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Multilinear Rough Fractional Integral on Product Morrey Spaces

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Abstract. We will study the boundedness of multilinear fractional integral operator $I_{\Omega,\alpha,m}$ with rough kernels $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s \leq \infty$ on product Morrey spaces. We find for the operator $I_{\Omega,\alpha,m}$ necessary and sufficient conditions on the parameters of the boundedness on product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to Morrey spaces $L^{q,\lambda}(\mathbb{R}^n)$.

Key Words and Phrases: product Morrey spaces, multilinear rough fractional integral.

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1. Introduction

The classical Morrey spaces, introduced by Morrey [9] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations. The boundedness of fractional integral operators on the classical Morrey spaces was studied by Adams [1], Chiarenza and Frasca *et al.* [2].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. Also for $\overrightarrow{x} = (x_1,\ldots,x_m) \in \mathbb{R}^{mn}$ and $r > 0$, we denote by $B(\overrightarrow{x},r)$ the open ball centered at $\overrightarrow{x} \in \mathbb{R}^{mn}$ of radius r, and $B(\overrightarrow{x},r)$ We denote by $\overrightarrow{f}$ the m-tuple $(f_1,f_2,\ldots,f_m)$, $\overrightarrow{y} = (y_1,\ldots,y_m)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

Definition 1. *Let $1 \leq p < \infty$, $0 \leq \lambda \leq n$, $[t]_1 = \min\{1,t\}$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, and by $WL_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms*

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}, \qquad \|f\|_{WL_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))}$$

respectively.

In 1999, Kenig and Stein [8] studied the following multilinear fractional integral

$$I_{\alpha,m}(\overrightarrow{f})(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\dots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{nm-\alpha}}dy_1dy_2\dots dy_m,$$

and showed that $I_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n)\times L_{p_2}(\mathbb{R}^n)\times\dots\times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q=1/p_1+\dots+1/p_m-\beta/n>0$ for each $p_i>1(i=1,\dots,m)$. If some $p_i=1$, then $I_{\alpha,m}$ is bounded $L_{p_1}(\mathbb{R}^n)\times L_{p_2}(\mathbb{R}^n)\times\dots\times L_{p_m}(\mathbb{R}^n)$ to $L_{q,\infty}(\mathbb{R}^n)$. Obviously, the multilinear fractional integral $I_{\alpha,m}$ is a natural generalization of the classical fractional integral $I_\alpha\equiv I_{\alpha,1}$.

Let $1<s\le\infty$, $\Omega\in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$. The multi-sublinear fractional maximal operator $\mathcal{M}_{\alpha,m}$ with rough kernels Ω is defined by

$$\mathcal{M}_{\alpha,m}(\overrightarrow{f})(x)=\sup_{r>0}\frac{1}{r^{nm-\alpha}}\int_{B(\overrightarrow{0},r)}|\Omega(\overrightarrow{y})|\prod_{j=1}^{m}|f_i(x-y_i)|d\overrightarrow{y},\qquad 0\le\alpha<nm.$$

If $m=1$, then $M_{\Omega,\alpha}\equiv\mathcal{M}_{\Omega,\alpha,1}$ is the fractional maximal operator with rough kernel Ω. When $m=1$ and $\Omega\equiv1$, then $M_\alpha\equiv\mathcal{M}_{1,\alpha,1}$ is the classical fractional maximal operator.

In [7] we proved the boundedness of the multi-sublinear fractional maximal operator with rough kernels $\mathcal{M}_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, if $p>s'$, $1<p_1,\dots,p_m<\infty$, $1/q=1/p_1+\dots+1/p_m-\alpha/(mn-\lambda)$ and from the space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $WL^{q,\lambda}(\mathbb{R}^n)$, if $p=s'$, $1\le p_1,\dots,p_m<\infty$ and $1/q=1/p_1+\dots+1/p_m-\alpha/(n-\lambda)$ and at least one exponent p_i, $1\le i\le m$ equals one.

In this work, we prove the boundedness of the multilinear fractional integral operator with rough kernels $I_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, if $p>s'$, $1<p_1,\dots,p_m<\infty$, $1/q=1/p_1+\dots+1/p_m-\alpha/(mn-\lambda)$ and from the space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $WL^{q,\lambda}(\mathbb{R}^n)$, if $p=s'$, $1\le p_1,\dots,p_m<\infty$ and $1/q=1/p_1+\dots+1/p_m-\alpha/(n-\lambda)$ and at least one exponent p_i, $1\le i\le m$ equals one.

