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# Explicit Form of Laplace-Stieltjes Transform of Joint Distribution of the First Passage Time of Some Level "$a$" ($a > 0$) and Overshoots Across this Level by a Complex Semi-Markov Walk Process with Reflecting Screen at Zero

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**Abstract.** In the paper, by the probability-statistical method we find explicit form of the Laplace-Stieltjes transform of joint distribution of the first passage time of some level "$a$" ($a > 0$) and overshoot across this level by a complex semi-Markov walk process with a reflecting screen at zero.

**Key Words and Phrases**: Laplace-Stieltjes transform, probability space, semi-Markov walk process.

**2010 Mathematics Subject Classifications**: 60A10, 60J25, 60D10

## 1. Introduction

In the paper [1, p. 61-63] asymptotic behavior of random walks in random medium with a delaying screen was considered. In [2, p. 160-165] random walk was studied in a strip. In the paper [3, p. 26-51] asymptotic expansion of distribution was found. In the paper [4, p. 61-63], various semi-Markov processes with a delaying screen and functional of these processes were studied. In [5, p. 77-84] the Laplace transform of distribution of the lower boundary functional of semi-Markov walk process with a delaying screen at zero was found. In [6, p. 49-60] the Laplace transform of ergodic distribution of semi-Markov walk process with a negative drift, non-negative jumps and a delaying screen at zero, was found.

In the present paper we study joint distribution of the first passage moment of some level "$a$" ($a > 0$) and the overshoot across this level by a complex semi-Markov walk process with a reflecting screen at zero.

## 2. Mathematical statement of the problem

Let on probability space $(\Omega,\, F,\, P(\cdot))$ be given the sequence $\left\{\xi_k^+, \eta_k^+, \xi_k^-, \eta_k^-\right\}_{k=1,\infty}$, where

 

$\xi_k^+, \eta_k^+, \xi_k^-, \eta_k^-$ are identically distributed between themselves positive random variables are identically.

Denote

$$S_k = \left| S_{k-1} + ... + \eta^+_{\nu(\tau_{k-1})} + \eta^+_{\nu(\tau_k)} - \eta_k^- \right|, \tag{1}$$

where $S_0 = z$,

$$X^{\pm}(t) = \sum_{i=1}^{\nu^{\pm}(t)} \eta_i^{\pm}, \tag{2}$$

$$\tau_k^{\pm} = \sum_{i=1}^{k} \xi_k^{\pm}; k = 1, 2, ..; \ \tau_0^{\pm} = 0, \tag{3}$$

where

$$v^{\pm}(t) = \min \left\{ k : \sum_{i=1}^{k+1} \xi_k^{\pm} > t \right\}. \tag{4}$$

The process $X(t) = S_{k-1} + ... + \eta^+_{\nu(\tau_{k-1})} + \eta^+_{\nu(\tau_k)} - \eta_k^-$ if $\tau^{\pm}_{k-1} < t < \tau^{\pm}_k$ is called a complex semi-Markov walk process with a reflecting screen at zero. One of the realizations of the process $X(t)$ is of the form

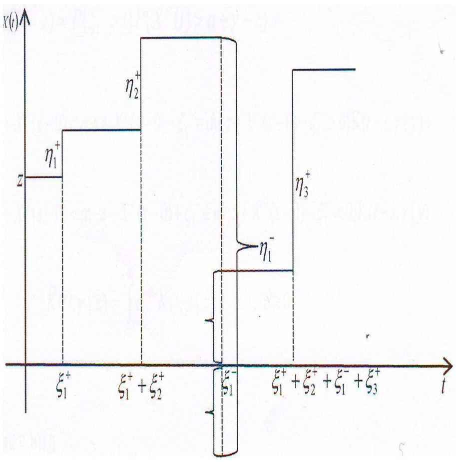

**Fig.** $v^{\pm}(t)$ **is the number of positive or negative jumps for time** $t$**.**

Our goal is to find the explicit form of the Laplace-Stieltjes transform of joint distribution of the first passage moment and overshoot of the level $a\,(a > 0)$.

