with rough kernel from a parabolic local generalized Morrey space to another one, including also the case of weak boundedness.

Note that we deal not exactly with the parabolic metric, but with a general anisotropic metric ρ of generalized homogeneity, the parabolic metric being its particular case, but we keep the term "parabolic in the title and text of the paper, following the existing tradition, see for instance [4].

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For $x \in \mathbb{R}^n$ and $r > 0$, we denote the open ball centered at x of radius r by $B(x,r)$, its complement by ${}^{\complement}B(x,r)$ and $|B(x,r)|$ will stand for the Lebesgue measure of $B(x,r)$.

Let P be a real $n \times n$ matrix, whose all the eigenvalues have positive real part. Let $A_t = t^P$ $(t > 0)$, and set

$$\gamma = \mathrm{tr}P.$$

Then, there exists a quasi-distance ρ associated with P such that (see [5])

(a) $\rho(A_t x) = t\rho(x)$, $t > 0$, for every $x \in \mathbb{R}^n$;

(b) $\rho(0) = 0$, $\rho(x) = \rho(-x) \geq 0$

and $\rho(x - y) \leq k(\rho(x - z) + \rho(y - z))$;

(c) $dx = \rho^{\gamma-1}d\sigma(w)d\rho$, where $\rho = \rho(x), w = A_{\rho^{-1}}x$

and $d\sigma(w)$ is a measure on the unit ellipsoid $S_\rho = \{w : \rho(w) = 1\}$.

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss (see [5]) and a homogeneous group in the sense of Folland-Stein (see [7]).

In the standard parabolic case $P_0 = \mathrm{diag}(1, \ldots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

The balls $\mathcal{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x - y) < r\}$ with respect to the quasidistance ρ are ellipsoids. For its Lebesgue measure one has

$$|\mathcal{E}(x,r)| = v_\rho r^\gamma,$$

where v_ρ is the volume of the unit ellipsoid. By ${}^{\complement}\mathcal{E}(x,r) = \mathbb{R}^n \setminus \mathcal{E}(x,r)$ we denote the complement of $\mathcal{E}(x,r)$.

Everywhere in the sequel $A \lesssim B$ means that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

1.1. Parabolic local generalized Morrey spaces

In the doctoral thesis [8], 1994 by Guliyev (see, also [9], [1]-[3]) introduced the local Morrey-type space $LM_{p\theta,w}$ given by

$$\|f\|_{LM_{p\theta,w}} = \left\| w(r) \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)},$$

where w is a positive measurable function defined on $(0,\infty)$. If $\theta = \infty$, it denotes $LM_{p,w} \equiv LM_{p\infty,w}$. In [8] Guliyev intensively studied the classical operators in the local Morrey-type space $LM_{p\theta,w}$ (see also the book [9] (1999)), where these results were

presented for the case when the underlying space is the Heisenberg group or a homogeneous group, respectively. Note that, the generalized local (central) Morrey spaces $LM_{p,\varphi}(\mathbb{R}^n) = LM_{p,\varphi}^{\{0\}}(\mathbb{R}^n)$ introduced by Guliyev in [8] (see also, [9], [1]-[3]).

We define the parabolic local Morrey space $LM_{p,\lambda,P}(\mathbb{R}^n)$ via the norm

$$\|f\|_{LM_{p,\lambda,P}} = \sup_{t>0} \left(t^{-\lambda} \int_{\mathcal{E}(0,t)} |f(y)|^p dy \right)^{1/p} < \infty,$$

where $1 \le p \le \infty$ and $0 \le \lambda \le \gamma$.

If $\lambda = 0$, then $LM_{p,0,P}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$; if $\lambda = \gamma$, then $LM_{p,\gamma,P}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$; if $\lambda < 0$ or $\lambda > \gamma$, then $LM_{p,\lambda,P} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WLM_{p,\lambda,P}(\mathbb{R}^n)$ the weak parabolic Morrey space of functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\lambda,P}} = \sup_{t>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(0,r))} < \infty,$$

where $WL_p(\mathcal{E}(0,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(\mathcal{E}(0,r))} = \sup_{t>0} t \,|\{y \in \mathcal{E}(0,r) : |f(y)| > t\}|^{1/p} .$$

Note that $WL_p(\mathbb{R}^n) = WLM_{p,0,P}(\mathbb{R}^n)$,

$$LM_{p,\lambda,P}(\mathbb{R}^n) \subset WLM_{p,\lambda,P}(\mathbb{R}^n) \ \text{ and } \ \|f\|_{WLM_{p,\lambda,P}} \le \|f\|_{LM_{p,\lambda,P}} .$$

If $P = I$, then $LM_{p,\lambda}(\mathbb{R}^n) \equiv LM_{p,\lambda,I}(\mathbb{R}^n)$ is the local Morrey space.

