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Atomic Decomposition in a Direct Sum of Banach Spaces and Their Application to Discontinuous Differential Operators

T.B. Gasymov*, Ch.M. Hashimov

Abstract. An atomic decomposition is considered in Banach space. A method for constructing an atomic decomposition of Banach space, proceeding from atomic decomposition of subspaces is presented. Some relations between them are established. The proposed method is used in the study of the frame properties of systems of eigenfunctions and associated functions of discontinuous differential operators.

Key Words and Phrases: p-frames, $\tilde{X}$-frames, conjugate systems to $\tilde{X}$.
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1. Introduction

One of the commonly used methods for solving differential equations is the method of separation of variables (Fourier method). This method yields the appropriate spectral problem (usually with respect to the space variables). To justify the method is very important the question of the expansion of functions of a certain class on eigenfunctions of the spectral problem. That is why many mathematicians have been paying so much attention lately to the study of frame properties (such as completeness, minimality, basicity, atomic decomposition) of the systems of special functions, mostly eigenfunctions and associated functions of differential operators. Various methods have been developed for establishing these properties. For more information we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9]. In case of discontinuous differential operator, there arise the systems of eigenfunctions that cannot be treated for frameness by the earlier methods. To shed some light on this situation, we consider the following model spectral problem for second order discontinuous differential operator

$$-y''(x) = \lambda y(x)\,,\ x \in (-1,\,0) \cup (0,\,1)\,, \tag{1}$$

with the boundary conditions

$$y(-1) = y(1) = 0; y(-0) = y(+0)\,, \tag{2}$$

*Corresponding author.

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$$y'(-0) - y(+0) = \lambda m y(0).$$

This spectral problem has two sets of eigenfunctions [9]:

$$u_n^{(1)}(x) = \sin \pi n x, x \in [-1, 1], n \in N,$$

and

$$\tilde{u}_n^{(2)}(x) = \begin{cases} \sin \pi n x + 0\left(\frac{1}{n}\right), & x \in [-1, 0], \\ -\sin \pi n x + 0\left(\frac{1}{n}\right), & x \in [0, 1], \quad n \in N. \end{cases}$$

Such spectral problems arise when solving the problem of a loaded string fixed at both ends with a load placed in the middle of the string by the Fourier method [10, 11]. The use of this method requires the study of basis properties of the double system $\left\{u_n^{(1)}; \tilde{u}_n^{(2)}\right\}_{n\in N}$ in corresponding spaces of functions (usually in the Lebesgue or Sobolev spaces). Of course, it should be started with the basis properties of the system $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n\in N}$, which is the principal part of the asymptotics of the system $\left\{u_n^{(1)}; \tilde{u}_n^{(2)}\right\}_{n\in N}$, where

$$u_n^{(2)} = \begin{cases} \sin \pi n x, & x \in [-1, 0), \\ -\sin \pi n x, & x \in [0, 1]. \end{cases}$$

This is usually followed by the application of various perturbation methods. This approach is well studied (see, e.g., the articles [5, 6, 7, 8, 9, 13, 14, 15, 16] and the monographs [12, 17, 18, 19, 20, 21, 22, 23]). On the other hand, it is not difficult to see that the principal part $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n\in N}$ is not a standard (in other words, classical) system. It turns out that the form of the system $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n\in N}$ is not special, i.e. it can be derived from the general case. The general approach to these systems allows introducing a new approach for constructing bases with a lot of applications in the spectral theory of differential operators. It should be noted that some problems of an atomic decomposition and frames with respect to the specific systems have been previously studied in [27, 28, 29, 30, 31].

In this work we consider an abstract approach to the above problem. We consider a direct expansion of a Banach space with respect to subspaces. We offer a method for constructing an atomic decomposition for a space proceeding from atomic decomposition for subspaces.

