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On the statistical type convergence and fundamentality in metric spaces

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Abstract. The concept of $\mathscr{F}$-fundamentality, generated by some filter $\mathscr{F}$ is introduced in metric spaces. Its equivalence to the concept of $\mathscr{F}$-convergence is proved in metric spaces. This convergence generalizes many kinds of convergence, including the well-known statistical convergence.

Key Words and Phrases: $\mathscr{F}$-convergence, $\mathscr{F}$- fundamentality, statistical convergence

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1. Introduction

The idea of statistical convergence (*stat-convergence*) was first proposed by A.Zigmund [1] in his famous monograph where he talked about "almost convergence". The first definition of it was given by H. Fast [2] and H. Steinhaus [3]. Later, this concept has been generalized in many directions. It is impossible to list all the related papers. More details on this matter and its applications can be found in [4-15, 24]. It should be noted that the methods of non-convergent sequences have long been known and they include e.g. Cesaro method, Abel method, etc. These methods are used in different areas of mathematics. For the applicability of these methods it is very important that the considered space has a linear structure. Therefore, the study of statistical convergence in metric spaces is of special scientific interest. Different aspects of this problem have been studied in [16, 17]. Statistical convergence is currently actively used in many areas of mathematics such as summation theory [7, 8, 19], number theory [11, 13], trigonometric series [1], probability theory [8], measure theory [12], optimization [20], approximation theory [21, 22], fuzzy theory [26], etc.

It should be noted that the concept of statistical fundamentality (*stat-fundamentality*) was first introduced by J.A. Fridy [4] who proved its equivalence to *stat-convergence* with respect to numerical sequences. This issue was raised in [10] concerning uniform space $(X;U)$. It was proved that if the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is *stat-convergent*, then it is *stat-fundamental*. The problem of the validity of converse statement was also raised in [10].

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Stat-convergence was generalized by many mathematicians (see [10, 15, 23, 24, 25]). The concepts of *I-convergence* and *I^*-convergence* were introduced in [23]. These kinds of convergences generalizes many previously known convergences, including the well-known *stat-convergence.* In present paper we introduce the concepts of *$\mathscr{F}$-convergence and $\mathscr{F}$-fundamentality,* generated by some filter $\mathscr{F} \subset 2^{\mathbb{N}}$. Their equivalence is proved. *$\mathscr{F}$-convergence* generalizes many kinds of convergence, related to such concepts as statistical density, logarithmic density, uniform density, etc. More details on these concepts can be found in [23].

2. Needful information

We will use the standard notation. $\mathbb{N}$ will be a set of all positive integers; $\mathbb{R}$ will be a set of real numbers; $\chi_M(\cdot)$ is the characteristic function of M; $(X;\rho)$ is a metric space. $O_\varepsilon(a)$ is an open ball centered at a with radius ε, i.e. $O_\varepsilon(a) \equiv \{x \in X : \rho(x;a) < \varepsilon\}$. 2^M will be a set of all subsets M; $\bar{M}$ will stand for the closure of M; $|A| = card\, A$ is the number of elements of A. $M^C = N\backslash M$. $\wedge$ will be a quantifier which means "and".

Let us recall the definition of asymptotic (statistical) density of $A \subset \mathbb{N}$. Assume

$$\delta_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k),$$

and let $\delta_*(A) = \liminf\limits_{n\to\infty} \delta_n(A)$, $\delta^*(A) = \limsup\limits_{n\to\infty} \delta_n(A)$. $\delta_*(A)$ and $\delta^*(A)$ are called lower and upper asymptotic density of the set A, respectively. If $\delta_*(A) = \delta^*(A) = \delta(A)$, then $\delta(A)$ is called asymptotic (or statistical) density of A. It should be noted that the statistical convergence is determined by means of this concept, namely, the consequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called statistically convergent to x, if $\delta(A_\varepsilon) = 0$, for $\forall\varepsilon > 0$, where $A_\varepsilon \equiv \{n \in \mathbb{N} : \rho(x_n;x) \geq \varepsilon\}$.

Let us also recall the definitions of the ideal and the filter.

A family of sets $I \subset 2^{\mathbb{N}}$ is called an ideal if: α) $\emptyset \in I$; β) $A;B \in I \Rightarrow A \cup B \in I$; γ) $(A \in I \wedge B \subset A) \Rightarrow B \in I$.

A family $\mathscr{F} \subset 2^{\mathbb{N}}$ is called a filter on X, if :

i) $\emptyset \notin \mathscr{F}$;

ii) from $A;B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$;

iii) from $A \in \mathscr{F} \wedge (A \subset B) \Rightarrow B \in \mathscr{F}$.

Filter, satisfying the condition

iv) if $A_1 \supset A_2 \supset ... \wedge A_n \in \mathscr{F}, \forall n \in \mathbb{N} \Rightarrow \exists\{n_m\}_{m\in\mathbb{N}} \subset \mathbb{N}; n_1 < n_2 < ... : \cup_{m=1}^{\infty}((n_m, n_{m+1}] \cap A_m) \in \mathscr{F}$ is called a monotone closed filter.

Filter $\mathscr{F}$ satisfying the following condition is called a right filter:

v) $F^C \in \mathscr{F}$, for any finite subset $F \subset \mathbb{N}$.

An ideal I is called non-trivial if $I \neq \emptyset \wedge I \neq X$. $I \subset 2^{\mathbb{N}}$ is a non-trivial ideal if and only if $\mathscr{F} = \mathscr{F}(I) = \{X\backslash A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^{\mathbb{N}}$ is called admissible if and only if $I \supset \{\{x\} : x \in X\}$.

In the sequel, we assume that $(X;\rho)$ is a metric space with metric ρ, and $I \subset 2^{\mathbb{N}}$ is some non-trivial ideal.

Definition 1 [23]. *The sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called I-convergent to $x \in X$ (I-$\lim\limits_{n\to\infty} x_n = x$),* if $A_\varepsilon \in I$, $\forall\varepsilon > 0$, *where* $A_\varepsilon = \{n \in \mathbb{N} : \rho(x_n; x) \geq \varepsilon\}$.

Let $I_d \equiv \{A \subset \mathbb{N} : d(A) = 0\}$. I_d is an ideal on $\mathbb{N}$. I_d*-convergence* means the statistical convergence.

It should be noted that if I is an admissible ideal, then the usual convergence in X implies I*-convergence* in X.

Definition 2. *The sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called I^*-convergent to $x \in X$, if $\exists M \in \mathscr{F}(I)$ (i.e. $\mathbb{N}\backslash M \in I$), $M = \{m_1 < m_2 < ... < m_k < ...\}$: $\lim\limits_{k\to\infty} \rho(x_{m_k}; x) = 0$.*

In the following discussion we will need the following interesting results of [23].

Theorem 1 [23]. *Let I be an admissible ideal. If I^*-$\lim\limits_{n\to\infty} x_n = x \Rightarrow I$-$\lim\limits_{n\to\infty} x_n = x$.*

The converse is not always true, it depends on the structure of space $(X;\rho)$, namely, we have

Theorem 2 [23]. *Let $(X;\rho)$ be a metric space. (i) If X has no accumulation point, then I-convergence and I^*-convergence coincide for each admissible ideal $I \subset 2^{\mathbb{N}}$; (ii) If X has an accumulation point ξ, then there exists an admissible ideal $I \subset 2^{\mathbb{N}}$ and a sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$: I-$\lim\limits_{n\to\infty} y_n = \xi$, but I^*-$\lim\limits_{n\to\infty} y_n$ does not exist.*

3. Main results

Let $(X;\rho)$ be some complete metric space and $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter. Accept the following

Definition 3. *Let $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter. The sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called $\mathscr{F}$-convergent to $x \in X$ ($\mathscr{F}$-$\lim\limits_{n\to\infty} x_n = x$), if $A_\varepsilon \in \mathscr{F}$, $\forall\varepsilon > 0$, where $A_\varepsilon \equiv \{n \in \mathbb{N} : x_n \in O_\varepsilon(x)\}$.*