We introduce the parabolic local generalized Morrey spaces following the known ideas of defining local generalized Morrey spaces ([8, 11, 12] etc).

Definition 1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. The space $LM_{p,\varphi,P} \equiv LM_{p,\varphi,P}(\mathbb{R}^n)$, called the parabolic local generalized Morrey space, is defined by the norm*

$$\|f\|_{LM_{p,\varphi,P}} = \sup_{t>0} \varphi(0,t)^{-1} \,|\mathcal{E}(0,t)|^{-\frac{1}{p}} \,\|f\|_{L_p(\mathcal{E}(0,t))}.$$

Definition 2. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. The space $WLM_{p,\varphi,P} \equiv WLM_{p,\varphi,P}(\mathbb{R}^n)$, called the weak parabolic local generalized Morrey space, is defined by the norm*

$$\|f\|_{WLM_{p,\varphi,P}} = \sup_{t>0} \varphi(0,t)^{-1} \,|\mathcal{E}(0,t)|^{-\frac{1}{p}} \,\|f\|_{WL_p(\mathcal{E}(0,t))}.$$

If $P = I$, then $LM_{p,\varphi}(\mathbb{R}^n) \equiv LM_{p,\varphi,I}(\mathbb{R}^n)$ and $WLM_{p,\varphi}(\mathbb{R}^n) \equiv WLM_{p,\varphi,I}(\mathbb{R}^n)$ are the generalized local Morrey space and the weak generalized local Morrey space, respectively.

Definition 3. *Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi,P}^{\{x_0\}} \equiv LM_{p,\varphi,P}^{\{x_0\}}(\mathbb{R}^n)$ the parabolic generalized local Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi,P}^{\{x_0\}}} = \|f(x_0 + \cdot)\|_{LM_{p,\varphi,P}}.$$

Also by $WLM_{p,\varphi,P}^{\{x_0\}} \equiv WLM_{p,\varphi,P}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi,P}^{\{x_0\}}} = \|f(x_0 + \cdot)\|_{WLM_{p,\varphi,P}} < \infty.$$

According to this definition, we recover the space $LM_{p,\lambda,P}^{\{x_0\}}(\mathbb{R}^n)$ under the choice $\varphi(0, r) = r^{\frac{\lambda - \gamma}{p}}$:

$$LM_{p,\lambda,P}^{\{x_0\}}(\mathbb{R}^n) = LM_{p,\varphi,P}^{\{x_0\}}(\mathbb{R}^n)\bigg|_{\varphi(x_0,r) = r^{\frac{\lambda - \gamma}{p}}}.$$

Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue surface measure $d\sigma$ and Ω be A_t-homogeneous of degree zero, i.e. $\Omega(A_t x) \equiv \Omega(x)$, $x \in \mathbb{R}^n$, $t > 0$. The parabolic fractional integral $I_{\Omega,\alpha}^P f$ by with rough kernels, $0 < \alpha < \gamma$, of a function $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$I_\alpha^P f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x - y)\, f(y)}{\rho(x - y)^{\gamma - \alpha}} dy.$$

We prove the boundedness of the parabolic integral operator $I_{\Omega,\alpha}^P$ with rough kernel from one parabolic local generalized Morrey space $LM_{p,\varphi_1,P}^{\{x_0\}}(\mathbb{R}^n)$ to another one $LM_{q,\varphi_2,P}^{\{x_0\}}(\mathbb{R}^n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $LM_{1,\varphi_1,P}^{\{x_0\}}(\mathbb{R}^n)$ to the weak space $WLM_{q,\varphi_2,P}^{\{x_0\}}(\mathbb{R}^n)$, $1 \leq q < \infty$, $1 - 1/q = \alpha/\gamma$.

2. Preliminaries

Let v be a weight on $(0, \infty)$. We denote by $L_{\infty,v}(0, \infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)|g(t)|$$

and write $L_\infty(0, \infty) \equiv L_{\infty,1}(0, \infty)$. Let $\mathfrak{M}(0, \infty)$ be the set of all Lebesgue-measurable functions on $(0, \infty)$ and $\mathfrak{M}^+(0, \infty)$ its subset of all nonnegative functions. By $\mathfrak{M}^+(0, \infty; \uparrow)$ we denote the cone of all functions in $\mathfrak{M}^+(0, \infty)$ non-decreasing on $(0, \infty)$ and introduce also the set

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0, \infty; \uparrow) : \lim_{t \to 0+} \varphi(t) = 0 \right\}.$$

Let u be a non-negative continuous function on $(0, \infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0, \infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t, \infty)}, \quad t \in (0, \infty).$$

The following theorem was proved in [2].