2. Notation and needful information

We will use the standard notation. N will be a set of all positive integers; $L[M]$ will denote the linear span of the set M and $\overline{M}$ will stand for the closure of M; X^* will denote a space conjugate to X; $L(X_1, X_2)$ will be a space of linear bounded operators from X_1 to X_2 with $L(X, X) = L(X)$; D_T will denote a domain of the operator T and R_T will be the range of T; $KerT$ will stand for the kernel of the operator T; $< x, f >= f(x)$ will denote the value of the functional f at the point x; Banach space will be referred to as

B-space; $\|\cdot\|_X$ will denote a norm in X; $\Leftrightarrow$ will mean "if and only if"; $1:n \equiv \{1;...;n\}$; δ_{ij} will be the Kronecker symbol.

We will also use the concept of the space of coefficients. We define it as follows. Let $\vec{x} \equiv \{x_n\}_{n\in N} \subset X$ be a non-degenerate system in a B-space X, i.e. $x_n \neq 0$, $\forall n \in N$. Define

$$\mathscr{K}_{\vec{x}} \equiv \left\{ \{\lambda_n\}_{n\in N} : \textit{ the series } \sum_{n=1}^{\infty} \lambda_n x_n \textit{ is convergent in } X \right\}.$$

Introduce the norm in $\mathscr{K}_{\vec{x}}$:

$$\left\|\vec{\lambda}\right\|_{\mathscr{K}_{\vec{x}}} = \sup_m \left\| \sum_{n=1}^{m} \lambda_n x_n \right\|, \quad \text{where} \quad \vec{\lambda} = \{\lambda_n\}_{n\in N}.$$

With respect to the usual operations of addition and multiplication by a complex number, $\mathscr{K}_{\vec{x}}$ is a B- space. Take $\forall \vec{\lambda} \in \mathscr{K}_{\vec{x}}$ and consider the operator $T : \mathscr{K}_{\vec{x}} \to X$:

$$T\vec{\lambda} = \sum_{n=1}^{\infty} \lambda_n x_n, \ \vec{\lambda} = \{\lambda_n\}_{n\in N}.$$

Denote by $\{e_n\}_{n\in N} \subset \mathscr{K}_{\vec{x}}$ a canonical system in $\mathscr{K}_{\vec{x}}$, where $e_n = \{\delta_{nk}\}_{k\in N}$. It is absolutely clear that $Te_n = x_n$, $\forall n \in N$. The following statement is true.

Statement 1. *Space of coefficients $\mathscr{K}_{\vec{x}}$ is a B-space with the canonical basis $\{e_n\}_{n\in N}$. Moreover, the system $\vec{x}$ forms a basis for $X \Leftrightarrow T$ performs an isomorphism between $\mathscr{K}_{\vec{x}}$ and X.*

Let's recall some concepts and facts from the frames theory . First, let us give a definition of atomic decomposition in Banach spaces.

Definition 1. *Let X be a B-space and $\mathscr{K}$ be a B- space of the sequences of scalars. Let $\{f_k\}_{k\in N} \subset X$, $\{g_k\}_{k\in N} \subset X^*$. Then $\left(\{g_k\}_{k\in N} ;\ \{f_k\}_{k\in N}\right)$ is an atomic decomposition of X with respect to $\mathscr{K}$, if :*

(i) $\{g_k(f)\}_{k\in N} \in \mathscr{K}$, $\forall f \in X$;

(ii) $\exists A, B > 0 : A\|f\|_X \le \left\|\{g_k(f)\}_{k\in N}\right\|_{\mathscr{K}} \le B\|f\|_X$, $\quad \forall f \in X$;

(iii) $f = \sum_{k=1}^{\infty} g_k(f)\, f_k$, $\ \forall f \in X$.

The concept of the frame is a generalization of the concept of an atomic decomposition.