Let us introduce the concept of $\mathscr{F}$*-fundamentality.*

Definition 4. *The sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called $\mathscr{F}$-fundamental, if $\forall\varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} : \Delta_{n_\varepsilon} \in \mathscr{F}$, where $\Delta_{n_\varepsilon} \equiv \{n \in \mathbb{N} : x_n \in O_\varepsilon(x_{n_\varepsilon})\}$.*

Assume that $\exists\mathscr{F}$-$\lim\limits_{n\to\infty} x_n = x$. Let $\varepsilon > 0$ be an arbitrary number. Consequently, $A_{\varepsilon/2} \in \mathscr{F}$. From the condition i) in the definition of filter above it follows that $A_{\varepsilon/2} \neq \emptyset$. Take $\forall n_\varepsilon \in A_{\varepsilon/2} : \rho(x_{n_\varepsilon}; x) < \frac{\varepsilon}{2}$. From the relation

$$\rho(x_n; x_{n_\varepsilon}) \leq \rho(x_n; x) + \rho(x; x_{n_\varepsilon}) < \frac{\varepsilon}{2} + \rho(x_n; x),$$

it directly follows that

$$\left\{n \in \mathbb{N} : \rho(x_n; x) < \frac{\varepsilon}{2}\right\} \subset \{n \in \mathbb{N} : \rho(x_n; x_{n_\varepsilon}) < \varepsilon\}.$$

Hence $\Delta_{n_\varepsilon} \in \mathscr{F}$, i.e. the sequence $\{x_n\}_{n\in\mathbb{N}}$ *is $\mathscr{F}$-fundamental* in X.

Now, vice versa, let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be *$\mathscr{F}$-fundamental.* From *$\mathscr{F}$- fundamentality* it follows that $\exists n_j \in \mathbb{N} : K_j \in \mathscr{F}$, where $K_j \equiv \{n \in \mathbb{N} : \rho(x_n; x_{n_j}) \leq 2^{1-j}\}$, $j = 1, 2$.

By the definition of filter we obtain $K_1 \cap K_2 \in \mathscr{F}$. Put $M_1 \equiv \overline{O_1(x_{n_1})} \cap \overline{O_{2^{-1}}(x_{n_2})}$. It is obvious that $x_n \in M_1$, $\forall n \in (K_1 \cap K_2) \equiv K_{(1)}$. Thus, $\exists n_3 \in \mathbb{N} : K_3 \in \mathscr{F}$, where $K_3 \equiv \left\{n : \rho(x_n; x_{n_3}) \le 2^{-2}\right\}$. Let $K_{(2)} = K_{(1)} \cap K_3$. It is clear that $K_{(2)} \in \mathscr{F}$. Now let $M_2 \equiv M_1 \cap \overline{O_{2^{-2}}(x_{n_3})}$. Denote by $d_\rho(M)$ the diameter of the set M, i.e.

$$d_\rho(M) = \sup_{x,y \in M} \rho(x;y).$$

Continuing in the same way, we obtain the nested sequence of closed sets $\{M_n\}_{n\in\mathbb{N}} \subset X$: $M_1 \supset M_2 \supset ...$; whose diameters tend to zero: i.e. $d_\rho(M_n) \le 2^{-n+1} \to 0$, $n \to \infty$. Moreover, $K_{(n)} \in \mathscr{F}$, where $K_{(n)} \equiv \{n \in \mathbb{N} : x_n \in M_n\}$. Take $\forall \tilde{x}_n \in M_n$, $\forall n \in \mathbb{N}$. We have

$$\rho(\tilde{x}_n; \tilde{x}_{n+p}) \le d_\rho(M_n) \to 0,\ n \to \infty,\ \forall p \in \mathbb{N}.$$

Hence, the sequence $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ is fundamental in X. Let $\lim_{n\to\infty} \tilde{x}_n = x$. It is absolutely clear that $x \in \cap_n M_n$, i.e. $\cap_n M_n$ is non-empty. From $d_\rho(M_n) \to 0$, $n \to \infty$, it directly follows that $\cap_n M_n \equiv \{x\}$, i.e. $\cap_n M_n$ consists of one element. Let us show that $\mathscr{F}\text{-}\lim_{n\to\infty} x_n = x$. Take $\forall \varepsilon > 0$. Take $n_\varepsilon \in \mathbb{N} : d_\rho(M_{n_\varepsilon}) < \varepsilon$. Let $y \in M_{n_\varepsilon}$ be an arbitrary element. So

$$\rho(y, x) \le d_\rho(M_{n_\varepsilon}) < \varepsilon.$$

Consequently, $M_{n_\varepsilon} \subset O_\varepsilon(x)$. We have $K_{(n_\varepsilon)} \in \mathscr{F}$, where $K_{(n_\varepsilon)} = \{n \in \mathbb{N} : x_n \in M_{n_\varepsilon}\}$. So, $K_{(n_\varepsilon)} \subset \{n \in \mathbb{N} : x_n \in O_\varepsilon(x)\}$, it is clear that $\{n \in \mathbb{N} : x_n \in O_\varepsilon(x)\} \in \mathscr{F} \Rightarrow \mathscr{F}\text{-}\lim_{n\to\infty} x_n = x$. Thus, we have proved the following theorem.

Theorem 3. *Let $(X;\rho)$ be complete metric space and $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter. The sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is $\mathscr{F}$-convergent in X if and only if it is $\mathscr{F}$-fundamental in X.*

It is easy to see that $\mathscr{F}\text{-}\lim_{n\to\infty} x_n$ is unique if it exists. In fact, let $\mathscr{F}\text{-}\lim_{n\to\infty} x_n$ has two values $y_1 \ne y_2$. Take $\forall \varepsilon \in \left(0, \frac{1}{2}\rho(y_1; y_2)\right)$. Let $A_k \equiv \{n \in \mathbb{N} : \rho(x_n; y_k) < \varepsilon\}$, $k = 1, 2$. It is clear that $A_k \in \mathscr{F}$, $k = 1, 2 \Rightarrow A_1 \cap A_2 \in \mathscr{F}$. As $A_1 \cap A_2 = \emptyset \notin \mathscr{F}$, the obtained contradiction proves that $y_1 = y_2$.

Let us consider the sequence $\left\{K_{(n)}\right\}_{n\in\mathbb{N}}$, constructed in the proof of Theorem 1. We have $K_{(1)} \supset K_{(2)} \supset ... \wedge K_{(n)} \in \mathscr{F}$, $\forall n \in \mathbb{N}$. Then from the condition iv) in the definition of filter we have

$$\exists \{n_m : n_1 < n_2 < ...\} \ : \cup_{m=1}^{\infty} \left((n_m, n_{m+1}] \cap K_{(m)}\right) \in \mathscr{F}.$$

Assume

$$\mathbb{N}_0 \equiv \left\{k \in \mathbb{N} :\ k \in (n_m, n_{m+1}] \cap K^C_{(m)},\ m \in \mathbb{N}\right\} \cup [1, n_1],$$

where $M^C \equiv \mathbb{N} \backslash M$. Define

$$y_k = \begin{cases} x, & k \in \mathbb{N}_0\ ; \\ x_k, & otherwise, \end{cases}$$

where $\mathscr{F}\text{-}\lim_{n\to\infty} x_n = x$. Take $\forall \varepsilon > 0$. If $k \in \mathbb{N}_0$, then $\rho(y_k; x) = \rho(x; x) < \varepsilon$. If $k \notin N_0$, then $\exists m : n_m < k \le n_{m+1} \wedge k \notin K^c_{(m)} \Rightarrow k \in K_{(m)} \Rightarrow x_k \in M_m$ $(M_1 \supset M_2 \supset ...$

is a sequence from Theorem 1) $\Rightarrow \rho(x_k;x) \le d_\rho(M_m) < \varepsilon$ for sufficiently great values of m (as $x \in M_m$, $\forall m \in \mathbb{N}$). Hence, we have $\lim_{k\to\infty} y_k = x$. Let us show that $\tilde{K} \equiv \{k \in \mathbb{N} : x_k = y_k\} \in \mathscr{F}$. In fact, it is clear that

$$\cup_{m=1}^{\infty} \left((n_m, n_{m+1}] \cap K_{(m)}\right) \subset \tilde{K},$$

holds. Then from the condition iii) in the definition of filter we get $\tilde{K} \in \mathscr{F}$. Thus, if $\mathscr{F}\text{-}\lim_{n\to\infty} x_n = x$, then $\exists \tilde{K} \in \mathscr{F}$: $\lim_{n\to\infty} y_n = x$ and $x_n = y_n$, $\forall n \in \tilde{K}$.

