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Nanoemitters in a defect layer embedded in photonic crystals: synthesis and optical characterization

Hong P. N., Guennouni-Assimi Z., Farha R. , Faure M-C., Goldmann M., Marcillac W., Coolen L., Maitre A., Schwob C., Benalloul P.

Abstract. Manipulation of the emission of nanocrystals embedded in photonic crystals can provide single photon source for quantum information.
Artificial opals are 3D photonic crystals whose synthesis is based on self-assembly of dielectric spheres. This cost-efficient and versatile method does not require a high technological platform and leads to nanostructured samples over cm range. In order to obtain light confinement inside opals, several fabrication methods have been used to create a defect. Artificial opals with a planar defect can be considered as a good "model system" to study the modification of the optical properties of nanoemitters in a photonic crystal.
We will present two efficient and reliable methods to engineer a defect between two silica opals: by sputtering a controlled amount of silica or by the transfer of a monolayer of silica spheres of different diameter by the Langmuir-Blodgett or Schaefer technique. The optical properties of the prepared samples were characterized by transmission and specular reflection spectra. Tunable and highly transmitted and reflected optical modes were evidenced, in good agreement with Finite Difference Time Domain simulations (FDTD).
Colloidal II-VI nanocrystals are efficient and stable emitters which can emit single photons. These nanoemitters were introduced in the defect. The collected fluorescence of these nanocrystals presents an emission diagram which is modified by the photonic crystal and especially by the defect layer.

Key Words and Phrases: Photonic Crystals, Defect Layer, Photoluminescence, Specular Reflectivity

1. Introduction

Quantum information requires a source of single photons emitted at a very high rate in connection with their short decay time [28]. 3D-photonic crystals can control and manipulate the fluorescence properties of embedded quantum emitters such as their decay time and their propagation direction. Quantum information application as others like low-threshold lasers requires insertion of a controlled defect inside this kind of structure. Indeed this disruption of the photonic crystal periodicity can create permitted optical frequency bands within the band gap: light whose frequency is included in the corresponding pass band is then localized in the defect allowing effects such as wave guiding and confinement for an enhanced emission.

Photonic crystals (PCs) in the visible and near infra-red ranges are characterized by a periodic dielectric constant at wavelength scale. Many approaches have been proposed to fabricate 3-D periodically modulated dielectric materials. Some of them come from more or less well established microelectronic industry processes such as semiconductor layer-by-layer nanomachining [12], layer-by-layer direct laser writing [33, 9, 22], interference lithography using a single diffraction mask [10], holographic lithography with at least 4 laser beams [8], [21] or two-beam only but with multi-exposure [15]. Alternatively to these methods which have known different sophisticated modifications proposed by several groups, bottom-up techniques taking advantage of the spontaneous self-organisation of spherical colloidal particles have been investigated. Various self-assembly techniques have been studied as they provide a low cost and relatively easy protocol to obtain artificial opals. These cheap and versatile methods do not require a high technological platform and can lead to nanostructured samples over cm range. Well orientated faced centered cubic (fcc) crystals have been formed by gravity [1], [19, 20] or controlled sedimentation onto patterned substrate [7]. Different convective methods such as vertical evaporation [14], induced by a temperature gradient [40] or by isothermal heating evaporation-induced self assembly (IHEISA) at a heating temperature of the solvent very close to its boiling point (method indicated for large balls) [41], have provided large mono-domain opals. Langmuir-Blodgett (LB) technique is a layer-by-layer procedure for the preparation of large area synthetic opals. Solid films of particles are transferred from the water surface onto substrates. The results suggest that a successful synthesis of ordered monolayers of monodisperse silica spheres with the LB technique depends critically on the hydrophilic/hydrophobic balance [34], [29]. Each technique has received large attention and several groups have proposed modifications and improvements.

Silica spheres opals have been the object of a large number of papers even if these opals do not present a complete photonic bandgap due to the low index difference between silica and air. Nevertheless the pseudo photonic band gap, called stop-band, affects the propagation of light through the opal and causes transmission dips or reflectance peaks in the sample spectra depending on the direction.

2. Synthesis of silica spheres and opals

The opals were synthesized from home-made silica spheres. The key point to fabricate high crystallographic quality opals over large monodomains is the low size dispersion of the spheres. These ones were obtained from a "multi-steps" synthesis protocol derived from the Stober method [32] which consists in the hydrolysis and condensation of a silica precursor (tetraethylorthosilicate - TEOS) in alcoholic medium, using ammonia as a catalyst. The precursor is added in several steps in order to first create silica seeds and then make them grow. With this method followed by a centrifugation step to select the particles in size, the obtained dispersion size was of the order of 3 to 5%. The diameter of the spheres was between 250 and 550 nm so that the stop band of the corresponding opals appeared in the visible part of the spectrum. The procedure is described on Figure 1.

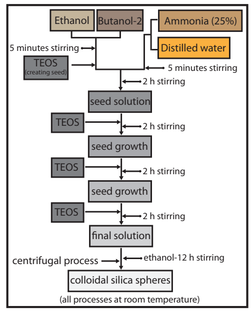

Figure 1: Process Used to Synthesize Silica Spheres.

The opals were synthetized using the convection self-assembly protocol [14] as this procedure gives the best face centered cubic structure (fcc) with the largest monodomains. In the case of direct opals, the photonic band gaps are incomplete: the corresponding stop bands can be evidenced as peaks in the optical reflection spectra [5, 23, 38, 13, 4, 6, 39, 24].

Specular reflection spectra

The opal specular reflection spectra were measured at various incidence angles. The incidence beam was provided by a halogen lamp connected to an optical fiber (core 600 m), mounted on a goniometer arm with a collimator (focal length 12.7 mm) and a diaphragm (diameter 0.6 mm). The reflected beam was collected by a symmetric collimated fiber (with 1 mm diaphragm) and analyzed by a spectrometer (resolution 1.5 nm). The overall goniometer resolution was 1°. All the spectra were normalized by the incident light spectrum. The stop band shifted to lower wavelengths as the incidence angle increased (Figure 2) [2].

The dependence of the wavelength of the stop band as a function of the considered propagation direction can be treated in two ways. A first way is to calculate numerically the three-dimensional band structure of the opal [17]. A second way is to model the opal by a lattice of scattering points (located at the center of the spheres) in a homogeneous medium with an effective index n_{eff}. At a given incident light angle, the opal stop band wavelength is then provided by the Bragg's law which expresses the condition for constructive interferences between beams reflected by parallel planes of scattering points [5, 23, 3, 30, 35]. The Bragg's law can be considered as a first approximation as the refractive index of silica and air are not too different. The second peak in the specular reflection spectra for incidence angles around 45° takes evidence that the prepared opals have the fcc structure with low defect density [2].

The presence of the stop-band in these photonic crystals is of great interest for nano-emitters fluorescence manipulation. If nano-emitters, whose emission frequency is in the forbidden frequencies band, are embedded in the photonic crystal, their fluorescence will be inhibited in certain directions, leading to modifications of their emission diagram. Moreover, thanks to the Fermi golden rule, the changes in the local density of states (LDOS)

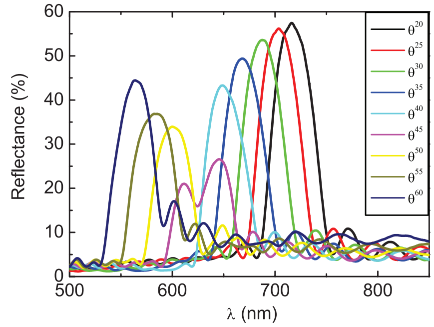

Figure 2: Specular Reflection versus the Incidence Angle.

induced by the photonic crystal will affect their spontaneous emission rate with respect to the case of nano-emitters in a homogeneous medium (more precisely their emission rate will decrease). As nano-emitters, we have selected colloidal II-VI nanocrystals which are stable and present high quantum efficiency at room temperature. They are good candidates for single photon sources. For example, C. Vion et al. [39] showed an increase of the lifetime of about 10% for CdTeSe nanocrystals infiltrated in an opal made by sedimentation method compared with these nanocrystals in a homogeneous medium with the same refractive index (Figure 3). A way to obtain larger effects is to use inverted opals as they present high refractive contrast and complete photonic bandgap [18, 31, 25]. As an example, lifetime reductions up to 30% have been measured for CdSe Qdots infiltrated in titanium oxide inverted opals [16]. But the production of good quality inverted opals over large scale is not yet totally mastered.

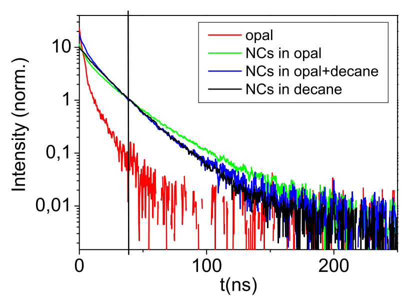

Figure 3: Luminescence Decay of Nanocrystals Embedded in an Opal (green line), Dissolved in Decane (black line), in the Opal Infiltrated with Decane (blue line). Self-luminescence of the Opal (red line).

3. Planar defect

The behaviour of nano-emitters in photonic crystals is somehow well understood. To go further in controlling the light emission and modifying the decay rate of nano-emitters, one has to introduce defects in the photonic structure. A controlled defect in the structure will disrupt the periodicity and create a cavity in which the emission will be confined and the electromagnetic field enhanced as the local density of states. This defect can be obtained through the omission or the displacement of one or more holes in the case of planar photonic crystals for which the periodic variation of the dielectric constant is provided by drilling holes of a few hundred nanometers in diameter in a semiconductor matrix. For 3D-photonic crystals, a planar defect can be obtained. In this case, thanks to the defect, emission in preferred directions (creation of a pass-band inside the bandgap) and enhancement of the spontaneous emission rate should be observed. An interesting feature of 3D-photonic crystals is the potential ability to control light propagation in the three directions in space. One can expect to provide resonant coupling in one direction (the one on which the periodicity has been disrupted by the planar defect) as well as wave-guiding in the two other directions.

To engineer planar defects into colloidal PCs different bottom-up routes have been proposed by different groups. Planar defects have been fabricated by combining convective colloidal self-assembly and chemical vapour deposition to introduce a silica thin layer between two opal PCs [27, 37]. While these defects are passive, the Ozin group has also proposed active and tunable planar defects by incorporating polyelectrolyte multilayers having refractive index and thickness tuned by exposure to different solvent vapour pressures [36], UV-exposure as well as temperature. In this latter case, they obtain a full reversibility [11]. Another way to introduce a planar defect has been proposed by incorporating a monolayer of microspheres prepared by the Langmuir-Blodgett technique between two opal films with spheres of different diameters prepared by convective self-assembly. The LB technique supplies a well-defined single layer whereas it is difficult to realize such objective by using other self-assembly methods [43, 42].

We have selected two efficient and reliable methods to engineer a defect between two silica opals: by the transfer of a monolayer of silica spheres of different diameter by the Langmuir-Blodgett and Langmuir-Schaefer techniques or by sputtering a controlled amount of silica. The first one consists in the transfer of a compact monolayer composed with functionalized spheres of diameter larger than the one of the opal by the Langmuir-Blodgett (vertical transfer) or the Langmuir-Schafer (horizontal transfer) techniques (Figure 4). If the first one is widely used to create opals layer by layer, the second one is usually devoted to transfer monolayers of small particles (gold nanoparticles or nano-emitters for example) and is rarely used for particles of this size. After the synthesis of a second opal on the transferred monolayer, the final structure was characterized by specular reflection spectroscopy. For both methods, a close to zero minimum of reflection, evidence of the defect mode, appeared in the stop-band. These results will be discussed on samples prepared by the second method. On Figure 5 one can see the good arrangement of the silica spheres of the sample.

The second method developed to create the defect layer consists in sputtering a certain amount of silica on the first opal. The sputtering conditions were chosen to grow the sputtered silica in a columnar way on the top of the spheres. Therefore, the obtained defect layer is made of elongated beads which keep very well the periodicity in the plane of the first opal and so permit a very well-ordered deposition of the second one. The structural quality is evidenced through SEM images of the edge of the sandwich structure (Figure 6).

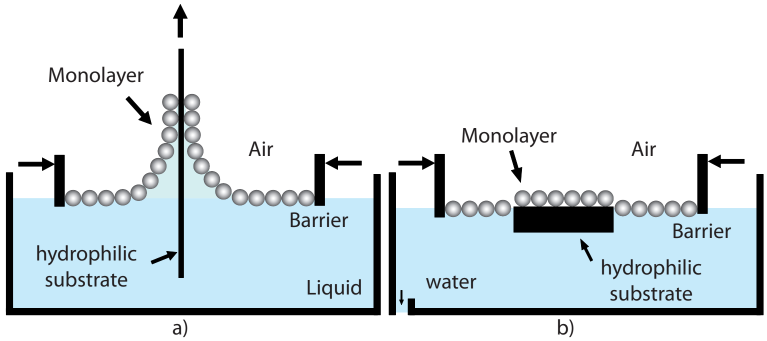

Figure 4: Langmuir Blodgett (a) and Langmuir Schaefer (b) Techniques.

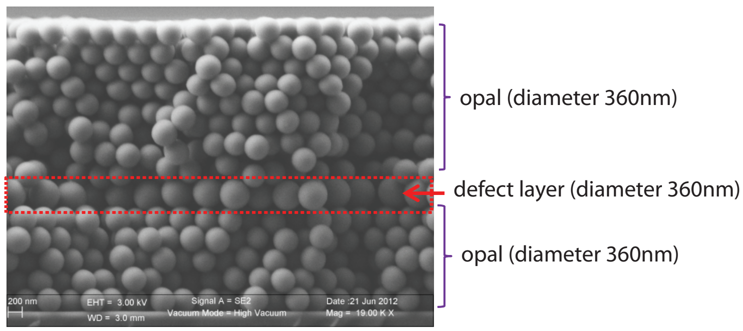

Figure 5: SEM Image of the Edge of the Sample with a Planar Defect Made by the LB Technique.

Several sandwich structures were prepared with different sputtered silica thicknesses. For a 123 nm-sputtered silica layer, the specular reflection spectra were performed for specular angles between 20° and 50° by step of 5°. Figure 7a shows the reflection spectra for 20° and 35° incidence angles. It appeared that, thanks to the defect, a pass-band was created in the stop-band. The spectral width of the pass-band was of the order of 25 nm. The value of the reflection minimum was almost zero. The corresponding contrast between reflected and transmitted optical modes, on the order of 90%, is very high compared to the best results reported in the literature. The spectral positions of the reflection minima for different incidence angles are plotted on Figure 7b (circles). The two extreme curves on Figure 7b correspond to the limits of the stop band taken to be the wavelengths closest to the maximum for which the reflection goes to zero on each side (see arrows on Figure 7 for 20° angle). By varying the angle from 20° to 50°, the defect mode spectral position goes from 717 nm to 598 nm, leading to a high tunability of almost 120 nm. Consequently, this kind of sandwich structure should be very suitable to allow wavelength-selective excitation and so to address selectively different emitters.

Finite Difference Time Domain (FDTD) Calculations

The experimental reflection and transmission spectra were compared with FDTD simulations using MEEP tool [26]. This method consists in calculating the propagation of the electromagnetic field from a given light source through a computed structure from the

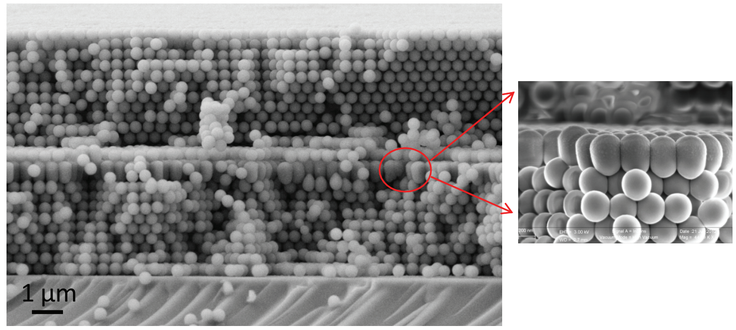

Figure 6: SEM Image of the Sandwich Structure.

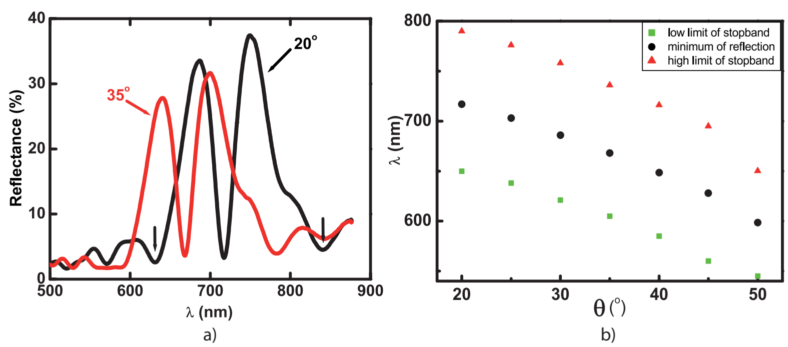

Figure 7: a) Experimental Reflection Spectra of the 123 nm-Defect Thickness Sample for 20° (black line) and 35° (red line) Incidence Angles. b) Position of the Reflexion Minima Versus the Incidence Angles (circles). The Triangles Correspond to the Limit of the Stop Band for High Wavelengths, the Squares to the Limit of the Stop Band for Low Wavelengths, Both Deduced from the Experimental Spectra.

Maxwell equations. For this, space and time are divided into a regular grid. The basic mechanism of this method is to discretize Maxwell's equations and to solve them in the leapfrog manner: the electric field vector components in a volume of space are solved at a given in time; then the magnetic field vector components in the same spatial volume are solved at a given time further; and the process is repeated until the desired transient or steady-state electromagnetic field behaviour is fully evolved.

The 0°-transmission spectra of the samples were simulated and compared with the experimental ones. For the 123 nm- thickness sample, a very good agreement was obtained (Figure 8) for the position of the defect mode, emphasizing the fact that the computed structure, simulated without any fitting parameter, is relevant. The position of the defect mode for various thicknesses of sputtered silica - 41, 87, 123 and 273 nm - was studied. The corresponding transmission spectra were measured and the simulations performed. Simulations were also run for other thicknesses between 150 and 450 nm. The results were summarized on Figure 9 which gives the spectral position of the defect mode versus the sputtered silica thickness. By controlling the amount of sputtered silica, one can monitor the position of the defect mode from the edge to the middle of the stop band.

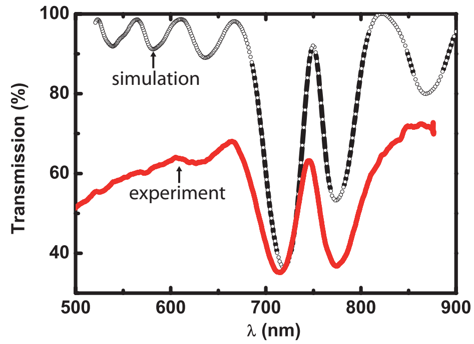

Figure 8: Experimental (red line) and Calculated (black line) 0°-Transmission Spectra of 123 nm-Defect Thickness Sample.

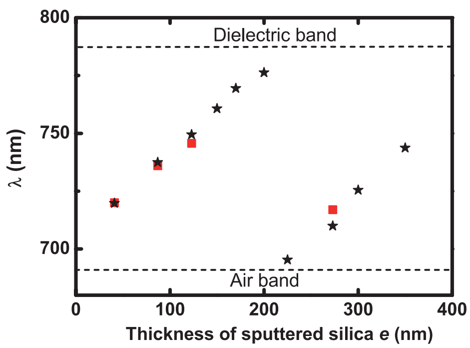

Figure 9: Maximum Wavelength of the Defect Mode for Different Thicknesses of Sputtered Silica: Simulations (star black point), Measurements (rectangle red point). The Dashed Horizontal Lines Correspond to the Edges of the Stopband Determined from the Experimental Spectra.

4. Modification of spectra and emission diagrams of CdTeSe nanocrystals

The prepared opal samples with a sputtered layer defect exhibit a pass-band around 700 nm for 20°. Therefore, CdSeTe nanocrystals which emit a band round 705 nm with a width of the order of 70 nm (Invitrogen Qdot 705 ITK) were selected. To introduce these nanoemitters in the defect layer, the procedure was the following: a drop of nanocrystals was deposited on the surface of the first opal covered by the sputtered silica (123 nm thick). Then, in order to protect these nanocrystals, a PMMA film (Polymethyl Methacrylate) was spin coated and then a heat treatment at 150°C was performed. Finally, the second opal was fabricated.

In order to control the quality of the photonic crystal after these procedures, the specular reflection spectra at the incidence angle of 20° of the samples with and without the nanocrystals and PMMA are compared on Figure 10. If the position of the defect mode is the same for both cases, unfortunately the defect mode is not as deep with the nano-emitters as with just the defect layer. That is a problem we will have to solve in the future. The nanocrystals were excited by a cw laser diode at a wavelength of 473 nm and the photoluminescence spectra were recorded for different detection angles by step of 2.5°.

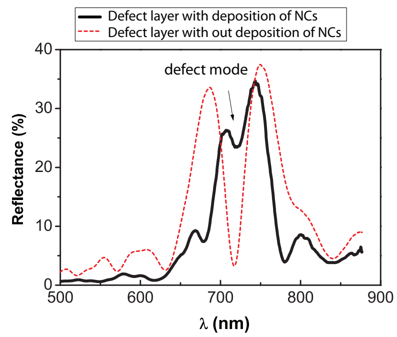

Figure 10: Reflection Spectra of the Sandwich Structure with the Defect Thickness of 123 m in the Cases of Samples with and without NCs.

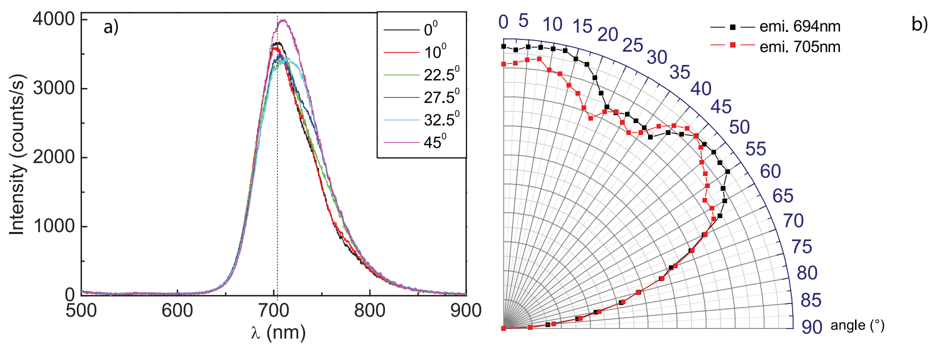

Figure 11: a) Luminescence Spectra of NCs Embedded in the Planar Defect of the Sandwich Structure with the Defect Thickness of 123 m. b) Diagrams of Radiation in Polar Representation at Two Different Wavelengths.

Depending on the collection angle, the maximum wavelength of the emission spectrum is shifted from 705 nm to 715 nm for incidence angles from 10° to 32.5°. Besides, the intensity of the emission spectra also varied (Figure 11a). The radiation diagrams are plotted as a function of the detector angle at wavelengths of 705 and 694 nm (Figure 11b). One can see a decrease of the intensity of the emission in the angle range 25°- 45° with a small increase around 30 and 35°. That corresponds to the stop-band and the pass-band localizations for these angles. In addition, the lambertian emission is kept for angles higher than 65° for which the stop band is completely out of the emission band of the nanoemitters. These results can be discussed just by considering the filter effect of the opal with a planar defect. The analysis of the emission decay modification is necessary to evidence the action of the photonic crystals on the emission of the nanocrystals.

5. Conclusion

Artificial opals with a planar defect can be considered as a good "model system" to study the modification of the optical properties of nanoemitters in a photonic crystal. We have prepared silica opals by the convective method and introduced in the middle of the opal a planar defect by the Langmuir techniques or by sputtering some amount of silica. This defect opens a pass-band in the stop-band of the photonic crystal. Nanocrystals for which the emission band is located in the stop-band of the opal were embedded in the defect layer. The collected fluorescence of these nanocrystals presents an emission diagram which is modified by the photonic crystal and especially by the defect layer. Preliminary experiments on the fluorescence of nanocrystals embedded in an opal showed that it is possible to record the emission decay of a single nanoemitter. This is the first step towards the study of the photonic environment on a single photon emitted by a single nanocrystal embedded within the opal defect layer.

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Paul Benalloul
Institut des NanoSciences de Paris, 75005 Paris, France
E-mail: paul.benalloul@insp.jussieu.fr

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On the estimation of the informativity of remote ground sensing network of urban air in Baku

Ismailov F. I., Zabidov Z. J., Abdurahmanov C. A. and Ismailli Sh. F.

Abstract. In this paper we present a method of estimating the informational value of the rational planning of remote ground sensing network of urban air. To solve this problem we use multi-year actinometric data collected in the city of Baku and its suburbs. This problem is considered from the statistical point of view on the basis of the method of optimal modeling.

Key Words and Phrases: remote sensing, optimal interpolation, network, informativity, urban air

2000 Mathematics Subject Classifications: 07.07.Df

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1. Introduction

In these days of intensive growth of cities, a lot of attention is paid to remote sensing methods of ecological control for urban air. These methods provide most efficient, comprehensive and reliable information about the variations of air pollution on the entire length of the city [1-3].

In this paper we present a method of estimating the informativity of the rational planning of remote ground sensing network of urban air. To solve this problem, we use multi-year actinometric data collected in the city of Baku and its suburbs [2]. This problem is considered from the statistical point of view on the basis of method of optimal modeling [4].

We consider the field of admissible values of observation errors and interpolation. Observation errors are associated with the measurement accuracy and working conditions of aktino photometer. This accuracy is determined by the degree of measurement error. Interpolation errors are associated with the field of maximum allowable angular distances between informative observation points on the daytime sky. We link this distance to the instability of urban air in different parts of the extended area of the city.

Thus, we conduct an assessment for informative overlapping fields of remote sensing of urban air in the city of Baku and its suburbs.

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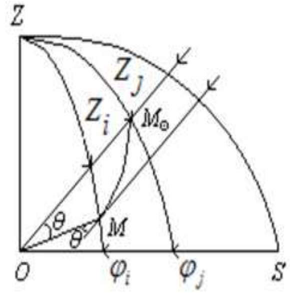
2. Methods of solution

We will describe the distribution of brightness of the daytime sky with the brightness function $f_\lambda(\theta)$, which depends on the angular distance θ between the Sun and the point of observation of the sky and the level of urban air pollution (in what follows the index of the wavelength λ will be omitted) [3].



Figures 1. Geometry of light scattering. M_Θ and M observed points; Z-zenith, S-south.

We define the distance between any two points of observation on the celestial sphere in angular units by the cosine formula of spherical geometry (Figure 1):

$$\cos\rho_{i_j} = \cos Z_i \cos Z_j + \sin Z_i \sin Z_j . \cos(\varphi_i - \varphi_j), \tag{1}$$

where Z_i and Z_j are the zenith angles of the observed points on the daytime sky, while φ_i and φ_j are the azimuthal angles between points i and j.

The basic criterion for the problem of the rational planning of remote ground sensing network of urban air is the accuracy of interpolation from points of observation into some other points.

Let us consider the region of space where correlation function $\mu(\rho)$ of the values of $f(\theta)$ can be considered homogeneous and isotropic. At a certain position of the sun, this function will then depend only on the distance ρ between the observed points of the celestial sphere.

Correlation function $\mu(\rho)$ can be determined directly from the observations as follows:

$$\mu(\rho) = \frac{\mu^*(\rho)}{1+\eta^2}, \tag{2}$$

where $\mu^*(\rho)$is the correlation function of the true values of $f(\theta)$, η is the measure of the observational errors, and the correlation function $\mu(\rho)$ tends to $\mu(0) = 1/(1+\eta^2)$ with the decrease in distance ρ.

Let it be required to restore the value of the function $f(\theta_0)$ at any point θ_0 using its measured values in several discrete points by interpolation with the help of the formula

$$f(\theta_0) = \sum_i^n a_i f(\theta_i), \tag{3}$$

where a_i are the weighting coefficients ($\sum_{i=1}^{n} a_i = 1$).

The solution of our problem comes down to the definition of the measure of the errors of interpolation defined as follows [4]:

$$\varepsilon^2 = 1 - \sum_{i=0}^{n} a_i \mu_{0_i}. \tag{4}$$

The coefficients here are defined by the equations

$$\sum_{j=1}^{n} a_j \mu_{i_j} + a_i \eta_i^2 = \mu_{0_i}. \tag{5}$$

In order to meet the requirements of rational placing of remote sensing network, the magnitude of interpolation error will be determined from the condition

$$\varepsilon_m \sim \eta, \tag{6}$$

where ε_m are the maximum allowable values of interpretation errors.

Let's use relations (3) and (5) for the interpolations in the center of a segment, of an equilateral triangle and of a square. Obtained expressions for the calculation of coefficients a_i and measures of interpolation errors ε are presented in Table 1.

Table 1. Formulas for interpolation errors

	weighting coefficients	optimal interpolation
1. on two points	$a_1 = \frac{\mu(\frac{\rho}{2})}{1+\eta^2+\mu(\rho)}$	$\varepsilon_1 = 1 - \frac{2\mu^2(\frac{\rho}{2})}{1+\eta^2+\mu(\rho)}$
2. on three points	$a_2 = \frac{\mu(\frac{\rho}{\sqrt{3}})}{1+\eta^2+2\mu(\rho)}$	$\varepsilon_2 = 1 - \frac{3\mu^2(\frac{\rho}{\sqrt{3}})}{1+\eta^2+2\mu(\rho)}$
3. on four points	$a_3 = \frac{\mu(\frac{\rho}{\sqrt{2}})}{1+\eta^2+2\mu(\rho)+\mu(\rho\sqrt{2})}$	$\varepsilon_3 = 1 - \frac{4\mu^2(\frac{\rho}{\sqrt{2}})}{1+\eta^2+2\mu(\rho)+\mu(\rho\sqrt{2})}$

Urban air in Baku is very unstable in the west of the city, more stable in the south, and middling in the north and the east. Therefore, we use the different correlation functions to describe statistical instability in different parts of the city; see Table 2. In this table, functions $\mu(\rho)$ depend on the parameters $\mu(0)$ or on the measure of the observation errors η as well as on the scale of correlation ρ_0 which, in turn, depends on the area of distribution of urban air.

Table 2. Correlation function of variations of scattering functions of the urban air of Baku in different directions

	Correlation functions	φ_i-φ_j	$\mu(0)$	ρ_0
1. Western direction	$\mu(\rho) = \mu(0).(1\text{-}\rho/\rho_0).\exp(\text{-}\rho/\rho_0)$	80°	0,75	0.85
2. Southern direction	$\mu(\rho) = \mu(0).(1\text{-}\rho/\rho_0)$	120°	0,75	1.18
3. Northern and eastern directions	$\mu(\rho) = \mu(0).\exp(\text{-}\rho/\rho_0)$	160°	0,75	1.37

3. Results of Calculations

Let's consider the dependence of weighting coefficients a_i and interpolation errors ε on the dimensionless parameter ρ/ρ_0.

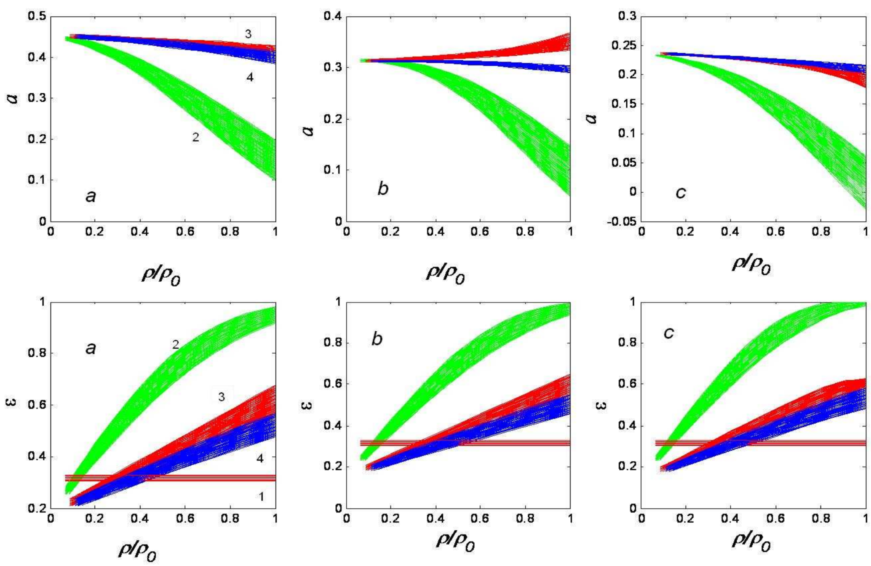


Figure 2. The dependence of the weighting coefficients a_i and errors of interpolation ε on dimensionless parameter ρ/ρ_0 for correlation functions of Table 2: a - on two points, b - on three points and c - on four points of interpolation; curves denote: 1 - the maximum permissible values of the errors of interpolation on condition, 2 - western direction, 3- southern direction, 4 - northern and eastern directions.

In view of the statistical nature of the variability of initial parameters η and ρ_0, we will assume that these parameters can vary within some intervals defined both by casual changes in the measuring device and by the differences in the variations of a sky brightness in different parts of the city. Multi-year actinometrical data [3, 5, 6] show that the changes in η and $\rho_0$can be considered in the intervals

$$\eta = \eta^{\min} + 10\% \cdot \eta^{\min}, \rho_0 = \rho_0^{\max} - 30\% \cdot \rho_0^{\max}. \quad (7)$$

Results of calculation taking into account (7) are presented graphically in Figure 2. In this Figure, the changes in a_i and ε as well as in the maximum values of interpretation error ε_m occur within certain areas which form corresponding strips.

Table 3 shows the areas of intersection S (km^2) between the fields of interpolation errors values ε and their possible maximum values ε_m depending on the relation ρ/ρ_0. Changes in S (km^2) and in ε characterize the informativity of ground stations providing remote sensing of urban air.

Table 3. Areas of crossing S (km^2) sets of ε and ε_m

$\mu(\rho) = \mu(0)$.(1-ρ/ρ_0).exp(-ρ/ρ_0)	ρ/ρ_0	0,109-0,207	0,1130-0,235	0,121-0,217
	S	1,659.10-3	1,854.10-3	1,681.10^{-3}
$\mu(\rho) = \mu(0)$.(1-ρ/ρ_0)	ρ/ρ_0	0,294-0,562	0,355-0,643	0,329-0,598
	S	5,537.10^{-3}	5,784.10^{-3}	5,481.10^{-3}
$\mu(\rho) = \mu(0)$.exp(-ρ/ρ_0)	ρ/ρ_0	0,352-0,682	0,422-0,756	0,388-0,705
	S	2,87.10^{-3}	4,829.10^{-3}	4,512.10^{-3}

It follows from Figure 2 that the admissible distances between observation points turn out to be considerably bigger in case of interpolation in the centre of an equilateral triangle and a square than in case of interpolation along a straight line. Areas of change S (2) are getting restricted with an increase in the number of points of interpolation. Besides, the difference between three or four points interpolation is not significant. Therefore, three points interpolation is recommended to use when estimating the density of a network in case of optimal interpolation.

Conclusion1. The estimation of informativity of remote ground sensing of urban air in the city of Baku is done with significant difference of the distribution of the brightness of the cloudless sky in different parts of the city, namely, 1) in the west, 2) in the north and the east, and 3) in the south.