Definition 2. *Let X be a B-space and $\mathscr{K}$ be a B-space of the sequences of scalars. Let $\{g_k\}_{k\in N} \subset X^*$ and be a bounded operator. Then $\left(\{g_k\}_{k\in N} ; S\right)$ forms a Banach frame for X with respect to $\mathscr{K}$, if:*

(i) $\{g_k(f)\}_{k\in N} \in \mathscr{K}$, $\forall f \in X$;

(ii) $\exists A, B > 0 : A\|f\|_X \le \left\|\{g_k(f)\}_{k\in N}\right\|_{\mathscr{K}} \le B\|f\|_X$, $\quad \forall f \in X$;

(iii) $S\ \left[\{g_k(f)\}_{k\in N}\right] = f$, $\ \forall f \in X$.

It is true the following

Proposition 1. *Let X be a B-space and $\mathscr{K}$ a B-space of the sequences of scalars with canonical basis $\{\delta_n\}_{n\in N}$. Let $\{g_k\}_{k\in N} \subset X^*$ and $S \in L(\mathscr{K};;X)$. Then the following statements are equivalent:*

(i) $\left(\{g_k\}_{k\in N}\,;S\right)$ is a Banach frame for X with respect to $\mathscr{K}$;

(ii)$\left(\{g_k\}_{k\in N}\,;\{S\,(\delta_k)\}_{k\in N}\right)$ is an atomic decomposition of X with respect to $\mathscr{K}$.

More information about the above facts can be found in [17, 18, 19, 20, 21, 22, 23, 24].

In the sequel, we will use the following construction and some obvious facts. Let the following direct sum hold

$$X = X_1 \oplus \ldots \oplus X_m,$$

where X_i, $i=\overline{1,\,m}$, are some B-spaces. For convenience, we will represent the elements of the space X in the form of a vector

$$x \in X \quad \Leftrightarrow \quad x = (x_1, x_2, ..., x_m)\,,$$

where $x_k \in X_k$, $k=\overline{1,m}$. The norm in X will be defined by the formula

$$\|x\|_X = \sqrt{\sum_{i=1}^{m} \|x_i\|_{X_i}^2}.$$

Then we have $X^* = X_1^* \oplus \ldots \oplus X_m^*$ (see [13]), and for $\vartheta \in X^*$ and $x \in X$ it holds

$$< x,\, \vartheta >= \sum_{i=1}^{m} < x_i,\, \vartheta_i >,$$

where $\vartheta = (\vartheta_1, ..., \vartheta_m)$ and

$$\|\vartheta\|_{X^*} = \sqrt{\sum_{i=1}^{m} \|\vartheta_i\|_{X_i^*}^2}.$$

Let some system $\left\{u_n^{(i)}\right\}_{n\in N} \subset X_i$ be given for every $i \in 1 : m$. Consider the following system in the space X :

$$u_{in}^0 = \left(\underbrace{0, ..., 0, u_n^{(i)}}_{i}, 0, ..., 0\right)\ ,\ \ i=\overline{1,\,m}\,;\ n\in N.$$

Let the pair $\left(\{u_{in}^0\}_{i=\overline{1,\,m};n\in N}\,;\{\vartheta_{in}\}_{i=\overline{1,\,m};n\in N}\right)$ be an atomic decomposition of X with respect to the space of coefficients $\mathscr{K}$, i.e. $\forall x \in X$ has a decomposition of the form

$$x = \sum_{i=1}^{m}\sum_{n=1}^{\infty} \vartheta_{in}\,(x)\, u_{in}^0, \tag{3}$$

moreover, the following inequality holds

$$A\,\|\{\vartheta_{in}\,(x)\}\|_{\mathscr{K}} \le \|x\|_X \le B\,\|\{\vartheta_{in}\,(x)\}\|_{\mathscr{K}}\,. \tag{4}$$