Conversely, let $\lim_{n\to\infty} y_n = x$ and $\tilde{K} \equiv \{n : x_n = y_n\} \in \mathscr{F}$. Take $\forall \varepsilon > 0$. Then $\exists n_\varepsilon \in \mathbb{N} : \rho(y_n;x) < \varepsilon$, $\forall n \ge n_\varepsilon$. We have $\{n \in \mathbb{N} : n \ge n_\varepsilon\} \cap \tilde{K} \subset \{n \in \mathbb{N} : \rho(x_n;x) < \varepsilon\}$. It is clear that $\left(\{n \in \mathbb{N} : n \ge n_\varepsilon\} \cap \tilde{K}\right) \in \mathscr{F}$. Then from the condition iii) in the definition of filter it follows $\{n \in \mathbb{N} : \rho(x_n;x) < \varepsilon\} \in \mathscr{F}$. Thus, the following theorem is true.

Theorem 4. *Let $(X;\rho)$ be a metric space and $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter. Then: 1) if $\mathscr{F}$ is a monotone close and $\mathscr{F}\text{-}\lim_{n\to\infty} x_n = x$, then $\exists \{y_n\}_{n\in\mathbb{N}} \subset X : \lim_{n\to\infty} y_n = x \wedge \{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{F}$;* 2) if *$\mathscr{F}$ is a right filter and $\lim_{n\to\infty} y_n = x \wedge (\{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{F})$, then $\mathscr{F}\text{-}\lim_{n\to\infty} x_n = x$.*

The Theorems 1;2 imply the following

Corollary 1. *Let $(X;\rho)$ be a complete metric space, $\mathscr{F} \subset 2^{\mathbb{N}}$ be some monotone close and right filter. Then the following statements are equivalent to each other:*
α) $\exists \mathscr{F}\text{-}\lim_{n\to\infty} x_n = x$; β) $\{x_n\}_{n\in\mathbb{N}}$ is $\mathscr{F}$-fundamental; γ) $\exists \lim_{n\to\infty} y_n = x \wedge (\{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{F})$.

The Theorem 2 immediately implies the following

Corollary 2. *Let $(X;\rho)$ be a metric space and $\mathscr{F} \subset 2^{\mathbb{N}}$ be a right filter. If $\exists \mathscr{F}\text{-}\lim_{n\to\infty} x_n = x$, then $\exists \{n_k : n_1 < n_2 < ...\} \in \mathscr{F}$: $\lim_{k\to\infty} x_{n_k} = x$.*

4. Filters

I. An ordinary convergence. $\mathscr{F} \equiv \{M \subset \mathbb{N} : M^C \equiv \mathbb{N}\backslash M$ *is a finite set*$\}$. *$\mathscr{F}$-convergence,* generated by this filter, coincides with the ordinary convergence.

II. Statistical convergence. Assume $\mathscr{F}_\delta \equiv \{M \subset \mathbb{N} : \delta(M) = 1\}$. $\mathscr{F}_\delta$ is a filter. It is easy to see that $\mathscr{F}_\delta$ is a right filter. Let us show that $\mathscr{F}_\delta$ is a monotone close filter. Let $K_1 \supset K_2 \supset ... \wedge (\delta(K_n) = 1, \forall n \in \mathbb{N})$. It is clear that $\delta(K_n^c) = 0$, $\forall n \in \mathbb{N}$. Therefore $\exists \{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$; $n_1 < n_2 < ...$:

$$\frac{1}{n} |I_n \cap K_m^c| < \frac{1}{m}, \ \forall n \ge n_m.$$

Let $\mathbb{N}_0 = \tilde{\mathbb{N}}_0 \cup I_n$, where $\tilde{\mathbb{N}}_0 \equiv \{k \in \mathbb{N} : n_m < k \le n_{m+1} \wedge (k \in K_m^c)\}$. It is obvious that $\delta(\mathbb{N}_0) = \delta\left(\tilde{\mathbb{N}}_0\right)$. Take $\forall n \in \mathbb{N}$. Then $\exists m \in \mathbb{N} : n_m < n \le n_{m+1}$. Without loss of generality, we may suppose that $n > n_1$. Let us show that

$$\left(I_n \cap \tilde{\mathbb{N}}_0\right) \subset \left(I_n \cap K_m^c\right). \qquad (1)$$

Let $k \in \left(I_n \cap \tilde{\mathbb{N}}_0\right) \Rightarrow \exists m_0 \leq m : n_{m_0} < k \leq n_{m_0+1} \wedge \left(k \in K^c_{m_0}\right) \Rightarrow k \in K^c_m$. So, the inclusion (1) is true. Consequently

$$\frac{1}{n}\left|I_n \cap \tilde{\mathbb{N}}_0\right| \leq \frac{1}{n}\left|I_n \cap K^c_m\right| < \frac{1}{m}\,. \tag{2}$$

From (2) it directly follows that $\delta\left(\tilde{\mathbb{N}}_0\right) = 0$. As a result, $\delta(\mathbb{N}_0) = 0 \Rightarrow \delta(\mathbb{N}^c_0) = 1 \Rightarrow \mathbb{N}^c_0 \in \mathscr{F}_\delta$. In the sequel, it should be pointed out that $\mathbb{N}^c_0 \equiv \{k \in \mathbb{N} : n_m < k \leq n_{m+1} \wedge (k \in K_m)\}$. Thus, $\mathscr{F}_\delta$ is a monotone close filter. Fulfilment of condition v) by $\mathscr{F}_\delta$ is obvious. Then, the statement of Corollary 1 is true with respect to $\mathscr{F}_\delta$-*convergence.* So,we get the validity of

Statement 1. *Filter $\mathscr{F}_\delta$, generated by statistical density, is a monotone close and right filter.*

III. Logarithmic convergence. Let $M \subset \mathbb{N}$. Assume

$$l_n(M) = \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_M(k)}{k},$$

where $s_n = \sum_{k=1}^{n}\frac{1}{k}$. If $\exists \lim_{n\to\infty} l_n(M) = l(M)$, then $l(M)$ is called a logarithmic density of the set M. Let $\mathscr{F}_l \equiv \{M \subset \mathbb{N} : l(M) = 1\}$. The following lemma is true.

Lemma 1. *If* $l(M_k) = 1,\ k = 1, 2 \Rightarrow l(M_1 \cap M_2) = 1$.

Proof. We have

$$M_1 \cap M_2 = (M_1 \cup M_2) \setminus [(M_2 \backslash M_1) \cup (M_1 \backslash M_2)]\,.$$

Consequently

$$M_1 \cap M_2 \cap I_n = [(M_1 \cup M_2) \cap I_n] \setminus [((M_2 \backslash M_1) \cup (M_1 \backslash M_2)) \cap I_n]\,. \tag{3}$$

From

$$((M_2 \backslash M_1) \cap I_n) \subset (M^c_1 \cap I_n)\,,$$

we get

$$\frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{M_2\backslash M_1}(k) \leq \frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{M^c_1}(k)\,. \tag{4}$$

It is absolutely clear that, if $l(M) = 1$, then $l(M^c) = 0$. Then from (4) we obtain $l(M_2 \backslash M_1) = 0$. Similarly, we have $l(M_1 \backslash M_2) = 0$. So

$$((M_2 \backslash M_1) \cup (M_1 \backslash M_2)) \cap I_n = ((M_2 \backslash M_1) \cap I_n) \cup ((M_1 \backslash M_2) \cap I_n)\,.$$

It is clear that

$$l((M_2 \backslash M_1) \cup (M_1 \backslash M_2)) = 0. \tag{5}$$

It is easy to see that $l(M_1 \cup M_2) = 1$. From (3) we have

$$\frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{M_1\cap M_2}(k) = \frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{M_1\cup M_2}(k) - \frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{(M_2\backslash M_1)\cup(M_1\backslash M_2)}(k).$$

Taking into account (5) we get $l(M_1 \cap M_2) = 1$. Lemma is proved.