2. Three points optimal interpolation is recommended to estimate the density of a network of remote ground sensing of urban air in Baku.

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Ismailov Fazil Ismail
Institute of Ecology of the National Academy of Sciences (NAS) of Azerbaijan, Baku, Azerbaijan
E-mail: isfazil@yandex.ru

Zabidov Zakir Jumshud
Institute of Mathematics and Mechanics of NAS of Azerbaijan
E-mail: zakir_zabidov@mail.ru

Abdurahmanov Chingiz Ahmad
Institute of Ecology of the National Academy of Sciences (NAS) of Azerbaijan, Baku, Azerbaijan
E-mail: isfazil@box.ru

Ismailli Shakhriyar Fazil
Institute of Ecology of the National Academy of Sciences (NAS) of Azerbaijan, Baku, Azerbaijan
E-mail: isfazil@box.ru

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Boundary control by the displacement for the telegraph equation with a variable coefficient and the Neumann boundary condition

Kritskov L. V., Abdukarimov M. F.

Abstract. A problem of the boundary control by the displacement at the point $x = 0$ with the Neumann condition at the point $x = l$ is considered for the process which is described by the telegraph equation with a variable coefficient on the finite interval $0 \leqslant x \leqslant l$. For the critical time period $T = 2l$ a necessary and sufficient condition for the existence of a unique boundary function $\mu(t) = u(0, t)$ which transfers the process from any initial state at $t = 0$ to any terminal state at $t = T$ is given.

Key Words and Phrases: Boundary control, telegraph equation with a variable coefficient

2000 Mathematics Subject Classifications: 49K20, 35L05

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1. Introduction

In this paper we study a problem of the boundary control by the displacement at one endpoint for the process which is described by the one-dimensional telegraph equation with a variable coefficient

$$\mathcal{L}u \equiv u_{tt}(x,t) - u_{xx}(x,t) - q(x,t)u(x,t) = 0, \ \ 0 < x < l, \tag{1}$$

assuming that, on the other endpoint $x = l$, the homogeneous Neumann condition $u_x(l,t) \equiv 0$ holds for all $t \in [0,T]$. The coefficient $q(x,t)$ in (1) is supposed to be a bounded and measurable function in the rectangle $Q_T = [0 \leqslant x \leqslant l] \times [0 \leqslant t \leqslant T]$.

The goal of this paper is to obtain the existence of a unique boundary control at the end-point $x = 0$: $\mu(t) = u(0,t)$ that transfers the process from any initial state $\{u(x,0) = \varphi(x), u_t(x,0) = \psi(x)\}$ at $t = 0$ to any terminal state $\{u(x,T) = \varphi_1(x), u_t(x,T) = \psi_1(x)\}$ at $t = T$ in the case when $T = 2l$. It is supposed that $u(x,t)$ satisfies Eq. (1) in the generalized sense (see Section 1) and has a finite energy.

Investigations of a similar problem for the one-dimensional wave equation ($q(x,t) \equiv 0$) in [1-3] showed that the time period $T = 2l$ of the boundary control's action at one endpoint is critical in the following sense. When $T = 2l$ the boundary control is defined uniquely for a rather wide class of initial and terminal data, while in the case $T > 2l$ the boundary control is not unique and in the case $T < 2l$ the control's existence demands the initial and terminal data satisfy restrictive additional conditions.

Note also that certain boundary control problems for Eq. (1) with a constant coefficient $q(x,t) \equiv -c^2$ are studied in [4-6]. Existence of the boundary control for general hyperbolic equations is considered in [7-12] in the case when the time period T exceeds its critical value.

This paper is the development of results announced in [13].

2. Main definitions

In order to define the notion of a generalized solution, we use the classes $\widehat{W}_2^1(Q_T)$ and $\widehat{W}_2^2(Q_T)$ which are quiet natural for hyperbolic equations (see, e.g., [1,2]). Let us consider the following three problems for Eq. (1) in the rectangle Q_T:

– the initial boundary-value problem I with conditions

$$u(0,t) = \mu(t),\ u_x(l,t) \equiv 0 \quad \text{for} \quad 0 \leqslant t \leqslant T, \tag{2}$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \quad \text{for} \quad 0 \leqslant x \leqslant l \tag{3}$$

where $\mu(t) \in W_2^1[0,T]$, $\varphi(x) \in W_2^1[0,l]$, $\psi(x) \in L_2[0,l]$ and the compatibility condition $\mu(0) = \varphi(0)$ is satisfied;

– the initial boundary-value problem II with conditions (2) for $0 \leqslant t \leqslant T$ and conditions

$$u(x,T) = \varphi_1(x),\ u_t(x,T) = \psi_1(x) \quad \text{for} \quad 0 \leqslant x \leqslant l, \tag{4}$$

where $\mu(t) \in W_2^1[0,T]$, $\varphi_1(x) \in W_2^1[0,l]$, $\psi_1(x) \in L_2[0,l]$ and the compatibility condition $\mu(T) = \varphi_1(0)$ is satisfied;

– the boundary control problem III with the Neumann condition $u_x(l,t) \equiv 0$ for $0 \leqslant t \leqslant T$, with the initial data (3) and the terminal data (4) where $\varphi(x), \varphi_1(x) \in W_2^1[0,l]$, $\psi(x), \psi_1(x) \in L_2[0,l]$.

The function $u(x,t)$ is called *the solution from the class* $\widehat{W}_2^1(Q_T)$ *to the problem* I if it belongs to this class and the identity

$$\int_0^l \int_0^T u(x,t)\mathcal{L}\Phi(x,t)\,dxdt + \int_0^l [\varphi(x)\Phi_t(x,0) - \psi(x)\Phi(x,0)]\,dx - \int_0^T \mu(t)\Phi_x(0,t)\,dt = 0 \tag{5}$$

holds for any test function $\Phi(x,t) \in \widehat{W}_2^2(Q_T)$ which satisfies the conditions $\Phi(0,t) = \Phi_x(l,t) \equiv 0$ for $0 \leqslant t \leqslant T$ and $\Phi(x,T) = \Phi_t(x,T) \equiv 0$ for $0 \leqslant x \leqslant l$.

Analogously the function $u(x,t)$ is called *the solution from the class* $\widehat{W}_2^1(Q_T)$ *to the problem* II if it belongs to this class and the identity similar to (5) holds* for any test

*The second integral in (5) should be substituted by $-\int_0^l [\varphi_1(x)\Phi_t(x,T) - \psi_1(x)\Phi(x,T)]\,dx$.

function $\Phi(x,t) \in \widehat{W}_2^2(Q_T)$ which satisfies the conditions $\Phi(0,t) = \Phi_x(l,t) \equiv 0$ for $0 \leqslant t \leqslant T$ and $\Phi(x,0) = \Phi_t(x,0) \equiv 0$ for $0 \leqslant x \leqslant l$.

The solution to the initial boundary value problem I $u(x,t)$ is called *the solution from the class* $\widehat{W}_2^1(Q_T)$ *to the boundary control problem* III if the boundary function $\mu(t)$ enables the first terminal condition (4) to hold pointwise and the second terminal condition (4) to hold almost everywhere on $[0,l]$.

It is easy to check (see [4]) that if a function $u(x,t)$ is the solution from $\widehat{W}_2^1(Q_T)$ to the problem I then the function $u_1(x,t) = u(x,T-t)$ gives a solution from the same class to the problem II with the coefficient $q(x,t)$ in (1) substituted by $q(x,T-t)$, the function $\mu(t)$ in (2) – by $\mu(T-t)$ and with $\varphi_1(x) = \varphi(x)$, $\psi_1(x) = -\psi(x)$ in (4).

3. Auxiliary statements

Applying the technique of [13, c.163-165] one can obtain the following

Assertion 1. *Let* $T > 0$ *and the coefficient* $q(x,t)$ *in Eq. (1) be bounded and measurable in* Q_T. *Then both initial boundary-value problems I and II have at most one solution from the class* $\widehat{W}_2^1(Q_T)$.

In what follows we use the notation $\underline{\mu}(t)$ for the function that coincides with $\mu(t)$ for $t \geq 0$ and vanishes for all $t < 0$. Hence $\underline{\mu}(t) \in W_2^1[-\varepsilon, T]$ for any $\varepsilon > 0$.

Assertion 2. *Let* $T \leqslant 2l$. *Then the unique solution from* $\widehat{W}_2^1(Q_T)$ *to the problem I with* $\varphi(x) \equiv 0$ *for* $x \in [0,l]$, $\psi(x) = 0$ *a.e. on* $[0,l]$ *and an arbitrary function* $\mu(t) \in W_2^1[0,T]$ *such that* $\mu(0) = 0$, *satisfies the relation*

$$u(x,t) = \underline{\mu}(t-x) + \frac{1}{2}\int\limits_0^t \int\limits_{|x-t+\tau|}^{l-|x+t-\tau-l|} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau +$$

$$+ \int\limits_0^{\max\{0,x+t-l\}} \int\limits_{2l-x-t+\tau}^{l} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau \tag{6}$$

in the case $0 \leqslant t \leqslant l$ *and the relation*

$$u(x,t) = \underline{\mu}(t-x) + \underline{\mu}(t+x-2l) + \frac{1}{2}\int\limits_0^{t-l} \int\limits_{|2l-x-t+\tau|}^{l-|t-x-l-\tau|} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau +$$

$$+\frac{1}{2}\int\limits_{t-l}^{t} \int\limits_{|x-t+\tau|}^{l-|t+x-l-\tau|} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau + \int\limits_{\max\{0,t-x-l\}}^{x+t-l} \int\limits_{l-x+|l-t+\tau|}^{l} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau \tag{7}$$

in the case $l \leqslant t \leqslant T$.

Proof. In order to obtain relations (6) and (7), it is sufficient to construct the solution from the class $\widehat{W}_2^1(Q_T)$ to the following initial boundary-value problem for the inhomogeneous wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t), \quad (x,t) \in Q_T, \\ u(x,0) = 0, \ u_t(x,0) = 0 \text{ for } 0 \leqslant x \leqslant l, \\ u(0,t) = \mu(t), \ u_x(l,t) = 0 \text{ for } 0 \leqslant t \leqslant T \end{cases} \tag{8}$$

and take $f(x,t) = q(x,t)u(x,t)$.

The solution to (8) is a sum of the solution $\underline{\mu}(t-x) + \underline{\mu}(t+x-2l)$ to the similar problem for the homogeneous wave equation (see Assertion 2 in [3]) and the solution to the problem (8) with zero initial and boundary data. The latter solution is given by the well-known integral $F(x,t) = \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)\,d\xi d\tau$ in which the integrand coincides with $f(x,t)$ inside Q_T and is the odd extension of $f(x,t)$ over the boundary $x=0$ and its even extension over the boundary $x=l$. One can easily show that if $f(x,t) \in L_2(Q_T)$ then $F(x,t)$ is a unique solution from $\widehat{W}_2^1(Q_T)$ to the corresponding problem. Using the symmetric properties of the continued function $f(x,t)$ one can refine the bounds of integration for $F(x,t)$ that immediately leads to relations (6) and (7). ∎

Assertion 3. *Let $T \leqslant 2l$. Then the solution $u(x,t)$ from $\widehat{W}_2^1(Q_T)$ to the problem I with $\varphi(x) \equiv 0$ for $x \in [0,l]$, $\psi(x) = 0$ a.e. on $[0,l]$ and an arbitrary function $\mu(t) \in W_2^1[0,T]$ such that $\mu(0) = 0$, is defined uniquely; moreover, $u(x,t) \equiv 0$ in the domain $\{(x,t) \mid 0 \leqslant t \leqslant l, t \leqslant x \leqslant l\} \bigcap Q_T$.*

Proof. All the values $T \in (0,2l]$ are discussed similarly. Thus for the sake of brevity let us confine ourselves to the case $T = 2l$. Let the rectangle Q_{2l} be subdivided into seven parts by the characteristic lines starting from its vertices. In these domains:

– the triangle $\Delta_1 = \{(x,t) \mid 0 \leqslant t \leqslant l/2, \ t \leqslant x \leqslant l-t\}$ which is adjacent to the lower base of Q_{2l},

– the triangles $\Delta_2 = \{(x,t) \mid 0 \leqslant t \leqslant l, \ 0 \leqslant x \leqslant l/2 - |t-l/2|\}$ and $\Delta_3 = \{(x,t) \mid 0 \leqslant t \leqslant l, \ l/2 + |t-l/2| \leqslant x \leqslant l\}$ which are adjacent to the lateral sides of Q_{2l} in its lower half,

– the square $\Delta_4 = \{(x,t) \mid l/2 \leqslant t \leqslant 3l/2, \ |t-l| \leqslant x \leqslant l - |t-l|\}$,

– the triangles $\Delta_5 = \{(x,t) \mid l \leqslant t \leqslant 2l, \ 0 \leqslant x \leqslant l/2 - |t-3l/2|\}$ and $\Delta_6 = \{(x,t) \mid l \leqslant t \leqslant 2l, \ l/2 + |t-3l/2| \leqslant x \leqslant l\}$ which are adjacent to the lateral sides of Q_{2l} in its upper half, and

– the triangle $\Delta_7 = \{(x,t) \mid 3l/2 \leqslant t \leqslant 2l, \ 2l-t \leqslant x \leqslant t-l\}$ which is adjacent to the upper base of Q_{2l},

let us assign $u_j(x,t) = u(x,t)$ for $(x,t) \in \Delta_j$, $j = \overline{1,7}$, and consider Eqs. (6) and (7) successively for the points $(x,t) \in \Delta_1, \Delta_2, ..., \Delta_7$ as integral equations for obtaining $u_1(x,t)$, $u_2(x,t), ..., u_7(x,t)$.

In the case $j=1$ Eq. (6) leads to the following equation for $u_1(x,t)$:

$$u_1(x,t) = \frac{1}{2} \iint\limits_{D_1} q(\xi,\tau) u_1(\xi,\tau)\,d\xi d\tau \tag{9}$$

where $D_1(x,t)=\{(\xi,\tau)\mid 0\leqslant\tau\leqslant t,\ x-t+\tau\leqslant\xi\leqslant x+t-\tau\}$.

Rewriting Eq. (9) in the operator form $u_1(x,t)=[\mathcal{N}_1u_1](x,t)$ one can note that the operator $\mathcal{N}_1$ is bounded in $L_\infty(\Delta_1)$ and its powers $\mathcal{N}_1^k$ satisfy the estimate[†]

$$\left|[\mathcal{N}_1^k\chi](x,t)\right|\leqslant\|q\|_\infty^k\frac{t^{2k}}{(2k)!}\sup_{(x,t)\in\Delta_1}|\chi(x,t)|,\quad (x,t)\in\Delta_1. \tag{10}$$

Thus Eq. (9) is a homogeneous Volterra-type integral equation of the second kind and has only the trivial solution.

As $u_1(x,t)\equiv 0$ and $\underline{\mu}(t-x)\equiv 0$ for $(x,t)\in\Delta_3$, the function $u_3(x,t)$ satisfies the equation similar to (9):

$$u_3(x,t)=\frac{1}{2}\left(\iint\limits_{D_3'}+\iint\limits_{D_3''}\right)q(\xi,\tau)u_3(\xi,\tau)\,d\xi d\tau \tag{11}$$

where the quadrangle $D_3'(x,t)=\{(\xi,\tau)\mid 0\leqslant\tau\leqslant t,\ (l+x-t)/2+|\tau-(l-x+t)/2|\leqslant\xi\leqslant(l+x+t-\tau)/2-|(\tau-x-t+l)/2|\}$ and the triangle $D_3''(x,t)=\{(\xi,\tau)\mid 0\leqslant\tau\leqslant x+t-l,\ (3l-x-t)/2+|\tau-(x+t-l)/2|\leqslant\xi\leqslant l\}$ lay entirely in Δ_3.

Since the integral operator $\mathcal{N}_3$ in the right-hand side of (11) is bounded in $L_\infty(\Delta_3)$ and satisfies the estimate

$$\left|[\mathcal{N}_3^k\chi](x,t)\right|\leqslant(2\|q\|_\infty)^k\frac{t^{2k}}{(2k)!}\sup_{(x,t)\in\Delta_3}|\chi(x,t)|,\quad (x,t)\in\Delta_3, \tag{12}$$

one similarly obtains that $u_3(x,t)\equiv 0$.

As in the domain Δ_2: $u_1(x,t)\equiv 0$, Eq. (6) there takes the form

$$u_2(x,t)=\mu(t-x)+\frac{1}{2}\iint\limits_{D_2}q(\xi,\tau)u_2(\xi,\tau)\,d\xi d\tau \tag{13}$$

where $D_2(x,t)=\{(\xi,\tau)\mid(t-x)/2\leqslant\tau\leqslant t,\ |x-t+\tau|\leqslant\xi\leqslant(x+t)/2-|\tau-(x+t)/2|\}\subset\Delta_2$. As one rewrites Eq. (13) in the operator form $u_2(x,t)=\mu(t-x)+[\mathcal{N}_2u_2](x,t)$, it is clear that $\mu(t-x)$ is bounded in Δ_2 and the bounded in $L_\infty(\Delta_2)$ operator $\mathcal{N}_2$ satisfies the estimate

$$\left|[\mathcal{N}_2^k\chi](x,t)\right|\leqslant(\|q\|_\infty/2)^k\frac{t^{2k}}{(2k-1)!!}\sup_{(x,t)\in\Delta_2}|\chi(x,t)|,\quad (x,t)\in\Delta_2. \tag{14}$$

Therefore Eq. (13) has the solution which equals the absolutely convergent Neumann series $u_2(x,t)=(I+\sum_{k=1}^{\infty}\mathcal{N}_2^k)\mu(t-x)$ and is bounded in Δ_2.

Now let $(x,t)\in\Delta_4$. As $u_1(x,t)\equiv 0$, $u_3(x,t)\equiv 0$, Eqs. (6) and (7) yield

$$u_4(x,t)=\mu(t-x)+\frac{1}{2}\iint\limits_{D_4'}q(\xi,\tau)u_2(\xi,\tau)\,d\xi d\tau+\frac{1}{2}\iint\limits_{D_4}q(\xi,\tau)u_4(\xi,\tau)\,d\xi d\tau \tag{15}$$

[†]By $\|q\|_\infty$ we denote the norm ess $\sup_{(x,t)\in Q_{2l}}|q(x,t)|$.

where $D_4'(x,t) = \{(\xi,\tau) \mid (t-x)/2 \leqslant \tau \leqslant (l+t-x)/2,\ |x-t+\tau| \leqslant \xi \leqslant l/2 - |\tau - l/2|\}$, $D_4(x,t) = \{(\xi,\tau) \mid l/2 \leqslant \tau \leqslant t,\ (l-t+x)/2 + |(l+t-x)/2 - \tau| \leqslant \xi \leqslant (t+x)/2 - |(t+x)/2 - \tau|\}$. Since $D_4' \subset \Delta_2$, the first integral term on the right-hand side of (15) is the known bounded function. As $D_4 \subset \Delta_4$, Eq. (15) can be treated as an integral equation for $u_4(x,t)$ in which the operator $[\mathcal{N}_4\chi](x,t) \equiv \frac{1}{2}\iint_{D_4} q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ in the right-hand side is bounded in $L_\infty(\Delta_4)$ and satisfies the estimate

$$\left|[\mathcal{N}_4^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/2)^k \,\frac{(t-x)^k}{2k!} \sup_{(x,t)\in\Delta_4} |\chi(x,t)|, \quad (x,t) \in \Delta_4. \tag{16}$$

Similar to (13), the estimates (16) yield that Eq. (15) has a bounded in Δ_4 solution $u_4(x,t)$.

The solution $u(x,t)$ in the remaining domains Δ_5, Δ_6, Δ_7 is constructed using the same approach. Let us list only the related integral equations and corresponding estimates.

In the triangle Δ_5, one obtains the equation

$$u_5(x,t) = \mu(t-x) + \frac{1}{2}\iint\limits_{D_5'} q(\xi,\tau)u_4(\xi,\tau)\,d\xi d\tau + \frac{1}{2}\iint\limits_{D_5} q(\xi,\tau)u_5(\xi,\tau)\,d\xi d\tau \tag{17}$$

where $D_5'(x,t) = \{(\xi,\tau) \mid (t-x)/2 \leqslant \tau \leqslant (l+t+x)/2,\ (t-x-l)/2 + |(t-x+l)/2 - \tau| \leqslant \xi \leqslant (t+x)/2 - |(t+x)/2 - \tau|\} \subset \Delta_4$, $D_5(x,t) = \{(\xi,\tau) \mid (t-x+l)/2 \leqslant \tau \leqslant t,\ |x-t+\tau| \leqslant \xi \leqslant (t+x-l)/2 - |(t+x+l)/2 - \tau|\} \subset \Delta_5$, and the integral operator $[\mathcal{N}_5\chi](x,t) \equiv \frac{1}{2}\iint_{D_5} q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ satisfies the estimate

$$\left|[\mathcal{N}_5^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/4)^k \,\frac{(t-x-l)^k}{k!} \sup_{(x,t)\in\Delta_5} |\chi(x,t)|, \quad (x,t) \in \Delta_5. \tag{18}$$

In the triangle Δ_6, one obtains the equation

$$u_6(x,t) = \mu(t-x) + \mu(t+x-2l) + \frac{1}{2}\left(\iint\limits_{D_{62}'} + \iint\limits_{D_{62}''}\right) q(\xi,\tau)u_2(\xi,\tau)\,d\xi d\tau +$$

$$+\frac{1}{2}\left(\iint\limits_{D_{64}'} + \iint\limits_{D_{64}''}\right) q(\xi,\tau)u_4(\xi,\tau)\,d\xi d\tau + \frac{1}{2}\left(\iint\limits_{D_6'} + \iint\limits_{D_6''}\right) q(\xi,\tau)u_6(\xi,\tau)\,d\xi d\tau \tag{19}$$

where $D_{62}'(x,t) = \{(\xi,\tau) \mid (t+x-2l)/2 \leqslant \tau \leqslant (t+x-l)/2,\ |x+t-2l-\tau| \leqslant \xi \leqslant l/2 - |l/2 - \tau|\} \subset \Delta_2$, $D_{62}''(x,t) = \{(\xi,\tau) \mid (t-x)/2 \leqslant \tau \leqslant (l+t-x)/2,\ |x-t+\tau| \leqslant \xi \leqslant l/2 - |l/2 - \tau|\} \subset \Delta_2$, $D_{64}'(x,t) = \{(\xi,\tau) \mid l/2 \leqslant \tau \leqslant (2l-x+t)/2,\ (l+x-t)/2 + |(l-x+t)/2 - \tau| \leqslant \xi \leqslant l - |l-\tau|\} \subset \Delta_4$, $D_{64}''(x,t) = \{(\xi,\tau) \mid l/2 \leqslant \tau \leqslant (x+t)/2,\ (3l-x-t)/2 + |(x+t-l)/2 - \tau| \leqslant \xi \leqslant l - |l-\tau|\} \subset \Delta_4$, $D_6'(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant x+t-l,\ (4l-x-t)/2 + |(x+t)/2 - \tau| \leqslant \xi \leqslant l\} \subset \Delta_6$, $D_6''(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant$

$t,\ (2l+x-t)/2+|(2l-x+t)/2-\tau| \leqslant \xi \leqslant (l+x+t-\tau/)2-|(\tau-x-t+l)/2|\} \subset \Delta_6$, and the integral operator $[\mathcal{N}_6\chi](x,t) \equiv \frac{1}{2}(\iint_{D_6'}+\iint_{D_6''})q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ satisfies the estimate

$$\left|[\mathcal{N}_6^k\chi](x,t)\right| \leqslant (l\|q\|_\infty)^k\,\frac{(x+t-2l)^k}{k!}\sup_{(x,t)\in\Delta_6}|\chi(x,t)|, \quad (x,t)\in\Delta_6. \tag{20}$$

And finally, in the triangle Δ_7, the following equation holds:

$$\begin{aligned} u_7(x,t) &= \mu(t-x)+\mu(t+x-2l)+\frac{1}{2}\iint\limits_{D_{72}} q(\xi,\tau)u_2(\xi,\tau)\,d\xi d\tau + \\ &+\frac{1}{2}\left(\iint\limits_{D_{74}'}+\iint\limits_{D_{74}''}\right)q(\xi,\tau)u_4(\xi,\tau)\,d\xi d\tau+\frac{1}{2}\iint\limits_{D_{75}} q(\xi,\tau)u_5(\xi,\tau)\,d\xi d\tau + \\ &+\frac{1}{2}\left(\iint\limits_{D_{76}'}+\iint\limits_{D_{76}''}\right)q(\xi,\tau)u_6(\xi,\tau)\,d\xi d\tau+\frac{1}{2}\iint\limits_{D_7} q(\xi,\tau)u_7(\xi,\tau)\,d\xi d\tau \end{aligned} \tag{21}$$

where $D_{72}(x,t) = \{(\xi,\tau) \mid (t+x-2l)/2 \leqslant \tau \leqslant (x+t-l)/2,\ |x+t-2l-\tau| \leqslant \xi \leqslant l/2-|l/2-\tau|\} \subset \Delta_2$, $D_{74}'(x,t) = \{(\xi,\tau) \mid l/2 \leqslant \tau \leqslant (x+t)/2,\ (3l-x-t)/2+|(x+t-l)/2-\tau| \leqslant \xi \leqslant l-|l-\tau|\} \subset \Delta_4$, $D_{74}''(x,t) = \{(\xi,\tau) \mid (t-x)/2 \leqslant \tau \leqslant (3l)/2,\ (t-x-l)/2+|(t-x+l)/2-\tau| \leqslant \xi \leqslant l-|l-\tau|\} \subset \Delta_4$, $D_{75}(x,t) = \{(\xi,\tau) \mid (t-x+l)/2 \leqslant \tau \leqslant (2l-x+t)/2,\ |x-t+\tau| \leqslant \xi \leqslant l/2-|3l/2-\tau|\} \subset \Delta_5$, $D_{76}'(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant (x+t+l)/2,\ l/2+|(3l)/2-\tau| \leqslant \xi \leqslant (l+x+t-\tau)/2-|(t+x-l-\tau)/2|\} \subset \Delta_6$, $D_{76}''(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant x+t-l,\ (4l-x-t)/2+|(x+t)/2-\tau| \leqslant \xi \leqslant l\} \subset \Delta_6$, $D_7(x,t) = \{(\xi,\tau) \mid 3l/2 \leqslant \tau \leqslant t,\ (2l+x-t)/2+|(2l+t-x)/2-\tau| \leqslant \xi \leqslant (x+t-l)/2-|(x+t+l)/2-\tau|\} \subset \Delta_7$, and the integral operator $[\mathcal{N}_7\chi](x,t) \equiv \frac{1}{2}\iint_{D_7} q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ satisfies the estimate

$$\left|[\mathcal{N}_7^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/4)^k\,\frac{(t+x-2l)^k}{k!}\sup_{(x,t)\in\Delta_7}|\chi(x,t)|, \quad (x,t)\in\Delta_7. \tag{22}$$

Thus it is proved that Eqs. (6) and (7) have the bounded solution $u(x,t)$ in the rectangle Q_T. The next step is to study its smoothness.

As all the terms in the right-hand sides of (6) and (7) are continuous with respect to (x,t) in Q_T, the obtained solution $u(x,t)$ is also continuous in Q_T. For the sake of simplicity, let us introduce the bounded function $U(x,t) \equiv q(x,t)u(x,t)$ in Q_T and make its odd extension over $x=0$ and its even extension over $x=l$. Then Eqs. (6) and (7) can be coupled into one equation

$$u(x,t) = \underline{\mu}(t-x)+\underline{\mu}(t+x-2l)+\frac{1}{2}\int\limits_0^t\int\limits_{x-t+\tau}^{x+t-\tau} U(\xi,\tau)\,d\xi d\tau. \tag{23}$$

By the straightforward differentiation, Eq. (23) yields that, a.e. in Q_T,

$$u_x(x,t) = -\underline{\mu}'(t-x) + \underline{\mu}'(t+x-2l) + \frac{1}{2}\int\limits_0^t [U(x+t-\tau,\tau) - U(x-t+\tau,\tau)]\,d\tau, \quad (24)$$

$$u_t(x,t) = \underline{\mu}'(t-x) + \underline{\mu}'(t+x-2l) + \frac{1}{2}\int\limits_0^t [U(x+t-\tau,\tau) + U(x-t+\tau,\tau)]\,d\tau, \quad (25)$$

and therefore the derivatives $u_x(x,t)$ and $u_t(x,t)$ belong to $L_2(0 \leqslant x \leqslant l)$ for all $t \in [0,2l]$ and to $L_2(0 \leqslant t \leqslant 2l)$ for all $x \in [0,l]$. ■

Remark. The direct analysis of the Neumann series for the solutions of Eqs. (9), (11), (13), (15), (17), (19), (21) and the inequalities (10), (12), (14), (16), (18), (20), (22) lead to the estimate

$$\max_{(x,t)\in Q_T} |u(x,t) - u^*(x,t)| \leqslant C\|q\|_\infty \quad (26)$$

where $u^*(x,t) = \underline{\mu}(t-x) + \underline{\mu}(t+x-2l)$ is the solution from $\widehat{W}_2^1(Q_T)$ to the initial boundary-value problem I for the homogeneous wave equation with zero initial data in the case when $T \leqslant 2l$ (see the proof of Assertion 2).

Together with Eqs. (24) and (25), it means that

$$\|u_x - u_x^*\|_{L_2(Q_T)} + \|u_t - u_t^*\|_{L_2(Q_T)} \leqslant C\|q\|_\infty. \quad (27)$$

Thus, combining (26) and (27), one obtains the estimate

$$\|u - u^*\|_{W_2^1(Q_T)} \leqslant C\|q\|_\infty. \quad (28)$$

For the problem II the following similar statement holds.

Assertion 4. *Let $T \leqslant 2l$. Then the solution from $\widehat{W}_2^1(Q_T)$ to the problem II where $\varphi_1(x) \equiv 0$ for $x \in [0,l]$, $\psi_1(x) = 0$ a.e. on $[0,l]$ and an arbitrary function $\mu(t) \in W_2^1[0,T]$ such that $\mu(T) = 0$, is defined uniquely; moreover, $u(x,t) \equiv 0$ in the domain $\{(x,t) \mid T-l \leqslant t \leqslant T,\ T-t \leqslant x \leqslant l\} \bigcap Q_T$.*

Let us proceed with the proof of uniqueness for the solution to the boundary control problem III.

Assertion 5. *For any $T \in (0,2l]$, the boundary control problem III has at most one solution from $\widehat{W}_2^1(Q_T)$.*

Proof. Let us consider only[‡] the case $T = 2l$. Suppose that in this case the problem III has two solutions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ from the class $\widehat{W}_2^1(Q_{2l})$. Then their difference $u(x,t) = u^{(2)}(x,t) - u^{(1)}(x,t)$ gives a solution from the same class to the problem III with zero initial and terminal data. Let $\mu(t) = u(0,t)$. It follows from the definition of the class $\widehat{W}_2^1(Q_{2l})$ that $\mu(t) \in W_2^1[0,2l]$ and $\mu(0) = \mu(2l) = 0$.

[‡]As it can be easily obtained from what follows, in the case $T < 2l$ it is essential that the domains where the solutions considered in Assertions 2-4 vanish should have common points.

The function $u(x,t)$ is the solution from $\widehat{W}_2^1(Q_{2l})$ both to the problem I with zero initial data and to the problem II with zero terminal data coupled with the boundary condition $\mu(t) = u(0,t)$. It follows from Assertions 2-4 that $u(x,t)$ vanishes in the domain $\Delta_0 = \{(x,t) \mid 0 \leqslant t \leqslant 2l,\ l - |l-t| \leqslant x \leqslant l\}$. Let us show that $u(x,t)$ vanishes also in the remaining domain $Q_{2l} \setminus \Delta_0$.

Let t_1 be an arbitrary value in $[0,2l]$. The characteristic line $t - x = t_1$ that starts at the point $(0,t_1)$ intersects the characteristic line $t + x = 2l$ at the point $(l - t_1/2, l + t_1/2) \in \Delta_0$ where $u(x,t) = 0$. Thus Eqs. (6) and (7) yield

$$\mu(t_1) = -\frac{1}{2} \iint\limits_{D_0'(t_1)} q(\xi,\tau) u(\xi,\tau)\, d\xi d\tau, \tag{29}$$

where $D_0'(t_1) = \{(\xi,\tau) \mid t_1/2 \leqslant \tau \leqslant t_1/2 + l,\ \ |\tau - t_1| \leqslant \xi \leqslant l - |l-\tau|\}$.

Consider an arbitrary point $(x,t) \in Q_{2l} \setminus \Delta_0$. It follows from (6) and (7) that $u(x,t) = \mu(t-x) + \frac{1}{2}\iint_{D_0(x,t)} q(\xi,\tau)u(\xi,\tau)\, d\xi d\tau$, hence, by Eq. (29) with $t_1 = t - x$, one obtains the relation

$$u(x,t) = -\frac{1}{2} \iint\limits_{D_0'(t-x)\setminus D_0(x,t)} q(\xi,\tau) u(\xi,\tau)\, d\xi d\tau \equiv [\mathcal{N}_0 u](x,t); \tag{30}$$

here the domain $D_0'(t-x) \setminus D_0(x,t)$ is the triangle $\{(\xi,\tau) \mid (x+t)/2 \leqslant \tau \leqslant l + (t-x)/2,\ x + |t-\tau| \leqslant \xi \leqslant l - |l-\tau|\}$.