Let $\tau_a$ be the first passage moment of the level $a\,(a > 0)$ and $\gamma$ be an overshoot across this level.

We assume that $\xi_1^+$ has exponential distribution with the parameter $\lambda_+$.

Denote

$$\mathrm{K}(t, \gamma | X(0) = z) = P\{\tau_a < t, \gamma_a > a | X(0) = z\}$$

By total probability formula we have

$$\mathrm{K}(t, \gamma | X(0) = z) = P\{\tau_a < t, \gamma_a > \gamma; \xi_1^- > t | X(0) = z\} +$$

$$+\int_{s=0}^{t}\int_{y=0}^{a} P\{\xi_1^- \in ds; \sup_{0\le u\le s-0} X(u) < a; |X(s)| \in dy|X(0)=z\}\mathrm{K}(t-s,\gamma|y) =$$

$$= P\{\xi_1^- > t; z + X^+(t) > a+\gamma\} +$$

$$+\int_{s=0}^{t}\int_{y=0}^{a} P\{\xi_1^- \in ds; z+X^+(s-0) < a; |z+X^+(s-0)-\zeta_1^-| \in dy\}\mathrm{K}(t-s,\gamma|y)$$

In view of $\{|u|<\varepsilon\} = \{-\varepsilon < u < \varepsilon\}$ we have

$$\mathrm{K}(t,\gamma|X(0)=z) = P\{\xi_1^- > t\}P\{X^+(t) > a+\gamma-z\} +$$

$$+\int_{s=0}^{t}\int_{y=0}^{a} P\{\xi_1^- \in ds; z+X^+(s-0) < a; z+X^+(s-0)-$$

$$-\zeta_1^- \in dy; z+X^+(s-0)-\zeta_1^- > 0\}\mathrm{K}(t-s,\gamma|y) +$$

$$+\int_{s=0}^{t}\int_{y=0}^{a} P\{\xi_1^- \in ds; z+X^+(s-0) < a; -z-X^+(s-0)+$$

$$+\zeta_1^- \in dy; z+X^+(s-0)-\zeta_1^- < 0\}\mathrm{K}(t-s,\gamma|y)$$

So, we get an integral equation for $\mathrm{K}(t,\gamma|X(0)=z)$.

$$\mathrm{K}(t,\gamma|X(0)=z) = P\{\xi_1^- > t\}P\{X^+(t) > a+\gamma-z\} +$$

$$+\int_{s=0}^{t}\int_{y=0}^{a} P\{\xi_1^- \in ds; z+X^+(s-0) < a; z+X^+(s-0)-\zeta_1^- \in dy;$$

$$z+X^+(s-0)-\zeta_1^- > 0\}\mathrm{K}(t-s,\gamma|y) +$$

$$+\int_{s=0}^{t}\int_{y=0}^{a} P\{\xi_1^- \in ds; z+X^+(s-0) < a; -z-X^+(s-0)+\zeta_1^- \in dy;$$

$$z+X^+(s-0)-\zeta_1^- < 0\}\mathrm{K}(t-s,\gamma|y). \qquad (5)$$

Denote $\tilde{K}(\theta,\gamma|z) = \int_{t=0}^{\infty} e^{-\theta t} K(t,\gamma|z)$, $\theta > 0$

Then (5) takes the form

$$\tilde{K}(\theta,\gamma|z) = \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t; X^+(t) > a+\gamma-z\}dt +$$

$$+\int_{y=0}^{a} \tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty} d_y P\{X^+(t) < a-z;$$

$$z+X^+(t)-\zeta_1^- < y; z+X^+(t)-\zeta_1^- > 0\}dP\{\xi_1^- < t\} +$$

$$+\int_{y=0}^{a} \tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty} d_y P\{X^+(t) < a-z; -z-X^+(t)+\zeta_1^- < y;$$

$$z+X^+(t)-\zeta_1^- < 0\}dP\{\xi_1^- < t\} \qquad (6)$$

Make a change of variables $X^+(t)=h$. Then (6) takes the form

$$\tilde{K}(\theta,\gamma|z)=\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}dt-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}P\{X^+(t)<a+\gamma-z\}dt+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{\zeta_1^->z-y+h;\zeta_1^-<z+h\}d_t\times$$