Suppose

$$\vartheta_{in}=\left(\vartheta_{in}^{(1)};...;\vartheta_{in}^{(m)}\right)\in X^*,$$

where $\vartheta_{in}^{(k)}\in X_k^*, \forall k\in 1:m$. We have $(x=(x_1,...,x_m))$: $\vartheta_{in}(x)=\sum_{k=1}^{m}\vartheta_{in}^{(k)}(x_k)$, $\quad i=\overline{1,\,m}$; $n\in N$. Take $\forall k\in 1:m$, and let

$$x_k^0=\left(\underbrace{0;...;0;x_k}_{k};0;...;0\right).$$

We have

$$\vartheta_{in}\left(x_k^0\right)=\vartheta_{in}^{(k)}\left(x_k\right).$$

Then from (4) we obtain

$$A\left\|\left\{\vartheta_{in}^{(k)}\left(x_k\right)\right\}\right\|_{\mathscr{K}}\le\|x_k\|_{X_k}\le B\left\|\left\{\vartheta_{in}^{(k)}\left(x_k\right)\right\}\right\|_{\mathscr{K}}.$$

Paying attention to the decomposition (3), we obtain

$$x_k^0=(0,...,0,x_k,0,...,0)=\sum_{i=1}^{m}\left(\underbrace{0,...,0,\sum_{n=1}^{\infty}\vartheta_{in}^{(k)}\left(x_k\right)u_n^{(i)}}_{i},0,...,0\right)=$$

$$=\left(\sum_{n=1}^{\infty}\vartheta_{1n}^{(k)}\left(x_k\right)u_n^{(1)},...,\sum_{n=1}^{\infty}\vartheta_{kn}^{(k)}\left(x_k\right)u_n^{(k)},...,\sum_{n=1}^{\infty}\vartheta_{mn}^{(m)}\left(x_k\right)u_n^{(m)}\right)\Rightarrow$$

$$\sum_{n=1}^{\infty}\vartheta_{in}^{(k)}\left(x_k\right)u_n^{(i)}=\begin{cases}x_k, & i=k,\\ 0, & i\neq k.\end{cases}\tag{5}$$

As a result, we obtain that

$$\left(\left\{u_n^{(k)}\right\}_{n\in N};\left\{\vartheta_{kn}^{(k)}\right\}_{n\in N}\right),\tag{6}$$

is an atomic decomposition of X_k with respect to the space of coefficients $\mathscr{K}$, for every $k\in 1:m$.

Accept the following

Definition 3. *The pair* $\left(\{u_n\};\left\{\vec{\vartheta}_n\right\}\right)$ $\left(u_n\in X\wedge\vec{\vartheta}_n\in X^*\right)$ *is called an atomic decomposition of* X *with respect to* $\mathscr{K}^m$, *if the following conditions are fulfilled:*

i) $\left\{\vec{\vartheta}_n(x)\right\}\in\mathscr{K}^m,\quad\forall x\in X;$

ii) $\exists\,A;B>0:\ \ A\left\|\left\{\vec{\vartheta}_n(x)\right\}\right\|_{\mathscr{K}^m}\le\|x\|_X\le B\left\|\left\{\vec{\vartheta}_n(x)\right\}\right\|_{\mathscr{K}^m};$

iii) $x=\sum_{n=1}^{\infty}\vec{\vartheta}_n(x)\,u_n,\quad\forall x\in X.$

The following theorem is true.

Theorem 1. *i) Let the pair* $\left(\left\{u_{in}^{0}\right\}_{i=\overline{1,m};\, n\in N};\left\{\vartheta_{in}\right\}_{i=\overline{1,m};\, n\in N}\right)$ *be an atomic decomposition of* X *with respect to* $\mathscr{K}$, *where* $\vartheta_{in}=\left(\vartheta_{in}^{(1)};...;\vartheta_{in}^{(m)}\right)\in X^{*}$, $i\in 1:m;\ n\in N$. *Then the relation* (5) *holds and system* (6) *is an atomic decomposition of* X_k *with respect to* $\mathscr{K}$.
ii) Let the pair (6) *be an atomic decomposition of* X_k *for every* $k\in 1:m$ *with respect to* $\mathscr{K}$ *and the relation* (5) *holds. Then* $\left(\left\{\vec{\vartheta}_n\right\};\{u_n\}\right)$ *is an atomic decomposition of* X *with respect to* $\mathscr{K}^m$ *in the sense of Definition 3.*