Eq. (30) is the homogeneous Volterra-type equation of the second kind since the operator $\mathcal{N}_0$ in its right-hand side is bounded in $L_\infty(Q_{2l} \setminus \Delta_0)$ and satisfies the estimates $\left|[\mathcal{N}_0^k \chi](x,t)\right| \leqslant (l\|q\|_\infty/2)^k\, (t-x)^k/k!\, \sup_{(x,t)\in Q_{2l}\setminus\Delta_0} |\chi(x,t)|$. Thus Eq. (30) has only the trivial solution, and it follows from Eq. (29) that $\mu(t_1) = 0$ for all $t_1 \in [0,2l]$. ∎

4. Main results

First of all let us note a certain peculiarity of the boundary control problem III for the critical value $T = 2l$. It follows from [3] that the function

$$\overset{0}{u}(x,t) = \frac{1}{2}\begin{cases} \varphi(x+t) + \varphi(x-t) + \int\limits_{x-t}^{x+t} \psi(\xi)\,d\xi & \text{in } \Delta_1, \\ \varphi(x+t) + \varphi(0) + \int\limits_{0}^{x+t} \psi(\xi)\,d\xi + \varphi_1(t-x) - \varphi_1(0) + \int\limits_{0}^{t-x} \psi_1(\xi)\,d\xi & \text{in } \Delta_2, \\ \varphi(x-t) + \varphi(2l-x-t) + \int\limits_{x-t}^{l} \psi(\xi)\,d\xi + \int\limits_{2l-x-t}^{l} \psi(\xi)\,d\xi & \text{in } \Delta_3, \\ \varphi(2l-x-t) + \varphi(0) + \int\limits_{2l-x-t}^{l} \psi(\xi)\,d\xi + \int\limits_{0}^{l} \psi(\xi)\,d\xi + & \\ \qquad + \varphi_1(t-x) - \varphi_1(0) + \int\limits_{0}^{t-x} \psi_1(\xi)\,d\xi & \text{in } \Delta_4, \\ \varphi(2l-x-t) + \varphi(0) + \int\limits_{2l-x-t}^{l} \psi(\xi)\,d\xi + \int\limits_{0}^{l} \psi(\xi)\,d\xi + & \\ \qquad + \varphi_1(2l+x-t) - \varphi_1(0) + \int\limits_{2l+x-t}^{l} \psi_1(\xi)\,d\xi + \int\limits_{0}^{l} \psi_1(\xi)\,d\xi & \text{in } \Delta_5, \\ \varphi_1(x+t-2l) + \varphi_1(t-x) - \int\limits_{x+t-2l}^{l} \psi_1(\xi)\,d\xi - \int\limits_{t-x}^{l} \psi_1(\xi)\,d\xi & \text{in } \Delta_6, \\ \varphi_1(x+t-2l) + \varphi_1(x-t+2l) - \int\limits_{x+t-2l}^{x-t+2l} \psi_1(\xi)\,d\xi & \text{in } \Delta_7 \end{cases} \tag{31}$$

is a unique solution to the boundary control problem III for the homogeneous wave equation if and only if its data satisfy the relation

$$A_0 \equiv \varphi(0) + \int\limits_0^l \psi(\xi)\,d\xi = \varphi_1(0) - \int\limits_0^l \psi_1(\xi)\,d\xi \equiv B_0. \tag{32}$$

Similarly, in the inhomogeneous case one can prove that the function[§]

$$u(x,t) = \overset{0}{u}(x,t) + \frac{1}{2}\int\limits_0^t \int\limits_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)\,d\xi d\tau \tag{33}$$

is the unique solution to the boundary control problem III for the forced oscillations (see (8)) if and only if the relation

$$A_0 + \int\limits_0^l \int\limits_\tau^l f(\xi,\tau)\,d\xi d\tau = B_0 + \int\limits_l^{2l} \int\limits_{2l-\tau}^{l} f(\xi,\tau)\,d\xi d\tau \tag{34}$$

[§]The integrand in (33) is obtained by the extension of the right-hand side $f(x,t)$ outside Q_T similar to (8).

holds where A_0, B_0 are constants in the left-hand and in the right-hand sides of (32).

The boundary control problem III for the telegraph equation (1) is also governed by a similar condition which is necessary for the existence of its solution from $\widehat{W}_2^1(Q_T)$.

Theorem 1. *Let $T = 2l$. Then, for the existence of the solution from $\widehat{W}_2^1(Q_{2l})$ to the boundary control problem III, it is necessary to observe the following conditions:*

1) $\varphi(x)$, $\varphi_1(x) \in W_2^1[0,l]$, $\psi(x)$, $\psi_1(x) \in L_2[0,l]$,

2) the initial and terminal data satisfy the relation

$$A_0 + \int\limits_0^l \int\limits_\tau^{2l-\tau} \widetilde{q}_A^*(\xi,\tau)A(\xi,\tau)\,d\xi d\tau = B_0 + \int\limits_l^{2l} \int\limits_{2l-\tau}^{\tau} \widetilde{q}_B^*(\xi,\tau)B(\xi,\tau)\,d\xi d\tau, \tag{35}$$

where A_0, B_0 are the constants in (32), the values $A(\xi,\tau)$, $B(\xi,\tau)$ are computed via the initial and terminal data by the formulas

$$A(\xi,\tau) = \frac{1}{2}\left[\varphi(l-|\xi+\tau-l|) + \varphi(l-|\xi-\tau-l|) + \int\limits_{\xi-\tau}^{\xi+\tau} \psi(l-|\zeta-l|)\,d\zeta\right], \tag{36}$$

$$B(\xi,\tau) = \frac{1}{2}\left[\varphi_1(l-|\xi+\tau-3l|) + \varphi_1(l-|\xi-\tau+l|) - \int\limits_{\xi+\tau-2l}^{\xi-\tau+2l} \psi_1(l-|\zeta-l|)\,d\zeta\right] \tag{37}$$

and the kernels $\widetilde{q}_A^*(\xi,\tau)$ and $\widetilde{q}_B^*(\xi,\tau)$ of the integral operators are connected with the coefficient $q(\xi,\tau)$ in (1) by the relations

$$\begin{aligned}
&\widetilde{q}_A^*(\xi,\tau) = q(l-|\xi-l|,\tau)\sum_{k=0}^{\infty}\widetilde{q}_A^{(k)}(l,l;\xi,\tau), \qquad \widetilde{q}_A^{(0)}(x,t;\xi,\tau) \equiv 1/2;\\
&\widetilde{q}_A^{(k+1)}(x,t;\xi,\tau) = \tfrac{1}{2}\int\limits_\tau^t \int\limits_{\max(x-t+\tau_1,\xi+\tau-\tau_1)}^{\min(x+t-\tau_1,\xi-\tau+\tau_1)} q(l-|\xi_1-l|,\tau_1)\widetilde{q}_A^{(k)}(\xi_1,\tau_1;\xi,\tau)\,d\xi_1 d\tau_1,\\
&\widetilde{q}_B^*(\xi,\tau) = q(l-|\xi-l|,\tau)\sum_{k=0}^{\infty}\widetilde{q}_B^{(k)}(l,l;\xi,\tau), \qquad \widetilde{q}_B^{(0)}(x,t;\xi,\tau) \equiv 1/2;\\
&\widetilde{q}_B^{(k+1)}(x,t;\xi,\tau) = \tfrac{1}{2}\int\limits_t^\tau \int\limits_{\max(x+t-\tau_1,\xi-\tau+\tau_1)}^{\min(x-t+\tau_1,\xi+\tau-\tau_1)} q(l-|\xi_1-l|,\tau_1)\widetilde{q}_B^{(k)}(\xi_1,\tau_1;\xi,\tau)\,d\xi_1 d\tau_1.
\end{aligned} \tag{38}$$

Proof. Let the function $u(x,t)$ be the solution from $\widehat{W}_2^1(Q_{2l})$ to the boundary control problem III. Then it is also the solution to the problem I in the triangles Δ_1, Δ_3 and the solution to the problem II in the triangles Δ_6, Δ_7. Mimicking the proof of Assertion 2, we construct the integral relations for the function $u(x,t)$ in the domains Δ_1, Δ_3, Δ_6 and Δ_7.

Let $(x,t) \in \Delta_1 \bigcup \Delta_3$. Denoting by $\overset{0}{u}_1(x,t)$ and $\overset{0}{u}_3(x,t)$ the solutions (31) to the problem I for the homogeneous wave equation in Δ_1 and Δ_3 respectively, one obtains the equations

$$u(x,t) = \overset{0}{u}_1(x,t) + \frac{1}{2}\iint\limits_{\Omega_1} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau, \quad (x,t) \in \Delta_1, \tag{39}$$

$$u(x,t) = \overset{0}{u}_3(x,t) + \frac{1}{2}\left(\iint\limits_{\Omega_3'} + \iint\limits_{\Omega_3''}\right) q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau, \quad (x,t) \in \Delta_3, \tag{40}$$

where $\Omega_1(x,t) = D_1(x,t)$, $\Omega_3'(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant t,\ x-t+\tau \leqslant \xi \leqslant (l+x+t-\tau)/2 - |(\tau - x - t + l)/2|\}$ and $\Omega_3''(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant x+t-l,\ 2l - x - t + \tau \leqslant \xi \leqslant l\}$.

Let the functions $u(x,t)$, $q(x,t)$ in (40) and the initial data $\varphi(x)$, $\psi(x)$ in $\overset{0}{u}_3(x,t)$ be continued evenly over $x = l$ to the domain $Q_{2l}' = [l \leqslant x \leqslant 2l] \times [0 \leqslant t \leqslant 2l]$. Denote these new functions by $\widetilde{u}(x,t)$, $\widetilde{q}(x,t)$, $\widetilde{\varphi}(x)$, $\widetilde{\psi}(x)$ respectively. Thus $\overset{0}{u}_3(x,t) = \frac{1}{2}[\widetilde{\varphi}(x+t) + \widetilde{\varphi}(x-t) + \int\limits_{x-t}^{x+t} \widetilde{\psi}(\xi)d\xi] \equiv \overset{0}{\widetilde{u}}_1(x,t)$ and using this continuation one can rewrite Eq. (40) in the following form

$$\widetilde{u}(x,t) = \overset{0}{\widetilde{u}}_1(x,t) + \frac{1}{2}\iint\limits_{\Omega_1} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\,d\xi d\tau, \quad (x,t) \in \Delta_3. \tag{41}$$

One can easily see that Eq. (41) transforms into Eq. (39) for $(x,t) \in \Delta_1$ and, by the symmetry, keeps its form if the point (x,t) belongs to the triangle which is mirror symmetric to the triangle $\Delta_1 \bigcup \Delta_3$ with respect to $x = l$. Thus the continued solution $\widetilde{u}(x,t)$ satisfies Eq. (41) for all $(x,t) \in \widetilde{\Delta}_1 = \{(x,t) \mid 0 \leqslant t \leqslant l,\ t \leqslant x \leqslant 2l - t\}$.

Similarly denoting by $\overset{0}{u}_6(x,t)$ and $\overset{0}{u}_7(x,t)$ the solutions (31) to the problem II for the homogeneous wave equation in Δ_6 and Δ_7 respectively, one obtains from the equations

$$u(x,t) = \overset{0}{u}_7(x,t) + \frac{1}{2}\iint\limits_{\Omega_7} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau,\ (x,t) \in \Delta_7, \tag{42}$$

$$u(x,t) = \overset{0}{u}_6(x,t) + \frac{1}{2}\left(\iint\limits_{\Omega_6'} + \iint\limits_{\Omega_6''}\right) q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau,\ (x,t) \in \Delta_6, \tag{43}$$

where $\Omega_7(x,t) = \{(\xi,\tau) \mid t \leqslant \tau \leqslant 2l,\ x+t-\tau \leqslant \xi \leqslant x-t+\tau\}$, $\Omega_6'(x,t) = \{(\xi,\tau) \mid t \leqslant \tau \leqslant 2l,\ x+t-\tau \leqslant \xi \leqslant (l+x-t+\tau)/2 - |(\tau + x - t - l)/2|\}$, $\Omega_6''(x,t) = \{(\xi,\tau) \mid l - x + t \leqslant \tau \leqslant 2l,\ 2l - x + t - \tau \leqslant \xi \leqslant l\}$, that the evenly extended solution $\widetilde{u}(x,t)$ for all $(x,t) \in \widetilde{\Delta}_7 = \{(x,t) \mid l \leqslant t \leqslant 2l,\ 2l - t \leqslant x \leqslant t\}$ satisfies the relation

$$\widetilde{u}(x,t) = \overset{0}{\widetilde{u}}_7(x,t) + \frac{1}{2}\iint\limits_{\Omega_7} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\,d\xi d\tau. \tag{44}$$

Here $\overset{0}{\widetilde{u}}_7(x,t) = \frac{1}{2}[\widetilde{\varphi}_1(x+t-2l) + \widetilde{\varphi}_1(x-t+2l) - \int\limits_{x+t-2l}^{x-t+2l} \widetilde{\psi}_1(\xi)d\xi]$ where $\widetilde{\varphi}_1(x), \widetilde{\psi}_1(x)$ are the terminal data which are extended evenly over $x = l$.

Since the function $u(x,t)$ (and therefore the function $\widetilde{u}(x,t)$) is continuous in Q_{2l} (as it belongs to $\widehat{W}_2^1(Q_{2l})$) the value $u(l,l) = \widetilde{u}(l,l)$ should be the same no matter whether it is computed from (41) or from (44). Therefore

$$\overset{0}{\widetilde{u}}_1(l,l) + \frac{1}{2}\iint\limits_{\Omega_1(l,l)} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\,d\xi d\tau = \overset{0}{\widetilde{u}}_7(l,l) + \frac{1}{2}\iint\limits_{\Omega_7(l,l)} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\,d\xi d\tau. \tag{45}$$

As $\overset{0}{\widetilde{u}}_1(l,l) = \overset{0}{u}_3(l,l) = \varphi(0) + \int_0^l \psi(\xi)d\xi = A_0$, $\overset{0}{\widetilde{u}}_7(l,l) = \overset{0}{u}_6(l,l) = \varphi_1(0) - \int_0^l \psi_1(\xi)d\xi = B_0$ and moreover $\Omega_1(l,l) = \Delta_1'$, $\Omega_7(l,l) = \Delta_7'$, Eq. (45) yields the relation

$$A_0 + \frac{1}{2}\iint\limits_{\Delta_1'} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\,d\xi d\tau = B_0 + \frac{1}{2}\iint\limits_{\Delta_7'} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\,d\xi d\tau. \tag{46}$$

Following an approach introduced in [14], Eq. (46) can be transformed to its final form (35) by expressing $\widetilde{u}(x,t)$ via $\overset{0}{\widetilde{u}}_1(x,t)$ in (41) and $\overset{0}{\widetilde{u}}_7(x,t)$ in (44) using the corresponding Neumann series and substituting the obtained expressions in the left-hand and the right-hand sides of (46).

Introducing the operators $[\mathcal{G}_j\widetilde{u}](x,t) = (1/2)\iint_{\Omega_j} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\,d\xi d\tau$, $j = 1,7$, one obtains

$$\widetilde{u}(x,t) = \overset{0}{\widetilde{u}}_1(x,t) + \sum_{k=1}^{\infty}[\mathcal{G}_1^k\overset{0}{\widetilde{u}}_1](x,t) \quad \text{for } (x,t)\in\widetilde{\Delta}_1 \tag{47}$$

and

$$\widetilde{u}(x,t) = \overset{0}{\widetilde{u}}_7(x,t) + \sum_{k=1}^{\infty}[\mathcal{G}_7^k\overset{0}{\widetilde{u}}_7](x,t) \quad \text{for } (x,t)\in\widetilde{\Delta}_7. \tag{48}$$

The series in the right-hand sides of (47) and (48) are absolutely convergent since the operators $\mathcal{G}_1$ and $\mathcal{G}_7$ satisfy the estimates:

$$\Big|[\mathcal{G}_1^k\chi](x,t)\Big| \leqslant (2\|q\|_\infty)^k\,\frac{t^{2k}}{(2k)!}\sup_{(x,t)\in\widetilde{\Delta}_1}|\chi(x,t)|, \tag{49}$$

$$\Big|[\mathcal{G}_7^k\chi](x,t)\Big| \leqslant (2l\|q\|_\infty)^k\,\frac{(2l-t)^k}{k!}\sup_{(x,t)\in\widetilde{\Delta}_7}|\chi(x,t)|. \tag{50}$$

Applying (47) and (48) in (46), changing the order of integration and taking into account that $A(x,t) = \overset{0}{\widetilde{u}}_1(x,t), B(x,t) = \overset{0}{\widetilde{u}}_7(x,t)$, $\widetilde{\varphi}(x) = \varphi(l-|x-l|)$, $\widetilde{\psi}(x) = \psi(l-|x-l|)$, $\widetilde{\varphi}_1(x) = \varphi_1(l-|x-l|)$, $\widetilde{\psi}_1(x) = \psi_1(l-|x-l|)$, $\widetilde{q}(x,t) = q(l-|x-l|,t)$, one obtains (35)–(38). ■

Let us show that the necessary condition (35) in Theorem 1 is also sufficient for the existence of the solution to the boundary control problem III.

Theorem 2. *Let $T = 2l$ and the condition 1 in Theorem 1 be satisfied. Then the relation (35) is sufficient for the existence of the unique solution from $\widehat{W}_2^1(Q_{2l})$ to the boundary control problem III.*

Proof. Let us define the function $u(x,t)$ in the triangles Δ_1 and Δ_3 as the solution to Eq. (41) constructed in the proof of Theorem 1 and in the triangles Δ_6 and Δ_7 – as the solution to Eq. (44). Due to the estimates (49) and (50) these equations have bounded solutions which are given by the series (47) and (48). As the right-hand sides of (41) and (44) are continuous these solutions are also continuous in $\Delta_1 \bigcup \Delta_3$ and $\Delta_6 \bigcup \Delta_7$. If the relation (35) holds true then, as it follows from the proof of Theorem 1, the relation (45) is satisfied and thus the function $u(x,t)$ is continuous in the union of domains $\Delta_1 \bigcup \Delta_3 \bigcup \Delta_6 \bigcup \Delta_7$.

Let $u_j(x,t)$ stand for the obtained solution $u(x,t)$ in Δ_j for $j = 1,3,6,7$ respectively and consider the remaining parts of the rectangle Q_{2l}.

For $(x,t) \in \Delta_4$ the following relation holds:

$$u(x,t) = \overset{0}{u}_4(x,t) +$$

$$+\frac{1}{2}\left[\iint\limits_{\Omega'_{41}} + \iint\limits_{\Omega''_{41}} + \iint\limits_{\Omega'_{43}} + \iint\limits_{\Omega''_{43}} - \iint\limits_{\Omega'_{46}} - \iint\limits_{\Omega''_{46}} - \iint\limits_{\Omega_{47}} - \iint\limits_{\Omega_4}\right] q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau \tag{51}$$

where $\overset{0}{u}_4(x,t)$ is the solution to the boundary control problem III for the homogeneous wave equation which is defined by (31) in the triangle Δ_4, and the integration domains are given by the inequalities: $\Omega'_{41}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant l/2,\ \tau \leqslant \xi \leqslant l-\tau\}$, $\Omega''_{41}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant (t+x-l)/2,\ 2l-x-t+\tau \leqslant \xi \leqslant l-\tau\}$, $\Omega'_{43}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant (x+t)/2,\ l/2+|\tau-l/2| \leqslant \xi \leqslant (l+x+t-\tau)/2 - |(l-x-t+\tau)/2|\}$, $\Omega''_{43}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant x+t-l,\ (3l-x-t)/2+|\tau-(x+t-l)/2| \leqslant \xi \leqslant l\}$, $\Omega'_{46}(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant l-x+t,\ (2l+x-t)/2+|\tau-(2l-x+t)/2| \leqslant \xi \leqslant l\}$, $\Omega''_{46}(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant (3l-x+t)/2,\ l/2+|\tau-(3l)/2| \leqslant \xi \leqslant (3l-x+t-\tau)/2 - |(\tau-l+x-t)/2|\}$, $\Omega_{47}(x,t) = \{(\xi,\tau) \mid (3l)/2 \leqslant \tau \leqslant 2l,\ 2l-\tau \leqslant \xi \leqslant (l-x+t)/2 - |\tau-(3l-x+t)/2|\}$, $\Omega_4(x,t) = \{(\xi,\tau) \mid (x+t)/2 \leqslant \tau \leqslant (2l-x+t)/2,\ x+|\tau-t| \leqslant \xi \leqslant l-|\tau-l|\}$.

Since $\Omega'_{41} \cup \Omega''_{41} \subset \Delta_1$, $\Omega'_{43} \cup \Omega''_{43} \subset \Delta_3$, $\Omega'_{46} \cup \Omega''_{46} \subset \Delta_6$, $\Omega_{47} \subset \Delta_7$, $\Omega_4 \subset \Delta_4$, all the integral terms in the right-hand side of (51), except the last one, are known and therefore the relation (1) can be treated as the integral equation for $u(x,t)$ in the domain Δ_4:

$$u(x,t) = F_4(x,t) - [\mathcal{G}_4 u](x,t) \tag{52}$$

where $[\mathcal{G}_4\chi](x,t) = (1/2)\iint_{\Omega_4} q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ while the function $F_4(x,t)$ is already known. The operator $\mathcal{G}_4$ is bounded in $L_\infty(\Delta_4)$ and satisfies the estimate

$$\Big|[\mathcal{G}_4^k\chi](x,t)\Big| \leqslant (l\|q\|_\infty/2)^k\,\frac{(t-x)^k}{k!}\sup_{(x,t)\in\Delta_4}|\chi(x,t)|. \tag{53}$$

This estimate yields that Eq. (52) has a bounded in Δ_4 solution. Let us denote it by $u_4(x,t)$. It follows from Eq. (51) that the function $u_4(x,t)$ is continuous in Δ_4.

On the common boarder of Δ_3 and Δ_4, i.e. for $x=t$, $l/2\leqslant t\leqslant l$, Eq. (51) transforms into the relation

$$u_4(t,t)=\overset{0}{u}_4(t,t)+\frac{1}{2}\sum_{k=1}^{2}\left[\iint\limits_{\Omega'_{4,2k-1}(t,t)}+\iint\limits_{\Omega''_{4,2k-1}(t,t)}\right]q(\xi,\tau)u_{2k-1}(\xi,\tau)\,d\xi d\tau. \tag{54}$$

Since $\overset{0}{u}_4(t,t)=\overset{0}{u}_3(t,t)$, $\Omega'_{41}(t,t)=\Omega'_3(t,t)\bigcap\Delta_1=\Delta_1$, $\Omega''_{41}(t,t)=\Omega''_3(t,t)\bigcap\Delta_1$, $\Omega'_{43}(t,t)=\Omega'_3(t,t)\bigcap\Delta_3$, $\Omega''_{43}(t,t)=\Omega''_3(t,t)\bigcap\Delta_3$, Eqs. (40) and (54) yield $u_4(t,t)=u_3(t,t)$.

On the common boarder of Δ_4 and Δ_6, i.e. for $x=2l-t$, $l\leqslant t\leqslant(3l)/2$, Eq. (51) transforms into the relation

$$u_4(2l-t,t)=\overset{0}{u}_4(2l-t,t)+\iint\limits_{\Delta_1}q(\xi,\tau)u_1(\xi,\tau)\,d\xi d\tau+\iint\limits_{\Delta_3}q(\xi,\tau)u_3(\xi,\tau)\,d\xi d\tau-$$

$$-\frac{1}{2}\left[\iint\limits_{\Omega'_{46}(2l-t,t)}+\iint\limits_{\Omega''_{46}(2l-t,t)}\right]q(\xi,\tau)u_6(\xi,\tau)\,d\xi d\tau-\frac{1}{2}\iint\limits_{\Omega_{47}(2l-t,t)}q(\xi,\tau)u_7(\xi,\tau)\,d\xi d\tau. \tag{55}$$

As $\overset{0}{u}_4(2l-t,t)=\overset{0}{u}_6(2l-t,t)+A_0-B_0$, Eq. (46) (which is equivalent to (35)) holds, and due to the relation (43), one comes to the equality $u_4(2l-t,t)=u_6(2l-t,t)$.

Similarly, for $(x,t)\in\Delta_2$ one obtains the equation

$$u(x,t)=\overset{0}{u}_2(x,t)+\frac{1}{2}\left[\iint\limits_{\Omega_{21}}-\iint\limits_{\Omega_{24}}-\iint\limits_{\Omega'_{26}}-\iint\limits_{\Omega''_{26}}-\iint\limits_{\Omega_{27}}-\iint\limits_{\Omega_{2}}\right]q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau \tag{56}$$

where $\overset{0}{u}_2(x,t)$ is the solution to the boundary control problem III for the homogeneous wave equation in the triangle Δ_2 (see (31)), and the integration domains are given by the inequalities $\Omega_{21}(x,t)=\{(\xi,\tau)\bigm| 0\leqslant\tau\leqslant(t+x)/2,\ \tau\leqslant\xi\leqslant x+t-\tau\}$, $\Omega_{24}(x,t)=\{(\xi,\tau)\mid l/2\leqslant\tau\leqslant(2l-x+t)/2,\ (l+x-t)/2+|\tau-(l-x+t)/2|\leqslant\xi\leqslant l-|l-\tau|\}$, $\Omega'_{26}(x,t)=\{(\xi,\tau)\mid l\leqslant\tau\leqslant(3l-x+t)/2,\ l/2+|\tau-(3l)/2|\leqslant\xi\leqslant(3l-x+t-\tau)/2-|(l-x+t-\tau)/2|\}$, $\Omega''_{26}(x,t)=\{(\xi,\tau)\mid l\leqslant\tau\leqslant l-x+t,\ (2l+x-t)/2+|\tau-(2l-x+t)/2|\leqslant\xi\leqslant l\}$, $\Omega_{27}(x,t)=\{(\xi,\tau)\mid (3l)/2\leqslant\tau\leqslant 2l,\ 2l-\tau\leqslant\xi\leqslant(l-x+t)/2-|\tau-(3l-x+t)/2|\}$, $\Omega_2(x,t)=\{(\xi,\tau)\mid (x+t)/2\leqslant\tau\leqslant(l-x+t)/2,\ x+|\tau-t|\leqslant\xi\leqslant l/2-|(\tau-l/2|\}$.

Since $\Omega_{21}\subset\Delta_1$, $\Omega_{24}\subset\Delta_4$, $\Omega'_{26}\cup\Omega''_{26}\subset\Delta_6$, $\Omega_{27}\subset\Delta_7$, $\Omega_2\subset\Delta_2$, all the integral terms on the right-hand side of (56), except the last one, are already known and therefore Eq. (56) is the integral equation of the form

$$u(x,t)=F_2(x,t)-[\mathcal{G}_2u](x,t) \tag{57}$$

for finding $u(x,t)$ in the domain Δ_2. Here the operator $[\mathcal{G}_2\chi](x,t) = (1/2)\iint_{\Omega_2} q(\xi,\tau)\chi(\xi,\tau)\, d\xi d\tau$ is bounded in $L_\infty(\Delta_2)$ and satisfies the estimate

$$\left|[\mathcal{G}_2^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/2)^k \frac{(t-x)^k}{k!} \sup_{(x,t)\in\Delta_2} |\chi(x,t)|. \tag{58}$$

Thus Eq. (57) has a bounded and continuous in Δ_2 solution $u(x,t) = u_2(x,t)$.

Eqs. (39), (51), (56) yield that on the boarder between Δ_1, Δ_2: $u_2(t,t) = u_1(t,t)$ and on the boarder between Δ_2, Δ_4: $u_2(l-t,t) = u_4(l-t,t)$.

Finally, for $(x,t)\in\Delta_5$ the following relation holds:

$$u(x,t) = \overset{0}{u}_5(x,t) +$$

$$+\frac{1}{2}\left[\iint\limits_{\Omega'_{51}} + \iint\limits_{\Omega''_{51}} + \iint\limits_{\Omega'_{53}} + \iint\limits_{\Omega''_{53}} - \iint\limits_{\Omega_{54}} - 2\iint\limits_{\Omega_{56}} - \iint\limits_{\Omega'_{57}} - \iint\limits_{\Omega''_{57}} - \iint\limits_{\Omega_5}\right] q(\xi,\tau)u(\xi,\tau)\, d\xi d\tau \tag{59}$$

where $\overset{0}{u}_5(x,t)$ is the solution to the boundary control problem III for the homogeneous wave equation in the triangle Δ_5 (see (31)), and the integration domains are given by the inequalities: $\Omega'_{51}(x,t) = \{(\xi,\tau) \mid 0\leqslant\tau\leqslant(x+t-l)/2,\ 2l-x-t+\tau\leqslant\xi\leqslant l-\tau\}$, $\Omega''_{51}(x,t) = \{(\xi,\tau) \mid 0\leqslant\tau\leqslant l/2,\ \tau\leqslant\xi\leqslant l-\tau\}$, $\Omega'_{53}(x,t) = \{(\xi,\tau) \mid 0\leqslant\tau\leqslant (x+t)/2,\ l/2+|\tau-l/2|\leqslant\xi\leqslant(l+x+t-\tau)/2-|(l-x-t+\tau)/2|\}$, $\Omega''_{53}(x,t) = \{(\xi,\tau) \mid 0\leqslant\tau\leqslant x+t-l,\ (3l-x-t)/2+|\tau-(x+t-l)/2|\leqslant\xi\leqslant l\}$, $\Omega_{54}(x,t) = \{(\xi,\tau) \mid (x+t)/2\leqslant \tau\leqslant(3l)/2,\ (x+t-l)/2+|\tau-(x+t+l)/2|\leqslant\xi\leqslant l-|\tau-l|\}$, $\Omega_{56}(x,t) = \{(\xi,\tau) \mid l\leqslant\tau\leqslant 2l,\ l/2+|\tau-(3l)/2|\leqslant\xi\leqslant l\}$, $\Omega'_{57}(x,t) = \{(\xi,\tau) \mid (3l)/2\leqslant\tau\leqslant 2l,\ 2l-\tau\leqslant\xi\leqslant\tau-l\}$, $\Omega''_{57}(x,t) = \{(\xi,\tau) \mid (3l)/2\leqslant\tau\leqslant 2l,\ (2l+x-t)/2+|\tau-(2l-x+t)/2|\leqslant\xi\leqslant\tau-l\}$, $\Omega_5(x,t) = \{(\xi,\tau) \mid (t+x+l)/2\leqslant\tau\leqslant(2l-x+t)/2,\ x+|\tau-t|\leqslant\xi\leqslant l/2-|\tau-(3l)/2|\}$.

Since $\Omega'_{51}\cup\Omega''_{51}\subset\Delta_1$, $\Omega'_{53}\cup\Omega''_{53}\subset\Delta_3$, $\Omega_{54}\subset\Delta_4$, $\Omega_{56}\subset\Delta_6$, $\Omega'_{57}\cup\Omega''_{57}\subset\Delta_7$, $\Omega_5\subset\Delta_5$, all the integral terms on the right-hand side of (59), except the last one, are already known and therefore Eq. (59) is the integral equation of the form

$$u(x,t) = F_5(x,t) - [\mathcal{G}_5 u](x,t), \tag{60}$$

for finding $u(x,t)$ in the domain Δ_5. The operator $[\mathcal{G}_5\chi](x,t) = (1/2)\iint_{\Omega_5} q(\xi,\tau)\chi(\xi,\tau)\, d\xi d\tau$ is bounded in $L_\infty(\Delta_5)$, and as it satisfies the estimate

$$\left|[\mathcal{G}_5^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/2)^k \frac{(2l-t-x)^k}{k!} \sup_{(x,t)\in\Delta_5} |\chi(x,t)|, \tag{61}$$

Eq. (60) has the bounded and continuous in Δ_5 solution $u(x,t) = u_5(x,t)$.

Applying Eqs. (42), (51) and (59) one can easily approve that on the boarder between Δ_5 and Δ_4: $u_5(t-l,t) = u_4(t-l,t)$, and, by virtue of Eq. (35), on the boarder between Δ_5 and Δ_7: $u_5(2l-t,t) = u_7(2l-t,t)$.

Thus the solutions to the integral equations (39), (40), (42), (43), (51), (56) and (59) define the continuous in Q_{2l} function $u(x,t)$ for which $u(x,t) = u_j(x,t)$ if $(x,t) \in \Delta_j$, $j = \overline{1,7}$.

Differentiating both parts of these integral equations with respect to x and t, one can easily show that the function $u(x,t)$ belongs to $\widehat{W}_2^1(Q_{2l})$ and $u_x(l,t) = 0$ for all $t \in [0,2l]$. The direct substitution of the integral equations for $u(x,t)$ in the identity (5) and smoothness arguments similar to those in the proof of Assertion 2, show that $u(x,t)$ is the acquired generalized solution to the boundary control problem III. ■

Remark 1. Estimates (49), (50), (53), (58), (61) and formulas that define the solutions $u_j(x,t)$, $j = \overline{1,7}$, to the corresponding integral equations in the form of the Neumann series (see, e.g., Eqs. (47), (48) for $j = 1$ and $j = 7$), yield the a priori estimate for the solution to the boundary control problem III

$$\|u(x,t)\|_{W_2^1(Q_{2l})} \leqslant C\left(\|\varphi\|_{W_2^1[0,l]} + \|\varphi_1\|_{W_2^1[0,l]} + \|\psi\|_{L_2[0,l]} + \|\psi_1\|_{L_2[0,l]}\right);$$

it claims that this solution is stable with respect to perturbations of initial and terminal data.

Remark 2. If Eq. (35) holds true then, generally speaking, the function $\overset{0}{u}(x,t)$ defined in (31) is not a solution from $\widehat{W}_2^1(Q_{2l})$ to the boundary control problem III for the homogeneous wave equation. Let us define the constant $\widetilde{C}_0 = \widetilde{C}_0(q)$ by the formula

$$\widetilde{C}_0 = -2\left(\int\limits_0^l \int\limits_\tau^{2l-\tau} \widetilde{q}_A^*(\xi,\tau)A(\xi,\tau)\,d\xi d\tau + 2\int\limits_l^{2l}\int\limits_{2l-\tau}^{\tau} \widetilde{q}_B^*(\xi,\tau)B(\xi,\tau)\,d\xi d\tau\right). \tag{62}$$

If one adds this constant $\widetilde{C}_0$ to the expressions that define the function $\overset{0}{u}(x,t)$ in the domains Δ_6 and Δ_7, the new function $\overset{0}{u}_*(x,t)$ becomes the generalized solution to the considered problem for the homogeneous wave equation but with a modified first terminal condition $\overset{0}{u}_*(x,2l) = \varphi_1(x) + \widetilde{C}_0$.

Applying Eqs. (36)–(38), one can easily show that if $\|q\|_\infty \to 0$ then the constant $\widetilde{C}_0$, defined in (62), vanishes while the function $\overset{0}{u}_*(x,t)$ transforms into $\overset{0}{u}(x,t)$.

Moreover, the estimates (49), (50), (53), (58), (61) and integral representations for partial derivatives of the solution $u(x,t)$ show that if $\|q\|_\infty \to 0$ then $\|u - \overset{0}{u}\|_{W_2^1(Q_{2l})} \to 0$ and respectively $\|\mu - \overset{0}{\mu}\|_{W_2^1[0,2l]} \to 0$ where $\overset{0}{\mu}(t) = \overset{0}{u}(0,t)$. In other words, the solution to the boundary control problem III is regular with respect the additive perturbation $q(x,t)u(x,t)$ of the wave operator in (1) with a bounded and measurable coefficient $q(x,t)$.

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Kritskov Leonid Vladimirovich
Moscow State University, Dpt. of Computational Mathematics and Cybernetics, 115172, Russian Federation, Moscow, Novospassky per., 3-2-151
E-mail: kritskov_leo@mail.ru

Abdukarimov Mahmadsalim Faizulloevich
Moscow State University,119991, Russian Federation, Moscow, Lenin Hills Street 1
*E-mail:*mahmadsalim_86@mail.ru

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Approximations of holomorphic functions by generalized Zygmund sums

Meremelia I., Savchuk V.

Abstract. We determine the asymptotic equality for the upper bounds of deviations of generalized Zygmund sums $Z_{n,\psi}(f)(z) = \widehat{f}_0 + \sum_{k=1}^{n-1}(1-\psi_n/\psi_k)\widehat{f}_k z^k$ on the functional classes $H_p^{\psi\phi}$ that are convolution of unit ball of the Hardy space H_p with kernels $\sum_{k=0}^{\infty}\psi_{k+1}\phi_{k+1}z^k$ in case when $\psi = \{\psi_k\}_{k=1}^{\infty}$ are the moment sequence. We give necessary and sufficient conditions on the sequence $\phi = \{\phi_k\}_{k=1}^{\infty}$ under which the sums $Z_{n,\psi}(f)$ approximate the class $H_p^{\psi\phi}$ with minimal possible error $|\psi_n|$.

Key Words and Phrases: Zygmund sums, Hardy space, Kolmogorov-Nikol'skii problem, Functions with positive real part.