$$\times P\{\xi_1^-<t\}d_hP\{X^+(t)<h\}+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\times$$

$$\times\int_{h=0}^{a-z}P\{\zeta_1^-<z+y+h;\zeta_1^->z+h\}d_tP\{\xi_1^-<t\}d_hP\{X^+(t)<h\}=$$

$$=\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}dt-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}\times$$

$$\times\sum_{k=0}^{\infty}P\{\sum_{i=1}^{\infty}\zeta_i^+<a+\gamma-z\}P\{\nu^+(t)=k\}dt+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{z-y+h<\zeta_1^-<z+h\}d_t\times$$

$$\times P\{\xi_1^-<t\}d_hP\{X^+(t)<h\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{z+h<\zeta_1^-<z+h+y\}d_t\times$$

$$\times P\{\xi_1^-<t\}d_hP\{X^+(t)<h\}.$$

Taking into account $X^+(t)=\sum_{i=1}^{\nu^+(t)}\eta_i^+$, from the last equation we have

$$\tilde{K}(\theta,\gamma|z)=$$

$$=\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}dt-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}\sum_{k=0}^{\infty}\times$$

$$\times P\{\sum_{i=1}^{k}\zeta_i^+<a+\gamma-z\}P\{\nu^+(t)=k\}dt-$$

$$-\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{\zeta_1^-<z-y+h\}d_tP\{\xi_1^-<t\}d_hP\{X^+(t)<h\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{\zeta_1^-<z+h+y\}d_tP\{\xi_1^-<t\}d_hP\{X^+(t)<h\}$$

From the fact that there should be $z-y+h>0$ or $h>\max(0,y-z)$, we have

$$\tilde{K}(\theta,\gamma|z)=$$

$$= \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} dt - \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} \sum_{k=0}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < a + \gamma - z\} P\{\nu^+(t) = k\} dt -$$

$$- \int_{y=0}^{z} \tilde{K}(\theta, \gamma | y) \int_{h=0}^{a-z} d_y P\{\zeta_1^- < -y + h + z\} \int_{t=0}^{\infty} e^{-\theta t} d_h \times$$

$$\times \sum_{k=0}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < h\} P\{\nu^+(t) = k\} d_t P\{\xi_1^- < t\} +$$

$$+ \int_{y=z}^{a} \tilde{K}(\theta, \gamma | y) \int_{h=y-z}^{a-z} d_y P\{\zeta_1^- < -y + h + z\} \int_{t=0}^{\infty} e^{-\theta t} d_h \times$$

$$\times \sum_{k=0}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < h\} P\{\nu^+(t) = k\} d_t P\{\xi_1^- < t\} +$$

$$+ \int_{y=0}^{a} \tilde{K}(\theta, \gamma | y) \int_{h=0}^{a-z} d_y P\{\zeta_1^- < y + h + z\} \int_{t=0}^{\infty} e^{-\theta t} d_h v \times$$

$$\times \sum_{k=0}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < h\} P\{\nu^+(t) = k\} d_t P\{\xi_1^- < t\}.$$

Simplify this equation. More exactly, taking into account

$$1 = \sum_{k=0}^{\infty} P\{\nu^+(t) = k\} = P\{\nu^+(t) = 0\} + P\{\nu^+(t) \geq 1\}$$

the last equation takes the following form

$$\tilde{K}(\theta, \gamma | z) = \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} dt -$$

$$- \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} \varepsilon(a + \gamma - z) P\{\nu^+(t) = 0\} dt -$$

$$- \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} \sum_{k=1}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < a + \gamma - z\} P\{\nu^+(t) = k\} dt -$$

$$- \int_{y=0}^{z} \tilde{K}(\theta, \gamma | y) \int_{h=0}^{a-z} d_y P\{\zeta_1^- < -y + h + z\} \int_{t=0}^{\infty} e^{-\theta t} d_h \varepsilon(h) P\{\nu^+(t) = 0\} d_t P\{\xi_1^- < t\} -$$