3. Main results

Let the following direct sum hold

$$X=X_1\dot{+}...\dot{+}X_m,$$

where $X_k,\ \ k=\overline{1,\,m}$—are some B-spaces. Consider the system $\{u_{in}\}_{n\in N}\subset X_i,\ \ i=\overline{1,\,m}$; and form

$$\vec{u}_{in}=\left(a_{i1}u_{1n};a_{i2}u_{2n};...;a_{im}u_{mn}\right),\ \ i=\overline{1,\,m};\ n\in N.$$

Let

$$A=(a_{ij})\ \ i,j=\overline{1,\,m};\quad \Delta=\det A.$$

We will need the following easy-to-prove lemma.

Lemma 1. *Let* $(\{u_n\};\{\vartheta_n\})$ *be an atomic decomposition of* X *with respect to* $\mathscr{K}$ *and* $T\in L(X)$ *be some automorphism. Then* $\left(\{Tu_n\};\left\{(T^{*})^{-1}\vartheta_n\right\}\right)$ *is also an atomic decomposition of* X *with respect to* $\mathscr{K}$.

Let $T_{ij}:X_i\to X_j$ be some operators. Consider the system

$$\sum_{i=1}^{m}a_{ij}T_{ij}x_i=y_j,\quad j=\overline{1,\,m}, \tag{7}$$

where $y_j\in X_j,\ j=\overline{1,\,m}$ are the given, and $x_i\in X_i,\ i=\overline{1,\,m}$ are the sought elements. Assume that the spaces $X_k,\ \ k=\overline{1,\,m}$, are pairwise isomorphic and T_{ij} performs a corresponding isomorphism. Besides, assume that the following conditions are satisfied:

α) $T_{ii}=I_i$, $T_{ij}=T_{ji}^{-1}$, $T_{jk}T_{ij}=T_{ik}$, $\forall i,j=\overline{1,m}$, *where* I_i *is the identity operator in* X_i.

Applying the operator $T_{j1}=T_{1j}^{-1}$ to the j-th equation in the system (7), we obtain the following system

$$\sum_{i=1}^{m}a_{ij}T_{i1}x_i=T_{j1}y_j,\quad j=\overline{1,\,m}.$$

Let $\tilde{x}_i = T_{i1}x_i, \;\; \tilde{y}_j = T_{j1}y_j$. It is clear that $\tilde{x}_i, \tilde{y}_j \in X_1$. As a result, we obtain the following system of linear equations in the space X_1:

$$\sum_{i=1}^{m} a_{ij}\tilde{x}_i = \tilde{y}_j \, , \, j = \overline{1, m}.$$

If the determinant of this system $\Delta = \det(a_{ij}) \neq 0$, then it is clear that this system is uniquely solvable with respect to the unknowns $\tilde{x}_i$. Then the system (7) is also uniquely solvable.

The following lemma is true.

Lemma 2. *Let the operators $T_{ij} : X_i \to X_j$ perform an isomorphism between X_i and X_j, the conditions $\alpha)$ be satisfied and $\Delta \neq 0$. Then the system* (7) *is uniquely solvable for $\forall y \in X, \;\; y = (y_1, \ldots, y_m)$ and, moreover, $\exists M > 0$:*

$$\|x\|_X \leq M \, \|y\|_X \, , \tag{8}$$

where $x = (x_1, \ldots, x_m)$.