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1. Introduction

Let $\mathcal{H}$ be a set of functions holomorphic in the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The Hardy space H_p, $1 \le p \le \infty$, is a set of all functions $f \in \mathcal{H}$ for which $\|f\|_p < \infty$, where

$$\|f\|_p := \begin{cases} \sup\limits_{0\le\varrho<1}\left(\displaystyle\int_0^{2\pi}\left|f(\varrho e^{it})\right|^p \frac{dt}{2\pi}\right)^{1/p}, & 1 \le p < \infty, \\ \sup\limits_{z\in\mathbb{D}}|f(z)|, & p = \infty. \end{cases}$$

We will denote by UH_p the unit ball of H_p.

Let $\psi = \{\psi_k\}_{k=1}^{\infty}$ be a sequence of complex numbers such that $|\psi_k| > 0$. We define generalized Zygmund sums for functions $f \in \mathcal{H}$ by

$$Z_{n,\psi}(f)(z) := \widehat{f}_0 + \sum_{k=1}^{n-1}\left(1 - \frac{\psi_n}{\psi_k}\right)\widehat{f}_k z^k,\ n \in \mathbb{N},$$

where $\widehat{f}_k := f^{(k)}(0)/k!$. (throughout this paper, we set empty sums equal to zero.)

In the 2π–periodic case the generalized Zygmund sums was introduced for the first time by Aljančić [1], [2]. They coincide with a classical Zygmund sums [15] in the case when $\psi_k = k^{-r}$, $r > 0$, and with the Fejér sums [4] when $\psi_k = k^{-1}$.

Denote by D^ψ the operator defined on $\mathcal{H}$ by the rule

$$D^\psi(f)(z) := \sum_{k=1}^{\infty} \frac{\widehat{f}_k}{\psi_k} z^{k-1}, \quad z \in \mathbb{D}.$$

If $\phi = \{\phi_k\}_{k=1}^{\infty}$ be another sequence of complex numbers such that $|\phi_k| > 0$, than

$$D^{\psi\phi}(f)(z) := D^\psi\left(zD^\phi(f)\right)(z) = \sum_{k=1}^{\infty} \frac{\widehat{f}_k}{\psi_k\phi_k} z^{k-1}, \quad z \in \mathbb{D}.$$

We assume in what follows that both sequences ψ and ϕ is such that sums of power series $\sum_{k=0}^{\infty} \psi_{k+1} z^k$ and $\sum_{k=0}^{\infty} \phi_{k+1} z^k$ defines a functions from $\mathcal{H}$.

By the class $H_p^{\psi\phi}$ we denote the set of functions $f \in \mathcal{H}$, for which $\|D^{\psi\phi}(f)\|_p \leq 1$. In particular, if $\phi_k = 1$ for all $k \in \mathbb{N}$, then

$$H_p^{\psi\mathbf{1}} = H_p^\psi := \left\{ f \in \mathcal{H} : \|D^\psi(f)\|_p \leq 1 \right\}.$$

The aim of the present work is to solve Kolmogorov–Nikolskii problem (K–N problem) for generalized Zygmund sums, that consists in finding the asymptotic formula for the quantity

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) := \sup\{\|f - Z_{n,\psi}(f)\|_p : f \in H^{\psi\phi}\}.$$

More precisely, we find a pair (μ, ν) of functions of natural argument such that $\nu(n) = o(\mu(n))$, $n \to \infty$, and

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) = \mu(n) + O(\nu(n)), \quad n \to \infty.$$

Generally speaking, the K–N problem with respect to component ν is not uniquely solved. So finding the solution $(\mu, 0)$, that is computation of the exact value of $\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right)$, is the most desirable.

For the classes of 2π-periodical real-valued functions the K–N problem for generalized Zygmund sums was investigated in many works (see review in [14], and [12], [8]). With respect to holomorphic functions such researches held much less. The first case of solving a K–N problem for holomorphic functions should be considered the theorem 1 in [13], from which in our notation for $\psi_k = k^{-1}$, $\phi_k = k^{-s}$, $s \in \mathbb{N}$, follows asymptotic equality

$$\mathcal{Z}_{n,\psi}(H_\infty^{\psi\phi}; H_\infty) = n^{-1} + O(n^{-s-1}), \quad n \to \infty.$$

Actually, it was shown in [10], that the value O in this equation is equal to zero, i.e. the following equality holds

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) = n^{-1} \quad \forall\, n \in \mathbb{N},\ 1 \leq p \leq \infty.$$

We generalize these two relations (Corollary 2) for the case when $\psi_k = k^{-r}$, $\phi_k = k^{-s}$, $r, s \geq 0$ and $r + s > 0$, namely, show that

$$\mathcal{Z}_{n,r}\left(H_p^{\psi\phi}; H_p\right) = \begin{cases} n^{-r} + O\left(n^{-r-s}\right), & 0 \leq s < 1, \\ n^{-r}, & s \geq 1, \end{cases} \quad n \in \mathbb{N}. \tag{1}$$

The first ratio in (1) follows from Theorem 1 and Corollary 1 of this paper, which are talking about pointwise approximation of individual functions $H_p^{\psi\phi}$ inside the disk $\mathbb{D}$ and about the solution of K–N problem for a value $\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right)$ in some important cases. The second ratio follows from the Theorem 3, which talks about the exact value of $\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right)$.

It is important to pay attention to the following fact. For any complex sequences ψ and ϕ such that $\psi_1 = \phi_1 = 1$ and $|\psi_k| > 0$, $|\phi_k| > 0$, $k = 2, 3, \ldots$, holds an inequality

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) \geq \|f^* - Z_{n,\psi}(f^*)\|_p = |\psi_n|, \tag{2}$$

where $f^*(z) = z$. Moreover, as it follows from the main result in [2], the ratio $\|f - Z_{n,\psi}(f)\|_p = o(|\psi_n|), n \to \infty$, can not be performed for any function $f \in H_p$ other than constant. Thus the order of $O(|\psi_n|)$ is the maximum order of the smallness of value $\mathcal{Z}_{n,\psi}\left(H_p^{\psi}; H_p\right)$. In connection with this naturally arise the question under what conditions for the sequence ψ the order of smallness is achieved. We show (Theorem 2) that it is sufficient to require for the ψ be a moment sequence in the sense of Hausdorff moment problem and satisfies condition $\psi_k = O(\psi_{2k})$, $k \in \mathbb{N}$.

In Theorem 3 we give a description of all sequences ϕ such that for a given sequence ψ holds an equality

$$\mathcal{Z}_{n,\psi}(H_\infty^{\psi\phi}; H_\infty) = |\psi_n|, \tag{3}$$

i.e. when generalized Zygmund sums $Z_{n\psi}$ approach the class $H_\infty^{\psi\phi}$ with minimum possible error.

2. The main results

Theorem 1. *Let* $1 \leq p \leq \infty$,

$$\psi_k = \int_0^1 \rho^{k-1} d\lambda(\rho), \quad k = 1, 2, \ldots, \tag{4}$$

where λ *be real-valued a bounded nondecreasing function on* $[0, 1]$ *such that* $\int_0^1 d\lambda = 1$, *and* ϕ *be a sequence of complex numbers, such that for all natural* n *beginning with some number* n_0

$$K_{n,\phi}(z) := \frac{1}{2} + \operatorname{Re} \sum_{k=1}^{\infty} \frac{\phi_{k+n}}{\phi_n} z^k \geq 0 \quad \forall\ z \in \mathbb{D}. \tag{5}$$

Then for every function $f \in H_p^{\psi\phi}$ the following equality holds for any natural $n \geq n_0$:

$$f(z) - Z_{n,\psi}(f)(z) = \psi_n z D^{\psi}(f)(z) + \varepsilon_n(z,f) \quad \forall\ z \in \mathbb{D}, \tag{6}$$

where

$$\|\varepsilon_n(\rho\ \cdot, f)\|_p \leq \rho^n |\phi_n| \left(\psi_n + \psi_{[\frac{n+1}{2}]}\right)\ \forall\ n \geq n_0,\ \rho \in [0,1],$$

and $[\cdot]$ is the integer part of number.

Corollary 1. *Let the conditions of Theorem 1 be satisfied and let the condition (5) hold for any $n \in \mathbb{N}$, $\psi_n = O(\psi_{2n})$, $\phi_1 = 1$ and $\phi_n = o(1)$. Then*

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) = \psi_n + O\left(|\phi_n|\psi_n\right), \quad n \to \infty. \tag{7}$$

Relation (7) is a solution of the K–N problem in these cases.

Theorem 2. *Let $1 \leq p \leq \infty$ and ψ be a sequence such as in the Theorem 1. Then*

$$\psi_n \leq \mathcal{Z}_{n,\psi}\left(H_p^{\psi}; H_p\right) \leq \psi_{[\frac{n+1}{2}]} \quad \forall\ n \in \mathbb{N}. \tag{8}$$

In the next statement we describe the set of all sequences ϕ such that

$$\mathcal{Z}_{n,\psi}(H_\infty^{\psi\phi}; H_\infty) = |\psi_n|. \tag{9}$$

Theorem 3. *Suppose $n \in \mathbb{N}$, $\psi = \{\psi_k\}_{k=1}^\infty$ and $\phi = \{\phi_k\}_{k=1}^\infty$ are sequences of complex numbers such that $\psi_1 = \phi_1 = 1$ and $|\psi_k| > 0$, $|\phi_k| > 0$. Equality (9) holds true if and only if*

$$M_{n,\psi,\phi}(z) := \frac{1}{2} + \operatorname{Re}\left(\sum_{k=1}^{n-1} \phi_{k+1} z^k + \sum_{k=n}^{\infty} \frac{\psi_{k+1}\phi_{k+1}}{\psi_n} z^k\right) \geq 0 \quad \forall\ z \in \mathbb{D}. \tag{10}$$

If inequality (10) is true, then the following relation holds for all $p \in [1,\infty)$:

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) = |\psi_n|. \tag{11}$$

Denote by H_p^{r+s} the class $H_p^{\psi\phi}$ when $\psi_k = k^{-r}$ and $\phi_k = k^{-s}$ and let $\mathcal{Z}_{n,r} := \mathcal{Z}_{n,\psi}$. Note that in such a case

$$D^{\psi\phi}(f)(z) = \sum_{k=0}^{\infty} (k+1)^{r+s} \widehat{f}_{k+1} z^k.$$

Corollary 2. *Let $1 \leq p \leq \infty$, $r, s \geq 0$ and $r + s > 0$. Then*

$$\mathcal{Z}_{n,r}\left(H_p^{r+s}; H_p\right) = \begin{cases} n^{-r} + O\left(n^{-(r+s)}\right), & 0 \leq s < 1, \\ n^{-r}, & s \geq 1, \end{cases} \qquad n \in \mathbb{N}.$$

3. Appendix

In this section we will show that relations (6) and (7), generally speaking, can not be the corollary of Theorem 3. Also we will formulate a simple condition under which the relation (10) are holds.

Proposition 1. *The sequence $\phi = \mathbf{1} := \{1\}_{k=1}^{\infty}$ satisfies the condition (5) for all $n \in \mathbb{N}$, but doesn't satisfy the condition (10) simultaneously for all $n \in \mathbb{N}$ whatever be the sequence ψ except the case $\psi = \mathbf{1} = \{1\}_{k=1}^{\infty}$.*

Proof. Indeed, for any $n \in \mathbb{N}$

$$K_{n,\mathbf{1}}(z) = M_{n,\mathbf{1},\mathbf{1}}(z) = \frac{1}{2} + \operatorname{Re}\sum_{k=1}^{\infty} z^k = \frac{1}{2}\operatorname{Re}\frac{1+z}{1-z} = \frac{1}{2}\frac{1-|z|^2}{|1-z|^2} \geq 0 \quad \forall\, z \in \mathbb{D}. \tag{12}$$

Suppose that the condition (10) holds for all natural n. Take an arbitrary function $g \in UH_{\infty}$ and with fixing any $n \in \mathbb{N}$ construct the sequence of functions $\{g_N\}_{N=0}^{\infty}$ by the rule

$$g_0 = g, \quad g_N(z) = \frac{1}{\pi}\int_0^{2\pi} g_{N-1}(e^{it}) M_{n,\psi,\mathbf{1}}(ze^{-it})dt, \;\; N = 1, 2, \ldots.$$

Clear that as a result of (10) $g_N \in UH_{\infty}$, $N = 0, 1, 2, \ldots$.

On the other hand, by direct computation easily convinced that

$$g_N(z) = \sum_{k=0}^{n-1} \widehat{g}_k z^k + \sum_{k=n}^{\infty}\left(\frac{\psi_{k+1}}{\psi_n}\right)^N \widehat{g}_k z^k \quad \forall\, z \in \mathbb{D}. \tag{13}$$

In particular, putting $g(z) = z^m$, $m \geq n$, we obtain the inequality

$$\left|\frac{\psi_{m+1}}{\psi_n}\right|^N = \|g_N\|_{\infty} \leq 1 \quad \forall\, N \in \mathbb{Z}_+ \;\forall\, m \geq n.$$

Because of the arbitrariness n this relation implies that

$$1 \geq \left|\frac{\psi_{n+1}}{\psi_n}\right| \geq \left|\frac{\psi_{n+2}}{\psi_n}\right| \geq \ldots \quad \forall\, n \in \mathbb{N}.$$

For a given n equate to the number 1 can be achieved only in a finite number of the first row correspondences, or at all at once. We need to consider only the first of this two cases. Therefore without losing generality we consider that

$$1 \geq |\psi_2| = \ldots = |\psi_n| > |\psi_{n+1}| > \ldots. \tag{14}$$

Based on the expansion (13) and inequalities (14) we obtain the ratio

$$\left|\sum_{k=0}^{n-1} \widehat{g}_k z^k\right| \leq |g_N(z)| + \left|\sum_{k=n}^{\infty}\left(\frac{\psi_{k+1}}{\psi_n}\right)^N \widehat{g}_k z^k\right| \leq 1 + \left|\frac{\psi_{n+1}}{\psi_n}\right|^N \frac{|z|^n}{1-|z|} \quad \forall\, z \in \mathbb{D}.$$

Hence when $N \to \infty$ it follows that

$$G_n := \sup\left\{\left|\sum_{k=0}^{n-1} \widehat{g}_k\right| : g \in UH_\infty\right\} \leq 1.$$

But as it was shown by E. Landau (see, for example, [7, p. 442]),

$$G_n = 1 + \sum_{k=1}^{n-1}\left(\frac{(2k-1)!!}{(2k)!!}\right)^2 > 1, \quad n \geq 2.$$

We have the contradiction. Hence, our assumption is incorrect, which proves the proposition 1.

◀

Proposition 2. *Let $n \in \mathbb{N}$, $\psi = \{\psi_k\}_{k=1}^{\infty}$ be any sequence of positive numbers decreasing to zero as $k \to \infty$ and $\phi = \{\phi_k\}_{k=1}^{\infty}$ be a sequence of complex numbers such that $|\phi_k| > 0$. Then for the condition (10) is sufficient to require that*

$$P_{n,\phi}(z) := \frac{1}{2} + \mathrm{Re}\sum_{k=1}^{n-1} \phi_{k+1} z^k \geq 0 \quad \forall\ z \in \partial\mathbb{D}\ \forall\ n \in \mathbb{N}. \tag{15}$$

Proof. Applying to the second sum in (10) the Abel transformation for series (this is correctly because of $\psi_n \left|\sum_{j=0}^{n} \phi_{j+1} z^j\right| \to 0, n \to \infty\ \forall\ z \in \mathbb{D}$), we get

$$\sum_{k=n}^{\infty} \frac{\psi_{k+1}\phi_{k+1}}{\psi_n} z^k = -\frac{\psi_{n+1}}{\psi_n}\sum_{k=0}^{n-1} \phi_{k+1} z^k + \frac{1}{\psi_n}\sum_{k=n}^{\infty}(\psi_{k+1} - \psi_{k+2})\sum_{j=0}^{k} \phi_{j+1} z^j.$$

Substituting this formula into expression of the function $M_{n,\psi,\phi}$ and taking into account that according to the maximum modulus principle $P_{n,\phi}(z) \geq 0\ \forall\ z \in \mathbb{D}$, we obtain

$$M_{n,\psi,\phi}(z) = -\frac{1}{2} + \frac{1}{\psi_n}\mathrm{Re}\left(\sum_{k=n-1}^{\infty} (\psi_{k+1} - \psi_{k+2})\sum_{j=0}^{k} \phi_{j+1} z^j\right) =$$

$$= -\frac{1}{2} + \frac{1}{\psi_n}\sum_{k=n-1}^{\infty} (\psi_{k+1} - \psi_{k+2})\left(P_{n,\phi}(z) + \frac{1}{2}\right) \geq$$

$$\geq -\frac{1}{2} + \frac{1}{2\psi_n}\sum_{k=n-1}^{\infty} (\psi_{k+1} - \psi_{k+2}) = 0 \quad \forall\ z \in \mathbb{D}.$$

◀

Let us note that in the case when all numbers ϕ_k are real the condition (15) is equivalent to the following

$$P_{n,\phi}(e^{it}) = \frac{1}{2} + \sum_{k=1}^{n-1} \phi_{k+1} \cos kt \geq 0 \quad \forall\, t \in [0,\pi] \ \forall\, n \in \mathbb{N}.$$

Detailed review of nonnegative trigonometric polynomials can be found in [7, ch. 4]. Currently the most general sufficient conditions for a real-valued sequence ϕ, for which $P_{n,\phi}(e^{it}) \geq 0$, are given in [3].

4. Proof of the results

Proofs of the Theorems 1 and 2 based on the following statement.

Lemma. *Suppose $1 \leq p \leq \infty$ and ψ are sequences of complex numbers defined by formula (4), where λ are a complex-valued function of bounded variation on $[0,1]$ such that $\int_0^1 d\lambda = 1$. Then for any function $f \in H_p^{\psi\phi}$ in every point $z \in \mathbb{D}$ and almost every point $z \in \mathbb{T}$*

$$f(z) - Z_{n,\psi}(f)(z) =$$

$$= \phi_n z^n \int_0^1 \int_0^{2\pi} D^{\psi\phi}(f)(\rho e^{it})\rho^{n-1} e^{-i(n-1)t} K_{n,\phi}(\rho e^{it} z)\frac{dt}{\pi}\ d\lambda(\rho^2) + \tag{16}$$

$$+\psi_n \sum_{k=1}^{n-1} \left(1 - |z|^{2(n-k)} \frac{\overline{\phi}_{2n-k}}{\phi_k} e^{2i \arg \phi_n}\right) \frac{\widehat{f}_k}{\psi_k} z^k \quad \forall\, n \in \mathbb{N}.$$

Proof. Consider the inner integral in (16). Denote for convenience $g(z) := D^{\psi\phi}(f)(\rho z)$, $c_k = \rho^k \phi_{k+n}/\phi_n$ and using the well-known identity (see [6, p. 515]), for any $z \in \mathbb{D}$ and $\rho \in [0,1)$ we obtain

$$\frac{1}{\pi} \int_0^{2\pi} D^{\psi\phi}(f)(\rho e^{it}) e^{-i(n-1)t} K_{n,\phi}(\rho e^{it} z) dt =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} g(e^{it}) e^{-i(n-1)t} \left(1 + 2\,\mathrm{Re} \sum_{k=1}^{\infty} c_k z^k e^{-ikt}\right) dt =$$

$$= \sum_{k=0}^{n-2} \widehat{g}_k \overline{c}_{n-k-1} \overline{z}^{n-k-1} + \sum_{k=n-1}^{\infty} \widehat{g}_k c_{k-n+1} z^{k-n+1} =$$

$$= \sum_{k=0}^{n-2} \frac{\widehat{f}_{k+1}\rho^k}{\psi_{k+1}\phi_{k+1}} \frac{\overline{\phi}_{2n-k-1}\rho^{n-k-1}}{\overline{\phi}_n} \overline{z}^{n-k-1} + \sum_{k=n-1}^{\infty} \frac{\widehat{f}_{k+1}\rho^k}{\psi_{k+1}\phi_{k+1}} \frac{\phi_{k+1}}{\phi_n} \rho^{k-n+1} z^{k-n+1} =$$

$$= \frac{1}{(\rho z)^{n-1}\phi_n} \left(\sum_{k=0}^{n-2} \frac{\widehat{f}_{k+1}}{\psi_{k+1}} \frac{\overline{\phi}_{2n-k-1}}{\phi_{k+1}} \rho^{2(n-1)} |z|^{2(n-k-1)} z^k + \sum_{k=n-1}^{\infty} \frac{\widehat{f}_{k+1}}{\psi_{k+1}} \rho^{2k} z^k\right).$$

By integrating the last equality with respect to the measure $d\lambda(\rho^2)$, and then reordering the change of integration and summation, we obtain

$$\frac{\phi_n z^n}{\pi}\int_0^1\int_0^{2\pi} D^{\psi\phi}(f)(\rho e^{it})\rho^{n-1}e^{-i(n-1)t}K_{n,\phi}(\rho e^{it}z)dt\ d\lambda(\rho^2) =$$

$$= z\left(\sum_{k=0}^{n-2}\frac{\psi_n}{\psi_{k+1}}\widehat{f}_{k+1}\frac{\overline{\phi}_{2n-k-1}}{\phi_{k+1}}|z|^{2(n-k-1)}z^k + \sum_{k=n-1}^{\infty}\widehat{f}_{k+1}z^k\right) =$$

$$= f(z) - Z_{n,\psi}(f)(z) - \sum_{k=1}^{n-1}\frac{\psi_n}{\psi_k}\widehat{f}_k z^k + \sum_{k=1}^{n-1}\frac{\psi_n}{\psi_k}\widehat{f}_k\frac{\overline{\phi}_{2n-k}}{\phi_k}|z|^{2(n-k)}z^k,$$

which proves the equality (16).

◀

Proof of Theorem 1. Set $g(z) = zD^{\psi}(f)(z)$,

$$U_{n,\phi}(g)(z) := \sum_{k=0}^{n-1}\left(1 - |z|^{2(n-k)}\frac{\overline{\phi}_{2n-k}}{\phi_k}e^{2i\arg\phi_n}\right)\widehat{g}_k z^k,$$

and denote by $I_{n,\psi,\phi}(f)(z)$ the integral in (16). Then the formula (16) can be rewritten in the following form:

$$f(z) - Z_{n,\psi}(f)(z) = \psi_n g(z) + \phi_n z^n I_{n,\psi,\phi}(f)(z) + \psi_n\left(U_{n,\phi}(g)(z) - g(z)\right) \qquad (17)$$

Evaluate the second and third summands in a righthand side of (17). According to the condition (5) $\int_0^{2\pi} K_{n,\phi}(\rho e^{it}z)dt = \pi$, by Holder inequality we have

$$|I_{n,\psi,\phi}(f)(z)|^p \le \int_0^1\int_0^{2\pi}\left|D^{\psi}(f)(\rho e^{it})\right|^p \rho^{n-1}K_{n,\phi}(\rho e^{it}z)\frac{dt}{\pi}\ d\lambda(\rho^2)\times$$

$$\times\left(\int_0^1\int_0^{2\pi}\rho^{n-1}K_{n,\phi}(\rho e^{it}z)\frac{dt}{\pi}\ d\lambda(\rho^2)\right)^{p/q} =$$

$$= \psi^{p/q}_{[\frac{n-1}{2}]}\int_0^1\int_0^{2\pi}\left|D^{\psi}(f)(\rho e^{it})\right|^p \rho^{n-1}K_{n,\phi}(\rho e^{it}z)\frac{dt}{\pi}\ d\lambda(\rho^2), \quad \frac{1}{p}+\frac{1}{q} = 1.$$

Therefore

$$\|I_{n,\psi,\phi}(f)(\rho\cdot)\|_p \le \psi^{1/p}_{[\frac{n-1}{2}]}\psi^{1/q}_{[\frac{n-1}{2}]} = \psi_{[\frac{n+1}{2}]}. \qquad (18)$$

Since $g \in H_p^{\phi}$ and ϕ satisfies the condition (5), then by result from [11, theorem 2],

$$\|U_{n,\phi}(g)(\rho\cdot) - g(\rho\cdot)\|_p \le \rho^n|\phi_n|\|g(\rho\cdot)\|_p \le \rho^n|\phi_n|. \qquad (19)$$

Combining (18), (19) and equality (17) the result follows.

◀

Proof of Corollary 1. By Cauchy formula we have the equality

$$D^{\psi}(f)(z) = \frac{1}{\pi}\int_0^{2\pi} D^{\psi\phi}(f)(e^{i\theta})K_{1,\phi}(ze^{i\theta})d\theta \quad \forall\ z \in \mathbb{D},$$

where $D^{\psi\phi}(f)(e^{i\theta})$ means nontangential boundary values of function $D^{\psi\phi}$ on the circle $\mathbb{T} := \{z : |z| = 1\}$, due to the fact that $D^{\psi\phi} \in H_p$.

Hence by Minkowski's inequality we have

$$\|D^{\psi}(f)\|_p \le \|D^{\psi\phi}(f)\|_p \le 1 \quad \forall\ f \in H_p^{\psi\phi}.$$

Thus, from the relation (6) and the last inequality follows an upper bound

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) \le \psi_n + O(|\phi_n|\psi_n),$$

which together with a lower bound (2) proved the Corollary 1.

◀

Proof of Theorem 2. Put in (16) $\phi_k = 1$, $k = 1, 2, \ldots$. By Proposition 1 the condition (5) is satisfied. Thus estimate (18) takes place.

Therefore, according to (18)

$$\|f_\rho - Z_{n,\psi}(f_\rho)\|_{L_p} \le \rho^n \|I_{n,\psi,\phi}(f_\rho)\|_p + \psi_n \sum_{k=1}^{n-1}\left(1 - \rho^{2(n-k)}\right)\frac{|\widehat{f}_k|}{\psi_k}\rho^k \le$$

$$\le \psi_{[\frac{n+1}{2}]} + \psi_n \|D^{\psi}(f)\|_p \sum_{k=1}^{n-1}\left(1 - \rho^{2(n-k)}\right)\rho^k.$$

Taken in these correspondences the limit when $\rho \to 1-$ and taking into account that for any function $f \in H_p$ $\|f\|_p = \lim_{\rho\to 1-}\|f_\rho\|_{L_p}$ (see, for example, [5, p. 55]), we obtain

$$\|f - Z_{n,\psi}(f)\|_p \le \psi_{[\frac{n+1}{2}]},$$

that together with (2) proves Theorem 2.

◀

Proof theorem 3. Let $D^{\psi\phi}(f)(e^{it})$, as before, denote the nontangential boundary value in a point e^{it} of function $D^{\psi\phi}(f)$.

Applying the Cauchy formula, easy to show that for any function $f \in H_p^{\psi\phi}$, $1 \le p \le \infty$,

$$f(z) - Z_{n,\psi}(f)(z) = \frac{z\psi_n}{\pi}\int_0^{2\pi} D^{\psi\phi}(f)(e^{i(\theta+t)})M_{n,\psi,\phi}(\rho e^{it})dt \quad \forall\ z \in \mathbb{D}, z = \rho e^{i\theta}. \tag{20}$$

Based on this formula, taking into account correspondence (2), it is easy to verify that the condition (10) is sufficient. Also provided (10) applying to the evaluation of integral in the right part the Minkowski's inequality, we obtain the equality (11).

To prove the necessity of conditions (10), we proceed as follows.

From the formula (20) considering the invariance of class $H^{\psi\phi}_{\infty}$ with respect to rotation of argument ($f \in H^{\psi\phi} \Rightarrow f(e^{i\theta}\cdot) \in H^{\psi\phi}_{\infty}\ \forall\ \theta \in [0, 2\pi]$), and also by the principle of maximum modulus, for any $z \in \mathbb{D}$ we obtain an inequality

$$|\psi_n||z|\mathcal{M}_n(|z|) = \sup\left\{|f(z) - Z_{n,\psi}(f)(z)| : f \in H^{\psi\phi}_{\infty}\right\} \le |\psi_n|,$$

where

$$\mathcal{M}_n(\rho) := \sup\left\{\left|\frac{1}{\pi}\int_0^{2\pi} F(e^{it})M_{n,\psi,\phi}(\rho e^{it})dt\right| : F \in UH_{\infty}\right\}.$$

So $\mathcal{M}_n(\rho) \le 1/\rho\ \forall\ \rho \in [0, 1)$.

On the other hand, according to the relations of duality for holomorphic functions (see, for example, [5, p. 129]) holds the equality

$$\mathcal{M}_n(\rho) = \min\{\|2M_{n,\psi,\phi}(\rho\cdot) - g_n(\rho, \cdot)\|_1 : g_n(\rho, \cdot) \in H^0_1\}, \quad \rho \in [0, 1), \tag{21}$$

where minimum is achieved for a unique function $w \mapsto g^*_n(\rho, w)$, $w \in \mathbb{D}$, from the space $H^0_1 := \{f \in H_1 : f(0) = 0\}$.

Thus

$$1 = \frac{1}{2\pi}\int_0^{2\pi} 2M_{n,\psi,\phi}(\rho e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi}\left(2M_{n,\psi,\phi}(\rho e^{it}) - g^*_n(\rho, e^{it})\right)dt \le \mathcal{M}_n(\rho).$$

Therefore,

$$1 \le \mathcal{M}_n(\rho) \le \frac{1}{\rho} \quad \forall\ \rho \in (0, 1). \tag{22}$$

Now show that the function $\rho \mapsto \mathcal{M}_n(\rho)$ is not decreasing on $[0, 1)$.

Let $0 \le \rho_1 < \rho_2 < 1$. By the Poisson's formula applying to the function $z \mapsto 2M_{n,\psi,\phi}(\rho_2 z) - g^*_n(\rho_2, z)$, we obtain

$$2M_{n,\psi,\phi}(\rho_1 e^{it}) - g^*_n\left(\rho_2, \frac{\rho_1}{\rho_2}e^{it}\right) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(2M_{n,\psi,\phi}(\rho_2 e^{i\theta}) - g^*_n(\rho_2, e^{i\theta})\right)\frac{\rho_2^2 - \rho_1^2}{|\rho_2 - \rho_1 e^{i(t-\theta)}|^2}dt.$$

Hence

$$\mathcal{M}_n(\rho_1) \le \left\|2M_{n,\psi,\phi}(\rho_1\cdot) - g^*_n\left(\rho_2, \frac{\rho_1}{\rho_2}\,\cdot\right)\right\|_1 \le \mathcal{M}_n(\rho_2).$$

Therefore, $\mathcal{M}_n(\rho) \nearrow$. Combining this fact with the equation $\lim_{\rho\to 1-}\mathcal{M}_n(\rho) = 1$, which follows from (22), we see that $\mathcal{M}_n(\rho) = 1$ for any $\rho \in [0, 1)$. It follows that the value in a righthand side of equality (21) also equals to 1. According to the theorem 2 in [9] it is possible if and only if the condition (10).

◄

Proof of Corollary 2. Suffices to show that sequences $\psi = \{k^{-r}\}_{k=1}^{\infty}$ and $\phi = \{k^{-s}\}_{k=1}^{\infty}$ satisfy conditions of Corollary 1 and Theorem 3 under the proper conditions on the parameter s.

Indeed, since

$$\psi_k = k^{-r} = \frac{1}{\Gamma(r)} \int_0^1 \rho^{k-1} \left(\ln \frac{1}{\rho}\right)^{r-1} d\rho,$$

and moreover $\psi_k = 2^r \psi_{2k}$, then ψ satisfies the conditions of Corollary 1.

For the sequence ϕ we have

$$K_{n,\phi}(z) = \frac{1}{2} + \operatorname{Re} \sum_{k=1}^{\infty} \frac{n^s}{(k+n)^s} z^k = n^s \left(\frac{a_0(z)}{2} + \sum_{k=1}^{\infty} a_k(z) \cos kx \right), \tag{23}$$

where $a_k(z) = |z|^k (k+n)^{-s}$, $x = \arg z$.

Since for each $z \in \mathbb{D}$ the sequence $\{a_k(z)\}_{k=1}^{\infty}$ is convex, and clearly, $a_k(z) \downarrow 0$, according to the well-known statement (see, for example, [16, p. 183]) a sum of series in righthand side of (23) is a nonnegative.

So $K_{n,\phi}$ satisfies the condition (5) for all $s \geq 0$. Hence holds true (7) for all $r \geq 0$ and $s \geq 0$.

If $s \geq 1$, then

$$P_{n,\phi}(z) = \frac{b_0(z)}{2} + \sum_{k=1}^{n-1} \frac{b_k(z) \cos kx}{k+1},$$

where $b_k(z) = |z|^k (k+1)^{1-s}$ and $x = \arg z$.

Since for each $z \in \overline{\mathbb{D}}$ coefficients $b_k(z)$, $k = 1, 2, \ldots$, are nonnegative and not increasing, then by the theorem of Rogosinskii–Szego (see, for example, [7, p. 330]), $P_{n,\phi}$ satisfies the condition (15) for all $n \in \mathbb{N}$ and $z \in \mathbb{D}$. Therefore, according to Proposition 2 the condition (10) is satisfied. Hence the condition (11) is also satisfied.

◄

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Iryna Meremelia
Institute of Mathematics of NAS of Ukraine, Kiev, Ukraine
E-mail: iramеremelya@gmail.com

Viktor Savchuk
Institute of Mathematics of NAS of Ukraine, Kiev, Ukraine
E-mail: savchuk@imath.kiev.ua

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On some classes of loaded equations and their applications

Nakhushev A.M., Nakhusheva V.A.

Abstract. An algorithm for calculating the concentration distribution of absorbing molecules along the laser beam, when the absorbing layer is a medium with fractal geometry, is offered. The algorithm is based on a loaded partial differential equation of the second order that changes its type at a critical time moment, when the concentration of molecules in absorbing medium reaches its maximum.

Key Words and Phrases: Loaded partial differential equations, laser beam, Bouguer-Lambert-Beer law, Tricomi equation, Lavrent'ev-Bitsadze equation, generalized exponential function, nonlocal boundary condition

2000 Mathematics Subject Classifications: 35R11, 93A30

Consider the system of the following three equations:

$$\partial_{0x}^{\alpha}u(\xi,t)+\sigma(t)\bar{v}(t)u(x,t)=0,\ 0<x<r,\ \sigma(t)>0; \tag{1}$$

$$k\partial_{\tau t}^{\beta}v(x,\eta)=-\frac{\partial w_1}{\partial x},\ \ \tau=\text{const}\geq 0,\ \ k=\text{const}>0; \tag{2}$$

$$w_1=-\frac{\partial}{\partial x}(av+b)v,\ \ a=\text{const}\geq 0,\ \ b=\text{const}>0. \tag{3}$$

Here $0<\alpha=\text{const}\leq 1$, $0<\beta=\text{const}\leq 1$, and t denotes dimensionless time,

$$\partial_{0x}^{\alpha}u(\xi,t)=D_{0x}^{\alpha-1}\frac{\partial u(\xi,t)}{\partial\xi}=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^x(x-\xi)^{-\alpha}\frac{\partial u(\xi,t)}{\partial\xi}d\xi;$$

$$\partial_{\tau t}^{\beta}v(x,\eta)=\text{sign}(t-\tau)D_{\tau t}^{\beta-1}\frac{\partial v(x,\eta)}{\partial\eta}=\frac{1}{\Gamma(1-\beta)}\int\limits_{\tau}^{t}|t-\eta|^{-\beta}\frac{\partial v(x,\eta)}{\partial\eta}d\eta,$$

where $\Gamma(z)$ is the Euler gamma-function;

$$\bar{v}(t)=\frac{1}{r}\int\limits_0^r v(x,t)dx.$$

For $\alpha = 1$, equation (1) is a differential form of the Bouguer-Lambert-Beer law for plane waves propagating in an absorbing medium along the ray $x \geq 0$ when the attenuation (absorption) coefficient ω^α, the concentration of absorbing particles $v = v(x,t)$ in the layer $0 \leq x \leq r$ and absorption cross-section $\sigma(t)$ are related by

$$\omega^\alpha = \sigma(t)\bar{v}(t). \tag{4}$$

Equation (3) expresses the second law of Fick when diffusion coefficient depends on the concentration linearly.