This lemma implies that $\mathscr{F}_l$ is a filter. If $M \subset \mathbb{N}$ is a finite set, then it is clear that $M^C \in \mathscr{F}_l$, i.e. $\mathscr{F}_l$ satisfies the condition v). Then it is absolutely clear that $l(M) = 0$. Let us show that $\mathscr{F}_l$ is a monotone close filter. Let $K_1 \supset K_2 \supset ... \wedge (l(K_n) = 1, \forall n \in \mathbb{N}) \Rightarrow l(K_n^c) = 0, \forall n \in \mathbb{N}$. Therefore

$$\exists \{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N},\, n_1 < n_2 < ... : \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_{K_m^c}(k)}{k} < \frac{1}{m},\ \forall n \geq n_m.$$

Similarly to the previous example, let $\mathbb{N}_0 = \tilde{\mathbb{N}}_0 \cup I_n$, where

$$\tilde{\mathbb{N}}_0 \equiv \{k \in \mathbb{N} : n_m \leq k \leq n_{m+1} \wedge (k \in K_m^c)\}.$$

It is clear that $l(\mathbb{N}_0) = l\left(\tilde{\mathbb{N}}_0\right)$. Let $n \in \mathbb{N} \Rightarrow \exists m \in \mathbb{N} : n_m < n \leq n_{m+1}$. As before, we assume that $n > n_1$. It is clear that (1) is true, i.e. .

$$\left(I_n \cap \tilde{\mathbb{N}}_0\right) \subset (I_n \cap K_m^c).$$

Hence

$$\frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_{\tilde{N}_0}(k)}{k} \leq \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_{K_m^c}(k)}{k} < \frac{1}{m},\ \forall n \geq n_m.$$

Consequently, $l\left(\tilde{\mathbb{N}}_0\right) = 0 \Rightarrow l(\mathbb{N}_0) = 0 \Rightarrow l(\mathbb{N}_0^c) = 1 \Rightarrow \mathbb{N}_0^c \in \mathscr{F}_l$. It is obvious that

$$\mathbb{N}_0^c \equiv \{k \in \mathbb{N} : n_m < k \leq n_{m+1} \wedge (k \in K_m)\}.$$

It directly follows that $\mathscr{F}_l$ is a right filter. Thus, we have proved

Statement 2. *Filter $\mathscr{F}_l$, generated by logarithmic density, is a monotone close and right filter.*

Note that, if $\exists\delta(M) \Rightarrow \exists l(M) \wedge l(M) = \delta(M)$. The converse is not generally true.

IV. Uniform convergence. Let $M \subset \mathbb{N} \wedge (t \in \mathbb{Z}_+; s \in \mathbb{N})$. Assume

$$M(t+1; t+s) = |n \in M : t+1 \leq n \leq t+s|.$$

Put

$$\beta_s(M) = \liminf_{t\to\infty} M(t+1; t+s),$$

$$\beta^s(M) = \limsup_{t\to\infty} M(t+1; t+s).$$

If $\lim\limits_{s\to\infty} \frac{\beta_s(M)}{s} = \lim\limits_{s\to\infty} \frac{\beta^s(M)}{s} = \beta(M)$, then the quantity $\beta(M)$ is called the uniform density of the set M. Let $\mathscr{F}_\beta \equiv \{M \subset \mathbb{N} : \beta(M) = 1\}$. Let us show that $\mathscr{F}_\beta$ is a filter. It is clear that

$$M(t+1;t+s) + M^c(t+1;t+s) = |\,[t+1,t+s]\,| = s.$$

Hence it directly follows that $\beta(M) = 1 \Leftrightarrow \beta(M^c) = 0$. $I_\beta \equiv \{M \subset \mathbb{N} : \beta(M) = 0\}$ is a non-trivial ideal [23]. Therefore, $\mathscr{F}_\beta$ is a filter. It is clear that $\mathscr{F}_\beta$ satisfies the condition v). Let us show that $\mathscr{F}_\beta$ is a monotone close filter. Let $K_1 \supset K_2 \supset ... \wedge (\beta(K_n) = 1, \forall n \in \mathbb{N})$ $\Rightarrow \beta(K_n^c) = 0, \forall n \in \mathbb{N} \Rightarrow \exists \{n_k\}_{k\in\mathbb{N}} \subset N,\ n_1 < n_2 < ... :$

$$\frac{\beta^s(K_m^c)}{s} < \frac{1}{m},\ \forall s \geq n_m.$$

As before, we set $\mathbb{N}_0 = \tilde{\mathbb{N}}_0 \cup I_{n_1}$, where $\tilde{\mathbb{N}}_0 \equiv \{k \in \mathbb{N} : n_m \leq k \leq n_{m+1} \wedge (k \in K_m^c)\}$. It is clear that $\beta(\mathbb{N}_0) = \beta\left(\tilde{\mathbb{N}}_0\right)$. Let $n > n_1$ be an arbitrary integer. Then $\exists m \in \mathbb{N} : n_m < n \leq n_{m+1}$. It is obvious that the inclusion

$$\left(I_n \cap \tilde{\mathbb{N}}_0\right) \subset (I_n \cap K_m^c),$$

is true in this case, too. From the arbitrariness of n we have

$$\left(\tilde{\mathbb{N}}_0 \cap [t+1;t+s]\right) \subset (K_m^c \cap [t+1;t+s]).$$

Consequently

$$\tilde{\mathbb{N}}_0(t+1;t+s) \leq K_m^c(t+1;t+s),$$

and as a result

$$\beta^s\left(\tilde{\mathbb{N}}_0\right) \leq \beta^s(K_m^c).$$

Thus

$$\frac{\beta^s\left(\tilde{\mathbb{N}}_0\right)}{s} \leq \frac{\beta^s(K_m^c)}{s} < \frac{1}{m},\ \ \forall s \geq n_m.$$

From this relation it directly follows

$$\beta\left(\tilde{\mathbb{N}}_0\right) = 0 \Rightarrow \beta(\mathbb{N}_0) = 0 \Rightarrow \beta(\mathbb{N}_0^c) = 1 \Rightarrow \mathbb{N}_0^c \in \mathscr{F}_\beta,$$

where

$$\mathbb{N}_0^c \equiv \{k \in \mathbb{N} : n_m < k \leq n_{m+1} \wedge (k \in K_m)\},$$

i.e. $\mathscr{F}_\beta$ is a monotone close filter. As a result, we obtain the validity of the following

Statement 3. *Filter $\mathscr{F}_\beta$, generated by the uniform convergence, is a monotone close and right filter.*

Following [23], number of such examples can be extended.

Remark 1. *Similar results can be obtained with respect to concepts of I-convergence and I^*-convergence.*

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Identities Involving the Terms of a Balancing-Like Sequence Via Matrices

P. K. Ray

Abstract. The goal of this paper is to establish some identities involving the terms of a newly introduced sequence $\{u_n\}_{n=0}^{\infty}$ called as the balancing-like sequence defined recursively by $u_n = 6au_{n-1} - u_{n-2}$ with initials $u_0 = 0, u_1 = 1$ via certain matrices.

Key Words and Phrases: Balancing numbers, Recurrence relation, Balancing matrix.
2000 Mathematics Subject Classifications: 11B37, 11B39

1. Introduction

It is well known that, the sequence of balancing numbers $\{B_n\}$ is defined recursively by the equation

$$B_n = 6B_{n-1} - B_{n-2},\ n \geq 2,$$

with initial conditions $B_0 = 0$ and $B_1 = 1$ [1]. Whereas the companion to these numbers is the sequence of Lucas-balancing numbers $\{C_n\}$ which is defined recursively by

$$C_n = 6C_{n-1} - C_{n-2},\ n \geq 2$$

with $C_0 = 1$ and $C_1 = 3$ [7, 8]. Both these numbers can also be extended negatively. The following results were established in [1].

$$B_{-n} = -B_n,\ \ C_{-n} = C_n.$$

The Binet's formulas for both balancing and Lucas-balancing numbers are respectively given by

$$B_n = \frac{\lambda_1^n - \lambda_1^n}{\lambda_1 - \lambda_2},\ \ C_n = \frac{\lambda_1^n + \lambda_1^n}{2}.$$

The generalizations for balancing numbers were done in different ways. To know in details about balancing numbers and their generalization, one can go through [2-5]. There is another way to generate balancing numbers through matrices. In [9], Ray introduced balancing matrix

$$Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix},$$

which is a second order matrix and whose entries are the first three balancing numbers 0, 1 and 6. He has also shown that

$$Q_B^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix},$$

for every integer n [9]. Later, he has deduced nice product formulas for both negatively and positively subscripted balancing and Lucas-balancing numbers [10, 12]. Ray has established many interesting identities for both balancing and Lucas-balancing numbers through matrices [9-15].