The validity of the estimate (8) follows immediately from the following representation for the solution of the system (7):

$$x_i = \sum_{j=1}^{m} b_{ij} T_{ji} y_j, \;\; i = \overline{1, \, m},$$

where b_{ij} are the elements of the inverse matrix A^{-1}.

Consider the operator $T : X \to X$ defined by the matrix $(a_{ij}T_{ij})_{i,j=1}^{m}$. Let all the conditions of Lemma 2 be satisfied. It follows from this lemma that $KerT = \{0\}\, , \;\; R_T = X$, and the estimate (8) means $T \in L(X)$. Then it follows from Banach's theorem on the inverse operator that T is an automorphism in X. So the following theorem is true.

Theorem 2. *Let $T_{ij} \in L(X_i, X_j)$ be an isomorphism, the conditions $\alpha)$ be satisfied and $\Delta \neq 0$. Then the operator $T : X \to X$ defined by the matrix $(a_{ij}T_{ij})_{i,j=1}^{m}$ is an automorphism in $X = X_1 \oplus \ldots \oplus X_m$.*

The following theorem is true.

Theorem 3. *Let the direct sum $X = X_1 \dot{+} ... \dot{+} X_m$ hold, the pair $\left(\{u_{in}\}_{n\in N}; \{\vartheta_{in}\}_{n\in N}\right)$ be an atomic decomposition of $X_i, \;\; i = \overline{1, m}$; with respect to $\mathscr{K}$, $T_{ij} \in L(X_i; X_j)$ be an isomorphism and $T_{ij}u_{in} = u_{jn}, \; \forall n \in N$, for $i \neq j$. Let $\det(a_{ij})_{i,j=\overline{1,m}} \neq 0$, and operators $T_{ij}; i, j = \overline{1, m}$; satisfy the condition $\alpha)$ and the operator $T \in L(X)$ defined by the matrix $(a_{ij}T_{ij})_{i,j=\overline{1,m}}$. Then the pair $\left(\left\{\left\{Tu_{in}^0\right\}_{n\in N}\right\}_{i=\overline{1,m}}; \left\{\left\{(T^*)^{-1}\vartheta_{in}^0\right\}_{n\in N}\right\}_{i=\overline{1,m}}\right)$, is also an atomic decomposition of X with respect to $\mathscr{K}^m$.*

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References

[1] D.L. Russel, *On exponential bases for the Sobolev spaces over an interval*, Journal of Math. Anal. and Appl., **87**, 1982, 528–550.

[2] A.M. Sedletskii, *Approximate properties of exponential systems in Sobolev spaces*, Vestn. MGU, Ser. Mat.,Mekh., **6**, 1999, 3–8.

[3] Z.G. Huseynov, A.M. Shykhammedov, *On bases of sines and cosines in Sobolev spaces*, Applied Mathematics Letters, **25(3)**, 2012, 275–278.

[4] B.T. Bilalov, T.B. Gasymov, *On basicity of a part of a systems with infinite defect*, Trans. of. NAS of Azerb., **XXVII(7)**, 2007, 53–59.

[5] B.T. Bilalov, *Bases of Exponentials, Sines, and Cosines*, Differ. Uravn., **39(5)**, 2003, 619–622.

[6] B.T. Bilalov, T.R. Muradov, *Defect bases of Banach spaces*, Proc. IMM of NASA, **XXII (XXX)**, 2005, 23–26.

[7] X. He, H. Volkmer, *Riesz bases of solutions of Sturm–Liouville equations*, J. Fourier Anal. Appl., **7(3)**, 2001, 297–307.

[8] A.A. Huseynli, *On the stability of basisness in L_p $(1 < p < +\infty)$ of cosines and sines*, Turk J Math., **35**, 2011, 47–54.

[9] T.B. Gasymov, Sh.J.Mammadova, *On convergence of spectral expansions for one discontinuous problem with spectral parameter in the boundary condition*, Trans. Of NAS of Azerb., **XXVI(4)** 2006, 103–116.