$$- \int_{y=0}^{z} \tilde{K}(\theta, \gamma | y) \int_{h=0}^{a-z} d_y P\{\zeta_1^- < -y + h + z\} \times$$

$$\times \int_{t=0}^{\infty} e^{-\theta t} d_h \sum_{k=1}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < h\} P\{\nu^+(t) = k\} d_t P\{\xi_1^- < t\} - \qquad (7)$$

$$-\int_{y=z}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=y-z}^{a-z}d_yP\{\zeta_1^- < -y+h+z\}\int_{t=0}^{\infty}e^{-\theta t}d_h\varepsilon(h)P\{\nu^+(t)=0\}d_tP\{\xi_1^- < t\}-$$

$$-\int_{y=z}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=y-z}^{a-z}d_yP\{\zeta_1^- < -y+h+z\}\times$$

$$\times\int_{t=0}^{\infty}e^{-\theta t}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_1^+ < h\}P\{\nu^+(t)=k\}d_tP\{\xi_1^- < t\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^- < y+h+z\}\int_{t=0}^{\infty}e^{-\theta t}d_h\varepsilon(h)P\{\nu^+(t)=0\}d_tP\{\xi_1^- < t\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^- < y+h+z\}\times$$

$$\times\int_{t=0}^{\infty}e^{-\theta t}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_1^+ < h\}P\{\nu^+(t)=k\}d_tP\{\xi_1^- < t\}$$

By virtue of $\varepsilon(h)=\begin{cases}0, h<0\\1, h>0\end{cases}$ (7) takes the form

$$\tilde{K}(\theta,\gamma|z)=\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^- > t\}dt-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^- > t\}P\{\nu^+(t)=0\}dt\varepsilon(a+\gamma-z)-$$

$$-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^- > t\}\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_1^+ < a+\gamma-z\}P\{\nu^+(t)=k\}dt-$$

$$-\int_{y=0}^{z}\tilde{K}(\theta,\gamma|y)d_yP\{\zeta_1^- < -y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^- < t\}-$$

$$-\int_{y=o}^{z}\tilde{K}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^- < -y+h+z\}d_h\times$$

$$\times\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+ < h\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^- < t\}-$$

$$-\int_{y=z}^{a}\tilde{K}(\theta,\gamma|y)d_yP\{\zeta_1^- < -y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^- < t\}-$$

$$-\int_{y=z}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=y-z}^{a-z}d_yP\{\zeta_1^- < -y+h+z\}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+ < h\}\times$$

$$\times\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^- < t\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)d_yP\{\zeta_1^- < y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^- < t\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^-<y+h+z\}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\times$$

$$\times\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}. \qquad (8)$$

Thus, when $\xi_1^+$, $\xi_1^-,\zeta_1^+,\zeta_1^-$ have exponential distribution, we get integral equation (8). When $\xi_1^+$ has exponential distribution $\xi_1^-,\zeta_1^+,\zeta_1^-$ have Erlang distribution of any order, and one can get an integral equation of type (8). Solve equation (8) in the case when $\xi_1^+$, $\xi_1^-,\zeta_1^+,\zeta_1^-$ have Erlang distribution of first order.

Denote

$$\tilde{\tilde{K}}(\theta,\chi|z)=\int_{\gamma=0}^{\infty}e^{-\chi\gamma}d_\gamma\tilde{K}(\theta,\gamma|z),\ \ \chi>0.$$