In this study, we will first introduce a new sequence which we call balancing-like sequence $\{u_n\}_{n=0}^{\infty}$ defined recursively by

$$u_n = 6au_{n-1} - u_{n-2}, \tag{1.1}$$

with initials $u_0 = 0, u_1 = 1$, where $n \geq 2$. Then we will define a second order matrix which we call balancing Q_B-like matrix whose entries are the first three balancing-like numbers $0, 1$, and $6a$. Later, we will show that the higher powers of this matrix also contain the balancing-like numbers. These matrices will be used to obtain identities involving the terms of a balancing-like sequence.$\{u_n\}_{n=0}^{\infty}$. From (1.1) we notice that, the first few terms of the balancing-like sequence are

$$0,\ 1,\ 6a,\ 36a^2-1,\ 216a^3-12a,\ 1296a^4-108a^2+1,$$
$$7776a^5-864a^3+18a,\ 46656a^6-6480a^4+216a^2-1.$$

Also observe that, for $a = 1$ the balancing-like numbers $\{u_n\}$ reduce to the balancing numbers.

2. Some identities involving balancing-like numbers

2.1. Binet's formula

In this section, we will establish Binet' formula for balancing-like numbers and the identity involving negatively subscripted balancing-like numbers.

Solving the homogenous recurrence relation (1.1), its characteristic equation $\lambda^2 - 6a\lambda + 1 = 0$ has the roots

$$v = 3a + \sqrt{9a^2-1}, \quad w = 3a - \sqrt{9a^2-1}.$$

The general solution of (1.1) is given by

$$u_n = c_1 v^n + c_2 w^n, \tag{2.1}$$

where c_1 and c_2 are arbitrary constants. Using the initial conditions given in (1.1), we obtain the following system of equations

$$u_0 = c_1 + c_2 = 0, \quad u_1 = c_1 v + c_2 w = 1.$$

Solving these two equations, we get

$$c_1 = \frac{1}{2\sqrt{9a^2-1}}, \quad c_2 = \frac{-1}{2\sqrt{9a^2-1}},$$

for $3a \neq \pm 1$. Therefore, equation (2.1) reduces to the following identity

$$u_n = \frac{v^n - w^n}{2\sqrt{9a^2-1}}, \tag{2.2}$$

which we call the Binet's formula of balancing-like numbers.

2.2. Negatively subscripted balancing-like numbers

We can also extend the sequence $\{u_n\}$ to the negative values of n. Since the product of v and w is one, therefore for $n \geq 0$, the Binet's formula (2.2) reduces to

$$u_{-n} = \frac{v^{-n} - w^{-n}}{2\sqrt{9a^2-1}} = \frac{w^n - v^n}{2\sqrt{9a^2-1}} = -u_n. \tag{2.3}$$

It is easy to show that, (2.3) holds for all negative integers n. Let $n < 0$, say $n = -m$, where $m > 0$. Using (2.3), we observe that

$$\begin{aligned} u_n - 6au_{n-1} + u_{n-2} &= u_{-m} - 6au_{-m-1} + u_{-m-2} \\ &= -u_m + 6au_{m+1} - u_{m+2} = u_{m+2} - u_{m+2} = 0. \end{aligned}$$

3. Balancing-like numbers via matrices

As stated before, the balancing Q_B matrix has introduced by Ray in [9]. In this section, we introduce balancing-like Q_B matrix, denoted by Q_u and defined by

$$Q_u = \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

whose entries are the first three balancing-like numbers $0, 1$ and $6a$, where a is arbitrary.

The following theorem shows that the sequence $\{u_n\}_{n=0}^{\infty}$ can also be generated by matrix multiplication.

Theorem 3.1. *If* $Q_u = \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}$ *as defined in (3.1), then for all positive integers* n, $Q_u^n = \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix}$.

Proof. This result can be proved by method of induction. The basis step is clear. In the inductive step, suppose the result holds for all integers $\leq n$. Using (1.1) and the hypothesis, we observe that

$$Q_u^{n+1} = Q_u^n \cdot Q_u = \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix} \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6au_{n+1} - u_n & -u_{n+1} \\ u_{n+1} & u_n \end{pmatrix} = \begin{pmatrix} u_{n+2} & -u_{n+1} \\ u_{n+1} & -u_n \end{pmatrix},$$

which completes the proof of the theorem.

We notice that, the balancing-like matrices also satisfy the same recurrence relation as that of balancing-like numbers, that is for $n \geq 1$ we have

$$Q_u^{n+1} = 6aQ_u^n - Q_u^{n-1}, \tag{3.2}$$

with initial conditions $Q_u^0 = I$ and $Q_u^1 = \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}$, where I is the identity matrix.

Since $Q_u^{m+n} = Q_u^m \cdot Q_u^n$, comparing the corresponding entries of both the matrices, we have the following identities:

$$\begin{aligned} &i)\ \ u_{m+n+1} = u_{m+1}u_{n+1} - u_m u_n \\ &ii)\ \ u_{m+n} = u_{m+1}u_n - u_m u_{n-1} = u_{n+1}u_m - u_n u_{m-1} \\ &iii)\ u_{m+n-1} = u_{m-1}u_{n-1} - u_m u_n. \end{aligned}$$

For $a = 1$, the similar results for balancing numbers are found in [7-9].

Theorem 3.1 can be used to obtain the following identities involving terms of the sequence $\{u_n\}$.

Corollary 3.1. *For $n \geq 1$, the following identities are valid:*

$$u_{2n+1} = u_{n+1}^2 - u_n^2 \ \text{ and } \ u_{2n} = u_n(u_{n+1} - u_{n-1}).$$

Proof. Since $Q_u^{2n} = (Q_u^n)^2$, we have

$$\begin{aligned} \begin{pmatrix} u_{2n+1} & -u_{2n} \\ u_{2n} & -u_{2n-1} \end{pmatrix} &= \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix} \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} u_{n+1}^2 - u_n^2 & u_n u_{n-1} - u_n u_{n+1} \\ u_n u_{n+1} - u_n u_{n-1} & -(u_n^2 - u_{n-1}^2) \end{pmatrix}. \end{aligned}$$

The results follow by equating corresponding rows and columns from both sides as the above matrix.

We notice that for $a = 1$, formulas for balancing numbers analogous to the last two identities are given in [7, 8]. In general, if k is a positive integer, then $Q_u^{kn} = (Q_u^n)^k$, that is

$$\begin{pmatrix} u_{kn+1} & -u_{kn} \\ u_{kn} & -u_{kn-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix}^k.$$

The determinants of the above matrices give the identity

$$u_{kn}^2 - u_{kn+1}u_{kn-1} = (u_n^2 - u_{n+1}u_{n-1})^k.$$

In fact, both sides of the above equation are equal to 1 as $\det Q_u = 1$. Therefore we have the following important identity for balancing-like numbers:

$$u_{kn}^2 - u_{kn+1}u_{kn-1} = 1, \tag{3.3}$$

which we call Cassini formula for balancing-like numbers. Recall that [8], the Cassini formula for balancing numbers is given by the identity

$$B_n^2 - B_{n+1}B_{n-1}.$$

This formula play a vital role to find many important identities for balancing numbers and their related sequences and can be obtained by setting $a = 1$ in (3.3). Since $det(Q_u) \neq 0$, for $n \geq 0$ the following identity is valid:

$$Q_u^{-n} = (Q_u^{-1})^n. \tag{3.4}$$

The following result can be easily shown by induction.

Theorem 3.2. *If* $Q_u = \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}$ *as defined in (3.1), then for* $n \geq 0$, $Q_u^{-n} =$ $= \begin{pmatrix} u_{-n+1} & -u_{-n} \\ u_{-n} & -u_{-n-1} \end{pmatrix}$.