Equation (2) may be interpreted as the continuity equation or as a fractal differential form of the mass conservation law, and it realizes the relationship between the concentration of absorbing particles with its density $w_1 = w_1(x,t)$ [1, p.26]. The order β of the time derivative with respect to t may depend on x.

By vitue of (4), equation (1) as may be written in the form follows:

$$\partial^\alpha_{0x} u(\xi,t) + \omega^\alpha u(x,t) = 0,\ 0 < x < r,\ t \geq 0. \tag{5}$$

Equation (5) is a loaded partial differential equation with partial derivative of order $\alpha \in]0,1]$ with respect to the spatial variable x.

The order α and the coefficient ω^α may be functions of time t. This equation generalizes the Bouguer-Lambert-Beer law for the intensity $u(x,t) \equiv I_\nu(x,t)$ of the radiation with given frequency ν at the point x and at time t when the absorber layer, $0 \leq x \leq r$, is the medium with fractal dimension that is less then or equal to α. It is the simplest case of an equation that is referred in [2, p. 242] as the generalized fractional oscillation equation.

Any solution $u = u(x,t)$ of (5) can be represented in the form

$$u(x,t) = u(0,t)\mathrm{Exp}_\alpha(-\omega x), \tag{6}$$

where

$$\mathrm{Exp}_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+\alpha k)}$$

is the generalized exponential function by terminology of V.A. Nakhusheva [3].

It follows from (6) that

$$u(r,t) = \varphi_0(t)\mathrm{Exp}_\alpha(-\omega r), \tag{7}$$

where $\varphi_0(t)$ is the intensity of radiation at the beginning of the route $x = 0$, and $\omega r = \tau_\nu$ is the optical depth (thickness) of the fractal layer $0 \leq x \leq r$.

From formula (7) with $\alpha = 1$, we the known representation of Bouguer law

$$u(r,t) = \varphi_0(t)\exp(-\omega r),$$

We assume that $u(x,t)$ satisfies the local boundary value condition

$$u(0,t) = \varphi_0(t),\ 0 \leq t \leq T, \tag{8}$$

and the function $v(x,t)$ satisfies non-local boundary condition (4).

Using (8), the spectral absorption

$$A_\alpha = \frac{\varphi_0(t) - u(r,t)}{\varphi_0(t)}$$

can be calculated by the formula $A_\alpha = 1 - \mathrm{Exp}_\alpha(-\omega r)$.

Due (2) and (3), the function $v(x,t)$ must be a solution of the equation.

$$k\partial^\beta_{\tau t} v(x,\eta) = \frac{\partial^2}{\partial x^2}[(av+b)v]. \tag{9}$$

We differentiate both sides of equation (9) with respect to the time variable t and, taking into account the equality

$$\frac{\partial}{\partial t}\partial^\beta_{\tau t} v(x,\eta) = \frac{\partial}{\partial t}\mathrm{sign}(t-\tau) D^{\beta-1}_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta} = D^\beta_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta},$$

we obtain the equation

$$kD^\beta_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta} = \frac{\partial^2}{\partial x^2}\left[(2av+b)\frac{\partial v}{\partial t}\right]. \tag{10}$$

The boundary value condition (4) gives us some justification to approximate equation (10) by the equation

$$kD^\beta_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta} = (2a\bar{v}+b)\frac{\partial^3 v(x,t)}{\partial x^2 \partial t}. \tag{11}$$

In equation (11) we introduce the new dependent variable

$$w(x,t) = \frac{\partial v(x,t)}{\partial t}. \tag{12}$$

Then, for $w = w(x,t)$, we obtain

$$D^\beta_{\tau t} w(x,\eta) = k_\alpha \frac{\partial^2 w(x,t)}{\partial x^2} \tag{13}$$

with coefficient

$$k_\alpha = \frac{2a\bar{v}+b}{k} = \frac{1}{k}\left[\frac{2a\omega^\alpha}{\sigma(t)} + b\right]. \tag{14}$$

Equation (11) can be approximately replaced by the equation

$$D^\beta_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta} = K(t)\frac{\partial^2 v(x,t)}{\partial x^2}, \tag{15}$$

where

$$K(t) = \frac{2a\bar{v}'(t)}{k} = \frac{2a}{k}\frac{d}{dt}\left[\frac{\omega^2}{\sigma(t)}\right]. \tag{16}$$

Let τ be a time moment when the average value of the concentration of molecules $\bar{v}(t)$ in absorbing medium $0 \le x \le r$ reaches its maximum, and let the function (16) can be represented in the form

$$K(t) = |t-\tau|^m \chi(t)\operatorname{sign}(\tau - t), \tag{17}$$

where $m = \text{const} \ge 0$, $\chi(t)$ is a continuous positive function defined on time interval $[0,T]$ with initial, $t = 0$, and estimated, $t = T$, time moments.

Condition (17) means that equation (15) for $\beta = 1$ is a partial differential equation of mixed type. At the model case, when $\chi(t) \equiv 1$, equation (15) takes the following form:

$$D^{\beta}_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta} = \operatorname{sign}(\tau - t)|t-\tau|^m \frac{\partial^2 v(x,t)}{\partial x^2}. \tag{18}$$

Equation (18) with $\beta = 1$ and $m = 1$ coincides with the Tricomi equation of hypersonic flow

$$(t-\tau)\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial^2 v(x,t)}{\partial t^2} = 0, \tag{19}$$

which is well known from the theory of gas dynamics, and, coincides with the Lavrent'ev-Bitsadze equation

$$\operatorname{sign}(t-\tau)\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial^2 v(x,t)}{\partial t^2} = 0, \tag{20}$$

when $\beta = 1$ and $m = 0$.

Using (12), we have

$$\bar{w}(t) = \bar{v}'(t) = \frac{d}{dt}\left[\frac{\omega^{\alpha}}{\sigma(t)}\right]. \tag{21}$$

Equation (13) may be approximated by the equation

$$k_{\alpha}\frac{\partial^2 w(x,t)}{\partial x^2} = D^{\beta}_{\tau t}\bar{w}(\eta), \tag{22}$$

where the coefficient k_α is uniquely defined by (14). Hence, we find

$$k_\alpha[w_x(x,t) - w_x(0,t)] = xD^{\beta}_{\tau t}\bar{w}(\eta),$$

$$k_\alpha[w(x,t) - w(r,t) - (x-r)w_x(0,t)] = \frac{1}{2}(x^2 - r^2)D^{\beta}_{\tau t}\bar{w}(\eta), \tag{23}$$

$$k_\alpha\left[\bar{w}(t) - w(r,t) + \frac{1}{2}rw_x(0,t)\right] = \frac{1}{3}r^2 D^{\beta}_{\tau t}\bar{w}(\eta), \tag{24}$$

where

$$w_x(x,t) = \frac{\partial w(x,t)}{\partial x}.$$

Consequently, the solution $w = w(x,t)$ of (22) may be determined uniquely, if we add to the nonlocal condition the local conditions on the edge of absorption layer $0 \le x \le r$

$$w_x(0,t) = \psi_1(t),\ w(r,t) = \psi_0(t),\ 0 \le t \le T, \tag{25}$$

where $\psi_1(t)$ and $\psi_0(t)$ are the given functions continuous on $[0, T]$.

It follows from (21), (23), (24) and (25) that

$$k_\alpha[w(x,t) - \psi_0(t) - (x-r)\psi_1(t)] = \frac{1}{2}(x^2 - r^2) D^\beta_{\tau t} \frac{d}{d\eta}\left[\frac{\omega^\alpha}{\sigma(\eta)}\right],$$

$$k_\alpha\left\{\frac{d}{dt}\left[\frac{\omega^\alpha}{\sigma(t)}\right] - \psi_0(t) + \frac{r}{2}\psi_1(t)\right\} = \frac{1}{3} r^2 D^\beta_{\tau t} \frac{d}{d\eta}\left[\frac{\omega^\alpha}{\sigma(\eta)}\right]. \tag{26}$$

The algorithm of calculation must involve the checkup of condition (26) for the input data (22).

Equation (18) can be approximated by the following equations:

$$D^\beta_{\tau t} \frac{\partial \bar{v}(\eta)}{\partial \eta} = \mathrm{sign}(\tau - t)|t-\tau|^m \frac{\partial^2 v(x,t)}{\partial x^2}, \tag{27}$$

$$D^\beta_{\tau t} \frac{\partial}{\partial \eta} \frac{1}{h_i} \det \begin{Vmatrix} v(x_i,\eta) & x_i - x \\ v(x_{i+1},\eta) & x_{i+1} - x \end{Vmatrix} +$$

$$+\mathrm{sign}(t-\tau)|t-\tau|^m \frac{\partial^2 v(x,t)}{\partial x^2} = 0, \tag{28}$$

where $x_i < x < x_{i+1}$, $i = 0, 1, ..., n$.

We use the method of reducing the Samarskii problem to the local boundary value problem, which is posed in the paper [4]. We introduce the new dependent variable in equation (27),

$$U(x,t) = \int_0^x v(\xi,t) d\xi. \tag{29}$$

Function (29) is a solution of partial differential equations of the first order

$$\frac{\partial U}{\partial x} = v(x,t),\ 0 \le x \le r, \tag{30}$$

and satisfies the local boundary value conditions

$$U(0,t) = 0,\ U(r,t) = r\bar{v}(t),\ 0 \le t \le T. \tag{31}$$

Therefore, equation (27) takes the following form:

$$D^\beta_{\tau t} \frac{\partial U(r,\eta)}{\partial \eta} = r\ \mathrm{sign}(\tau - t)|t-\tau|^m \frac{\partial^3 U}{\partial x^3}. \tag{32}$$

We add to condition (4) the boundary condition

$$v(0,t) = \psi_0(t),\ 0 \le t \le T, \tag{33}$$

where $\psi_0(t)$ is a given function continuous on $[0, T]$. Because of (30), from this condition it follows that

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \psi_0(t),\ 0 \leq t \leq T. \tag{34}$$

So, the problem is reduced to the following: find a solution $U = U(x,t)$ of (32) at any point x of the absorbing layer and at any time t from the initial $t = 0$ to the estimated $t = T$ time moments, which satisfies the boundary value conditions (31) and (34).

In the case of (28), the function $v = v(x,t)$ must satisfy the conditions (4) and (33).

Due to the fact that equations (15), (19) and (20) are loaded ones of mixed type, we can interpret the absorbing medium as a fractal input-output mixed system [5].

To study the boundary value problems for the equation (13), one can successfully use the Green function that is constructed in [6].

In conclusion, it should be mentioned that the work is based on the report that was made on in the International conference "Physics of Extreme States of Matter" [7].

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Adam M. Nakhushev
Institution of Russian Academy of Sciences Scientific Research Institute of Applied Mathematics and Automation of KBSC of RAS, Nalchik, Russia
E-mail: niipma@mail333.com

Viktoriya A. Nakhusheva
Institution of Russian Academy of Sciences Scientific Research Institute of Applied Mathematics and Automation of KBSC of RAS, Nalchik, Russia
E-mail: niipma@mail333.com

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Multipliers for Bochner (p, Y)- operator Bessel mapping in Banach spaces

Rahimi A.

Abstract. Multipliers have recently been introduced as operators for Bessel sequences and frames in Hilbert spaces. Also, it was extended for Banach frames, controlled frames, fusion frames, g-frames, Banach frames and p-frames. In this paper, we define the concept of multipliers for (p, Y)-operator Bessel sequences and we show some of its properties in view point of operator theory.

Key Words and Phrases: Bessel sequence, frame, multiplier operator, Bochner integral, (p, Y)-operator frame, Bochner (p, Y)-operator frame, reflexive Banach space
2000 Mathematics Subject Classifications: AMS Primary 42C50; Secondary 40A99,42C15.

1. Introduction

Frames for Hilbert spaces introduced by Duffin and Schaefer in 1952 and for Banach spaces discussed by Grochening, Han, Larson, Casazza and .... One of the most popular generalization of frames in Hilbert spaces is due to W. Sun under the name g-frames which generalizes pseudoframes, oblique frames, outer frames, bounded quasi-projectors, frame of subspaces. The concept of g-frames extended to Banach spaces under the name pg-frames. One of the extensions of g-frames is Von Neumann-Schatten frames that introduced by Sadeghi and Arefijamaal at [11]. The newer generalization is (p, Y)-operator frame that introduced by Huai-Xin Cao, Lan Li, Qing-Jiang Chen and Guo-Xing Ji [3].

Recently the concept of (p, Y)-operator frames and continuous frames combined to define Bochner (p, Y)-operator frames [6]. In this paper, we focus on this family and we define multipliers for this family of operators. Also multipliers of (p, Y)-operator frames investigate.

1.1. Multipliers of frames

In 1960, R. Schatten provided a detailed study of ideals of compact operators by using their singular decomposition. He investigated the operators of the form $\sum_k \lambda_k \varphi_k \otimes \psi_k$ where (ϕ_k) and (ψ_k) are orthonormal families. In [2], the orthonormal families were replaced with Bessel and frame sequences to define Bessel and frame multipliers.

Definition 1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, let $(\psi_k) \subseteq \mathcal{H}_1$ and $(\phi_k) \subseteq \mathcal{H}_2$ be Bessel sequences. Fix $m = (m_k) \in l^\infty$. The operator $\mathbf{M}_{m,(\phi_k),(\psi_k)} : \mathcal{H}_1 \to \mathcal{H}_2$ defined by*

$$\mathbf{M}_{m,(\phi_k),(\psi_k)}(f) = \sum_k m_k \langle f, \psi_k \rangle \phi_k,$$

is called the Bessel multiplier for the Bessel sequences (ψ_k) and (ϕ_k). The sequence m is called the symbol of $\mathbf{M}$.

Several basic properties of these operators were investigated in [2]. Mutipliers for Banach frames introduced in [7, 9].

2. (p, Y)- Operator Bessel Sequence

Let X and Y be Banach spaces with dual spaces X^* and Y^*, for $1 \le p < \infty$, put

$$\ell^p(Y) = \big\{ \{y_n\}_{n\in\mathbb{N}} : y_n \in Y, \|\{y_n\}_{n\in\mathbb{N}}\|_p = (\sum_{n\in\mathbb{N}} \|y_n\|^p)^{\frac{1}{p}} < \infty \big\},$$

and

$$\ell^\infty(Y) = \big\{ \{y_n\}_{n\in\mathbb{N}} : y_n \in Y, \|\{y_n\}_{n\in\mathbb{N}}\|_\infty = \sup_n \|y_n\| < \infty \big\}.$$

It is easy to show that $\ell^p(Y)$ is a Banach space with norm $\|.\|_p$. Let $\frac{1}{p} + \frac{1}{q} = 1$, for every $\{y_n^*\} \in \ell^q(Y^*)$ and every $\{y_n\} \in \ell^p(Y)$, the operator π_q defined by

$$\langle \{y_n\}, \pi\{y_n^*\} \rangle := \sum_{n\in\mathbb{N}} \langle y_n, y_n^* \rangle,$$

is an isometric linear isomorphism from $\ell^q(Y^*)$ onto $(\ell^p(Y))^*$.

Definition 2. *Let $1 \le p \le \infty$. A family $\mathcal{T} = \{T_i\}_{i\in\mathbb{N}}$ of bounded linear operators in $B(X, Y)$ is a (p, Y)- operator Bessel sequence for X, if there exists a positive constant B such that*

$$\|\{T_i x\}_{i\in\mathbb{N}}\|_p \le B\|x\| \quad \forall x \in X.$$

We denote by $B_X^p(Y)$ the set of all (p, Y)- operator Bessel sequences for X.
Let $\mathcal{T} = \{T_i\}_{i\in\mathbb{N}}$ be a (p, Y)- operator Bessel sequence for X, then the mapping $R_\mathcal{T} : X \to \ell^p(Y)$ defined by $R_\mathcal{T} x = \{T_i x\}_{i\in\mathbb{N}}$ is a linear bounded operator and $\|R_\mathcal{T}\|_{op} \le B$. Also the operator $S_\mathcal{T} : \ell^q(Y^*) \to X^*$ defined by $S_\mathcal{T}\{y_i^*\}_{i\in\mathbb{N}} = \sum_{i\in\mathbb{N}} T_i^* y_i^*$ is well-defined and bounded.

Definition 3. *Let $1 \le p \le \infty$. A family $\mathcal{T} = \{T_i\}_{i\in\mathbb{N}}$ of bounded linear operators in $B(X, Y)$ is a (p, Y)- operator frame X, if there exists positive constants $A \le B$ such that*

$$A\|x\| \le \|\{T_i x\}_{i\in\mathbb{N}}\|_p \le B\|x\| \quad \forall x \in X.$$

If $A = B$, then $\mathcal{T} = \{T_i\}_{i\in\mathbb{N}}$ is called tight (p, Y)- operator frame for X.

We denote by $F_X^p(Y)$ the set of all (p, Y)- operator frames for X.

The following proposition characterize $F_X^p(Y)$ in terms of the operators $R_{\mathcal{T}}$ and $R_{\mathcal{T}}^*$.

Proposition 1. *[3] Let $\mathcal{T} = \{T_i\}_{i\in\mathbb{N}}$ be a (p, Y)- operator Bessel sequence for X. Then*

1. *$\mathcal{T} = \{T_i\}_{i\in\mathbb{N}}$ is a (p, Y)- operator frame sequences for X if and only if $R_{\mathcal{T}}$ is bounded below if and only if $R_{\mathcal{T}}^*$ is surjective.*

2. *$\mathcal{T} = \{T_i\}_{i\in\mathbb{N}}$ is a (p, Y)- operator tight frame sequences for X if and only if $R_{\mathcal{T}}$ is a scaled isometry.*

3. Multipliers of (p, Y)- Operator Bessel

Multipliers for p-frames defined in [9] and generalized for Banach frames in [7, 8]. In this section, we define the concept of multiplier operators to (p, Y)- operator Bessel sequences.

Lemma 1. *Let X_1, X_2 and Y be reflexive Banach spaces and $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathcal{T} = \{T_n\}_{n\in\mathbb{N}}$ be a (p, Y)- operator Bessel sequences for X_1 and $\mathcal{Q} = \{Q_n\}_{n\in\mathbb{N}}$ be a (q, Y^*)- operator Bessel sequences for X_2^*. Let $m \in \ell^\infty$. The series $\sum_{i\in\mathbb{N}} m_i Q_i^* T_i x$ is unconditionally convergent for any $x \in X_1$ and the operator $\boldsymbol{M} := \sum_{i\in\mathbb{N}} m_i Q_i^* T_i x$ is well defined and $\|\boldsymbol{M}\|_{op} \leq \|m\|_\infty B_{\mathcal{Q}} B_{\mathcal{T}}$.*

Proof. For $n > m \geq 1$ and $x \in X_1$

$$\begin{aligned}
\|\sum_{i=m}^{n} m_i Q_i^* T_i x\| &= \sup_{f\in X_1^*, \|f\|=1} |\langle \sum_{i=m}^{n} m_i Q_i^* T_i x, f\rangle| \\
&= \sup_{f\in X_1^*, \|f\|=1} |\sum_{i=m}^{n} m_i \langle T_i x, Q_i f\rangle| \\
&\leq \|m\|_\infty \sup_{f\in X_1^*, \|f\|=1} \sum_{i=m}^{n} |\langle T_i x, Q_i f\rangle| \\
&\leq \|m\|_\infty \sup_{f\in X_1^*, \|f\|=1} (\sum_{i=m}^{n} \|T_i x\|^p)^{\frac{1}{p}} (\sum_{i=m}^{n} \|Q_i f\|^q)^{\frac{1}{q}} \\
&\leq \|m\|_\infty \sup_{f\in X_1^*, \|f\|=1} B_{\mathcal{Q}} B_{\mathcal{T}} \|x\| \|f\| \\
&= \|m\|_\infty B_{\mathcal{Q}} B_{\mathcal{T}} \|x\|.
\end{aligned}$$

Definition 4. *Let X_1, X_2 and Y be reflexive Banach spaces and $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathcal{T} = \{T_n\}_{n\in\mathbb{N}}$ be a (p, Y)- operator Bessel sequences for X_1 and $\mathcal{Q} = \{Q_n\}_{n\in\mathbb{N}}$ be a (q, Y^*)- operator Bessel sequences for X_2^*. Let $m \in \ell^\infty$. The operator*

$$\boldsymbol{M} = M_{m,\mathcal{Q},\mathcal{T}} : X_1 \to X_2,$$

defind by

$$M_{m,\mathcal{Q},\mathcal{T}}x = \sum_{i\in\mathbb{N}} m_i Q_i^* T_i x \quad \forall x \in X_1,$$

is called the $(\mathcal{T}, \mathcal{Q})$-multiplier operator with symbol m.

Consider the diagonal operator

$$D_m : \ell^p(Y) \to \ell^p(Y),$$

corresponding to sequence $m = \{m_i\}$, which is defined by

$$D_m\{z_i\}_{i\in\mathbb{N}} := \{m_i z_i\}_{i\in\mathbb{N}}, \quad \{z_i\}_{i\in\mathbb{N}} \in domD_m,$$

where

$$domD_m := \big\{\{z_i\}_{i\in\mathbb{N}} \in \ell^p(Y) : \{m_i z_i\}_{i\in\mathbb{N}} \in \ell^p(Y)\big\}.$$

Lemma 2. *If $m \in c_0$, the sequence converges to zero, then D_m is a compact operator.*

Proof. For a given $m \in c_0$ and $\varepsilon > 0$, there is a N such that $\|m - m^{(N)}\| \leq \varepsilon$ where $m^{(N)} = (m_1, m_2, ..., m_N, 0, 0, ...)$. For $\{z_i\}_{i\in\mathbb{N}} \in \ell^p(Y)$

$$\|(D_m - D_{m^{(N)}})\{z_i\}\| \leq \|m - m^{(N)}\|_\infty \|\{z_i\}\|_p,$$

so $\|D_m - D_{m^{(N)}}\|_\infty \leq \varepsilon$. The operator $D_{m^{(N)}}$ is finite rank operator. Therefore $D_{m^{(N)}}$ is converging to D_m in operator norm.

By a simple calculation, it is easy to show that

$$M_{m,\mathcal{Q},\mathcal{T}} = S_{\mathcal{Q}} D_m R_{\mathcal{T}}.$$

Theorem 1. *By above notations, if $m \in c_0$ then $M_{m,\mathcal{Q},\mathcal{T}}$ is a compact operator.*

4. Bochner (p, Y)- Operator Bessel Sequence

4.1. Bochner spaces

Let (Ω, Σ, μ) be a measure space, where μ is a positive measure. Recall that the function $F : \Omega \to X$ is Bochner measurable if there is a sequence $\{f_n\}$ of simple functions such that

$$\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\| = 0, a.e.[\mu].$$

The function $f : \Omega \to X$ is Bochner integrable if there is a sequence $\{f_n\}$ of simple integrable functions such that

$$\lim_{n\to\infty} \int_\Omega \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0,$$

and the integral $\int_E f(\omega)d\mu(\omega)$ is defined by

$$\int_E f(\omega)d\mu(\omega) := \lim_{n\to\infty} \int_E f_n(\omega)d\mu(\omega), \quad E \in \Sigma.$$

Definition 5. *If μ is a measure on (Ω, Σ) then X has the Radon-Nikodym property with respect to μ if for every countably additive vector measure γ on (Ω, Σ) with values in X which has bounded variation and is absolutely continuous with respect to μ, there is a Bochner integrable function $g : \Omega \to X$ such that*

$$\gamma(E) = \int_\Omega g(\omega) d\mu(\omega),$$

for every set $E \in \Sigma$.

A Banach space X has the Radon-Nikodym property if X has the Radon-Nikodym property with respect to every finite measure. It is known that reflexive spaces has Radon-Nikodym property.

Suppose X^* has the Radon-Nikodym property. For $0 \leq p \leq \infty$, the Bochner space $L^p(\mu, X)$ is defined to be the Banach space of (equivalence classes of) X -valued Bochner measurable functions f from Ω to X for which the norms

$$\|f\|_p := (\int_\Omega \|f(\omega)\|^p d\mu(\omega))^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty := esssup_{\omega \in \Omega} \|f(\omega)\|, \quad p = \infty.$$

For $1 \leq p < \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$, it has been shown that $L^q(\mu, X^*)$ is isometrically isomorphic with $(L^p(\mu, X))^*$ if and only if X^* has Radon-Nikodym property, [5].

4.2. Bochner operator frames

Continuous frames introduced by Ali, Antoine and Gazeau in [1] and some of its properties showed in [10] also (p, Y)-operator frames was introduced in [3]. In this section we combine continuous frames and (p, Y)-operator frames to define Bochner (p, Y)-operator frame in Banach spases.

Definition 6. *Let X and Y be Banach spaces, (Ω, Σ, μ) be a measure space, where μ is a positive measure. For $1 < p < \infty$, the mapping*

$$F : \Omega \to B(X, Y),$$

$$\omega \to F_\omega,$$

is Bochner (p, Y)-operator frame, if:

- *for any $x \in X$, the mapping $\omega \to F_\omega(x)$ is measurable;*
- *there exist A and B such that for any $x \in X$*

$$A\|x\| \leq (\int_\Omega \|F_\omega(x)\|^p d\mu)^{\frac{1}{p}} \leq B\|x\|.$$

If the right hand side of the above inequality holds, we call it Bochner (p, Y)-operator Bessel.

The set of all Bochner (p, Y)-operator Bessel mappings is denoted by $BB_X^p(Y)$ and Bochner (p, Y)- operator frames by $BF_X^p(Y)$. For $F \in BB_X^p(Y)$, we define the operator $R_F : X \to L^p(\mu, Y)$ by $x \to F_.(x)$ where

$$R_F(x) : \Omega \to Y,$$

$$R_F(x)(\omega) = F_\omega(x), \quad x \in X.$$

It follows from definition that for any $F \in BB_X^p(Y)$

$$\|R_F(x)\|_p = \|F_.(x)\|_p \leq B\|x\| \quad x \in X.$$

Like (p, Y)-operator frames (Proposition 1) we have:

Proposition 2. *For $F \in BB_X^p(Y)$,*

1. *$F \in BF_X^p(Y)$ if and only if R_F is bounded away from zero (bounded below) if and only if R_F^* is onto.*

2. *F is tight Bochner (p, Y)-operator frame if and only if $R_{\mathcal{F}}$ is a scaled isometry.*

5. Multipliers of Bochner (p, Y)- Operator Bessel

Multipliers of continuous frames was introduced by Balazs, Bayer and Rahimi in [4] and multipliers of p-Bessel sequences introduced in [9]. In this section we generalize it to Bochner (p, Y)-operator Bessel mappings.

Lemma 3. *Let X_1, X_2 and Y be reflexive Banach spaces, $\frac{1}{p} + \frac{1}{q} = 1$ and (Ω, Σ, μ) be a measure space, where μ is a positive measure. Let $m : \Omega \to \mathbb{C}$ be a measurable function, $F \in BF_{X_1}^p(Y)$ and $G \in BF_{X_2^*}^q(Y^*)$. If $m \in L^\infty(\Omega, \mu)$, then the integral*

$$\int_\Omega m(\omega)\langle F_\omega(x), G_\omega(g)\rangle d\mu(\omega),$$

is convergent for every $x \in X_1$ and $g \in X_2^*$.

Proof. The Cauchy-Schwarz's inequality shows that for such x and g, we have:

$$|\int_\Omega m(\omega)\langle F_\omega(x), G_\omega(g)\rangle d\mu(\omega)| \leq \|m\|_\infty \int_\Omega |\langle F_\omega(x), G_\omega(g)\rangle| d\mu(\omega) \leq \|m\|_\infty B_F B_G \|x\| \|g\|.$$

Definition 7. *Let X_1, X_2 and Y be reflexive Banach spaces, $\frac{1}{p}+\frac{1}{q}=1$ and (Ω, Σ, μ) be a measure space, where μ is a positive measure. Let $m : \Omega \to \mathbb{C}$ be a measurable function, $F \in BF^p_{X_1}(Y)$ and $G \in BF^q_{X_2^*}(Y^*)$. Let $m \in L^\infty(\Omega, \mu)$, the operator*

$$M_{m,F,G} : X_1 \to X_2,$$

weakly defined by

$$\langle M_{m,F,G}x, g\rangle = \int_\Omega m(\omega)\langle F_\omega(x), G_\omega(g)\rangle d\mu(\omega), \quad x \in X_1, g \in X_2^*,$$

is called Bochner (p, Y)-operator Bessel multiplier.

We use the following notation to be understood in weak sense as above:

$$M_{m,F,G}x := \int_\Omega m(\omega)G^*_\omega F_\omega(x)d\mu(\omega), \quad x \in X_1.$$

The proof of lemma (3) shows that $M_{m,F,G}$ is bounded and $\|M_{m,F,G}\| \leq \|m\|_\infty B_F B_G$. It is easy to prove that if $m(\omega) > 0$ a.e., then for every $F \in BB^p_X(Y)$ the multiplier $M_{m,F,F}$ is a positive operator.

By a simple calculation, it is easy to show for $m, m' \in L^\infty(\Omega, \mu)$ and Bochner (p, Y)-operator Bessel mappings F, F', G and G' we have:

$$M_{m,F,G} - M_{m',F,G} = M_{m-m',F,G};$$
$$M_{m,F,G} - M_{m,F',G} = M_{m,F-F',G};$$
$$M_{m,F,G} - M_{m,F,G'} = M_{m,F,G-G'}.$$

The above equalities result:

Proposition 3. *Let X_1, X_2 and Y be reflexive Banach spaces, $\frac{1}{p}+\frac{1}{q}=1$ and (Ω, Σ, μ) be a measure space, where μ is a positive measure. Let $m, m_n : \Omega \to \mathbb{C}$ be a measurable function, $F \in BF^p_{X_1}(Y)$ and $G \in BF^q_{X_2^*}(Y^*)$. Let $m, m_n \in L^\infty(\Omega, \mu)$ and $m_n \to m$ in L^∞ as $n \to \infty$, then $M_{m_n,F,G}$ converges to $M_{m,F,G}$ in the operator norm.*

Proof. The proof is obtained from the following inequality.

$$\|M_{m_n,F,G} - M_{m,F,G}\|_{op} = \sup_{x\in X_1, g\in X_2^*} |\langle M_{m-m_n,F,G}x, g\rangle| \leq \|m - m_n\|_\infty B_F B_G.$$

The Minkowski's inequality shows that for any $F, F' \in BF^p_{X_1}(Y)$, $F \pm F' \in BF^p_{X_1}(Y)$. So we can do perturbations on Bessel mappings.

Theorem 2. *Let X_1, X_2 and Y be reflexive Banach spaces, $\frac{1}{p}+\frac{1}{q}=1$ and (Ω, Σ, μ) be a measure space, where μ is a positive measure and $m \in L^\infty(\Omega, \mu)$. For $F \in BF^p_{X_1}(Y)$ and $G \in BF^q_{X_2^*}(Y^*)$,*

1. *$\{F^{(n)}\} \subseteq BF^p_{X_1}(Y)$, if $F^{(n)} \to F$ in a uniform strong sense then $M_{m,F^{(n)},G}$ converges to $M_{m,F,G}$.*
2. *$\{G^{(n)}\} \subseteq BF^q_{X_2^*}(Y^*)$, if $G^{(n)} \to G$ in a uniform strong sense then $M_{m,F,G^{(n)}}$ converges to $M_{m,F,G}$.*

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Asghar Rahimi
Department of Mathematics, University of Maragheh , P. O. Box 55181-83111, Maragheh, Iran.
E-mail: rahimi@maragheh.ac.ir

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On frame properties of degenerate system of exponents in Hardy classes

Sadigova S.R.

Abstract. Part of the classical system of exponents with degenerate coefficient is considered. Frame properties of this system in Hardy classes are studied in case when the coefficient may not satisfy the Muckenhoupt condition.

Key Words and Phrases: system of exponents, degeneration, frames, Hardy class, the Muckenhoupt condition

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1. Introduction

In [1], N.K.Bari raised the issue of the existence of normalized basis for L_2 which is not a Riesz basis. The first example was given by K.I.Babenko [2]. He proved that the degenerate system of exponents $\left\{ |x|^{\alpha} e^{int} \right\}_{n\in\mathbb{Z}}$ with $|\alpha| < \frac{1}{2}$ forms a basis for $L_2(-\pi,\pi)$ but is not a Riesz basis when $\alpha \neq 0$. This result has been generalized by V.F.Gaposhkin [3]. In [4], the condition on the weight ρ was found which makes the system $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ form a basis for the weight space $L_{2,\rho}(-\pi,\pi)$ with a norm $\|f\|_{2,\rho} = \left(\int_{-\pi}^{\pi} |f(t)|^2 \rho(t)\, dt\right)^{\frac{1}{2}}$. All the above-mentioned works consider the cases when the weight or the degenerate coefficient satisfies the Muckenhoupt condition (see, for example, [5]). Basis properties of the linear phase systems of exponents and sines in weighted Lebesgue spaces have been studied in [6-8], while those of the systems of exponents with degenerate coefficients have been studied in [9;10]. In case when the Muckenhoupt condition does not hold, then these systems have finite defects: some parts of them are complete and minimal, but they (i.e. these parts) do not form a basis. The question then arises: are these systems frames?

Note that when solving many problems in mathematical physics and mechanics by Fourier method (see, e.g. [11-13]), one has to deal with the systems of the form $\{\rho(t)\sin[(n+\alpha)t+\beta]\}_{n\in\mathbb{N}}$, where $\rho(t)$ is a degenerate coefficient, $\alpha,\beta\in\mathbb{R}$ are real parameters, and $\mathbb{N}$ is the set of natural numbers. Hence there comes a necessity to study the frame properties of these systems in various function spaces. This is directly related to the study of the same issue for a part of the system of exponents $\left\{\rho(t)\, e^{int}]\right\}_{n\geq 0}$

($\left\{\rho\left(t\right)e^{int}\right]\}_{n\leq m}$) in Hardy classes. Our work is dedicated to the study of frame properties of these systems in Hardy classes in the case when the weight $\rho(t)$ is given in the form of power function. Note that the similar issues concerning degenerate systems of sines, cosines and exponents in Lebesgue spaces have been studied in [14;15].

2. Needful information

We will use some concepts and facts from the theory of frames and the standard notation. $\exists$ will mean "there exist(s)"; $\mathbb{Z}_+ = \{0\}\bigcup\mathbb{N}$; $\mathbb{Z} = -\mathbb{N}\bigcup\mathbb{Z}_+$; δ_{nk} will be the Kronecker symbol, and $\overline{(\cdot)}$ will stand for conjugation.

Let X be some Banach space with a norm $\|\cdot\|_X$, and X^* denote its conjugate with the corresponding norm $\|\cdot\|_{X^*}$ By $L\left[M\right]$ we denote the linear span of the set $M\subset X$, and $\overline{M}$ will stand for the closure of M.