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Boundedness of multilinear fractional integral operator $\mathcal{M}_{\Omega,\alpha,m}$ on product Morrey spaces

In this part, we investigate the boundedness of multilinear fractional integral operator $I_{\Omega,\alpha,m}$ on product Morrey spaces.

Spanne and Adams obtained two remarkable results on Morrey spaces (see Definition 1.1 of the Morrey spaces in Section 1) for I_α. Their results can be summarized as follows.

Theorem 1. *[5, 10] (Spanne, but published by Peetre) Let $0<\alpha<n$, $0\le\lambda<n-\alpha p$, $1/q=1/p-\alpha/n$ and $\mu/q=\lambda/p$. Then for $p>1$, the operator I_α are bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ and for $p=1$, I_α is bounded from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\mu}(\mathbb{R}^n)$.*

Theorem 2. *[1, 4] Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n - \alpha p$.*

(i) If $p > 1$, then condition $1/p - 1/q = \alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of the operator I_α from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.

(ii) If $p = 1$, then condition $1 - 1/q = \alpha/(n-\lambda)$ s necessary and sufficient for the boundedness of the operator I_α from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$.

If $\lambda = 0$, then the statement of Theorems 1 and 2 reduces to the well known Hardy-Littlewood-Sobolev inequality.

When $m \ge 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, in [6] was find out $\mathcal{M}_{\Omega,m}$ also have the same properties by providing the following multi-version result of the Chiarenza and Frasca [2].

Theorem 3. *[6] Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \ldots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \quad for \quad 0 \le \lambda_j < n. \tag{1}$$

(i) If $p > s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{L^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{WL^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

Lemma 1. *[11] Let $0 < \alpha < mn$, $1 \le s' < mn/\alpha$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$ and $f \in L^{p_1}(\mathbb{R}^n) \times \ldots \times L^{p_m}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ for any $x \in \mathbb{R}^n$*

$$\Big| I_{\Omega,\alpha,m}\boldsymbol{f}(x) \Big| \le C \Big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\boldsymbol{f}(x)\Big]^{\frac{1}{2}} \Big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\boldsymbol{f}(x)\Big]^{\frac{1}{2}}. \tag{2}$$

Lemma 2. *[7] Let $0 < \alpha < mn$, $1 \le s' < mn/\alpha$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \ldots, p_m > 1$ and $f \in L^1_{\rm loc}(\mathbb{R}^n) \times \ldots \times L^1_{\rm loc}(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$*

$$\mathcal{M}_{\Omega,\alpha,m}\boldsymbol{f}(x) \le C_0 \prod_{j=1}^{m} \Big[M_{\frac{\alpha s'}{m}}(f_j^{s'})(x)\Big]^{\frac{1}{s'}} \le C_0 \prod_{j=1}^{m} \Big[M_{\frac{\alpha s' p_j}{mp}}(f_j^{\frac{s' p_j}{p}})(x)\Big]^{\frac{p}{s' p_j}}, \tag{3}$$

where $C_0 = \frac{\|\Omega\|_{L_s(\mathbb{S}^{mn-1})}}{(mn)^{\frac{1}{s}}}$.

When $m \geq 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, we find out $I_{\Omega,\alpha,m}$ also have the same properties by providing the following multi-version of the Theorem 2.

Theorem 4. *Let* $0 < \alpha < mn$, $1 < s \leq \infty$ *and* $\Omega \in L^s(\mathbb{S}^{mn-1})$. *Let also* $\sum_{j=1}^m \frac{\lambda_j}{p_j} = \frac{\lambda}{p}$, $\frac{1}{p_j} - \frac{1}{q_j} = \frac{\alpha}{m(n-\lambda_j)}$ *and* $0 \leq \lambda_j < n - \frac{\alpha p_j}{m}$, $j = 1, \ldots, m$.

(i) If $p > s'$ *and* $\sum_{j=1}^m \frac{\lambda_j}{q_j} = \frac{\lambda}{q}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* $I_{\Omega,\alpha,m}$ *from product Morrey space* $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$. *Moreover, there exists a positive constant* C *such that for all* $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|I_{\Omega,\alpha,m}\boldsymbol{f}\|_{L^{q,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = s'$ *and* $\lambda \sum_{j=1}^m \frac{1}{p_j q_j} = \sum_{j=1}^m \frac{\lambda_j}{p_j q_j}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* $I_{\Omega,\alpha,m}$ *from product Morrey space* $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to the weak Morrey space* $WL^{q,\lambda}(\mathbb{R}^n)$. *Moreover, there exists a positive constant* C *such that for all* $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|I_{\Omega,\alpha,m}\boldsymbol{f}\|_{WL^{q,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

Proof.