Since Theorem 3.2 is a special case of Theorem 3.1 with n negative, the identities proved in Theorem 3.1 also holds for $n < 0$.

At the end of the section, we now establish the following two important results.

Theorem 3.3. *If* $\binom{n}{k}$ *denote the usual notation for combination, then for* $n \geq 0$

$$u_{2n+1} = \sum_{k=0}^{n} (-1)^{k+1}(6a)^k \binom{n}{k} u_{k+1} \quad \text{and} \quad u_{2n} = \sum_{k=0}^{n} (-1)^{k+1}(6a)^k \binom{n}{k} u_k.$$

Proof. By (3.2), $Q_u^2 = 6aQ_u - I$. Both sides of this relation raised to a power n yields the following expression

$$Q_u^{2n} = (-I + 6a)^n.$$

For $n \geq 0$, the binomial expansion of the right hand side expression gives the following:

$$Q_u^{2n} = \sum_{k=0}^{n} (-1)^{k+1}(6a)^k \binom{n}{k} u_k.$$

Therefore by virtue of Theorem 3.1, we get

$$\begin{pmatrix} u_{2n+1} & -u_{2n} \\ u_{2n} & -u_{2n-1} \end{pmatrix} = \sum_{k=0}^{n} (-1)^{k+1}(6a)^k \binom{n}{k} \begin{pmatrix} u_{k+1} & -u_k \\ u_k & -u_{k-1} \end{pmatrix}.$$

Equating the corresponding entries, we get the desired result.

By (3.4) we observe that, $Q_u^{-2} = -I + 6aQ_u^{-1}$ and present following result.

Theorem 3.4. *For $n \geq 0$, we have*

$$u_{-2n+1} = \sum_{k=0}^{n} (-1)^{k+1}(-6a)^k \binom{n}{k} u_{-k+1} \quad \text{and} \quad u_{-2n} = \sum_{k=0}^{n} (-1)^{k+1}(-6a)^k \binom{n}{k} u_{-k}.$$

Proof. The proof is analogous to that of Theorem 3.3.

Remark 3.1. *All these identities we have proven so far in this paper, can be obtained directly by using the Binet's formula for balancing-like numbers given in (2.2). However, the matrix method is noticeably simpler.*

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Features Of Regression Modeling Of Solar Radiation With Different Types Of Functions

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Abstract. The paper investigates the characteristics of solar radiation process as stochastic processes. It was found that due to solar radiation multi factorial process modelling should be based on real statistical data. Time series of solar radiation was considered as a functional depending on four components:trend,more or less regular fluctuations relative to trend, seasonal fluctuations and non-systematic random effect. It is shown that non-systematic random effect can be defined as the difference between the statistics and the amount of trend and seasonality.Researches were conducted to determine the trend type on the basis of polynomial regression analysis and the Gaussian function, and seasonal variations on the basis of the Fourier series.

Key Words and Phrases: Solar radiation, time series, regression model, Gaussian function

1. Introduction

Simulation of solar radiation is necessary for design and operation of automatic control for photovoltaic systems. The ultimate goal of simulation is to calculate the dependence of the total amount of solar radiation on solar panels on geographic latitude, meteorological factors, day of year, time of day and the angle of inclination of a surface [1, 2].

Among techniques of analysis and calculation of solar radiation developed in recent years, the most fully studied one is the cloudless sky technique [3, 4].

However, a number of studies have shown that these methods give inaccurate results because of the significant influence of clouds on the amount of insolation caused by the weakening of direct radiation and an increase in diffuse fraction in most cases. Reduction of direct radiation due to increasing cloudiness is not compensated by dissipated insolation. Consequently, the total insolation, as well as the direct one, decreases with increasing cloudiness.

Simulation of the formation of clouds is a multi factorial problem. In this problem, cloudiness is usually taken as a random value with the distributive law that corresponds to real statistical data. For example, in [1] experimental observations are best described by the beta-law with the corresponding parameters of distribution. However, these results

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were based on average daily chart of the progress of solar radiation at the average cloudiness conditions. Optimal control of photovoltaic power system requires hourly values of cloudiness rate. This is not possible using the classical techniques as cloud formation is an uncontrollable time-varying process [5].

Radiation is a random variable below in Fig. 1. It is usually considered as a statistical phenomenon that develops over time according to the laws of the theory of probability. The sequence of observations is a time series, analysis of which can provide a stochastic model with a minimal number of parameters allowing calculation of the probability that some future value of insolation will lie within a certain range and at the same time adequately describing the process under study.

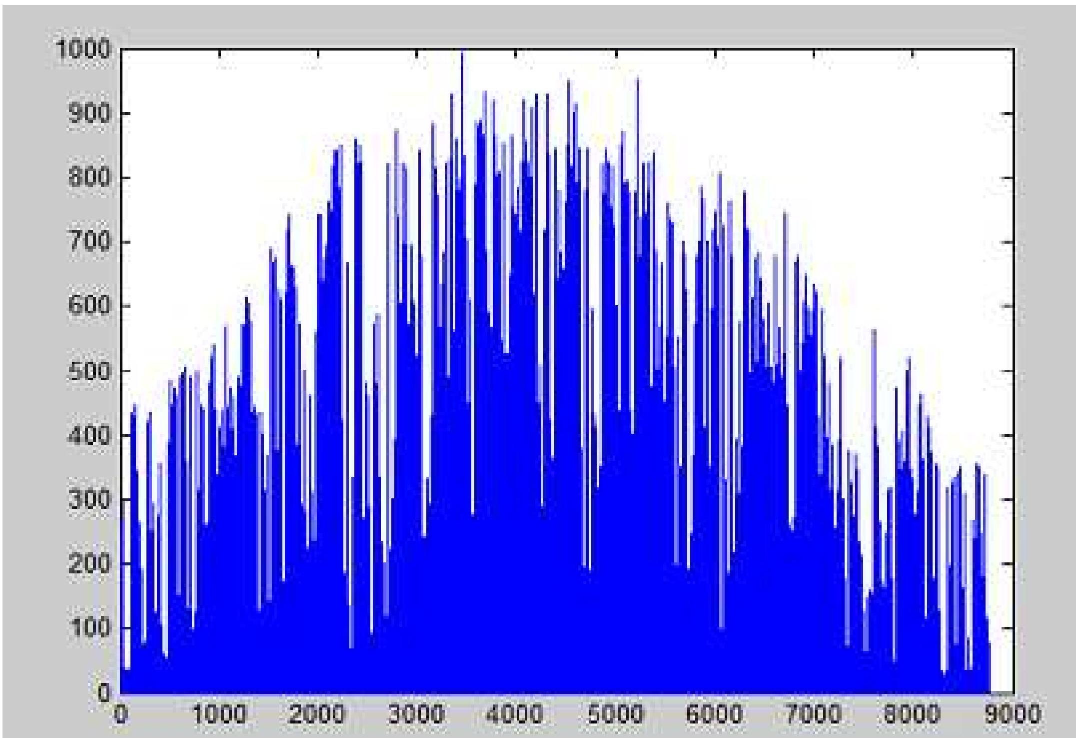


Figure 1. Hourly radiation in Baku in 2012 (X-axis shows the number of measurements, Z-axis shows total radiation in MJ/h).

Statistical approach is often used in the analysis of time series of solar radiation. Unlike other statistical objects, this time series has a characteristic feature: the observations are made sequentially in time and depend heavily on external influence. There is an unsystematic random effect of external factors in each measurement, which significantly complicates the process of simulation. If we consider several series of measurements as a multidimensional complex then we have to take into account the statistical relationship between these variables.

2. Setting objectives

It should be noted that in the general case, the time series of solar radiation is considered as a probability functional depending on four components: a trend, more or less regular fluctuations related to the trend, seasonal fluctuations and non-systematic random effect. Usually a time series is the sum of these four components. Let us consider each of them separately. The existence of a trend is explained by the presence of permanent forces operating uniformly in about the same direction of more or less regular fluctuations related to the trend which are happening due to the influence of regular perturbations that appear randomly. It is easier to present the components that are generated cyclically, such as daily and seasonal variations of solar radiation. But it should be noted that the values of these components are subject to daily and yearly changes. Nevertheless, a qualitative picture remains, which means that the seasonal effect has a trend. In general, if you can determine the trend and seasonal variations and subtract them from the data, then you have a fluctuating time series, which may represent a purely random fluctuation in one marginal case and fluent vibrational change in another.