Then (4) takes the form

$$\tilde{\tilde{K}}(\theta,\chi|z)=-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}P\{\nu^+(t)=0\}dt\int_{\gamma=0}^{\infty}d_\gamma\varepsilon(a+\gamma-z)-$$

$$-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}\sum_{k=1}^{\infty}P\{\nu^+(t)=k\}\int_{\gamma=0}^{\infty}e^{-\chi\gamma}d_\gamma P\{\sum_{i=1}^{k}\zeta_1^+<a+\gamma-z\}-$$

$$-\int_{y=0}^{z}\tilde{\tilde{K}}(\theta,\gamma|y)d_yP\{\zeta_1^-<-y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}-$$

$$-\int_{y=o}^{z}\tilde{\tilde{K}}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^-<-y+h+z\}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\times$$

$$\times\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}-$$

$$-\int_{y=z}^{a}\tilde{\tilde{K}}(\theta,\gamma|y)d_yP\{\zeta_1^-<-y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}-$$

$$-\int_{y=z}^{a}\tilde{\tilde{K}}(\theta,\gamma|y)\int_{h=y-z}^{a-z}d_yP\{\zeta_1^-<-y+h+z\}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\times$$

$$\times\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}+$$

$$+\int_{y=0}^{a}\tilde{\tilde{K}}(\theta,\gamma|y)d_yP\{\zeta_1^-<y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}+$$

$$+\int_{y=0}^{a}\tilde{\tilde{K}}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^-<y+h+z\}d_h\times$$

$$\times\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}.$$

Now let

$$P\{\xi_1^{\pm} < t\} = \begin{cases} 0, t < 0 \\ 1 - e^{-\lambda_{\pm} t}, \lambda_{\pm} > 0, t > 0 \end{cases}$$

$$\zeta_1^{\pm} < x\} = \begin{cases} 0, x < 0 \\ 1 - e^{-\mu_{\pm} x}, x > 0, \mu_{\pm} > 0 \end{cases}$$

Then we get

$$\tilde{\tilde{K}}(\theta,\chi|z) = -\frac{e^{(a-z)\chi}}{\lambda_+ + \lambda_- + \theta} +$$
$$+\frac{\lambda_+\mu_+}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\chi+\mu_+)(\lambda_+ + \lambda_- + \theta)\right)} e^{-\frac{\mu_+(\lambda_- + \theta)(a-z)}{\lambda_+ + \lambda_- + \theta}} +$$
$$+\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta} e^{-\mu_- z}\int_{y=0}^{z} \tilde{\tilde{K}}(\theta,\chi|y) e^{\mu_- y} dy +$$
$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)\right)} \times$$
$$\times \left(e^{\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(a-z)\right)} - 1\right) e^{-\mu_- z}\int_{y=0}^{z} \tilde{\tilde{K}}(\theta,\chi|y) e^{\mu_- y} dy +$$
$$+\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta} e^{-\mu_- z}\int_{y=z}^{a} \tilde{\tilde{K}}(\theta,\chi|y) e^{\mu_- y} dy +$$
$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)\right)} e^{-\mu_- z} \times$$
$$\times \int_{y=z}^{a} (e^{(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(a-z)} - e^{(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(y-z)}) \tilde{\tilde{K}}(\theta,\chi|y) e^{\mu_- y} dy +$$
$$+\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta} e^{-\mu_- z}\int_{y=0}^{a} \tilde{\tilde{K}}(\theta,\chi|y) e^{-\mu_- y} dy +$$
$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)\right)} \times$$
$$\times \left(e^{\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(a-z\right)} - 1\right) e^{-\mu_- z}\int_{y=0}^{a} \tilde{\tilde{K}}(\theta,\chi|y) e^{-\mu_- y} dy. \qquad (9)$$

Having multiplied the both sides by $e^{\mu_- z}$ and differentiated with respect to $z$, we get

$$e^{\mu_- z}\left[\mu_- \tilde{\tilde{K}}(\theta,\chi,z) + \tilde{\tilde{K}}'(\theta,\chi,z)\right] = -\frac{(\mu_- - \chi)e^{a\chi}}{\lambda_+ + \lambda_- + \theta} e^{(\mu_- - \chi)z} +$$
$$+\frac{\lambda_+\mu_+(\mu_-(\lambda_+ + \lambda_- + \theta) + \mu_+(\lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2\left(\lambda_+\mu_+ - (\chi+\mu_+)(\lambda_+ + \lambda_- + \theta)\right)} e^{-\frac{\mu_+(\lambda_- + \theta)a}{\lambda_+ + \lambda_- + \theta} + \left(\frac{\mu_+(\lambda_- + \theta)}{\lambda_+ + \lambda_- + \theta} + \mu_-\right)z} +$$
$$+\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}\tilde{\tilde{K}}(\theta,\chi,z)e^{\mu_- z} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)\right)} \times$$