(i) *Sufficiency.* Following the method used in [3], we choose a small positive number ε with $0 < \varepsilon < \min\{\alpha, \frac{m(n-\lambda_j)}{p_j} - \alpha, \frac{n-\lambda}{p} - \alpha\}$. One can then see from the condition of Theorem 4 that $1 \leq s' < p_j < \frac{m(n-\lambda_j)}{\alpha+\varepsilon}$ and $1 \leq s' < p_j < \frac{m(n-\lambda_j)}{\alpha-\varepsilon}$, and we let

$$\frac{1}{\widetilde{q}_1} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - \frac{\alpha+\varepsilon}{n-\lambda} = \frac{1}{p} - \frac{\alpha+\varepsilon}{n-\lambda},$$

and

$$\frac{1}{\widetilde{q}_2} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - \frac{\alpha-\varepsilon}{n-\lambda} = \frac{1}{p} - \frac{\alpha-\varepsilon}{n-\lambda}.$$

Now if each $p_j > s'$, then from [7], Theorem 1.1(i) implies that

$$\|\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}\|_{L^{q,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}, \quad \|\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}\|_{L^{q,\lambda}} \quad \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

A simple calculation yields $\frac{q}{2\widetilde{q}_1} + \frac{q}{2\widetilde{q}_2} = 1$. Hence, using Lemma 1, the Holder inequality and the above inequalities, we have

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda}\int_{B(x,t)} |I_{\Omega,\alpha,m}f(y)|^q dy\right)^{1/q}$$

$$\le C \sup_{x\in\mathbb{R}^n,t>0} \Big(\frac{1}{t^\lambda}\int_{B(x,t)} \Big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\Big]^{\frac{q}{2}} \Big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\Big]^{\frac{q}{2}} dy\Big)^{\frac{1}{q}}$$

$$\le C \sup_{x\in\mathbb{R}^n,t>0} \Big(\frac{1}{t^\lambda}\int_{B(x,t)} \Big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\Big]^{\widetilde{q}_1} dy\Big)^{\frac{1}{2\widetilde{q}_1}} \sup_{x\in\mathbb{R}^n,t>0} \Big(\frac{1}{t^\lambda}\Big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\Big]^{\widetilde{q}_2} dy\Big)^{\frac{1}{2\widetilde{q}_1}}$$

$$\le C\|\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}\|^{1/2}_{L^{\widetilde{q}_1,\lambda}} \|\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}\|^{1/2}_{L^{\widetilde{q}_2,\lambda}} = C\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}},$$

Necessity. Suppose that $I_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L_{q,\lambda}(\mathbb{R}^n)$. Define $\mathbf{f}_\varepsilon(x) = \big(f_1(\varepsilon x),\ldots,f_m(\varepsilon x)\big)$ for $\varepsilon > 0$. Then it is easy to show that

$$I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y) = \varepsilon^{-\alpha} I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y). \tag{4}$$

Thus

$$\begin{aligned}\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{L^{q,\lambda}} &= \varepsilon^{-\alpha} \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{B(x,t)} |I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)|^q dy\Big)^{1/q}\\ &= \varepsilon^{-\alpha-n/q} \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{B(\varepsilon x,\varepsilon t)} |I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{1/q}\\ &= \varepsilon^{-\alpha-n/q+\lambda/q} \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{(\varepsilon t)^\lambda}\int_{B(\varepsilon x,\varepsilon t)} |I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{1/q}\\ &= \varepsilon^{-\alpha-(n-\lambda)/q}\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}}.\end{aligned}$$

Since $I_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}\times\ldots\times L^{p_m,\lambda_m}$ to $L^{q,\lambda}$, we have

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = \varepsilon^{\alpha+(n-\lambda)/q}\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{L^{q,\lambda}} \le C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m \|f_j(\varepsilon\cdot)\|_{L^{p_j,\lambda_j}}$$

$$= C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^{\lambda_j}}\int_{B(x,t)}|f_j(\varepsilon y)|^{p_j}dy\Big)^{1/p_j}$$

$$\begin{aligned}&= C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m \varepsilon^{-n/p_j} \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)}|f_j(y)|^{p_j}dy\Big)^{1/p_j}\\ &= C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m \varepsilon^{(\lambda_j-n)/p_j} \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{(\varepsilon t)^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)}|f_j(y)|^{p_j}dy\Big)^{1/p_j}\\ &= C\varepsilon^{\alpha+(n-\lambda)/q-(n-\lambda)/p}\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}},\end{aligned}$$

where C is independent of ε.