System $\{x_n\}_{n\in\mathbb{N}}\subset X$ *is said to be complete in* X *if* $\overline{L\left[\{x_n\}_{n\in\mathbb{N}}\right]} = X$. *It is called minimal in* X *if* $x_k\notin \overline{L\left[\{x_n\}_{n\neq k}\right]}$, $\forall k\in\mathbb{N}$.

System $\{x_n\}_{n\in\mathbb{N}}\subset X$ *is said to be uniformly minimal in* X *if* $\exists\delta>0$: $\inf\limits_{\forall u\in L\left[\{x_n\}_{n\neq k}\right]}\|x_k-u\|_X\geq\delta\left\|x_k\right\|_X$, $\forall k\in\mathbb{N}$.

System $\{x_n\}_{n\in\mathbb{N}}\subset X$ *is said to be a basis for* X *if* $\forall x\in X$, $\exists!\left\{\lambda_n\right\}_{n\in\mathbb{N}}\subset K$: $x=\sum_{n=1}^{\infty}\lambda_n x_n$.

If system $\{x_n\}_{n\in\mathbb{N}}\subset X$ *forms a basis for* X*, then it is uniformly minimal.*

We will also need some facts about frames.

Definition 1 *Let* X *be a Banach space and* $\mathcal{K}$ *be a Banach sequence space indexed by* $\mathbb{N}$. *Let* $\{f_k\}_{k\in\mathbb{N}}\subset X$, $\{g_k\}_{k\in\mathbb{N}}\subset X^*$. *Then* $\left(\{g_k\}_{k\in\mathbb{N}}\,,\ \{f_k\}_{k\in\mathbb{N}}\right)$ *is an atomic decomposition of* X *with respect to* $\mathcal{K}$ *if :*
(i) $\{g_k\left(f\right)\}_{k\in\mathbb{N}}\in\mathcal{K}$, $\forall f\in X$;
(ii) $\exists A,B>0$:$A\left\|f\right\|_X\leq\left\|\{g_k\left(f\right)\}_{k\in\mathbb{N}}\right\|_{\mathcal{K}}\leq B\left\|f\right\|_X$, $\forall f\in X$;
(iii) $f=\sum_{k=1}^{\infty}g_k\left(f\right)\,f_k$, $\forall f\in X$.

Definition 2. *Let* X *be a Banach space and* $\mathcal{K}$ *be a Banach sequence space indexed by* $\mathbb{N}$. *Let* $\{g_k\}_{k\in\mathbb{N}}\subset X^*$ *and* $S:\mathcal{K}\to X$ *be a bounded operator. Then* $\left(\{g_k\}_{k\in\mathbb{N}}\ ,\ S\right)$ *is a Banach frame for* X *with respect to* $\mathcal{K}$ *if :*
(i) $\{g_k\left(f\right)\}_{k\in\mathbb{N}}\in\mathcal{K}$, $\forall f\in X$;
(ii) $\exists A,B>0$:$A\left\|f\right\|_X\leq\left\|\{g_k\left(f\right)\}_{k\in\mathbb{N}}\right\|_{\mathcal{K}}\leq B\left\|f\right\|_X$, $\forall f\in X$;
(iii) $S\ \left[\{g_k\left(f\right)\}_{k\in\mathbb{N}}\right]=f$, $\forall f\in X$.

Proposition 1. *Let* X *be a Banach space and* $\mathcal{K}$ *a Banach sequence space indexed by* $\mathbb{N}$. *Assume that the canonical unit vectors* $\{\delta_k\}_{k\in\mathbb{N}}$ *constitute a basis for* $\mathcal{K}$ *and let* $\{g_k\}_{k\in\mathbb{N}}\subset X^*$ *and* $S:\mathcal{K}\to X$ *be a bounded operator. Then the following statements are equivalent:*

(i) $(\{g_k\}_{k\in\mathbb{N}},S)$ *is a Banach frame for* X *with respect to* $\mathcal{K}$.

(ii) $(\{g_k\}_{k\in\mathbb{N}},\{S\left(\delta_k\right)\}_{k\in\mathbb{N}})$ *is an atomic decomposition of* X *with respect to* $\mathcal{K}$.

From these statements we directly obtain that if any element in X can not be expanded with respect to the system $\{x_n\}_{n\in\mathbb{N}} \subset X$, then it doesn't form a frame for X.

More details about these and related facts can be found in [16-18].

We will also use the symbol " $\sim$ ". The expression $f \sim g$, $t \to a$, means that in sufficiently small neighborhood of the point $t = a$ there holds the inequality $0 < \delta \leq \left|\frac{f(t)}{g(t)}\right| \leq \delta^{-1} < +\infty$.

Let us recall the definition of the Hardy classes. By H_p^+ we denote the usual Hardy class of analytical functions inside the unit circle furnished with the norm

$$\|f\|_{H_p^+} = \sup_{0<r<1} \left(\int_{-\pi}^{\pi} \left| f\left(re^{it}\right)\right|^p dt \right)^{1/p}, \; p \geq 1.$$

Restriction of class H_p^+ to the unit circumference $\partial\omega$ will be denoted by L_p^+ . Spaces H_p^+ and L_p^+ are isomorphic and isometric.

3. Basicity

Consider the case when the system $E_+^0(\rho))$ forms a basis for H_p^+ with $E_+^{(k)}(\rho) \equiv \left\{\rho(t)\, e^{int}\right\}_{n\geq k}$. We will assume that the degenerate coefficient ρ is given in the form of power function

$$\rho(t) = \left(e^{it} - 1\right)^{\alpha_0} \prod_{k=1}^{r} \left(e^{it} - e^{it_k}\right)^{\alpha_k},$$

where $\{t_k\}_1^r \subset (-\pi, \pi] \setminus \{0\}$ are different points and $\{\alpha_k\}_0^r \subset \mathbb{R}$. By $\mathcal{M}_p$ we denote the class of weights $\nu(t)$ satisfying the Muckenhoupt condition (see e.g. [5])

$$\sup_{I\subset[-\pi,\pi]} \left(\frac{1}{|I|} \int_I \nu(t)\; dt \right) \left(\frac{1}{|I|} \int_I [\nu(t)]^{-\frac{1}{p-1}}\; dt \right)^{p-1} < +\infty,$$

where *sup* is taken over all intervals $I \subset [-\pi, \pi]$ and $|I|$ is the Lebesgue measure I. It is easy to see that $|\rho|^{\frac{1}{p}} \in \mathcal{M}_p$ if and only if the following inequalities are true

$$-\frac{1}{p} < \alpha_k < 1 - \frac{1}{p}, \;\; k = \overline{0, r}. \tag{1}$$

Consider the system $E_+^{(k)}(\rho) \equiv \left\{\rho(t)\, e^{int}\right\}_{n\geq k}$.

The following theorem is true.

Theorem 1. *Let the inequalities (1) be fulfilled. Then the system $E_+^{(0)}(\rho)$ forms a basis for L_p^+, $1 < p < +\infty$.*

4. Defect case

We will consider the case when $|\rho|^{\frac{1}{p}} \notin \mathcal{M}_p$. Let the following inequalities hold

$$1-\frac{1}{p} \leq \alpha_0 < 2-\frac{1}{p}\ ,\ \ -\frac{1}{p} < \alpha_k < 1-\frac{1}{p}\ ,\ \ k=\overline{1,r}. \tag{2}$$

Assume

$$H_p^+(0)=\{f\in H_p^+ : f(0)=0\}$$

Every functional $l \in \left(H_p^+\right)^*$ can be determined by $g \in L_q$ through the expression

$$l\left(f\right)=l_g\left(f\right)=\int_{-\pi}^{\pi}\left(g\left(e^{it}\right)+F\left(e^{it}\right)\right)f\left(e^{it}\right)dt\ ,\ \ \forall f\in H_p^+,$$

where $F \in H_q^+\left(0\right)$ is an arbitrary function. Consequently, zero functional is generated by zero function. Let us assume that the functional $l_g \in \left(L_p^+\right)^* = \left(H_p^+\right)^*$ cancels the system $E_+^{(0)}\left(\rho\right)$ out, i.e.

$$\int_{-\pi}^{\pi}\left(g\left(e^{it}\right)+F\left(e^{it}\right)\right)\rho\left(t\right)e^{int}dt=0\ ,\ \ \forall n\in\mathbb{Z}_+, \tag{3}$$

where $F \in H_q^+\left(0\right)$ is an arbitrary function. Take $\forall\beta \geq 0$:$(\alpha_0-\beta) \in \left(-\frac{1}{p}, 1-\frac{1}{p}\right)$ and assume

$$\rho\left(t\right)\equiv\tilde{\rho}\left(t\right)\left(e^{it}-1\right)^{\beta}.$$

Let $\tilde{g}\left(e^{it}\right) = g\left(e^{it}\right)\left(e^{it}-1\right)^{\beta}$ and $\tilde{F}\left(z\right) = F\left(z\right)\left(z-1\right)^{\beta}$. It is clear that $\tilde{g} \in L_q$ and $\tilde{F} \in H_q^+\left(0\right)$, as, $\beta \geq 0$ and $\tilde{F}\left(0\right)=0$. The relation (3) can be rewritten as follows

$$\int_{-\pi}^{\pi}\left(\tilde{g}\left(e^{it}\right)+\tilde{F}\left(e^{it}\right)\right)\tilde{\rho}\left(t\right)e^{int}dt=0\ ,\ \ \forall n\in\mathbb{Z}_+. \tag{4}$$

As $|\tilde{\rho}|^{\frac{1}{p}} \in \mathcal{M}_p$, it is clear that the system $E_+^{(0)}\left(\tilde{\rho}\right)$ forms a basis for L_p^+, and, moreover, is complete in L_p^+. Then from (4) it follows $\tilde{g}=0 \Rightarrow g=0$. As a result, we obtain that the system $E_+^{(0)}\left(\rho\right)$ is complete in L_p^+. In a similar way we prove that, with the conditions

$$\alpha_k > -\frac{1}{p},\ \ k=\overline{0,r}, \tag{5}$$

fulfilled, the system $E_+^{(0)}\left(\rho\right)$ is complete in L_p^+.

Let us show that if the inequalities (2) hold, then the system $E_+^{(1)}\left(\rho\right)$ is minimal in L_p^+. Consider the system

$$\left\{\overline{\rho^{-1}\left(t\right)}\left(e^{int}-1\right)\right\}_{n\geq 1}. \tag{6}$$

We have

$$\int_{-\pi}^{\pi} \rho(t)\, e^{int} \rho^{-1}(t) \left(e^{-imt} - 1\right) dt = \int_{-\pi}^{\pi} e^{i(n-m)t} dt - \int_{-\pi}^{\pi} e^{int} dt =$$
$$= 2\pi\delta_{nm}\,, \ \forall n, m \in \mathbb{N}.$$

The relations

$$e^{int} - 1 \sim t\,, \ \ t \to 0\,, \ \ e^{it} - e^{it_k} \sim t - t_k\,, \ t \to t_k,$$

imply

$$\left|\overline{\rho^{-1}(t)} \left(e^{int} - 1\right)\right|^q \sim |t|^{q(1-\alpha_0)} \prod_{k=1}^{r} |t - t_k|^{-\alpha_k q}\,,$$

on $(-\pi, \pi)$. Consequently

$$\left\|\overline{\rho^{-1}(t)} \left(e^{int} - 1\right)\right\|_q^q = \int_{-\pi}^{\pi} \left|\overline{\rho^{-1}(t)} \left(e^{int} - 1\right)\right|^q dt \sim$$
$$\int_{-\pi}^{\pi} |t|^{q(1-\alpha_0)} \prod_{k=1}^{r} |t - t_k|^{-\alpha_k q}\, dt.$$

Hence, from (2) we obtain that the system (6) belongs to L_q. As a result, the previous relations yield the minimality of system $E_+^{(1)}(\rho)$ in L_p^+. We can similarly prove that the system $E_+^{(1)}(\rho)$ is complete in L_p^+. So, under condition (2) the system $E_+^{(1)}(\rho)$ is complete and minimal in L_p^+. In this case the system $E_+^{(1)}(\rho)$ is not uniformly minimal in L_p^+, and, moreover, it doesn't form a basis for L_p^+. Consequently, the system $E_+^{(0)}(\rho)$ has a defect equal to 1. We can similarly prove that if the inequalities

$$k - \frac{1}{p} \leq \alpha_0 < k + \frac{1}{q}\,, \ -\frac{1}{p} < \alpha_k < \frac{1}{q}\,, \ k = \overline{1, r}, \tag{7}$$

are fulfilled, then the system $E_+^{(k)}(\rho)$ is complete and minimal in L_p^+, but it doesn't form a basis for L_p^+. Consequently, in this case the system $E_+^{(0)}(\rho)$ has a defect equal to (k). As a result, we get the validity of

Theorem 2. *Let the inequalities (7) hold. Then the system $E_+^{(0)}(\rho)$ has a defect equal to (k) in L_p^+. In addition, the system $E_+^{(k)}(\rho)$ is complete and minimal in L_p^+, but is not uniformly minimal in it, and, consequently, it doesn't form a basis for L_p^+.*

We also proved that, in case when the Muckenhoupt condition does not hold, any function from closure of the linear span of the system $E_+^{(0)}(\rho)$ can not be expanded with respect to this system, i.e. the following theorem is true.

Theorem 3. *Let the inequalities $-\frac{1}{p} < \alpha_i < \frac{1}{q}$, $i = \overline{1, r}$ hold. Then the system $E_+^{0}(\rho)$ forms a frame for L_p^+ if and only if $-\frac{1}{p} < \alpha_0 < \frac{1}{q}$.*

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Sadigova Sabina Rahib
Institute of Mathematics and Mechanics of NAS of Azerbaijan, B.Vahabzade 9, AZ 1141
E-mail: s_sadigova@mail.ru

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Necessary conditions of optimality in a problem of optimal control of moving sources for singular heat equation

Teymurov R.A.

Abstract. A problem of optimal control of processes described by a singular heat equation and systems of ordinary differential equations with moving sources is investigated in this paper. In spite of applied importance of problems with moving sources controls, they have not been studied enough so far [1-3],[7-8]. Sufficients conditions of Frechet differentiability of quality test and an expression for its gradient are obtained, necessary conditions of optimality in the form of point wise and integral maximum principles are established for an optimal control problem considered below.

Key Words and Phrases: moving sources, maximum principles, integral identity, reduced problem, necessary conditions of optimality

2000 Mathematics Subject Classifications: 35K20, 49K20, 49J20

1. Introduction

Practical examples of moving sources of influence are electronic, laser and ionic beams, an electric arch, the induction current raised by the moving inductor. The most widespread processes in which these sources are applied, processes of melting and metal refinement in metallurgy are; processes of heat treatment, welding and microprocessing in mechanical engineering and instrumentation; processes of manufacturing of semi-conductor and resistor elements in microelectronics; processes of activation, radiation and drying in biology, medicine, agriculture, etc. For the first time theoretical statement of problems optimal control of moving sources for systems with the distributed parameters was given in A.G.Butkovsky and L.M.Pustylnikovs works [2]. One of the main features of this systems is their nonlinearity concerning the control defining the law of movement of a sources. The problem of the moments becomes nonlinear. Thus, the method of the moments which is widely used for search of optimal control in linear systems with the distributed and concentrated parameters, becomes unsuitable for systems with moving sources. In this work the variation method to solve a problem of optimum control of moving sources for the heat conductivity processes described by totality of a parabolic type equation and ordinary differential equation with moving sources is considered. Considering that the received problem of optimum control wasnt studied earlier, for it questions of a correctness of the decision are investigated, uniqueness and existence theorems are proved, sufficient conditions of Frechet differentiability of criterion of quality are found and expression for its gradient is received. Necessary conditions of optimality in the form of point wise and integral maximum principles are established for the optimal control problems.

2. Problem statement

Let‘s consider a problem on minimization of the functional

$$J(\overline{\vartheta}) = \int_0^l [u(x,T) - y(x)]^2 dx + \alpha_1 \sum_{k=1}^{n} \int_0^T [p_k(t) - \tilde{p}_k(t)]^2 dt +$$

$$+\alpha_2 \sum_{m=1}^{r} \int_0^T [\vartheta_m(t) - \tilde{\vartheta}_m(t)]^2 dt, \tag{1}$$

on the set

$$V = \{\overline{\vartheta} = (p,\vartheta) : p = (p_1(t),...,p_n(t)) \in L_2^n(0,T), \vartheta = (\vartheta_1(t),...,\vartheta_r(t)) \in L_2^n(0,T),$$

$$0 \le p_i(t) \le A_i, 0 \le \vartheta_j(t) \le B_j, i = \overline{1,n}\ , j = \overline{1,r}\},$$

under conditions

$$u_t = a^2 u_{xx} + \sum_{k=1}^{n} p_k(t)\delta(x - s_k(t)), (x,t) \in \Omega = \{0 < x < l, 0 < t \le T\}, \tag{2}$$

$$u_x \mid_{x=0} = g_1(t),\ u_x \mid_{x=l} = g_2(t),\ 0 < t \le T, \tag{3}$$

$$u(x,o) = \varphi(x), 0 \le x \le l, \tag{4}$$

$$\dot{s}_k(t) = f_k(s(t),\vartheta(t),t),\ 0 < t \le T,\ s_k(o) = s_{k0}, k = \overline{1,n}, \tag{5}$$

where $s_{k0} \in [0,l]$,α_1,α_2 $\ge$0, $\alpha_1 + \alpha_2 > 0$, a, l, T, $A_i > 0$, $i = \overline{1,n}$,$B_j > 0, j = \overline{1,r}$ are the given numbers; $s_k(t) = s_k(t;\vartheta) \in C(0,T)$, $0 \le s_k(t) \le l, k = \overline{1,n}$ is a solution of problem (5) corresponding to the control $\vartheta = \vartheta(t) = (\vartheta_1(t),\vartheta_2(t),...,\vartheta_r(t)) \in L_2^r(0,T)$; the functions $f_k\,(s,\vartheta,t)\,,\ k = \overline{1,n}$, are continuous and have continuous derivatives with respect to s and ϑ for $(s,\vartheta,t) \in E^n \times E^r \times [0,T]$; $g_1(t), g_2(t) \in L_2(0,T)$, $\varphi(x) \in L_2(0,l)$, $y(x) \in L_2(0,l)$, $\delta\,(\cdot)$ is a Dirac function; $\omega = (\tilde{p}(t),\tilde{\vartheta}(t))$, $\tilde{p}(t) = (\tilde{p}_1(t),\tilde{p}_2(t),...,\tilde{p}_n(t)) \in L_2^n(0,T)$, $\tilde{\vartheta}(t) = (\tilde{\vartheta}_1(t),\tilde{\vartheta}_2(t),...,\tilde{\vartheta}_r(t)) \in L_2^r(0,T)$ are the given functions.

For the sake of brevity, we denote by $H = L_2^n(0,T) \times L_2^r(0,T)$ a Hilbert space of pairs $\bar{\vartheta} = (p(t),\vartheta(t))$ with scalar product $< \bar{\vartheta}^1, \bar{\vartheta}^2 >_H = \int_0^T [p^1(t)p^2(t) + \vartheta^1(t)\vartheta^2(t)]dt$, and the norm $\|\bar{\vartheta}\|_H = \sqrt{(< \bar{\vartheta}, \bar{\vartheta} >_H)} = \sqrt{(\|p\|_{L_2}^2 + \|\vartheta\|_{L_2}^2)}$.

3. Correctness of problem statement

Definition. A problem of finding the function $(u(x,t),s(t)) = (u(x,t;\overline{\vartheta}),s(t;\vartheta))$ satisfying the conditions (2) – (5) for the given control $\overline{\vartheta} \in V$ is said to be a reduced problem. Under the solution of reduced problem (2) – (5) corresponding to the control $\overline{\vartheta} = (p(t),\vartheta(t)) \in V$ we understand the function $(u(x,t),s(t))$ from $(V_2^{1,0}(\Omega),C[0,T])$, where the function $u=u(x,t)$ satisfies the integral identity

$$\int_0^l\int_0^T[-u\eta_t+a^2u_x\eta_x]dxdt = a^2\int_0^T[g_2(t)\eta(l,t)-g_1(t)\eta(0,t)]\,dt+\int_0^l\varphi(x)\eta(x,0)dx+$$

$$+\sum_{k=1}^n\int_0^T p_k(t)\eta(s_k(t),t)dt, \tag{6}$$

for $\forall\eta=\eta(x,t)\in W_2^{1,1}(\Omega)$ and $\eta(x,T)=0$, and the function $s_k(t)$ satisfies the integral equation

$$s_k(t)=\int_0^t f_k(s(\tau),\vartheta(\tau),\tau)d\tau+s_{k0}, 0\le t\le T, k=\overline{1,n}. \tag{7}$$

It follows from the results of the papers [5-6] that for each fixed $\overline{\vartheta}\in V$, the reduced problem (2) – (5) has a unique solution from $(V_2^{1,0}(\Omega),C[0,T])$. Let the conditions in the problem (1) – (5) be fulfilled. Then problem (1) – (5) has a unique solution [7] :

Theorem 1. *There exists a dense subset K of the space H, such that for each $\omega\in K$ and $\alpha_i>0(i=\overline{1,2})$ problem (1)-(5) has a unique solution.*

4. Differentiability of functional and necessary conditions of optimality

Let $\psi=\psi(x,t)$ be a solution from $V_2^{1,0}(\Omega)$ of the problem

$$\psi_t+a^2\psi_{xx}=0,\ \ (x,t)\in\Omega, \tag{8}$$

$$\psi_x|_{x=0}=\psi_x|_{x=\ell}=0,\ \ t\in[0,T), \tag{9}$$

$$\psi(x,T)=2[u(x,T)-y(x)],\ x\in[0,\ell], \tag{10}$$

conjugated to (1) – (5), where $u\,(x,T)$ is a solution of reduced problem (1) – (5) for $t=T$,and $q=q(t)$ is a solution of the conjugated problem

$$\dot q_k(t)=-\sum_{i=1}^n\frac{\partial f_i}{\partial s_k}q_i(t)+\psi_x(s_k(t),t)p_k(t),\ 0\le t<T,\ q_k(T)=0,k=\overline{1,n}. \tag{11}$$

from $C\,[0,T]$.

The function $\psi = \psi(x,t)$ satisfies the integral identity

$$\int_0^l \int_0^T [\psi \eta_{1t} + a^2 \psi_x \eta_{1x}] dxdt = 2 \int_0^l [u(x,T) - Y(x)] \eta_1(x,T) dx, \tag{12}$$

for $\forall \eta_1 = \eta_1(x,t) \in W_2^{1,1}(\Omega)$ and $\eta_1(x,0) = 0$, and the function $q_k(t)$ satisfies the integral identity

$$q_k(t) = \int_t^T \left[\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(\tau) - p_k(\tau) \psi_x(s_k(\tau), \tau) \right] d\tau, 0 \le t \le T, k = \overline{1,n}. \tag{13}$$

The conjugated problem (8) – (11) is a mixed problem for a linear parabolic equation. If in relations (8) – (11), instead of the variable t we take a new independent variable $\tau = T - t$, we get a boundary value problem of the same types as (2) – (5). Therefore, it follows from the facts established for problem (2) – (5) that for each given $\overline{\vartheta} = (p(t), \vartheta(t)) \in V$ problem (8) – (11) has a unique solution from $(V_2^{1,0}(\Omega), C[0,T])$.

Let $\Delta\overline{\vartheta} = (\Delta p, \Delta\vartheta) \in V$ be an increment of the control on the element $\overline{\vartheta} = (p, \vartheta) \in V$ such that $\overline{\vartheta} + \Delta\overline{\vartheta} \in V$. Denote $u \equiv u(x,t;\bar{\vartheta})$, $s_k \equiv s_k(t;\bar{\vartheta})$, $\Delta u(x,t) \equiv \equiv u(x,t;\bar{\vartheta} + \Delta\bar{\vartheta}) - u(x,t,\bar{\vartheta})$, $\Delta s_k \equiv \Delta s_k(t) = s_k(t;\bar{\vartheta} + \Delta\overline{\vartheta}) - s_k(t;\overline{\vartheta})$, $p_k = p_k(t)$, $\Delta p_k = \Delta p_k(t)$.

It follows from (2)-(5) that $\Delta u(x,t)$ is a generalized solution of the boundary value problem

$$\Delta u_t = a^2 \Delta u_{xx} + \sum_{k=1}^n [(p_k + \Delta p_k)\delta(x - (s_k + \Delta s_k)) - p_k \delta(x - s_k)], \ \ (x,t) \in \Omega, \tag{14}$$

$$\Delta u_x|_{x=0} = \Delta u_x|_{x=l} = 0, \ \ t \in [0,T], \tag{15}$$

$$\Delta u|_{t=0} = 0, \ \ x \in [0,l], \tag{16}$$

and functions $\Delta s_k(t), \ k = \overline{1,n}$, are the solutions of the Cauchy problem

$$\Delta \dot{s}_k(t) = f_k(s + \Delta s, \vartheta + \Delta\vartheta, t) - f_k(s, \vartheta, t), \quad \Delta s_k(0) = 0, \ k = \overline{1,n}. \tag{17}$$

It follows from (6) that the function $\Delta u(x,t)$ satisfies the integral identity

$$\int_0^l \int_0^T [-\Delta u \eta_t + a^2 \Delta u_x \eta_x] dxdt = \sum_{k=1}^n \int_0^T [(p_k(t) + \Delta p_k)\eta(s_k(t) + \Delta s_k, t) -$$

$$-p_k(t)\eta(s_k(t), t)]dt, \tag{18}$$

for $\forall \eta = \eta(x,t) \in W_2^{1,1}(\Omega), \ \eta(x,T) = 0$.

The function

$$H(t,s,\psi,q,\overline{\vartheta}) = -\left\{\sum_{k=1}^{n}[-f_k(s(t),\vartheta(t),t)q_k(t) + \psi(s_k(t),t)p_k(t) + \right.$$
$$\left. + \ \alpha_1\left(p_k(t)-\tilde{p}_k(t)\right)^2\right] + \alpha_2\sum_{m=1}^{r}\left(\vartheta_m(t)-\tilde{\vartheta}_m(t)\right)^2\Bigg\}, \tag{19}$$

is said to be Hamilton-Pontryagin function of problem (1)-(5). Now, we state sufficient conditions of Frechet differentiability of functional (1) and find an expression for its gradient.

Theorem 2. *Let the function $f(s,\vartheta,t)$ be continuous in totality of all its arguments together with all its partial derivatives with respect to variables s,ϑ for $(s,\vartheta,t)\in E^n\times E^r\times[0,T]$ and the following conditions*

$$\begin{array}{l} |f_k(s+\Delta s,\vartheta+\Delta\vartheta,t)-f_k(s,\vartheta,t)|\le L(|\Delta s|+|\Delta\vartheta|),\\ |f_{ks}(s+\Delta s,\vartheta+\Delta\vartheta,t)-f_{ks}(s,\vartheta,t)|\le L(|\Delta s|+|\Delta\vartheta|),\\ |f_{k\vartheta}(s+\Delta s,\vartheta+\Delta\vartheta,t)-f_{k\vartheta}(s,\vartheta,t)|\le L(|\Delta s|+|\Delta\vartheta|),\ k=\overline{1,n}, \end{array}$$

be fulfilled for all $(s+\Delta s,\vartheta+\Delta\vartheta,t)$, $\quad(s,\vartheta,t)\in E^n\times E^r\times[0,T]$, where $L=const\ge 0$.

Then the functional (1) is Frechet differentiable and the expression

$$J'(\bar{\vartheta}) = -\frac{\partial H}{\partial\bar{\vartheta}} \equiv \left(-\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial \vartheta}\right), \tag{20}$$

where

$$\begin{array}{l} \frac{\partial H}{\partial p} = \left(\frac{\partial H}{\partial p_1},\frac{\partial H}{\partial p_2},...,\frac{\partial H}{\partial p_n}\right), \quad \frac{\partial H}{\partial \vartheta} = \left(\frac{\partial H}{\partial \vartheta_1},\frac{\partial H}{\partial \vartheta_2},...,\frac{\partial H}{\partial \vartheta_r}\right),\\ \frac{\partial H}{\partial p_k} = -\psi(s_k(t),t) - 2\alpha_1\left(p_k(t)-\tilde{p}_k(t)\right), k=\overline{1,n},\\ \frac{\partial H}{\partial \vartheta_m} = \sum_{k=1}^{n}\frac{\partial f_k(s(t),\vartheta(t),t)}{\partial\vartheta_m}q_k(t) - 2\alpha_2\left(\vartheta_m(t)-\tilde{\vartheta}_m(t)\right), m=\overline{1,r}, \end{array}$$

is valid for its gradient.

Proof. Consider the increment of the functional

$$\begin{array}{l} \Delta J \equiv J(\bar{\vartheta}+\Delta\bar{\vartheta}) - J(\bar{\vartheta}) = 2\int_0^l[u(x,T)-y(x)]\Delta u(x,T)dx + \int_0^l|\Delta u(x,T)|^2dx + \\ + \sum_{k=1}^{n}\left\{2\alpha_1\int_0^T[p_k(t)-\tilde{p}_k(t)]\Delta p_k(t)dt + \alpha_1\int_0^T|\Delta p_k|^2dt\right\} + \\ + \sum_{m=1}^{r}\left\{2\alpha_2\int_0^T\left[\vartheta_m(t)-\tilde{\vartheta}_m(t)\right]\cdot\Delta\vartheta_m(t)dt + \alpha_2\int_0^T|\Delta\vartheta_m|^2dt\right\} \end{array} \tag{21}$$

where $\bar{\vartheta}=(p,\vartheta)\in V$, $\bar{\vartheta}+\Delta\bar{\vartheta}\in V$, $\Delta u(x,T)\equiv u(x,T;\bar{\vartheta}+\Delta\bar{\vartheta})-u(x,T;\bar{\vartheta})$, $u\equiv u(x,T;\bar{\vartheta})$. Prove that

$$\begin{array}{l} 2\int_0^l[u(x,T)-y(x)]\Delta u(x,T)dx = \sum_{k=1}^{n}\left\{\int_0^T\psi(s_k(t),t)\ \Delta p_k(t)dt + \right.\\ \left. + \sum_{m=1}^{r}\int_0^T\frac{\partial f_k(s(t),\vartheta(t),t)}{\partial\vartheta_m}q_k(t)\Delta\vartheta_m(t)dt\right\} + R_1, \end{array} \tag{22}$$

where $R_1 = \sum_{k=1}^{n} \int_0^T \psi_x(s_k(t),t)\Delta p_k(t)\Delta s_k(t)dt$.

If we set $\eta_1 = \Delta u\,(x,t)$, in (12), $\eta = \upsilon\,(x,t)$ in (18), and then subtract the obtained relations, we have

$$\int_0^l \int_0^T [\psi \Delta u_t + a^2 \psi_x \Delta u_x] dxdt = 2\int_0^l [u(x,T) - y(x)]\Delta u(x,T)dx,$$

$$\int_0^l \int_0^T [-\Delta u \psi_t + a^2 \psi_x \Delta u_x] dxdt = \sum_{k=1}^{n} \int_0^T [(p_k + \Delta p_k)\psi(s_k + \Delta s_k, t) - p_k \psi(s_k,t)]dt,$$

$$\int_0^l 2[u(x,T) - y(x)]\Delta u(x,T)dx = \sum_{k=1}^{n} \int_0^T [(p_k + \Delta p_k)\psi(s_k + \Delta s_k, t) - p_k \psi(s_k,t)]dt. \quad (23)$$

It follows from (17) that the function $\Delta s_k(t)$ satisfies the integral identity

$$\int_0^T \left[\Delta s_k(t)\dot{\theta}_k(t) + \Delta f_k(s(t),\vartheta(t),t)\theta_k(t)\right] dt = 0, \quad (24)$$

for $\forall \theta_k(t) \in C[0,T]$,$\theta_k(T) = 0, k = \overline{1,n}$.

It follows from (11) that the function $q_k(t)$ satisfies the integral identity

$$\int_0^T \left[\dot{\theta}_{1k}(t)q_k(t) - \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial s_k} q_i(t) - \psi_x(s_k(t),t)p_k(t)\right)\theta_{1k}(t)\right] dt = 0, \quad (25)$$

for $\forall \theta_{1k}(t) \in C[0,T]$,$\theta_{1k}(0) = 0, k = \overline{1,n}$.

In the same way, if we set $\theta_{1k} = \Delta s_k$ in (25), $\theta_k = q_k$ in (24) and then sum the obtained relations, we have

$$\int_0^T \left[\Delta \dot{s}_k(t)q_k(t) - \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial s_k} q_i(t) - \psi_x(s_k(t),t)p_k(t)\right)\Delta s_k(t)\right] dt = 0,$$

$$\int_0^T [\dot{q}_k(t)\Delta s_k(t) + \Delta f_k(s(t),\vartheta(t),t)q_k(t)]\, dt = 0,$$

$$[\Delta s_k(t)q_k(t)]\,\Big|_{t=0}^{t=T} = \int_0^T \left[\left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial s_k} q_i(t) - \psi_x(s_k(t),t)p_k(t)\right)\Delta s_k(t) - \Delta f_k q_k(t)\right] dt.$$

Considering conditions of the theorem, we can represent the function $\Delta f_k = \Delta f_k(s(t),\vartheta(t),t)$ in the form

$$\Delta f_k = \sum_{i=1}^{n} \frac{\partial f_k}{\partial s_i}\Delta s_i + \sum_{m=1}^{r} \frac{\partial f_k}{\partial \vartheta_m}\Delta \vartheta_m + R_2,$$

where $R_2 = o\left(\sqrt{\|\Delta s\|^2_{L_2(0,T)} + \|\Delta\vartheta\|^2_{L_2(0,T)}}\right)$ as $\|\Delta s\|_{L_2(0,T)} \to 0$, and $\|\Delta\vartheta\|_{L_2(0,T)} \to 0$.

Then, from the last equality we have:

$$[\Delta s_k(t) q_k(t)]\,\big|_{t=0}^{t=T} = \int_0^T \left[\left(\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(t) - \psi_x(s_k(t),t) p_k(t)\right)\Delta s_k(t) - \right.$$

$$\left. - \sum_{m=1}^r \frac{\partial f_k}{\partial \vartheta_m} \Delta\vartheta_m(t) q_k(t) - \sum_{i=1}^n \frac{\partial f_k}{\partial s_i} \Delta s_i(t) q_k(t)\right] dt + R_2.$$

From(17) and (11) we get

$$\int_0^T \psi_x(s_k(t),t) p_k(t) \Delta s_k(t) dt = -\sum_{m=1}^r \int_0^T \frac{\partial f_k}{\partial \vartheta_m} \Delta\vartheta_m(t) q_k(t) dt -$$

$$-\sum_{i=1}^n \int_0^t \left[\frac{\partial f_k}{\partial s_i} q_k(t) \Delta s_i(t) - \frac{\partial f_i}{\partial s_k} q_i(t) \Delta s_k(t)\right] dt + R_2. \tag{26}$$

It is clear that under the assumptions made above, the expansion

$\psi(s_k + \Delta s_k, t) = \psi(s_k, t) + \psi_x(s_k(t),t)\Delta s_k + o\left(\|\Delta s\|_{C[0,T]}\right)$ as $\|\Delta s\|_{C[0,T]} \to 0$,
is valid.