$$\times\left[-\frac{\lambda_+\mu_+-(\mu_++\mu_-)(\lambda_++\lambda_-+\theta)}{\lambda_++\lambda_-+\theta}e^{(\frac{\lambda_+\mu_+}{\lambda_++\lambda_-+\theta}-\mu_+-\mu_-)(a-z)}\times\right.$$

$$\left.\times\int_{y=0}^{z}\tilde{\tilde{K}}(\theta,\chi|y)e^{\mu_-y}dy+\left(e^{\left(\frac{\lambda_+\mu_+}{\lambda_++\lambda_-+\theta}-\mu_+-\mu_-\right)(a-z)}-1\right)\tilde{\tilde{K}}(\theta,\chi,z)e^{\mu_+z}\right]+$$

$$-\frac{\lambda_-\mu_-}{\lambda_++\lambda_-+\theta}\tilde{\tilde{K}}(\theta,\chi,z)e^{\mu_+z}+$$

$$-\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\lambda_-+\theta)(\lambda_+\mu_+-(\mu_++\mu_-)(\lambda_++\lambda_-+\theta))}\left(e^{\left(\frac{\lambda_+\mu_+}{\lambda_++\lambda_-+\theta}-\mu_+-\mu_-\right)(a-z)}-1\right)\times$$

$$\times\tilde{\tilde{K}}(\theta,\chi,z)e^{\mu_-z}-\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\lambda_-+\theta)(\lambda_+\mu_+-(\mu_++\mu_-)(\lambda_++\lambda_-+\theta))}\times$$

$$\times\frac{\lambda_+\mu_+-(\mu_++\mu_-)(\lambda_++\lambda_-+\theta)}{\lambda_++\lambda_-+\theta}e^{-\left(\frac{\lambda_+\mu_+}{\lambda_++\lambda_-+\theta}-\mu_+-\mu_-)z\right)}\times$$

$$\times\left[\int_{y=z}^{a}(e^{\left(\frac{\lambda_+\mu_+}{\lambda_++\lambda_-+\theta}-\mu_+-\mu_-)a\right)}-e^{\left(\frac{\lambda_+\mu_+}{\lambda_++\lambda_-+\theta}-\mu_+-\mu_-)y\right)})\tilde{\tilde{K}}(\theta,\chi|y)e^{\mu_-y}dy\right]-$$

$$-\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\lambda_-+\theta)(\lambda_+\mu_+-(\mu_++\mu_-)(\lambda_++\lambda_-+\theta))}\frac{\lambda_+\mu_+-(\mu_++\mu_-)(\lambda_++\lambda_-+\theta)}{\lambda_++\lambda_-+\theta}\times$$

$$\times e^{-\left(\frac{\lambda_+\mu_+}{\lambda_++\lambda_-+\theta}-\mu_+-\mu_-)(a-z)\right)}\int_{y=0}^{a}\tilde{\tilde{K}}(\theta,\chi|y)e^{-\mu_-y}dy. \tag{10}$$

We differentiate the obtained equation by $z$. As a result, we get a second order inhomogeneous equation with constant coefficients

$$\tilde{\tilde{K}}''(\theta,\chi,z)+(\mu_-+\frac{\mu_+\lambda_+}{\lambda_++\lambda_-+\theta})\tilde{\tilde{K}}'(\theta,\chi,z)+\left[\frac{\lambda_+\mu_+\mu_-}{\lambda_++\lambda_-+\theta}+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\lambda_-+\theta)^2}\right]\tilde{\tilde{K}}(\theta,\chi,z)=$$

$$=\frac{(\mu_--\chi)(\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2}e^{(a-z)\chi}. \tag{11}$$