Theorem 1. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t, \infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0, \infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty, v_1}(0, \infty)$ to $L_{\infty, v_2}(0, \infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|_{L_\infty(\cdot, \infty)}^{-1} \right) \right\|_{L_\infty(0, \infty)} < \infty. \tag{1}$$

We are going to use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s) w(s) ds, \; 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0, \infty)$.

The following theorem in the case $w = 1$ was proved in [3].

Theorem 2. *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0, \infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w^* g(t) \leq C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{2}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s) ds}{\operatorname*{ess\,sup}\limits_{s < \tau < \infty} v_1(\tau)} < \infty. \tag{3}$$

Moreover, if C^* is the minimal value of C in (2), then $C^* = B$.

Remark 1. *In (2) and (3) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.*

3. Parabolic fractional integral operator with rough kernels in the spaces $LM_{p,\varphi,P}^{\{x_0\}}$

In [10] was proved the (p, p)-boundedness of the operator M_Ω^P and the (p, q)-boundedness of the operator $M_{\Omega, \alpha}^P$.

Theorem 1. *[10] Let $\Omega \in L_s(S_\rho)$, $1 < s \leq \infty$, be A_t-homogeneous of degree zero. Then the operator M_Ω^P is bounded in the space $L_p(\mathbb{R}^n)$, $p > s'$.*

Theorem 2. *[10] Suppose that $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$, is A_t-homogeneous of degree zero. Let $1 \leq p < \frac{\gamma}{\alpha}$ and $1/p - 1/q = \alpha/\gamma$. Then the fractional integration operator I_α^P is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$ and from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$.*

The following lemma is valid.

Lemma 1. *Suppose that $x_0 \in \mathbb{R}^n$, $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$, is A_t-homogeneous of degree zero. Let $1 \leq p < \frac{\gamma}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. Then for all $f \in L_p^{\text{loc}}$ there hold the inequalities*

$$\|I_{\Omega,\alpha}^P f\|_{L_q(\mathcal{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} t^{-\frac{\gamma}{q}-1} \|f\|_{L_p(\mathcal{E}(x_0,t))} dt, \quad p > 1$$

and

$$\|I_{\Omega,\alpha}^P f\|_{WL_q(\mathcal{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} t^{-\frac{\gamma}{q}-1} \|f\|_{L_1(\mathcal{E}(x_0,t))} dt, \quad p = 1. \tag{1}$$

Proof. For a given ball $\mathcal{E} = \mathcal{E}(x_0, r)$ f, we represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2k\mathcal{E}}(y), \quad f_2(y) = f(y)\chi_{\complement(2k\mathcal{E})}(y), \quad r > 0,$$

and have

$$\|I_{\Omega,\alpha}^P f\|_{L_q(\mathcal{E})} \leq \|I_{\Omega,\alpha}^P f_1\|_{L_q(\mathcal{E})} + \|I_{\Omega,\alpha}^P f_2\|_{L_q(\mathcal{E})}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, by the boundedness of $I_{\Omega,\alpha}^P$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that

$$\|I_{\Omega,\alpha}^P f_1\|_{L_q(\mathcal{E})} \leq \|I_{\Omega,\alpha}^P f_1\|_{L_q(\mathbb{R}^n)} \leq C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2k\mathcal{E})}.$$

Observe that the conditions $x \in \mathcal{E}$, $y \in \complement(2k\mathcal{E})$ imply

$$\frac{1}{2k}\rho(x_0 - y) \leq \rho(x - y) \leq \frac{3k}{2}\rho(x_0 - y).$$

We then get

$$|I_{\Omega,\alpha}^P f_2(x)| \leq 2^{\gamma-\alpha} c_1 \int_{\complement(2k\mathcal{E})} \frac{|f(y)||\Omega(x-y)|}{\rho(x_0-y)^{\gamma-\alpha}} dy.$$