[10] A.N. Tikhonov, A.A.Samarskii, *Equations of Mathematical Physics*, Izd. MGU, Moscow, 1999.

[11] F.V. Atkinson, *Discrete and Continuous Boundary Problems*, Moscow, Mir, 1968.

[12] T. Kato, *Theory of perturbations of linear operators*, Moscow, Mir, 1972, 738 p.

[13] I.C. Gohberg, A.S.Markus, *On stability of bases of Banach and Hilbert spaces*, Izv. Akad. Nauk. Mold. SSR., **5**, 1962, 17–35.

[14] V.D. Milman, *Geometric theory of Banach spaces*, Part I. The theory of basis and minimal systems. Uspekhi Mat. Nauk, **25:3(153)**, 1970, 113-174.

[15] B.T. Bilalov, *On the basis property of systems of exponentials, cosines, and sines in* $L\_\{p\}$, Dokl. Math., **365(1)**, 1999, 7–8.

[16] B.T. Bilalov, *On bases for some systems of exponentials, cosines, and sines in* $L\_\{p\}$, Dokl. Math., **379(2)**, 2001, 7–9.

[17] R.M. Young, *An Introduction to Nonharmonic Fourier series*, Springer, 1980, 246 p.

[18] O. Christensen, *An Introduction to Frames and Riesz bases*, Springer, 2003, 440 p.

[19] O. Christensen, *Frames, Riesz bases and discrete Gabor-wavelet expansions*, Bull Amer. Math. Soc., **38**, 2001, 273–291.

[20] O. Christensen, D.T. Stoeva, *p-frames in separable Banach spaces*, Advances in Computational Mathematics, **18**, 2003, 17–126.

[21] Ch. Heil, *A Basis Theory Primer*, Springer, 2011, 534 p.

[22] I. Singer, *Bases in Banach spaces*, v.1, Springer, 1970, 673 p.

[23] I. Singer, *Bases in Banach spaces*, v.2, Springer, 1981, 880 p.

[24] B.T. Bilalov, Z.V. Mamedova, *On the frame properties of some degenerate trigonometric system*, Dokl. Acad. Nauk, **LXVIII(5)**, 2012, 14–18.

[25] V.A. Il'in, *Necessary and sufficient conditions for the basis property and the equiconvergence with a trigonometric series of spectral expansions in systems of exponentials*, Dokl. Akad. Nauk SSSR, **273(4)**, 789–793.

[26] V.A. Marchenko, *Spectral theory of the Sturm-Liouville problem*, Naukova Dumka, Kiev, 1972.

[27] B.T. Bilalov, Ch.M. Hashimov, *On Decomposition In Banach Spaces*, Proceedings of the IMM, NAS of Azerbaijan, **40(2)**, 2014, 97–106.

[28] S.R. Sadigova, Z.A. Kasumov, *On atomic decomposition for Hardy classes with respect to degenerate exponential systems*, Proceedings of the IMM, NAS of Azerbaijan, **40(1)**, 2014, 55–67.

[29] B.T. Bilalov, F.A. Guliyeva, *t-frames and their Noetherian perturbation*, Complex Anal. Oper. Theory, **9**, 2015, 1609–1631.

[30] B.T. Bilalov, F.A. Guliyeva, *Noetherian perturbation of frames*, Pensee Journal, **75(12)**, 2013, 425–431.

[31] B.T. Bilalov, F.A. Guliyeva, *On The Frame Properties of Degenerate System of Sines*, Journal of Function Spaces and Applications, **2012**, 2012, Article ID 184186, 12 pages.

Telman B. Gasymov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az1141, Baku, Azerbaijan
Baku State University, Az1148, Baku, Azerbaijan
E-mail: telmankasumov@rambler.ru

Chingiz M. Hashimov
Ganja State University, Ganja, Azerbaijan
E-mail: chingiz.heshimov.88@mail.ru

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