Considering this formula in (23), we get

$$2\int_0^l [u(x,T) - y(x)]\Delta u(x,T) dx = \sum_{k=1}^n \int_0^T [\psi_x(s_k(t),t) p_k(t) \Delta s_k(t) +$$
$$+\psi(s_k(t),t)\Delta p_k(t) + \psi_x(s_k(t),t)\Delta p_k(t)\Delta s_k(t)]\, dt + o\left(\|\Delta s\|_{C[0,T]}\right).$$

In view of the fact that

$$\sum_{k=1}^n \sum_{i=1}^n \left[\frac{\partial f_k}{\partial s_i} q_k(t) \Delta s_i(t) - \frac{\partial f_i}{\partial s_k} q_i(t) \Delta s_k(t)\right] = 0,$$

from the last equality and the relation (26) we get

$$2\int_0^l [u(x,T) - y(x)]\Delta u(x,T) dx = \sum_{k=1}^n \int_0^T \left[-\sum_{m=1}^r \frac{\partial f_k}{\partial \vartheta_m} q_k(t) \Delta\vartheta_m(t) + \right.$$
$$\left. + \psi(s_k,t)\Delta p_k\right] dt + R_3, \tag{27}$$

where

$$R_3 = \sum_{k=1}^n \int_0^T [\psi_x(s_k(t),t)\Delta p_k(t) \Delta s_k(t)]\, dt + R_2 + o(\|\Delta s\|_{C[0,T]}).$$

It is proved in (13) that the estimation

$$\|\Delta u(x,T)\|_{L_2(0,l)} \leq c_1 \left\|\Delta\bar{\vartheta}\right\|_H \tag{28}$$

holds for the function $\Delta u(x,t)$ and in 6.3 of [6] it is established that the estimation

$$\|\Delta s\|_{L_2(0,T)} \leq c_2 \left\|\Delta\vartheta\right\|_{L_2(0,T)}, \tag{29}$$

where $c_1 \geq 0, c_2 \geq 0$ are some constants, follows for the solution of problem (17).

Taking into account the estimation (29) in the expressions for R_1 and R_3, we get $R_3 = o\left(\left\|\Delta\overline{\vartheta}\right\|_H\right)$.

Considering these estimations in (21) and (22), we have:
$\Delta J(\bar{\vartheta}) = \sum_{k=1}^{n}\left(J_1(k) + \sum_{m=1}^{r} J_2(k,m)\right) + o\left(\left\|\Delta\bar{\vartheta}\right\|_H\right)$, as $\left\|\Delta\bar{\vartheta}\right\|_H \to 0$,
where

$$J_1(k) = \int_0^T \left[\psi(s_k(t),t) + 2\alpha_1\left(p_k(t) - \tilde{p}_k(t)\right)\right]\Delta p_k(t)dt,$$

$$J_2(k,m) = \int_0^T \left[-\frac{\partial f_k(s(t),\vartheta(t),t)}{\partial\vartheta_m}q_k(t) + 2\alpha_2\left(\vartheta_m(t) - \tilde{\vartheta}_m(t)\right)\right]\Delta q_m(t)dt.$$

Hence, allowing for expression of Hamilton-Pontryagin function, we get

$$\Delta J(\bar{\vartheta}) = \left(-\frac{\partial H}{\partial\bar{\vartheta}}, \Delta\bar{\vartheta}\right)_H + o\left(\left\|\Delta\bar{\vartheta}\right\|_H\right) \text{ as } \left\|\Delta\bar{\vartheta}\right\|_H \to 0,$$

that shows Frechet differentiability of functional (1) and validity of formula (20). The Theorem 2 is proved.

Now, let's get necessary conditions, i.e. control optimality conditions for problem (1)-(5).

Theorem 3. *Let all the conditions of theorem 1 be fulfilled and* $(u^*(x,t), s^*(t))$, $(\psi^*(x,t), q^*(t))$ *be solutions of problems* (2)-(5) *and* (8)-(11), *respectively, for* $\overline{\vartheta} = \overline{\vartheta}^* \in V$. *Then for optimality of the control* $\overline{\vartheta}^* = (p^*(t), \vartheta^*(t))$ *the condition*

$$H(t,s^*,\psi^*,q^*,\bar{\vartheta}^*) = \max_{\bar{\vartheta}\in V} H(t,s^*,\psi^*,q^*,\bar{\vartheta}), \tag{30}$$

should be fulfilled for $\forall(x,t) \in \Omega$.

Proof. Assume that $\bar{\vartheta}^* = (p^*, \vartheta^*)$ is an optimal control. Assume the contrary, i.e. assume there are a control $\tilde{\vartheta} = \bar{\vartheta}^* + h \cdot \Delta\bar{\vartheta} \in V$and the number $\beta > 0$ such that

$$H(t,s^*,\psi^*,q^*,\tilde{\vartheta}) - H(t,s^*,\psi^*,q^*,\bar{\vartheta}^*) \geq \beta > 0, \tag{31}$$

where $h > 0$ is some number, $\tilde{\vartheta} = (\tilde{p},\tilde{\vartheta}) \equiv (p^* + h\Delta p, \vartheta^* + h\Delta\vartheta)$.

If in (31) we take into account formula (20), we get

$$h\sum_{i=1}^{2}\left(\frac{\partial J(\breve{\vartheta}_i)}{\partial\bar{\vartheta}}, \Delta\bar{\vartheta}\right)_H \leq -\beta < 0,$$

where $\breve{\vartheta}_1 = (h\theta_0\Delta p, \tilde{\vartheta})$, $\breve{\vartheta}_2 = (p^*, h\theta_1\Delta\vartheta)$, $\theta_i \in (0,1)$, $i = \overline{0,1}$ are some numbers. Hence, from the finite increment formula we have

$$J(\tilde{\vartheta}) - J(\bar{\vartheta}^*) = h\sum_{i=1}^{2}\left(\frac{\partial J(\widehat{\vartheta}_i)}{\partial\bar{\vartheta}}, \Delta\bar{\vartheta}\right)_H \leq -\beta + h\cdot 0(\left\|\Delta\bar{\vartheta}\right\|_H), \tag{32}$$

where $\widehat{\vartheta}_1 = (h\gamma_0\Delta p, \tilde{\vartheta})$, $\widehat{\vartheta}_2 = (p^*, h\gamma_1\Delta\vartheta)$, $\gamma_i \in (0,1)$, $i = \overline{0,1}$ are some numbers.

Let $0<h1<h$ be such a number that $-\beta + h_1 o\left(\left\|\Delta\overline{\vartheta}\right\|_H\right) < 0$. Assume $\tilde{\tilde{\vartheta}} = (\tilde{\tilde{p}}, \tilde{\tilde{\vartheta}}) = (p^* + h_1\Delta p, \vartheta^* + h_1\Delta\vartheta)$. Reasoning as in the getting of inequality (32), we have

$$J(\tilde{\tilde{\vartheta}}) - J(\bar{\vartheta}^*) \leq -\beta + h_1 o\left(\left\|\Delta\bar{\vartheta}\right\|_H\right) < 0.$$

This contradicts to the optimality of the control $\overline{\vartheta}^*$. Hence we get the validity of relation (30). The Theorem 3 is proved.

Using formula (20) and taking into account the expression of Hamilton-Pontryagin function, by the known theorem ([6], p.28) we get the validity of the following theorem:

Theorem 4. *Let the conditions of Theorem 1 be fulfilled. Then for the optimality of the control* $\overline{\vartheta}^* = (p^*(t), \vartheta^*(t)) \in V$ *, the condition*

$$\begin{aligned}&\textstyle\int_0^T \sum_{k=1}^n \left[(\psi^*(s_k^*(t),t) + 2\alpha_1(p_k^*(t) - \tilde{p}_k(t)), p_k(t) - p_k^*(t)) + \right.\\ &\left. + \textstyle\sum_{m=1}^r \left(-\frac{\partial f_k(s^*(t),\vartheta^*(t),t)}{\partial\vartheta_m} q_k^*(t) + 2\alpha_2(\vartheta_m^*(t) - \tilde{\vartheta}_m(t)), \vartheta_m(t) - \vartheta_m^*(t)\right)\right] dt \geq 0,\end{aligned}$$

should be fulfilled for $\forall\bar{\vartheta} = (p(t), \vartheta(t)) \in V$. *Here* $\psi^*(s_k^*(t),t), q_k^*(t)$ *are the solutions of problems* (8)-(10) *and* (11), *respectively, for* $\overline{\vartheta} = \overline{\vartheta}^*(p^*(t), \vartheta^*(t))$.

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Rafig A. Teymurov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
*E-mail:*rafiqt@mail.ru

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Spin-Top Model of galaxies and the universe

Veziroglu T. N.

Abstract. The most accepted model of the universe is the Big Bang Model or Theory. However, it does not explain many features of the universe and galaxies, which are being observed. A new model called "Spin-Top Model" is proposed to replace the Big Bang Model, which explains the observed features of the universe and galaxies better than the Big Bang Model.

1. Introduction

Presently, the most accepted model of the universe is the Big Bang Model. However, this model cannot explain many features of the universe, which have been observed. According to the Big Bang Model, there should be a homogenous distribution of galaxies; the observations show it to be otherwise. The Big Bang Model cannot explain why there is a string of galaxies as observed by the Hubble telescope. According to the Big Bang Model, galaxies should have spread out from each other evenly, and hence there could be no collisions between them; observations show that there are collisions between galaxies. Big Bang Theory cannot explain why heavenly bodies are rotating around themselves, around each other, and why galaxies are rotating. Big Bang Theory cannot explain why more galaxies have more left hand spins than the right hand spins. Big Bang Theory cannot explain why the universe is not spherical but flat.

A new model of galaxies and the universe, the Spin-Top Model has been developed to explain the features explained by the Big Bang Model, as well as the features of galaxies and the universe, which cannot be explained by the Big Bang Model.

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2. Spin-Top Model of Galaxies

Figure 1 is a sketch of the Spin-Top Model of Galaxies. It consists of a galaxy seed (which we shall call g-seed) spinning with a left hand spin* in the form of an oblate ellipsoid with very high densities, pressures and temperatures, surrounded by hot gases. It is well known that spinning fluids, under the influence of their own gravitational forces,

*Note that if one looks at the galaxy from below, the spin looks to be a right hand spin!

take the forms of oblate ellipsoids. Out of the g-seed, galaxy spirals extend out which are also spinning with a left hand spin. The spirals contain solar systems, asteroids, heavenly bodies, particulate matter and clouds of gas, including hydrogen and helium. The outer boundary of the galaxy is also in the shape of an oblate ellipsoid.

During the original creation of the galaxy, g-seed explodes by thermonuclear reactions spilling out various nuclear particles and subparticles, which quickly join to form mostly hydrogen atoms, as well as some helium atoms. Nuclear particles, and hydrogen and helium atoms will have velocities imparted by the spinning g-seed and the explosion. These velocities will have radial components, angular components, and relatively smaller axial components. The axial components will be upward if the particle in question comes out of the upper part (i.e., above the galaxy plane) of the g-seed, and will be downward if the particle in question comes from the lower part (i.e., below the galaxy plane) of the g-seed. These spinning gases, which are moving outwardly, will start to coalesce because of the gravitational forces, forming galaxy spirals, just like rotating humid air coalesces into hurricane spirals. Within the spirals, as a result of gravitational effects, the hydrogen gas will coalesce to form stars (as well as hydrogen/helium planets), which in turn will generate all the elements to form “rocky” planets, moons and the star systems (or the solar systems), as well as asteroids and other heavenly bodies. It should be noted that, because of the angular velocity effects, as one moves out in radial directions, the spins (or rotations) of the heavenly bodies and systems will be in the opposite directions to that of the galaxy.

Eventually, the outward motion of the heavenly bodies within the galaxy spirals (i.e., star systems, other heavenly bodies, asteroids and gas clouds) will slow down because of the gravitational forces pulling them towards the galaxy center, and the radial velocity component

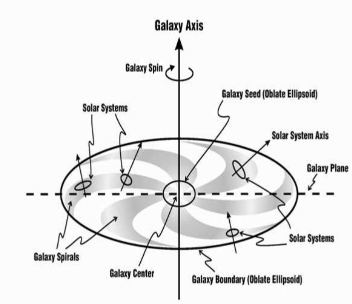

Fig. 1 - Spintop Model of Galaxy

will reverse itself together with the axial velocity components, and they will start moving towards the galaxy center. Note that at this time the outermost envelope of the galaxy will have been defined, and it will be an oblate ellipsoid because of the velocities imparted to them at the time of the g-seed explosion. Then, the heavenly systems and bodies will start their travels towards the center of the galaxy, and begin reforming the g-seed. When about half or more of the matter in spirals reaches the galaxy center, the g-seed will consist of space debris (made up of stars, planets, other heavenly bodies and gases) on the outside surface. As one moves to the center of the g-seed, densities, pressures and temperatures will rise, and as a result molecules will be crushed into atoms, and atoms will be crushed into nuclei. When the pressures and temperatures in the center of the g-seed reaches the critical values, a new explosion will ensue. Note that some of the heavenly systems and bodies will still be within the galaxy boundaries, and in the process of returning towards the galaxy center.

As a result of the new explosion of the g-seed, again galaxy spirals, star systems, other heavenly bodies, and gas clouds will be formed. The newly created heavenly matter will be moving away from the galaxy center, while the heavenly matter produced as a result of the previous explosion will be returning towards the galaxy center. These will cause some collisions between the outgoing heavenly bodies and systems, and the incoming heavenly bodies and systems. Such collisions will result in some astray heavenly bodies. The supernova explosions regularly observed are very likely be the g-seed explosions, rejuvenating the galaxies.

The above described galaxy activities will be repeated for eons to come.

3. Spin-Top Model of Universe

Figure 2 is a sketch of the Spin-Top Model of the Universe. It consists of a universe seed (which we shall call u-seed) spinning with a right hand spin[†] in the form of an oblate ellipsoid with extremely high densities, pressures and temperatures, surrounded by hot gases. Out of the u-seed, universe spirals extend out which are also spinning with a right hand spin. The universe spirals contain galaxies, novas, supernovas, heavenly bodies, particulate matter and clouds of gas, including hydrogen and helium. The outer boundary of the universe is also in the shape of an oblate ellipsoid.

During the original creation of the universe, the u-seed explodes by thermonucleon reactions (which is expected to be many orders of magnitude greater than thermonuclear reactions) spilling out various nuclear particles and subparticles, which quickly join to form mostly hydrogen atoms, as well as some helium atoms. Nuclear particles, and hydrogen and helium atoms will have velocities imparted by the spinning u-seed and the explosion. These velocities will have radial components, angular components, and relatively smaller axial components. The axial components will be upward if the particle in question comes out of the upper part (i.e., above the universe plane) of the u-seed, and will be downward if the particle in question comes from the lower part (i.e., below the universe plane) of the u-seed. These spinning gases, which are moving outwardly, will start to coalesce because of the gravitational forces, forming universe spirals. Within the universe spirals, as a result of gravitational effects, the hydrogen gas will coalesce to form galaxies. It should be noted that, because of the angular velocity effects, the spins (or rotations) of the galaxies will be in opposite directions to that of the universe. As described in the

[†]Note that if one looks at the universe from below, spin looks to be a left hand spin.

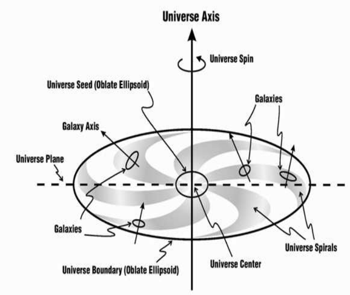

Fig. 2 - Spintop Model of the Universe

last section, within the galaxies stars and hydrogen/helium planets will form. In turn, stars will generate all the elements to form "rocky" planets, moons and the star systems (or the solar systems), as well as asteroids and other heavenly bodies.

Eventually, the outward motions of heavenly bodies within the universe spirals (i.e., galaxies, all the heavenly bodies and gas clouds) will slow down because of the gravitational forces pulling them towards the universe center, and the radial velocity components will reverse itself together with the axial velocity components, and they will start moving towards the universe center. Note that at this time the outermost envelope of the universe will have been defined, and it will be an oblate ellipsoid because of the velocities imparted to them at the time of the u-speed explosion. Then, the galaxies will start their travels towards the center of the universe, and begin reforming the u-seed. When about half or more of the matter in universe spirals reaches the universe center, the u-seed will consist of space debris (made up of stars, planets, other heavenly bodies and gases) on the outside surface. As one moves to the center of the u-seed, densities, pressures and temperatures will rise, and as a result molecules will be crushed into atoms, atoms will be crushed into nuclei, and nuclei will be crushed into nucleons. When the pressures and temperatures in the center of the u-seed reaches the critical values, a new explosion will ensue. Note that some of the galaxies will still be within the universe boundaries, and in the process of returning towards the universe center.

As a result of the new explosion of the u-seed, again universe spirals, galaxies, star systems, other heavenly bodies, and gas clouds will be formed. The newly created heavenly matter will be moving away from the universe center, while the heavenly matter produced as a result of the previous explosion will be returning towards the universe center. These will cause some collisions between the outgoing galaxies and the incoming galaxies. Such collisions in some cases will result in the birth of bigger galaxies by the combination of the colliding galaxies, or dismemberment of one or both of the colliding galaxies. The universe has many such examples.

The above described universe activities will be repeated for eons to come.

4. The Next Step

The next step will be for the theoretical physicists to incorporate the Spin-Top Model (i.e., conservation relationships of mass, momentum and energy for the spinning u-seed) into the equations describing the workings of the universe.

5. The Big Question

Of course, the big question is how a Spinning Universe Seed had been created in the first place. Answer to this might come from the **Uber-Humans** to be created as a result of natural selection by the mutations in the human genetic code over millions of years.

6. Conclusion

A new model for galaxies, i.e., the spin-top model of galaxies, and a new model for the universe, i.e., the spin-top model of the universe, have been developed, which are in better agreement with the observed features of the galaxies and the universe than the big bang model.

T. Nejat Veziroglu
Emeritus Professor, University of Miami President, International Association for Hydrogen Energy 5794 SW 40 St. #303, Miami, FL 33155, USA
E-mail: veziroglu@iahe.org

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Markov type integral inequality for Pseudo-integrals

Daraby B.

Abstract. In this paper, generalizations of the Markov type integral inequalities for pseudo-integrals are proved. There are considered two cases of the real semiring with pseudo-operations: One, when pseudo-operations are defined by monotone and continuous function g (then the pseudo-integrals reduces g-integral), and the second with a semiring $([a, b], max, \odot)$, where the pseudo-multiplication $\odot$ is generated.

Key Words and Phrases: Non-additive measure, Chebyshev type inequality, Pseudo-addition, Pseudo-multiplication, Pseudo-integral

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1. Introduction

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, \infty]$ endowed with pseudo-addition $\bigoplus$ and with pseudo-multiplication $\odot$ (see [13, 17, 20]). Based on this structure there where developed the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform and etc. Pseudo-analysis would be an interesting topic to generalize an inequality from the framework of the classical analysis to that of some integrals which contain the classical analysis as special cases [1, 2, 4, 5, 6, 8, 15, 18, 19, 20].

The well-known Markov inequality is a part of the classical mathematical analysis. The following inequality is a classical Markov type inequality [9]:

$$\mu\{x \in A : f(x) \geq c\} \leq \frac{1}{c} \int_A f d\mu,$$

where f is a non-negative integrable function and $c > 0$. A. Flores-Franulič et al. have proved Markov type inequalities for fuzzy integrals in [7].

In this paper, we generalize their works for pseudo-integrals. In special case, if in Markov type inequalities for pseudo-integrals we put $\oplus = max$ and $\odot = min$, then we get the Markov type inequality for Sugeno integrals [3].

The paper is organized as follows: Section 2 and 3 contain some of preliminaries, such as pseudo-operations and pseudo-analysis as well as integrals. In Section 4, We have proved generalizations of the Markov type inequality for pseudo-integrals. Finally, a conclusion is given in Section 5.

2. Preliminaries

In this section, we are going to review some well-known definitions of pseudo-operations. We refer to [10, 11, 12, 13, 14, 17].

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Definition 2.1. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, nondecreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.2. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$ is a semiring (see [10, 14]). In this paper we consider semirings with the continuous operations those that are discussed in [2, 13, 16].In this paper we consider semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) Suppose that $x \oplus y = sup(x, y), \odot$ is arbitrary and is not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $sup(x, y) = y$.

(b) Suppose that $x \oplus y = inf(x, y), \odot$ is arbitrary and is not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = b$ and the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $inf(x, y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo operations are given with $x \oplus y = g^{-1}(g(x) + g(x))$ and $x \odot y = g^{-1}(g(x)g(x))$.
If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = 1$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = 1$. If the generator g is increasing (respectively decreasing), then the operation $\oplus$ induces the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \leq g(y)$.

Case III: Both operations are idempotent. We have

(a) Suppose that $x \oplus y = sup(x, y), x \odot y = inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation sup induces the usual order ($x \preceq y$ if and only if $sup(x, y) = y$).

(b) Suppose that $x \oplus y = inf(x, y), x \odot y = sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces an order opposite to the usual order ($x \preceq y$ if and only if $inf(x, y) = y$).

Let X be a non-empty set. Let $\mathbb{A}$ be a σ-algebra of subsets of a set X.

Definition 2.3. A set function $m : \mathbb{A} \to [a, b]_+$ (or semiclosed interval) is a $\oplus$-measure if there hold:

(i) $m(\phi) = \mathbf{0}$ (if $\oplus$ is not idempotent);

(ii) m is $\sigma - \oplus -$(decomposable) measure, i.e.

$$m\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \bigoplus_{i=1}^{\infty} m(A_i)$$

holds for any sequence $A_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathbb{A}$.

We suppose that $([a,b],\oplus)$ and $([a,b],\odot)$ are complete lattice ordered semigroups. Further, suppose that $[a,b]$ is endowed with a metric d compatible with sup and inf, i.e. $lim_{n\to\infty} sup x_n = x$ and $lim_{n\to\infty} inf x_n = x$, imply $lim_{n\to\infty} d(x_n, x) = 0$, and which satisfies at least one of the following conditions:

(a) $d(x \oplus y, x' \oplus y') \leq d(x, x') + d(y, y')$,

(b) $d(x \oplus y, x' \oplus y') \leq max\{d(x, x'), d(y, y')\}$.

Both conditions (a) and (b) imply:

$d(x_n, y_n) \to 0 \Rightarrow d(x_n \oplus z, y_n \oplus z) \to 0$.

Metric d is also monotonic, i.e.,

$x \leq z \leq y \Rightarrow d(x, y) \geq sup\{d(y, z), d(x, z)\}$.

Let f and g be two functions defined on X and with values in a semiring $([a,b],\oplus,\odot)$. Then for any $x \in X$ and for any $\lambda \in [a,b]$ we define $(f \oplus g)(x) = f(x) \oplus g(x)$, $(f \odot g)(x) = f(x) \odot g(x)$ and $(\lambda \odot f)(x) = \lambda \odot f(x)$.

Definition 2.4. The characteristic function with values in a semiring $([a,b],\oplus,\odot)$ is defined by

$$\chi_A(x) = \begin{cases} \mathbf{1}, & \text{if } x \in A, \\ \mathbf{0}, & \text{if } x \notin A. \end{cases}$$

Where $\mathbf{0}$ is zero element for $\oplus$ and $\mathbf{1}$ is unit element for $\odot$.

Definition 2.5. An elementary (measurable) function is a mapping $e : X \to [a,b]$ that has the following representation:

$$e = \bigoplus_{i=1}^{n} a_i \odot \chi_{A_i} \qquad ,$$

where $a_i \in [a,b]$ and sets $A_i \in \mathbb{A}$ are pairwise disjoint if $\oplus$ is nonidempotent.

Definition 2.6 ([11]). Let ϵ be a positive real number and $B \subset [a,b]$. A subset $\{l_i^\epsilon\}$ of the set B is a ϵ-net on B if for each $x \in B$ there exists l_i^ϵ such that $d(l_i^\epsilon, x) \leq \epsilon$. If we, also, have $l_i^\epsilon \preceq x$, then we call $\{l_i^\epsilon\}$ a lower ϵ-net. If $l_i^\epsilon \preceq l_{i-1}^\epsilon$ holds, then $\{l_i^\epsilon\}$ is monotone.

Definition 2.7. Let $m : \mathbb{A} \to [a,b]$ be a $\oplus$-measure.

(i) The pseudo-integral of an elementary function $e : X \to [a,b]$ with respect to m is defined by

$$\int_X^{\oplus} e \odot dm = \bigoplus_{i=1}^{n} a_i \odot m(A_i).$$

(ii) The pseudo-integral of a bounded measurable function $f : X \to [a,b]$, (if $\oplus$ is not idempotent we suppose that for each $\epsilon > 0$ there exists a monotone ϵ-net in $f(X)$) is defined by

$$\int_X^{\oplus} f(x) \odot dm = lim_{n\to\infty} \int_X^{\oplus} e_n(x) \odot dm,$$

where $(e_n)_{n\in\mathbb{N}}$ is a sequence of elementary functions such that $d(e_n(x), f(x)) \to 0$ uniformly as $n \to \infty$. For more details see [14, 16].

3. Two important cases: generated and max-plus semirings

We shall consider the semiring $([a,b], \oplus, \odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a,b] \to [0,\infty]$. Then the pseudo-integral for a function $f : [c,d] \to [a,b]$ reduces on the g-integral [12, 13],

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\Big(\int_c^d g(f(x))dx\Big).$$

Now easily we can obtain the properties listed in the following proposition.

Proposition 3.1 ([16]). Let $(X, \mathcal{F}, \mu, \mathbb{R}_+^-, \oplus, \odot)$ is a pseudo-space and $f, g \in \mathcal{F}$, then:
(1) If $f = 0$ on A a.e., then $\int_A^{\oplus} f d\mu = 0$.
(2) If $\mu(A) = 0$, then $\int_A^{\oplus} f d\mu = 0$.
(3) $\int_A^{\oplus} a d\mu \geq a \odot \mu(A)$.
(4) If $f \leq g$ on A, then $\int_A^{\oplus} f d\mu \leq \int_A^{\oplus} g d\mu$.
(5) If $A \subset B$, then $\int_A^{\oplus} f d\mu \leq \int_B^{\oplus} f d\mu$.

Second case is when the semiring is of the form $([a, b], max, \odot)$. Then the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = sup_{x\in\mathbb{R}}\Big(f(x) \odot \psi(x)\Big),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition [11]. We shall denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = ess\ sup_{\mu}(x \mid x \in A) = sup\{a \mid \mu(\{x \mid x \in A, x > a\}) > 0\}.$$

We have by [11]:

Theorem 3.1. *Let m be a sup-measure on $([0, \infty], \mathbb{B}([0, \infty]))$, where $\mathbb{B}([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $m(A) = ess\ sup_{\mu}(\psi(x)|x \in A)$, and $\psi : [0, \infty] \to [0, \infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$–measure on $([0, \infty), \mathbb{B})$, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in (0, \infty)$, such that $lim_{\lambda\to\infty} m_\lambda = m$.*

For any continuous function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [11].

Theorem 3.2. *Let $([0, \infty]$, sup, $\odot)$ be a semiring with $\odot$ generated by some increasing generator g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in [a, b]$. Let m be the same as in Theorem 3.1. Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is generated by $g^\lambda, \lambda \in (0, \infty)$, such that for every continuous function $f : [0, \infty] \to [0, \infty]$*

$$\int^{sup} f \odot dm = lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = lim_{\lambda\to\infty}(g^\lambda)^{-1}\Big(\int g^\lambda(f(x))dx\Big).$$

4. Main results

Lemma 4.1. *Let $g : [a, b] \to [0, \infty]$ be a continuous and increasing function, then for any non-negative integrable function $f : [c, d] \to [a, b]$ the inequality*

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e^2} \int_A^{\oplus} f^2 d\mu, \tag{4.1}$$

holds where $A = [c, d]$ *and* $e \in [a, b]$.

Proof. Let us consider $A^* = \{x \in A : f(x) \geq e\}$. We must show that:

$$\int_A^{\oplus} f^2 d\mu \geq e^2.\mu(A^*).$$

As $A^* \subseteq A$, then by (5) of Proposition 3.1 we have

$$\int_A^{\oplus} f^2 d\mu \geq \int_{A*}^{\oplus} f^2 d\mu. \tag{4.2}$$

Since $f(x) \geq e$ for all $x \in A^*$, we have

$$(f)^2 \geq (e)^2.$$

Since g is an increasing function, then $g(f^2) \geq g(e^2)$. Therefore by (4) of Proposition 3.1 we have

$$\int_{A^*} g(f^2) d\mu \geq \int_{A^*} g(e^2) d\mu.$$

Since inverse of increasing function is increasing, so g^{-1} is also increasing. It follows that

$$\begin{aligned} g^{-1}\Big(\int_{A^*} g(f^2) d\mu\Big) &\geq g^{-1}\Big(\int_{A^*} g(e^2) d\mu\Big) \\ &= g^{-1} g(e^2).\mu(A^*) \\ &= e^2.\mu(A^*) \end{aligned}$$

i.e.

$$\begin{aligned} \int_{A^*}^{\oplus} f^2 d\mu &= g^{-1}\Big(\int_{A^*} g(f^2) d\mu\Big) \\ &\geq e^2.\mu(A^*). \end{aligned}$$

From (4.2) we have

$$\begin{aligned} \int_A^{\oplus} f^2 d\mu &\geq \int_{A^*}^{\oplus} f^2 d\mu \\ &\geq e^2.\mu(A^*). \end{aligned}$$

Consequently

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e^2} \int_A^{\oplus} f^2 d\mu,$$

which completes the proof.

The following result is generalization of the Markov type inequality for pseudo-integrals.

Theorem 4.2. *If $g : [a,b] \to [0,\infty]$ is a continuous and increasing function, then for every non-negative integrable function $f : [c,d] \to [a,b]$, the inequality*

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e}\int_A^{\oplus} f d\mu$$

holds, where $e \in [a,b]$ and $A = [c,d]$.

Proof. As $f \geq 0$ and $\{x \in A^* : f(x) \geq e\} = \{x \in A : f(x) \geq e\}$, by Lemma 4.1 we have

$$\begin{aligned} \mu(\{x \in A^* : f(x) \geq e\}) &= \mu(\{x \in A : f(x) \geq e\}) \\ &\leq \frac{1}{(e)^2}\int_A^{\oplus} (f(x))^2 d\mu \\ &= \frac{1}{e}\int_A^{\oplus} f d\mu, \end{aligned}$$

which implies that the Theorem 4.2 holds.

Example 4.3. Let $f(x) = x$, for all $x \in [1,2]$ and $g : [1,2] \to [0,\infty]$ be defined as $g(x) = e^x$. Taking $A = [1,2]$ and $e = \frac{3}{2}$, we have

$$\begin{aligned} \mu(\{x \in A : f(x) \geq e\}) &= \mu(\{x \in [1,2] : x \geq \frac{3}{2}\}) \\ &= \mu([\frac{3}{2}, 2]) \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \int_A^{\oplus} f d\mu &= \int_A^{\oplus} x d\mu \\ &= g^{-1}(\int_1^2 g(x)dx) \\ &= g^{-1}(e^2 - e) \\ &= \ln(e^2 - e). \end{aligned}$$

Therefore

$$\mu(\{x \in A : f(x) \geq e\}) = frac12 \leq Ln(e^2 - e) = \frac{1}{e}\int_A^{\oplus} f d\mu.$$

In the sequel, we generalize the Markov inequality by the semiring $([a,b], max, \odot)$, where $\odot$ is generated.

Theorem 4.4. *Let $f : [c, d] \to [a, b]$ be a non-negative integrable function. If $\odot$ is represented by an increasing multiplicative generator g and m is the same as in Theorem 3.1, then the inequality*

$$m(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e} \int_A^{\sup} f \odot dm$$

holds, where $A = [c, d]$ and $e \in [a, b]$.

Proof. Suppose that $A^* = \{x \in A : f(x) \geq e\}$. Theorem 3.2 implies that

$$\begin{aligned}
\int_{[c,d]}^{\sup} f \odot dm &= \lim_{\lambda\to\infty} \int_{[c,d]}^{\oplus_\lambda} f \odot dm_\lambda \\
&= \lim_{\lambda\to\infty} (g^\lambda)^{-1}\Big(\int_c^d g^\lambda(f(x))dx\Big) \\
&\geq \lim_{\lambda\to\infty} (g^\lambda)^{-1}\Big(\int_{A^*} g^\lambda(f(x))dx\Big) \\
&\geq \lim_{\lambda\to\infty} (g^\lambda)^{-1}\Big(\int_{A^*} g^\lambda(e)dx\Big) \\
&= \lim_{\lambda\to\infty} (g^\lambda)^{-1} g^\lambda(e).m(A^*) \\
&= e.m(A^*),
\end{aligned}$$

therefore

$$m(A^*) \leq \frac{1}{e} \int_{[c,d]}^{\sup} f \odot dm.$$

This completes the proof.

Note that the third important case $\oplus = max$ and $\odot = min$ for Theorem 4.2 has been studied in [3] and the pseudo-integral in such a case yields the Sugeno integral.

5. Conclusion

We have proved the Markov type inequalities for pseudo-integrals. There are two classes of pseudo-integrals. One of them concerning the pseudo-integrals based on a function reduces to the g-integral, where pseudo-addition and pseudo-multiplication are defined by a monotone and continuous function g. The other one concerns the pseudo-integrals based on a semiring $([a, b], max, \odot)$, where $x \odot y$ is generated by $g^{-1}\big(g(x)g(y)\big)$.

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Bayaz Daraby

Department of Mathematics, University of Maragheh, P. O. Box 55181-83111, Maragheh, Iran

E-mail: bayazdaraby@yahoo.com, bdaraby@maragheh.ac.ir

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Bushell-Okrasiaski type inequality for pseudo-integrals

Daraby B., Arabi L. and Firuzi S.

Abstract. In this paper, we prove Bushell-Okrasiaski inequality at decreasing case for two classes of pseudo-integrals. One of them, classes with pseudo-integrals where pseudo-operations are defined via a monotone and continuous generator function. The other one concerns the pseudo-integrals based on a semiring with an idempotent addition and a pseudo-multiplication generator.

Key Words and Phrases: B-O type inequality, Convolution type inequality, Pseudo-addition, Pseudo-multiplication, Pseudo-integral

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1. Introduction

Not long ago, H. Román-Flores et al. analyzed an interesting type of geometric inequalities for the Sugeno integrals with some applications to convex geometry in [12]. More precisely, a Prékopa-Leindler type inequality for fuzzy integrals was proven, and subsequently used for the characterization of some convexity properties of fuzzy measures.

In this paper, we use Pseudo-analysis for the generalization of the classical analysis, where instead of the field of the numbers a semiring is defined on a real interval $[a, b] \subset [-\infty, \infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$. Thus it would be an interesting topic to generalize an inequality from the classical analysis as special cases. We prove generalizations of the Bushell-Okrasinski's type inequality for pseudo-integrals.