To simulate the process of changes in solar radiation, each of these components has to be examined individually. As it was already mentioned, the first three components contain the oscillating process. Therefore it is of interest to determine the number of turning points, i.e., peak number, trough and the distribution of the distance between them.

3. Preliminary Analysis

Let's consider a finite number of values n of solar radiation q_1, q_2, ... q_n and let

$$X_i = \begin{cases} 1, & if \quad q_i < q_{i+1} > q_{i+2} \quad or \quad q_i > q_{i+1} < q_{i+2}, \\ 0, & otherwise. \end{cases}$$

Then the number of turning points is equal to

$$p = \sum_{i=1}^{n-2} X_i.$$

The expectancy of any of number of turning points is equal to $\frac{2}{3}(n-2)$. Research show that the daily number of turning points for the time series of solar radiation is 2, i.e. there is one maximum and one minimum for each one-day series of measurements, which corresponds to $n \geq 5$. This means that for a study of a time series we need at least 6 values of solar radiation daily.

As it was already mentioned, the first three components can be considered as deterministic components. Various methods are used for simulation of deterministic components of insolation, including regression and harmonic analysis. Experimental or actinometrical and meteorological observations are used. In some cases of international practice, computer databases are widely used to build a model, such as US NASA SSE and Swiss

METEONORM. The comparison of the values of intensity of solar radiation in NASA SSE database with those of Russian meteorological stations revealed the possibility of using NASA SSE program in Russia. In [4], the same comparison was made on the basis of METEONORM. It was concluded in result of comparing the values of the METEONORM and NASA SSE databases of solar radiation that they are useful for designing solar power plants in the absence of reliable values of ground-based observation stations, as well as for feasibility calculations for their construction.

In current studies on the modeling of various processes, including the modeling of solar radiation, approaches to extrapolation of the trend of the time series by analytic function, building of multivariate regression or auto regression models and its extended version - method of group accounting of arguments, are common. In addition, techniques based on wavelet transform and techniques of time series prediction based on neural network technology are popular.

4. Regression models

Tabular data of solar radiation by month for the city of Baku for 2012 are used for building regression models of solar radiation. Research was conducted on Microsoft Office EXCEL. Polynomial functions of various degrees, ranging from third degree, are accepted as a regression model (Table 1).

(Table 1)

Model	R^2	Relative errors
$-4.1586t^2 + 52.24t - 10.386$	$R^2 = 0.9119$	max=50%, min=0.9% average=16.2%
$0.0411t^3 - 4.9591t^2 + 56.571t - 15.99$	$R^2 = 0.9127$	max=45, min=0.1% average=15.9%
$0.1225t^4 - 3.1446t^3 + 22.434t^2 - 30.351t + 60.465$	$R^2 = 0.9674$	max=17.5%, min=0.4% average=6.5%
$0.0075t^5 - 0.1207t^4 - 0.267t^3 + 7.4326t^2 + 2.4175t + 38.409$	$R^2 = 0.9689$	max=16.2%, min=0.2% average=6.9%
$-0.0008t^6 + 0.0388t^5 - 0.5933t^4 + 3.1937t^3 - 5.2008t^2 + 23.364t + 26.788$	$R^2 = 0.9691$	max=17%, min=0.9% average=7.4%
Fourier model (see Table 2)	$R^2 = 0.9245$	max=6.2%, min=1.3% average=2.8%
Exponential model $\mathrm{Exp}(-[t-6)/\sigma_{max}]^2$ / $\sigma_0\sigma_{max}=21.7$ $\sigma_0=0.006$	$R^2 = 0.9134$	max=22.8%, min=0.2% average=9.73%

As shown in Table 1, you can take a polynomial of order 4 as the polynomial regression model. A similar study can be conducted for daily changes in solar radiation.

5. Simulation Using a Gaussian Function

As stated above, number of turning points is 2 for the time series of solar radiation, i.e. there is one maximum and one minimum for each one-day series of measurements and it is the same for monthly and daily average data. Therefore, we considered the possibility of modeling time series of solar radiation with the use of a Gaussian function:

$$Q\left(t\right)=\frac{1}{\sigma_0}\exp^{-\left[\frac{t-t_{cp}}{\sigma_{\max}}\right]^2}$$

We consider t_{cp} as a median (average) of the timeline, and the parameters σ_0 and σ_{max} are determined using the method of least squares.

Suppose we are given the values of the time series of solar radiation for $t_iQ\left(t_i\right))$ $i=1,2,\dots n$, where n is the number of measurements. We use the following notation:

$$\text{in }\ \left(Q\left(t_i\right)\right)\left(t_i-t_{cp}\right)^2\ \text{ in }\ \left(Q\left(t_i\right)\right),$$

$$S_2=\sum_{i=1}^{n}\left(t_i-t_{cp}\right)^2,$$

$$S_4=\sum_{i=1}^{n}\left(t_i-t_{cp}\right)^4,$$

$$D=\left(S_2\right)^2-nS_4.$$

Parameters σ_0 and σ_{max} are determined by the following formulas:

$$\sigma_0=\exp^{-\frac{S_2S_{1_2}-S_4S_{1_0}}{}}$$

$$\sigma_{\max}=\sqrt{\frac{D}{nS_{1_2}-S_2S_{1_0}}}$$

As shown in Table 1, the model of solar radiation built with the use of a Gaussian function gives satisfactory results, despite the high value of maximum relative error. Studies show that a maximum relative error occurs at the point of internal rotation (Fig. 2, at $t=3$ in our case), which implies the impossibility of taking into account the point of internal rotation using a Gaussian function.

6. Simulation Using Harmonic Fourier Series

Because of the quasi-periodic nature of the time series of solar radiation, regression method of accounting for the trend and the description of periodic components using Fourier series are combined sometimes.

As is known, the Fourier series has the following form:

$$s(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(k\omega_1 t) + b_k \sin(k\omega_1 t))$$

Here $\omega_1 = 2\pi/T$ is a circular frequency corresponding to the period T of signal repetition. Multiple frequencies $k\omega_1$ are called harmonics, numbered in accordance with the index k. Frequency $\omega_k = k\omega_1$ is called the k^{th} harmonics of signal. The coefficients of $a_k u$ are calculated by the formulas:

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} s(t)\cos(k\omega_1 t)dt.$$

The constant a_0 is calculated using the general formula for a_k. u is the average value of the signal on the period.

$$\frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} s(t)dt$$

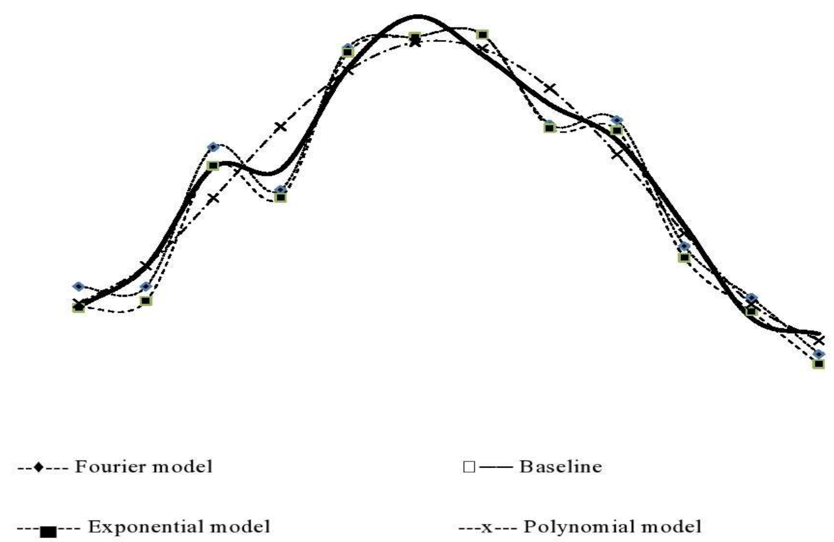
$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} s(t)\sin(k\omega_1 t)dt$$

The following values of the parameter are used when building the Fourier model of solar radiation by month for 2012 in Baku:

$T = 12; k = 6; \omega_1 = 0.5236.$

Table 2. Coefficients of the Fourier model of the solar radiation by month for 2012 in Baku

k	1	2	3	4	5	6
a_k	-8.87	3.175426	1.83	1.42E-14	4.70	0
b_k	-63.77	-2.5	-7.33	7.33	0.60	-2.25
R_m	64.38	4.04	7.56	7.33	4.74	2.25
φ_m	-0.14	0.90	0.24	-1.94E-15	-1.44	-3.14



Solar radiation in MgJ/h

Figure 2. Models of monthly solar radiation in 2012, Baku

As shown in Table 2, some of Fourier coefficients are equal to zero, which testifies to the no true association on the basis of mathematical statistics between the Fourier coefficients and the influencing factors. Reducing the frequency (increasing the number of harmonics oscillations) increases the accuracy of the model, but it does not fully take into account all the features of process (such as various gaps, steps and peaks, Fig. 2).