The roots of the appropriate characteristic equation are

$$k_{1;2}(\theta)=\frac{-(\mu_-+\frac{\mu_+\lambda_+}{\lambda_++\lambda_-+\theta})\pm\sqrt{(\mu_-+\frac{\mu_+\lambda_+}{\lambda_++\lambda_-+\theta})^2-4\left[\frac{\lambda_+\mu_+\mu_-}{\lambda_++\lambda_-+\theta}+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\lambda_-+\theta)^2}\right]}}{2}$$

The solution of equation (11) is

$$\tilde{\tilde{K}}(\theta,\chi,z)=\frac{(\mu_--\chi)((\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2(\chi+k_1(\theta))(\chi+k_2(\theta))}e^{\chi(a-z)}+$$

$$+C_1(\theta)e^{k_1(\theta)z}+C_2(\theta)e^{k_2(\theta)z}, \tag{12}$$

where $C_1(\theta)$ and $C_2(\theta)$ are constant with respect to $z$.

Find $C_1(\theta)$ and $C_2(\theta)$.

In (9), having substituted $z=a$, we get an equation with respect to $C_1(\theta)$ and $C_2(\theta)$

$$C_1(\theta)\left[e^{k_1 a}-\frac{\lambda_-\mu_-}{\lambda_++\lambda_-+\theta}e^{-\mu_- a}\left[\frac{1}{k_1(\theta)+\mu_-}\left(e^{(k_1(\theta)+\mu_-)a}-1\right)+\right.\right.$$
$$\left.\left.+\frac{1}{k_1(\theta)-\mu_-}\left(e^{(k_1(\theta)-\mu_-)a}-1\right)\right]\right]+$$
$$+C_2(\theta)\left[e^{k_2 a}-\frac{\lambda_-\mu_-}{\lambda_++\lambda_-+\theta}e^{-\mu_- a}\left[\frac{1}{k_2(\theta)+\mu_-}\left(e^{(k_2(\theta)+\mu_-)a}-1\right)+\right.\right.$$
$$\left.\left.+\frac{1}{k_2(\theta)-\mu_-}\left(e^{(k_{21}(\theta)-\mu_-)a}-1\right)\right]\right]=$$
$$=-\frac{(\mu_--\chi)((\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2(\chi+k_1(\theta))(\chi+k_2(\theta))}-$$
$$-\frac{\mu_++\chi}{\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta)}+\frac{\lambda_-\mu_-}{\lambda_++\lambda_-+\theta}\times$$
$$\times\frac{(\mu_--\chi)(\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2(\chi+k_1(\theta))(\chi+k_2(\theta))}\times$$
$$\times\left[\frac{1}{\mu_--\chi}e^{(\mu_--\chi)a}-\frac{1}{\mu_-+\chi}e^{-(\mu_-+\chi)a}-\frac{2\chi}{(\mu_-+\chi)(\mu_--\chi)}\right]e^{(\chi-\mu_-)a}.$$

In (10), having substituted $z=a$, we get an equation with respect to $C_1(\theta)$ and $C_2(\theta)$

$$C_1(\theta)\left[(\mu_-+k_1(\theta))e^{k_1(\theta)a}+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\lambda_-+\theta)^2}e^{-\mu_- a}\times\right.$$
$$\left.\times\left[\frac{1}{k_1(\theta)+\mu_-}\left(e^{(k_1(\theta)+\mu_-)a}-1\right)+\frac{1}{k_1(\theta)-\mu_-}\left(e^{(k_1(\theta)-\mu_-)a}-1\right)\right]\right]+$$
$$+C_2(\theta)\left[\left(\mu_-+k_2(\theta))e^{k_2(\theta)a}+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\lambda_-+\theta)^2}e^{-\mu_- a}\times\right.\right.$$
$$\left.\times\left[\frac{1}{k_2(\theta)+\mu_-}\left(e^{(k_2(\theta)+\mu_-)a}-1\right)+\frac{1}{k_2(\theta)-\mu_-}\left(e^{(k_2(\theta)-\mu_-)a}-1\right)\right]\right]=$$
$$=-\frac{(\mu_--\chi)^2((\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2(\chi+k_1(\theta))(\chi+k_2(\theta))}-\frac{\mu_--\chi}{\lambda_++\lambda_-+\theta}-$$
$$-\frac{\lambda_+\mu_+(\mu_+(\lambda_-+\theta)+\mu_-(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2((\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}-$$
$$-\frac{\lambda_+\lambda_-\mu_+\mu_-(\mu_--\chi)(\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^4(\chi+k_1(\theta))(\chi+k_2(\theta))}\times$$
$$\times\left[\frac{1}{\mu_--\chi}e^{(\mu_--\chi)a}-\frac{1}{\mu_-+\chi}e^{-(\mu_-+\chi)a}-\frac{2\chi}{(\mu_-+\chi)(\mu_--\chi)}\right]e^{(\chi-\mu_-)a}.$$