By Fubini's theorem we have

$$\int_{\complement(2k\mathcal{E})} \frac{|f(y)||\Omega(x-y)|}{\rho(x_0-y)^{\gamma-\alpha}} dy \approx \int_{\complement(2k\mathcal{E})} |f(y)||\Omega(x-y)| \int_{\rho(x_0-y)}^{\infty} \frac{dt}{t^{\gamma+1-\alpha}} dy$$

$$\approx \int_{2kr}^{\infty} \int_{2kr \leq \rho(x_0-y) < t} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{\gamma+1-\alpha}}$$

$$\lesssim \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{\gamma+1-\alpha}}.$$

Applying Hölder's inequality with $1/p + (\gamma - \alpha)/\gamma + (\alpha p - \gamma)/\gamma p = 1$ taken into account, we get

$$
\begin{aligned}
\int_{\complement_{(2k\mathcal{E})}} & \frac{|f(y)||\Omega(x-y)|}{\rho(x_0-y)^{\gamma-\alpha}} dy \\
&\lesssim \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \|\Omega(\cdot - y)\|_{L_{\frac{\gamma}{\gamma-\alpha}}(\mathcal{E}(x_0,t))} |\mathcal{E}(x_0,t)|^{\frac{\alpha}{\gamma}-\frac{1}{p}} \frac{dt}{t^{\gamma+1-\alpha}} \\
&\lesssim \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.
\end{aligned}
$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$
\|I_{\Omega,\alpha}^P f_2\|_{L_q(\mathcal{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{2}
$$

is valid. Thus

$$
\|I_{\Omega,\alpha}^P f\|_{L_q(\mathcal{E})} \lesssim \|f\|_{L_p(2k\mathcal{E})} + r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.
$$

On the other hand,

$$
\begin{aligned}
\|f\|_{L_p(2k\mathcal{E})} &\approx r^{\frac{\gamma}{q}} \|f\|_{L_p(2k\mathcal{E})} \int_{2kr}^{\infty} \frac{dt}{t^{\frac{\gamma}{q}+1}} \\
&\leq r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{3}
\end{aligned}
$$

Thus

$$
\|I_{\Omega,\alpha}^P f\|_{L_q(\mathcal{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.
$$

By Fubini's theorem and the Minkowski inequality, we get

$$
\begin{aligned}
\|I_{\Omega,\alpha}^P f_2\|_{L_q(\mathcal{E})} &\leq \left(\int_{\mathcal{E}} \left| \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{\gamma+1-\alpha}} \right|^q \right)^{\frac{1}{q}} \\
&\leq \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_q(\mathcal{E})} dy \frac{dt}{t^{\gamma+1-\alpha}} \\
&\lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_1(\mathcal{E}(x_0,t))} \frac{dt}{t^{\gamma+1-\alpha}} \\
&\lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.
\end{aligned}
$$

Finally, in the case $p = 1$ by the weak $(1,q)$-boundedness of $I_{\Omega,\alpha}^P$ and the inequality (3) it follows that

$$
\|I_{\Omega,\alpha}^P f_1\|_{WL_q(\mathcal{E})} \leq \|I_{\Omega,\alpha}^P f_1\|_{WL_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)}
$$

$$= \|f\|_{L_1(2k\mathcal{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_1(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{4}$$

Then from (2) and (4) we get the inequality (1).

Theorem 3. *Suppose that $x_0 \in \mathbb{R}^n$, $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero. Let $1 \le p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{\gamma}{q}+1}} dt \le C\,\varphi_2(x_0,r), \tag{5}$$

where C does not depend on x_0 and r. Then the operator $I_{\Omega,\alpha}^P$ is bounded from $LM_{p,\varphi_1,P}^{\{x_0\}}$ to $LM_{q,\varphi_2,P}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1,P}^{\{x_0\}}$ to $WLM_{q,\varphi_2,P}^{\{x_0\}}$ for $p = 1$.

Proof. By Lemma 1 and Theorem 2 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1(r) = \varphi_1(x_0,r)^{-1}r^{-\frac{\gamma}{p}}$ and $w(r) = r^{-\frac{\gamma}{q}}$ we have for $p > 1$

$$\|I_{\Omega,\alpha}^P f\|_{M_{q,\varphi_2,P}} \lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}$$

$$\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p}} \|f\|_{L_p(\mathcal{E}(x_0,r))} = \|f\|_{M_{p,\varphi_1,P}}$$

and for $p = 1$

$$\|I_{\Omega,\alpha}^P f\|_{WM_{q,\varphi_2,P}} \lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_1(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}$$

$$\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\gamma} \|f\|_{L_1(\mathcal{E}(x_0,r))} = \|f\|_{M_{1,\varphi_1,P}}.$$

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