If $(n-\lambda)/p < (n-\lambda)/q+\alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = 0$ as $\varepsilon \to 0$.

If $(n-\lambda)/p > (n-\lambda)/q+\alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = 0$ as $\varepsilon \to \infty$.

Therefore we get $(n-\lambda)/p = (n-\lambda)/q+\alpha$.

(ii) *Sufficiency.* If $p_i = s'$ for some i, we take $\eta^2 = \beta^{2-\frac{q}{\widetilde{q}_2}} \Big(\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}\Big)^{\frac{q}{\widetilde{q}_2}-1}$ for any $\beta > 0$, then applying Lemma 1 and Theorem 4 in [7], we get

$$\begin{aligned}
&\Big|\big\{y \in B(x,t) : \big|I_{\Omega,\alpha,m}\mathbf{f}(y)\big| > \beta\big\}\Big| \\
&\le C\Big|\big\{y \in B(x,t) : C\big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\big]^{\frac12}\big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\big]^{\frac12} > \beta\big\}\Big| \\
&\le C\Big|\big\{y \in B(x,t) : \sqrt{C}\big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\big]^{\frac12} > \eta\big\}\Big| \\
&+ \Big|\big\{y \in B(x,t) : \sqrt{C}\big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\big]^{\frac12} > \beta/\eta\big\}\Big| \\
&\le C\Big|\big\{y \in B(x,t) : \mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y) > C\eta^2\big\}\Big| + \Big|\big\{y \in B(x,t) : \mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y) > C\beta^2/\eta^2\big\}\Big| \\
&= Ct^\lambda\Big[\Big(\frac{1}{\eta^2}\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}\Big)^{\widetilde{q}_1} + \Big(\frac{\eta^2}{\beta^2}\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}\Big)^{\widetilde{q}_2}\Big] \\
&= Ct^\lambda\left(\frac{1}{\beta}\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}\right)^q.
\end{aligned}$$

Hence, we obtain the following inequality

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} &= \sup_{\beta>0}\beta \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\Big|\Big\{y \in B(x,t) : |I_{\Omega,\alpha,m}\mathbf{f}(y)| > \beta\Big\}\Big|\Big)^{\frac1p} \\
&\le C\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.
\end{aligned}$$

This is the conclusion (ii) of Theorem 4.

Necessity. Suppose that $I_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $WL_{q,\lambda}(\mathbb{R}^n)$. From equality (4) we get

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{WL^{q,\lambda}} &= \sup_{\tau>0}\tau \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{\big\{y\in B(x,t):I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y)>\tau\big\}} dy\Big)^{1/q} \\
&= \sup_{\tau>0}\tau \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{\big\{y\in B(x,t):I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\big\}} dy\Big)^{1/q}
\end{aligned}$$

$$= \varepsilon^{-\frac{n}{q}} \sup_{\tau>0} \tau \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^\lambda} \int_{\{y\in B(x,\varepsilon t): I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\}} dy\Big)^{1/q}$$

$$= \varepsilon^{-\alpha-\frac{n}{q}+\frac{\lambda}{q}} \sup_{\tau>0} \tau\varepsilon^\alpha \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{(\varepsilon t)^\lambda} \int_{\{y\in B(x,\varepsilon t): I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\}} dy\Big)^{1/q}$$

$$= \varepsilon^{-\alpha-(n-\lambda)/q} \|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}}.$$

By the boundedness of the operator $I_{\Omega,\alpha,m}$ from $L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$ to $WL^{q,\lambda}$, we have

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} &= \varepsilon^{\alpha+(n-\lambda)/q} \|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{WL^{q,\lambda}} \\
&\le C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \|f_j(\varepsilon\cdot)\|_{L^{p_j,\lambda_j}} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^{\lambda_j}} \int_{B(x,t)} |f_j(\varepsilon y)|^{p_j} dy\Big)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \varepsilon^{-n/p_j} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^{\lambda_j}} \int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy\Big)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \varepsilon^{(\lambda_j-n)/p_j} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{(\varepsilon t)^{\lambda_j}} \int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy\Big)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q-(n-\lambda)/p} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}},
\end{aligned}$$

where C is independent of ε.

If $(n-\lambda)/p < (n-\lambda)/q + \alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} = 0$ as $\varepsilon \to 0$.

If $(n-\lambda)/p > (n-\lambda)/q + \alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} = 0$ as $\varepsilon \to \infty$.

Therefore we get $(n-\lambda)/p = (n-\lambda)/q + \alpha$.

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