Thus, we get a system of linear algebraic equations with respect to $C_1(\theta)$and $C_2(\theta)$. Denote

$$S_1 = e^{-\mu_- a}\left[\frac{1}{k_1(\theta)+\mu_-}\left(e^{(k_1(\theta)+\mu_-)a}-1\right)+\frac{1}{k_1(\theta)-\mu_-}\left(e^{(k_1(\theta)-\mu_-)a}-1\right)\right],$$

$$S_2 = e^{-\mu_- a}\left[\frac{1}{k_2(\theta)+\mu_-}\left(e^{(k_2(\theta)+\mu_-)a}-1\right)+\frac{1}{k_2(\theta)-\mu_-}\left(e^{(k_2(\theta)-\mu_-)a}-1\right)\right],$$

$$A = -\frac{(\mu_- - \chi)((\mu_+(\lambda_- + \theta) + \chi(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2(\chi + k_1(\theta))(\chi + k_2(\theta))} -$$

$$-\frac{\mu_+ + \chi}{\mu_+(\lambda_- + \theta) + \chi(\lambda_+ + \lambda_- + \theta)} + \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}\times$$

$$\times\frac{(\mu_- - \chi)(\mu_+(\lambda_- + \theta) + \chi(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2(\chi + k_1(\theta))(\chi + k_2(\theta))}\times$$

$$\times\left[\frac{1}{\mu_- - \chi}e^{(\mu_- - \chi)a} - \frac{1}{\mu_- + \chi}e^{-(\mu_- + \chi)a} - \frac{2\chi}{(\mu_- + \chi)(\mu_- - \chi)}\right]e^{(\chi - \mu_-)a},$$

$$B = -\frac{(\mu_- - \chi)^2((\mu_+(\lambda_- + \theta) + \chi(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2(\chi + k_1(\theta))(\chi + k_2(\theta))} - \frac{\mu_- - \chi}{\lambda_+ + \lambda_- + \theta} -$$

$$-\frac{\lambda_+\mu_+(\mu_+(\lambda_- + \theta) + \mu_-(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2((\mu_+(\lambda_- + \theta) + \chi(\lambda_+ + \lambda_- + \theta))},$$

$$C_1(\theta) =$$

$$= \frac{A[(\mu_- + k_2(\theta))e^{k_2(\theta)a} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}S_2] - B[e^{k_2 a} - \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}S_2]}{(k_2 - k_1)e^{(k_1 + k_2)a} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}(S_2 e^{k_1 a} - S_1 e^{k_2 a}) + \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}(S_2(\mu_- + k_1(\theta))e^{k_1 a} - S_1(\mu_- + k_2(\theta))e^{k_2 a})},$$

$$C_2(\theta) =$$

$$= \frac{B[e^{k_1 a} - \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}S_1] - A[(\mu_- + k_1(\theta))e^{k_1(\theta)a} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}S_1]}{(k_2 - k_1)e^{(k_1 + k_2)a} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}(S_2 e^{k_1 a} - S_1 e^{k_2 a}) + \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}[S_2(\mu_- + k_1(\theta))e^{k_1 a} - S_1(\mu_- + k_2(\theta))e^{k_2 a}]}$$

Finally, we find the solution of equation (8).

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