The classical Bushell-Okrasinski [3] is a convolution type inequality. More precisely,

$$0^x (x-t)^{s-1} g(t)^s dt \leq \Big(\int_0^x g(t) \Big)^s, \qquad 0 \leq x \leq b, \tag{1.1}$$

holds for a continuous and increasing function $g : [0,1] \to [0,\infty)$ and $s \geq 1$, $b \leq 1$. This inequality was used by Bushell and Okrasinski [3] in the study of solutions of Volterra integral equations (see also [6]). Later on Walter and Weckesser [16] study some extensions of (1.1) and finally, after the change of variable $t = xs$, Malamud [5] analyze the B-O inequality (1.1) in the following new form:

$$s\int_0^1 (1-t)^{s-1} g(t)^s dt \leq \Big(\int_0^x g(t)\Big)^s.$$

H. Román-Flores et al [11] proved Bushell-Okrasinski type inequality for the Sugeno integrals as the following way:

Theorem 1.1. (Fuzzy B-O inequality). Let $g : [0,1] \to [0,\infty)$ be a continuous and decreasing function. Then

$$s\,⨍_0^1 (1-t)^{s-1} g(t)^s dt \geq \Big(⨍_0^1 g(t)dt\Big)^s,$$

holds for all $s \geq 2$.

2. Preliminaries

2.1. Pseudo-integrals

Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. A binary operation $\oplus$ on $[a,b]$ is pseudo-addition if it is commutative, non-decreasing(with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preceq x\}$. A binary operation $\odot$ on $[a,b]$ is Pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a,b]_+$, associative and with a unit element $1 \in [a,b]$, i.e., for each $x \in [a,b]$, $1 \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a,b], \oplus, \odot)$ is a semiring ([2, 4, 9, 15]). In this paper we will consider semirings with following continuous operations:

Case I. The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.

(b) $x \oplus y = \inf(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = b$ and the the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II. The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$ (additive generator of $\oplus$), i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \qquad \textit{and} \qquad x \odot y = g^{-1}(g(x).g(y)).$$

If the zero element for the pseudo-addition is a, we will consider increasing generators.Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$.

If the generator g is increasing (respectively decreasing), the operation $\oplus$ induce the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \preceq g(y)$.

Case III. Both operation are idempotent. We have

(a) $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation *sup* induces a usual order ($x \prec y$ if and only if $\sup(x, y) = y$).

(b) $x \oplus y = \inf(x, y)$, $x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces an order opposite to the usual order($x \preceq y$ if and only if $\inf(x, y) = y$).

2.2. Explicit forms of special Pseudo-integrals

We shall consider the semiring $([a, b], \oplus, \odot)$ for three (with completely different behaviour) cases, namely I(a), II, and III(a). Observe that the cases I(b) and III(b) are linked to the cases I(a) and III(a) by duality. First case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, case then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$X^{\oplus} f \odot dm = g^{-1}\left(\int_X (gof) d(gom)\right), \tag{2.1}$$

Where the integral applied on the right side is the standard Lebesgue integral. In spacial case, when $X = [c,d]$, $\mathcal{A} = \mathcal{B}(X)$ and $m = g^{-1}o\lambda$, λ the standard Lebesgue measure on $[c,d]$, then we use notation

$$\int_{[c,d]}^{\oplus} f(x)dx = \int_{X}^{\oplus} f \odot dm.$$

By (2.1)

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\big(\int_{c}^{d} g(f(x))dx\big),$$

i.e., we have recovered the g-integral (see[8, 9]).

Second case is when the semiring is of the form $([a,b], \sup, \odot)$, case I(a) and III(a). We will consider complete sup-measure m only and $\mathcal{A} = 2^x$, i.e., for any system $(A_i)_i \in I$ of measurable sets,

$$m(\underset{i\in I}{\cup} A_i) = \sup_{i\in I} m(A_i).$$

Recall that if X is countable (especially, if X is finite) then any σ-sup-measure m is complete and, moreover, $m(A) = \sup_{x\in A} \psi(X)$, where $\psi : X \to [a,b]$ is a density function given by $\psi(x) = m(\{x\})$. Then the pseudo-integral for a function $f : X \to [a,b]$ is given by

$$\int_{X}^{\oplus} f \odot dm = \sup_{x\in X}(f(x) \odot \psi(x)),$$

where function ψ defines sup-measure m.

Theorem 2.1. Let m be a sup-measure on $([0,\infty], \mathfrak{B}([0,\infty]))$, where $\mathfrak{B}([0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $m(A) = esssup_{\mu}(\psi(x)|x \in A)$, where $\psi : [0,\infty] \to [0,\infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measure on $([0,\infty[, \mathfrak{B})$, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in]0,\infty[$, such that $\lim_{\lambda\to\infty} m_\lambda = m$.

For any continuous function $f : [0,\infty] \to [0,\infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [7].

Theorem 2.2. Let $([0,\infty], \sup, \odot)$ be a semiring with $\odot$ generated by some increasing generator g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in [a,b]$. Let m be the same as in Theorem 2.1. Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measure, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in]0,\infty[$, such that for every continuous function $f : [0,\infty] \to [0,\infty]$

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = \lim_{\lambda\to\infty} (g^\lambda)^{-1}\big(\int g^\lambda(f(x))dx\big).$$

Now we recall generalization of the Jensen inequality for pseudo-integral that proved by E. Pap et al. on [10].

Theorem 2.3. Let $\Phi : [a,b] \to [a,b]$ be a convex and nondecreasing function. If a generator $g : [a,b] \to [a,b]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then for any measurable function $f : [0,1] \to [a,b]$ we have

$$\Phi\big(\int_{[0,1]}^{\oplus} f(x)dx\big) \leq \int_{[0,1]}^{\oplus} \Phi(f(x))dx.$$

Theorem 2.4. Let $\Phi : [a,b] \to [a,b]$ be a convex and nondecreasing function, and the pseudo-multiplication $\odot$ is represented by a convex and increasing generator g. Let m be the same as in Theorem 2.1. Then for any continuous function $f : [0,1] \to [a,b]$ we have

$$\Phi\big(\int_{[0,1]}^{\sup} f \odot dm\big) \leq \int_{[0,1]}^{\sup} \Phi(f) \odot dm.$$

Theorem 2.5. Let $u, v : [0,1] \to [a,b]$ be two measurable functions and let a generator $g : [a,b] \to [0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone functions, then the inequality

$$\int_{[0,1]}^{\oplus} (u \odot v)dx \geq \big(\int_{[0,1]}^{\oplus} udx\big) \odot \big(\int_{[0,1]}^{\oplus} vdx\big),$$

holds and the reserve inequality also holds whenever u and v are countermonotone functions.

3. Main results

In this section, we prove Bushell-Okrasiaski inequality for pseudo-integrals.

Theorem 3.1. (Pseudo Bushell-Okrasiaski inequality) Let $f : [0,1] \to]a,b[$ be a continuous and decreasing function. If a generator $g :]a,b[\to]a,b[$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then

$$\int_{[0,1]}^{\oplus} (1-t)^{s-1} \odot f^s(t)dt \geq \frac{1}{s} \odot \big(\int_{[0,1]}^{\oplus} f(t)dt\big)^s,$$

holds for all $s \geq 2$.

Proof. By the definition of pseudo-integral and pseudo-operations we have

$$\begin{aligned}\int_{[0,1]}^{\oplus}(1-t)^{s-1}\odot f^s(t)dt &= g^{-1}\Big(\int_0^1 g\big[(1-t)^{s-1}\odot f^s(t)\big]dt\Big)\\ &= g^{-1}\Big(\int_0^1 g\big[g^{-1}(g((1-t)^{s-1})g(f^s(t))\big]dt\Big)\\ &= g^{-1}\Big(\int_0^1 g((1-t)^{s-1})g(f^s(t))dt\Big).\end{aligned}$$

By classic Chebyshev's integral inequality ([14]), we have;

$$\begin{aligned}g^{-1}\Big(\int_0^1 g((1-t)^{s-1})g(f^s(t))dt\Big) &\geq g^{-1}\big[\Big(\int_0^1 g((1-t)^{s-1})dt\Big(\int_0^1 g(f^s(t))dt\Big)\big]\\ &= g^{-1}\big[gg^{-1}\Big(\int_0^1 g((1-t)^{s-1})dt\Big)gg^{-1}\Big(\int_0^1 g(f^s(t))dt\Big)\big]\\ &= g^{-1}\big[g\Big(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\Big)g\Big(\int_{[0,1]}^{\oplus}f^s(t)dt\Big)\big]\\ &= \Big(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\Big)\odot\Big(\int_{[0,1]}^{\oplus}f^s(t)dt\Big).\end{aligned}$$

By using the Theorem 2.3,

$$[0,1]^{\oplus}(1-t)^{s-1}\odot f^s(t)dt \geq \Big(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\Big)\odot\Big(\int_{[0,1]}^{\oplus}f(t)dt\Big)^s, \tag{3.1}$$

in the other hand by using the classic Jensen inequality ([13]), we can show that

$$\begin{aligned}\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt &= g^{-1}\Big(\int_0^1 g((1-t)^{s-1})dt\Big)\\ &\geq g^{-1}\Big(g\int_0^1(1-t)^{s-1}dt\Big)\end{aligned}$$

$$= \int_0^1 (1-t)^{s-1}dt = \frac{1}{s} \tag{3.2}$$

so by (3.1) and (3.2) we obtain that:

$$\int_{[0,1]}^{\oplus}(1-t)^{s-1}\odot f^s(t)dt \geq \frac{1}{s}\odot\Big(\int_{[0,1]}^{\oplus}f(t)dt\Big)^s.$$

Thereby, the theorem is proved.◄

Example 3.2. Let $g(x)=e^x$. The corresponding pseudo-operations are $x\oplus y=\ln(e^x+e^y)$ and $x\odot y=x+y$, the Theorem 3.1 reduces on the following inequality,

$$\ln\Big(\int_0^1 e^{(1-t)^{s-1}+f^s(t)}dt\Big)\geq\frac{1}{s}+\Big(\ln\Big(\int_0^1 e^{f(t)}dt\Big)\Big)^s.$$

In the sequel, we generalize the Bushell-Okrasiaski inequality by the semiring $([0,1], max, \odot)$, where $\odot$ is generated.

Theorem 3.3. (Pseudo Bushell-Okrasiaski inequality) Let $f : [0,1] \to]a,b[$ be a continuous and decreasing function, and $\odot$ is represented by a convex and increasing multiplication generator g and m be the same as in Theorem 2.1, then

$$\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \geq \frac{1}{s} \odot \big(\int_{[0,1]}^{\sup} f(t)dt \big)^s,$$

holds for all $s \geq 2$.

Proof. By Theorem 2.2 we have:

$$\begin{aligned}\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm &= \lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} (1-t)^{s-1} \odot f^s(t) \odot dm_\lambda \\ &= \lim_{\lambda\to\infty} (g^\lambda)^{-1} \big(\int_0^1 g^\lambda((1-t)^{s-1} \odot f^s(t))dt \big).\end{aligned}$$

Using the Theorem 2.5 so we have

$$\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \geq \lim_{\lambda\to\infty} \big[(g^\lambda)^{-1} \big(\int_0^1 g^\lambda((1-t)^{s-1})dt \big) \odot (g^\lambda)^{-1} \big(\int_0^1 g^\lambda(f^s(t))dt \big) \big]$$

$$= \big[\lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda((1-t)^{s-1})dt \big) \odot \big(\lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda((f^s(t))dt \big]$$

$$= \big(\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot dm \big) \odot \big(\int_{[0,1]}^{\sup} f^s(t) \odot dm \big).$$

Applying the Theorem 2.4, we obtain that:

$$\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \geq \big(\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot dm \big) \odot \big(\int_{[0,1]}^{\sup} f(t) \odot dm \big)^s. \tag{3.3}$$

Also we have:

$$\begin{aligned}\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot dm &= \lim_{\lambda\to\infty} \big(\int_{[0,1]}^{\oplus_\lambda} (1-t)^{s-1} \odot dm_\lambda \big) \\ &= \lim_{\lambda\to\infty} (g^\lambda)^{-1} \big(\int_0^1 g^\lambda((1-t)^{s-1})dt \big) \\ &\geq \lim_{\lambda\to\infty} (g^\lambda)^{-1} \big(g^\lambda \int_0^1 ((1-t)^{s-1})dt \big)\end{aligned}$$

$$= \lim_{\lambda\to\infty} \int_0^1 ((1-t)^{s-1})dt = \frac{1}{s}. \tag{3.4}$$

From (3.3) and (3.4) we have $\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \geq \frac{1}{s} \odot \big(\int_{[0,1]}^{\sup} f(t)dt \big)^s$. ◀

Example 3.4. Let $g^\lambda = e^{\lambda x}$ and $\psi(x)$ be from Theorem 2.1, then

$$x \odot_\lambda y = x + y \quad \text{and} \quad \lim_{\lambda\to\infty}\big(\frac{1}{\lambda}\ln(e^{\lambda x}+e^{\lambda y})\big) = max(x,y).$$

Therefore B-O type inequality from Theorem 3.3 reduces on

$$\underset{x\in[0,1]}{sup}\big[\big((1-x)^{s-1}+f^s(x)\big)+\psi(x)\big] \geq \frac{1}{s} + \big[\underset{x\in[0,1]}{sup}\big(f(x)+\psi(x)\big)\big]^s.$$

Note that third important case $\oplus$ =max and $\odot$ =min has been studied in [11] and the Pseudo-integrals in such a case yields the Sugeno integral.

4. Conclusion

We have proved the B-o integral type inequality for the pseudo-integral for two characteristic cases: generated and max-plus. For further investigation we continue to explore other integral inequalities for fuzzy integrals.
Open problem: Dose B-O type inequalities hold for the Chaquet integral?

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Bayaz Daraby
Department of Mathematics, University of Maragheh, P. O. Box 55181-83111, Maragheh, Iran
E-mail: bayazdaraby@yahoo.com

Leila Arabi
Department of Mathematics, University of Maragheh,Maragheh, Iran
*E-mail:*arabi_l88@yahoo.com

Samaneh Firuzi
Department of Mathematics, University of Maragheh,Maragheh, Iran
E-mail: samanefirouzi@gmail.com

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On a compactness criteria for multidimensional Hardy type operator in p-convex Banach function spaces

Bandaliev R.A.

Abstract. The main goal of this paper is to prove a criteria on compactness of multidimensional Hardy type operator from weighted Lebesgue spaces into p-convex weighted Banach function spaces. Analogously problem for the dual operator is considered.

Key Words and Phrases: Banach function spaces, weights, Hardy type operator, compactness

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1. Introduction

The investigation of Hardy operator in weighted Banach function spaces (BFS) have recently history. The goal of this investigations were closely connected with the found of criterion on the geometry and on the weights of BFS for validity of boundedness of Hardy operator in BFS. Characterization of the mapping properties such as boundedness and compactness were considered in the papers [8], [9], [13], [25] and e.t.c. More precisely, in [8] and [9] were considered the boundedness of certain integral operator in ideal Banach spaces. In [13] was proved the boundedness of Hardy operator in Orlicz spaces. Also, in [25] the compactness and measure of non-compactness of Hardy type operator in Banach function spaces was proved. But in this paper we consider the boundedness of Hardy operator in p-convex Banach function spaces and find a new type criterion on the weights for validity of Hardy inequality. Note that the notion of BFS was introduced in [26]. In particular, the weighted Lebesgue spaces, weighted Lorentz spaces, weighted variable Lebesgue spaces, variable Lebesgue spaces with mixed norm, Musielak-Orlicz spaces and e.t.c. is BFS.

In this paper, we establish an integral-type necessary and sufficient condition on weights, which provides the compactness of the multidimensional Hardy type operator from weighted Lebesgue spaces into p-convex weighted BFS. We also investigate the corresponding problems for the dual operator.

2. Preliminaries

Let $(\Omega,\,\mu)$ be a complete σ-finite measure space. By $L_0=L_0(\Omega,\mu)$ we denote the collection of all real-valued μ-measurable functions on Ω.

Definition 1. *[26, 24, 7] We say that real normed space X is a Banach function space (BFS) if:*

(P1) the norm $\|f\|_X$ is defined for every μ-measurable function f, and $f\in X$ if and only if $\|f\|_X<\infty$; $\|f\|_X=0$ if and only if $f=0$ a.e. ;

(P2) $\|f\|_X=\||f|\|_X$ for all $f\in X$;

(P3) if $0\le f\le g$ a.e., then $\|f\|_X\le\|g\|_X$;

(P4) if $0\le f_n\uparrow f$ a.e., then $\|f_n\|_X\uparrow\|f\|_X$ (Fatou property);

(P5) if E is a measurable subset of Ω such that $\mu(E)<\infty$, then $\|\chi_E\|_X<\infty$, where χ_E is the characteristic function of the set E;

(P6) for every measurable set $E\subset\Omega$ with $\mu(E)<\infty$, there is a constant $C_E>0$ such that $\int_E f(x)\,dx\le C_E\,\|f\|_X$.

Given a BFS X we can always consider its associate space X' consisting of those $g\in L_0$ that $f\cdot g\in L_1$ for every $f\in X$ with the usual order and the norm $\|g\|_{X'}=\sup\{\|f\cdot g\|_{L_1}:\ \|g\|_{X'}\le 1\}$. Note that X' is a BFS in $(\Omega,\,\mu)$ and a closed norming subspaces.

Let X be a BFS and ω be a weight, that is, positive Lebesgue measurable and a.e. finite functions on Ω. Let $X_\omega=\{f\in L_0:\ f\omega\in X\}$. This space is a weighted BFS equipped with the norm $\|f\|_{X_\omega}=\|f\,\omega\|_X$. (For more detail and proofs of results about BFS we refer the reader to [7] and [24].)

Let us recall the notion of p-convexity and p-concavity of BFS's.

Definition 2. *[33] Let X is a BFS. Then X is called p-convex for $1\le p\le\infty$ if there exists a constant $M>0$ such that for all $f_1,\dots,f_n\in X$*

$$\left\|\left(\sum_{k=1}^{n}|f_k|^p\right)^{\frac1p}\right\|_X\le M\,\left(\sum_{k=1}^{n}\|f_k\|_X^p\right)^{\frac1p}\quad \text{if } 1\le p<\infty,$$

or $\left\|\sup\limits_{1\le k\le n}|f_k|\right\|_X\le M\,\max\limits_{1\le k\le n}\|f_k\|_X$ *if $p=\infty$. Similarly X is called p-concave for $1\le p\le\infty$ if there exists a constant $M>0$ such that for all $f_1,\dots,f_n\in X$*

$$\left(\sum_{k=1}^{n}\|f_k\|_X^p\right)^{\frac1p}\le M\,\left\|\left(\sum_{k=1}^{n}|f_k|^p\right)^{\frac1p}\right\|_X\quad \text{if } 1\le p<\infty,$$

or $\max\limits_{1\le k\le n}\|f_k\|_X\le M\,\left\|\sup\limits_{1\le k\le n}|f_k|\right\|_X$ *if $p=\infty$.*

Remark 1. *Note that the notions of p-convexity, respectively p-concavity are closely related to the notions of upper p-estimate (strong ℓ_p- composition property), respectively lower p-estimate (strong ℓ_p-decomposition property) as can be found in [24].*

Now we reduce some examples of p-convex and respectively p-concave BFS. Let R^n be the n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$ and let Ω be a Lebesgue measurable subset in R^n and $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. The Lebesgue measure of a set Ω will be denoted by $|\Omega|$. It is well known that $|B(0,1)| = \dfrac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$, where $B(0,1) = \{x : x \in R^n \,;\, |x| < 1\}$.

Example 1.1. Let $1 \le q \le \infty$ and $X = L_q$. Then the space L_q is p-convex (p-concave) BFS if and only if $1 \le p \le q \le \infty$ ($1 \le q \le p \le \infty$.)

The proof implies from usual Minkowski inequality in Lebesgue spaces.

Example 1.2. The following Lemma shows that the variable Lebesgue spaces $L_{q(y)}(\Omega)$ is p-convex BFS.

Lemma 1. *[1] Let $1 \le p \le q(x) \le \overline{q} < \infty$ for all $y \in \Omega_2 \subset R^m$. Then the inequality*

$$\left\| \|f\|_{L_p(\Omega_1)} \right\|_{L_{q(\cdot)}(\Omega_2)} \le C_{p,q} \left\| \|f\|_{L_{q(\cdot)}(\Omega_2)} \right\|_{L_p(\Omega_1)}$$

is valid, where $C_{p,q} = \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + p\left(\frac{1}{\underline{q}} - \frac{1}{\overline{q}}\right)\right)(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty)$, $\underline{q} = ess \inf_{\Omega_2} q(x)$, $\overline{q} = ess \sup_{\Omega_2} q(x)$, $\Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2 : q(y) = p\}$, $\Delta_2 = \Omega_1 \times \Omega_2 \backslash \Delta_1$ *and* $f : \Omega_1 \times \Omega_2 \to R$ *is any measurable function such that*

$$\left\| \|f\|_{L_p(\Omega_1)} \right\|_{L_{q(\cdot)}(\Omega_2)} = \inf \left\{ \mu > 0 : \int\limits_{\Omega_2} \left(\frac{\|f(\cdot,y)\|_{L_p(\Omega_1)}}{\mu} \right)^{q(y)} dy \le 1 \right\} < \infty$$

and $\|f(\cdot,y)\|_{L_p(\Omega_1)} = \left(\int_{\Omega_1} |f(x,y)|^p \, dx \right)^{1/p}$.

Analogously, if $1 \le q(x) \le p < \infty$, then $L_{q(x)}(\Omega)$ is p-concave BFS.

Definition 3. *[31, 15]. Let $\Omega \subset R^n$ be a Lebesgue measurable set. A real function $\varphi : \Omega \times [0,\infty) \mapsto [0,\infty)$ is called a generalized φ-function if it satisfies:*

a) $\varphi(x,\cdot)$ is a φ-function for all $x \in \Omega$, i.e., $\varphi(x,\cdot) : [0,\infty) \mapsto [0,\infty)$ is convex and satisfies $\varphi(x,0) = 0$, $\lim_{t \to +0} \varphi(x,t) = 0$;

b) $\psi : x \mapsto \varphi(x,t)$ is measurable for all $t \ge 0$.

If φ is a generalized φ-function on Ω, we shortly write $\varphi \in \Phi$.

Definition 4. *[31, 15]. Let $\varphi \in \Phi$ and be ρ_φ defined by the expression*

$$\rho_\varphi(f) := \int\limits_\Omega \varphi(y, |f(y)|)\, dy \quad \text{for all} \quad f \in L_0(\Omega).$$

We put $L_\varphi = \{f \in L_0(\Omega) :\ \rho_\varphi(\lambda_0 f) < \infty \quad \text{for some} \quad \lambda_0 > 0\}$ and

$$\|f\|_{L_\varphi} = \inf\left\{\lambda > 0 :\ \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

The space L_φ is called Musielak-Orlicz space.

Let ω be a weight function on Ω, i.e., ω is a non-negative, almost everywhere positive function on Ω. In this work we considered the weighted Musielak-Orlicz spaces. We denote

$$L_{\varphi,\,\omega} = \{f \in L_0(\Omega) :\ \ f\omega \in L_\varphi\}.$$

It is obvious that the norm in this spaces is given by

$$\|f\|_{L_{\varphi,\,\omega}} = \|f\omega\|_{L_\varphi}.$$

Remark 2. *Let $\varphi(x,t) = t^{q(x)}$ in the Definition 4, where $1 \le q(x) < \infty$ and $x \in \Omega$. Then we have the definition of variable exponent weighted Lebesgue spaces $L_{q(x)}(\Omega)$ (see [15]).*

Example 1.3. The following Lemma shows that the Musielak-Orlicz spaces L_φ is p-convex BFS.

Lemma 2. *[4] Let $\Omega_1 \subset R^n$ and $\Omega_2 \subset R^m$. Let $(x,t) \in \Omega_1 \times [0,\infty)$, and $\varphi\left(x, t^{1/p}\right) \in \Phi$ for some $1 \le p < \infty$. Suppose $f : \Omega_1 \times \Omega_2 \mapsto R$. Then the inequality*

$$\left\|\|f(x,\cdot)\|_{L_p(\Omega_2)}\right\|_{L_\varphi} \le 2^{1/p} \left\|\|f(\cdot,y)\|_{L_\varphi}\right\|_{L_p(\Omega_2)}$$

is valid.

We note that the Lebesgue spaces with mixed norm, weighted Lorentz spaces and e.t.c. is p-convex (p-concave) BFS. Now we reduce more general result connected with Minkowski's integral inequality.

Let X and Y be BFSs on (Ω_1, μ) and (Ω_2, ν), respectively. By $X[Y]$ and $Y[X]$ we denote the spaces with mixed norm and consisting of all functions $g \in L_0(\Omega_1 \times \Omega_2,\, \mu \times \nu)$ such that $\|g(x,\,\cdot)\|_Y \in X$ and $\|g(\cdot,\,y)\|_X \in Y$. The norms in these spaces is defined as

$$\|g\|_{X[Y]} = \|\|g(x,\,\cdot)\|_Y\|_X\,, \quad \|g\|_{Y[X]} = \|\|g(\cdot,\,y)\|_X\|_Y\,.$$

Theorem 1. *[33] Let X and Y be BFSs with the Fatou property. Then the generalized Minkowski integral inequality*

$$\|f\|_{X[Y]} \le M\,\|f\|_{Y[X]}$$

holds for all measurable functions $f(x,y)$ if and only if there exists $1 \le p \le \infty$ such that X is p-convex and Y is p-concave.

It is known that $X[Y]$ and $Y[X]$ are BFSs on $\Omega_1 \times \Omega_2$ (see [24].)

3. Main results

We consider the multidimensional Hardy type operator and its dual operator

$$Hf(x) = \int\limits_{|y|<|x|} f(y)\,dy \quad \text{and} \quad H^*f(x) = \int\limits_{|y|>|x|} f(y)\,dy,$$

where $f \geq 0$ and $x \in R^n$.

Now we prove a two-weight criterion for multidimensional Hardy type operator acting from the p-concave weighted BFS to weighted Lebesgue spaces.

Theorem 2. *[5] Let $v(x)$ and $w(x)$ are weights on R^n. Suppose that X_w be a p-convex weighted BFSs for $1 \leq p < \infty$ on R^n. Then the inequality*

$$\|Hf\|_{X_w} \leq C\,\|f\|_{L_{p,\,v}} \tag{3.1}$$

holds for every $f \geq 0$ if and only if there is a $\alpha \in (0,1)$ such that

$$A(\alpha) = \sup_{t>0}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{|z|>t\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w} < \infty. \tag{3.2}$$

Moreover, if $C > 0$ is the best possible constant in (3.1), then

$$\sup_{0<\alpha<1}\frac{p'\,A(\alpha)}{(1-\alpha)\,\left[\left(\frac{p'}{1-\alpha}\right)^p + \frac{1}{\alpha(p-1)}\right]^{1/p}} \leq C \leq M \inf_{0<\alpha<1}\frac{A(\alpha)}{(1-\alpha)^{1/p'}}.$$

Example 3.1. *Let $n = 2$, $q(x) = q = const$, $x = (x_1, x_2) \in R^2$ and $1 < p \leq q < \infty$. Suppose that $v(x) = \dfrac{|x_1|^\beta}{|x|}$, $w(x) = |x|^\gamma$ and $\beta < \dfrac{1}{p'}$, and $\gamma = \beta - 1 - 2\left(\dfrac{1}{p'} + \dfrac{1}{q}\right)$. Then the condition of Theorem 2 is satisfied.*

For the dual operator, the below stated theorem is proved analogously.

Theorem 3. *[5] Let $v(x)$ and $w(x)$ are weights on R^n. Suppose that X_w be a p-convex weighted BFSs for $1 \leq p < \infty$ on R^n. Then the inequality*

$$\|H^*f\|_{X_w} \leq C\,\|f\|_{L_{p,\,v}} \tag{3.3}$$

holds for every $f \geq 0$ if and only if there is a $\gamma \in (0,1)$ such that

$$B(\gamma) = \sup_{t>0}\left(\int\limits_{|y|>t}[v(y)]^{-p'}\,dy\right)^{\frac{\gamma}{p'}}\left\|\chi_{\{|z|<t\}}(\cdot)\left(\int\limits_{|y|>|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\gamma}{p'}}\right\|_{X_w} < \infty.$$

Moreover, if $C>0$ is the best possible constant in (3.3) then

$$\sup_{0<\gamma<1}\frac{p'\,B(\gamma)}{(1-\gamma)\,\left[\left(\frac{p'}{1-\gamma}\right)^p+\frac{1}{\gamma(p-1)}\right]^{1/p}}\le C\le M\inf_{0<\gamma<1}\frac{B(\gamma)}{(1-\gamma)^{1/p'}}.$$

Corollary 1. *Note that Theorem 2 and Theorem 3 in the case $X_w=L_{\varphi,\,w}$, $\varphi\left(x,t^{1/p}\right)\in\Phi$ for some $1\le p<\infty$, $x\in R^n$ was proved in [4]. In the case $X_w=L_{q,\,w}$, $1<p\le q<\infty$, for $x\in(0,\infty)$, $\alpha=\dfrac{s-1}{p-1}$ and $s\in(1,\,p)$ Theorem 2 and Theorem 3 was proved in [35]. For $x\in R^n$ in the case $X_w=L_{q(x),\,w}$ and $1<p\le q(x)\le ess\sup\limits_{x\in R^n}q(x)<\infty$ Theorem 2 and Theorem 3 was proved in [3] (see also [2]). Also, in [6] the embeddings theorems between different variable Lebesgue spaces with measures was proved.*

Remark 3. *In the case $n=1$, $X_w=L_{q,\,w}$, $1<p\le q\le\infty$, at $x\in(0,\infty)$, for classical Lebesgue spaces the various variants of Theorem 2 and Theorem 3 were proved in [19], [11], [22], [23], [29], [30], [34] and etc. In particular, in the Lebesgue spaces with variable exponent the boundedness of Hardy type operator was proved in [14], [16], [18], [20], [21], [27], [28] and etc. For $X_w=L_{q(x),\,w}$, $1<p\le q(x)\le ess\sup\limits_{x\in[0,1]}q(x)<\infty$ and $x\in[0,1]$ the two-weighted criterion for one-dimensional Hardy operator was proved in [21]. Also, other type two-weighted criterion for multidimensional Hardy type operator in the case $X_w=L_{q(x),\,w}$, $1<p\le q(x)\le ess\sup\limits_{x\in R^n}q(x)<\infty$ and $x\in R^n$ was proved in [27] (see also [28]). In the papers [10] and [32] the inequalities of modular type for more general operators was proved. Also, in [12] the Hardy type inequalities with special power-type weights in Orlicz spaces was proved.*

Now we reduce a compactness criteria for multidimensional Hardy type operator from weighted Lebesgue spaces into p-convex weighted Banach function spaces.

Theorem 4. *Let $v(x)$ and $w(x)$ are weights on R^n. Suppose that X_w be a p-convex weighted BFSs for $1\le p<\infty$ on R^n. Then H is compact from $L_{p,v}$ to X_w if and only if the following two conditions are satisfied:*

(a) There exists an $\alpha\in(0,1)$ such that

$$A(\alpha)=\sup_{t>0}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{|z|>t\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}<\infty;$$

$$(b)\ \lim_{\gamma\to+0}\sup_{0<t<\gamma}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{t<|z|<\gamma\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0\ \text{ and}$$

$$\lim_{\delta\to\infty}\sup_{\delta<t<\infty}\left(\int\limits_{\delta<|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{|z|>t\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0;$$

(c) for every $\varepsilon\in(0,\infty)$ the following two alternatives hold:

$$\lim_{\beta\to\varepsilon+0}\left\|\chi_{\{\varepsilon<|z|<\beta\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0\ \ \textit{and}$$

$$\lim_{\beta\to\varepsilon-0}\left\|\chi_{\{\beta<|z|<\varepsilon\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0.$$

The proof of Theorem 4 follows from the general result of paper [17].

Now suppose that the space X_w is a BFS with absolute continuous norm. Then the condition (c) of Theorem is satisfied automatically. On the other words, we have the following Corollary.

Corollary 2. *Let $v(x)$ and $w(x)$ are weights on R^n. Suppose that X_w be a p-convex weighted BFSs for $1\le p<\infty$ on R^n. Then H is compact from $L_{p,v}$ to X_w if and only if the following two conditions are satisfied:*

(a) There exists an $\alpha\in(0,1)$ such that

$$A(\alpha)=\sup_{t>0}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{|z|>t\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}<\infty;$$

$$(b)\ \lim_{\gamma\to+0}\sup_{0<t<\gamma}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{t<|z|<\gamma\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0\ \ \textit{and}$$

$$\lim_{\delta\to\infty}\sup_{\delta<t<\infty}\left(\int\limits_{\delta<|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{|z|>t\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0.$$

Corollary 3. *Let $1 < p \le q(x) \le \overline{q} < \infty$ and $v(x)$ and $w(x)$ are weights on R^n. Then H is compact from $L_{p,v}$ to $L_{q(x),w}$ if and only if the following two conditions are satisfied:*

(a) There exists an $\alpha \in (0,1)$ such that

$$A(\alpha) = \sup_{t>0} \left(\int\limits_{|y|<t} [v(y)]^{-p'}\, dy \right)^{\frac{\alpha}{p'}} \left\| \left(\int\limits_{|y|<|\cdot|} [v(y)]^{-p'}\, dy \right)^{\frac{1-\alpha}{p'}} \right\|_{L_{q(\cdot),w}(|\cdot|>t)} < \infty;$$

$$(b) \lim_{\gamma\to+0} \sup_{0<t<\gamma} \left(\int\limits_{|y|<t} [v(y)]^{-p'}\, dy \right)^{\frac{\alpha}{p'}} \left\| \left(\int\limits_{|y|<|\cdot|} [v(y)]^{-p'}\, dy \right)^{\frac{1-\alpha}{p'}} \right\|_{L_{q(\cdot),w}(|\cdot|>t)} = 0 \quad and$$

$$\lim_{\delta\to\infty} \sup_{\delta<t<\infty} \left(\int\limits_{\delta<|y|<t} [v(y)]^{-p'}\, dy \right)^{\frac{\alpha}{p'}} \left\| \left(\int\limits_{|y|<|\cdot|} [v(y)]^{-p'}\, dy \right)^{\frac{1-\alpha}{p'}} \right\|_{L_{q(\cdot),w}(|\cdot|>t)} = 0.$$

Example 3.2. *Let $q(x) = q = const$ and $1 < p \le q < \infty$. Suppose that $v(x) = |x|^\beta$ and $w(x) = \begin{cases} |x|^{\gamma_1}, & for\ |x| < \frac{1}{2} \\ |x|^{\gamma_2}, & for\ |x| \ge \frac{1}{2}, \end{cases}$ and $\gamma_2 + n\left(\frac{1}{p'} + \frac{1}{q}\right) < \beta \le \min\left\{\frac{n}{p'},\ \gamma_1 + n\left(\frac{1}{p'} + \frac{1}{q}\right)\right\}$. Then the conditions of Corollary 3 are satisfied.*

Example 3.3. *Let $q(x) = q = const$, $x \in B(0,1)$ and $1 < p \le q < \infty$. Suppose that $v(x) = |x|^\beta$, $w(x) = |x|^\gamma$ and $\beta \le \min\left\{\frac{n}{p'},\ \gamma + n\left(\frac{1}{p'} + \frac{1}{q}\right)\right\}$ or $\gamma + n\left(\frac{1}{p'} + \frac{1}{q}\right) < \beta < \frac{n}{p'}$. Then the conditions of Corollary 3 are satisfied.*

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Rovshan A. Bandaliev
Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan
E-mail: bandaliyev.rovshan@math.ab.az