7. Model Analysis

Along with above-stated drawbacks, some more difficulties arise in applying these models in practice, such as oscillation measurements, which occur due to the noise generated by the environment or the equipment. Therefore, we performed an analysis of the time series of solar radiation with the use of smoothing methods before determining the deterministic components in the model. For many years, the method of moving averages has been one of the most common methods of following the trend. Simplicity of building and interpretation largely contributed to this. Moving averages is a method that smooth's series average by its current value and its immediate neighbors in the past and in the future.

Simple moving average or arithmetic moving average is numerically equal to the arithmetic mean of the values of the initial function for a specified period and is calculated by the formula:

$$Q_t = \frac{1}{n}\sum_{i=0}^{n-1} q_{t-i} = \frac{q_t + q_{t-1} + \dots + q_{t-i} + \dots + q_{t-n+2} + q_{t-n+1}}{n},$$

where Q_t is the value of simple moving average at point t, n is the number of values of the original function to calculate the moving average (smoothing interval; the wider the smoothing interval, the smoother the graph of the function), q_{t-i} is the initial value of the function at point $t-i$.

It is easy to establish some properties of these moving averages:

- Sum of the weights is equal to one. This had to be this way because, as we apply the procedure of weighing to a series whose terms are equal to the same constant, the average must be equal to the same constant;

- Scales are symmetric with regard to the median value;

- Because of the symmetry, of the weight, the trend values do not depend on the direction of timing.

Findings

1. Characteristic features of the process of solar radiation are studied and investigated. It is found that due to multifactorial process of solar radiation, the modeling must be based on real statistical data.

2. The series of solar radiation functions are considered and their dependence on four components: trend, more or less regular fluctuations relative to trend, seasonal fluctuations, and non-systematic random effect is established. It is shown that the non-systematic random effect could be defined as a difference between the statistics and the amount of trend and seasonal fluctuations.

3. Some research is conducted in order to determine the type of trend based on polynomial regression analysis and a Gaussian function, and some analysis of seasonal variations based on Fourier series. A series of Fourier harmonics in E-6 could be used as a model for seasonal fluctuations.

1. shoaling process in the Eastern Aral Sea is more dynamic than in the North and Western Aral Seas;

2. effective albedo of the shoaled areas varies between $0.5 \leq A \leq 0.85$.

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Growth Of Entire Functions With Respect To The Totality Of Variables

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Abstract. This work is focused on the entire functions of severable variables. A finite set of entire functions is considered. The relationship between the orders of these functions is established under some conditions. The inequalities concerning the upper and lower orders of these functions are obtained.

Key Words and Phrases: entire functions, several complex variables.

2000 Mathematics Subject Classifications: 32A05, 32A15

1. On the order of the system of entire functions of several complex variables

We consider entire functions of two complex variables represented by double power series.

Let

$$f(z_1, z_2) = \sum_{m_1, m_2}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}, \tag{1}$$

where $f(z_1, z_2) \in B(C^2)$ is a function of two complex variables z_1 and z_2.

It is known that

$$M(r_1, r_2) \equiv M(r_1, r_2; f) = \max_{|z_1| < r_i} |f(z_1, z_2)|, \quad i = 1, 2, \tag{2}$$

is the maximum of the modulus of the function $f(z_1, z_2)$ and

$$\max_{m_1, m_2} \{m_1, m_2\} = \nu(r_1, r_2) = (\nu_1(r_1, r_2; f), \nu_2(r_1, r_2; f)),$$

$$\mu(r_1, r_2; f) = \left| a_{v(r_1, r_2; f)} \right| r_1^{v_1(r_1, r_2; f)} r_2^{v_2(r_1, r_2; f)}. \tag{3}$$

It is proved in [1] that the functions $v_i(r_1, r_2; f) \quad (i = 1, 2)$ are increasing and continuous functions with an uncountable set of points of discontinuity with respect to each variable and $\mu(r_1, r_2; f)$ is an increasing and continuous function.

Lemma. *(M.M. Djrbashian [2]). In order for the series* (1) *to represent an entire function of variables z_1 and z_2, it is necessary and sufficient that the relation*

$$\lim_{n+m} \sqrt[n+m]{|a_{n,m}|} = 0, \tag{4}$$

hold.

By definition (see [3] and [4]), we have

$$\overline{\underline{\lim}}_{r_1+r_2\to\infty} \frac{\ln\ln M(r_1,r_2;f)}{\ln(r_1+r_2)} =^{\rho,}_{\lambda,} \tag{5}$$

$$\overline{\underline{\lim}}_{r_1+r_2\to\infty} \frac{\ln M(r_1,r_2;f)}{r_1^{\rho}+r_2^{\rho}} =^{T}_{t}, \quad (0<\rho<\infty). \tag{6}$$

Let

$$\overline{\underline{\lim}}_{m_1+m_2\to\infty} \frac{1}{e\rho}\left\{m_1^{m_1}m_2^{m_2}\left|a_{m_1,m_2}\right|\rho\right\}^{\frac{1}{m_1+m_2}} =^{T_1}_{t_1}, \tag{7}$$

$$\overline{\underline{\lim}}_{m_1+m_2\to\infty} \frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,m_2}\right|} =^{\rho_1}_{\lambda_1}. \tag{8}$$

It was also proved there that $\rho=\rho_1$, $\lambda=\lambda_1$ and $T_1=T$, $t=t_1$.

Let there be given a function

$$f(z_1,z_2) = \sum_{m_1,m_2=0}^{\infty} a_{m_1,m_2} z_1^{m_1} z_2^{m_2}, \tag{9}$$

and a system of entire functions

$$\left\{ f_k(z_1,z_2) = \sum_{m_1,m_2=0}^{\infty} a^{(k)}_{m_1,m_2} z_1^{m_1} z_2^{m_2} \right\}_{k=1}^{n}, \tag{10}$$

where $a_{m_1,m_2}, a^{(k)}_{m_1,m_2}$ $(k=1,2,...n)$ are the complex numbers and $f_k(z_1,z_2)\in B\left(C^2\right)$.

Theorem 1.1. *Let every function $f_k(z_1,z_2)\in B\left(C^2\right)$ in the system* (10) *be of regular growth. In order for the orders of these functions to be the same, it is necessary and sufficient that the condition*

$$\ln\left\{\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|\right\} = o\left\{\ln\left(m_1^{m_1}m_2^{m_2}\right)\right\}, \quad (k=1,2,...,n-1)$$

be satisfied as $m_1+m_2\to\infty$.

Proof. If the functions $f_k(z_1,z_2)$ $(k=1,2,....n)$ are of finite regular growth, then

$$\overline{\lim_{m_1+m_2\to\infty}} \frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a^{(k)}_{m_1,m_2}\right|^{-1}} = \rho_k = \lambda_k = \underline{\lim}_{m_1+m_2\to\infty} \frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a^{(k)}_{m_1,m_2}\right|^{-1}},$$

where $k = 1, 2, ...n$.

Let the functions $f_k(z_1, z_2) \quad (k = 1, 2, ...n)$ have the same order $\rho = \rho_1 = \rho_2 = ... = \rho_n = \lambda_1 = \lambda_2 = ... = \lambda_n = \lambda$, i.e.

$$\lim_{m_1+m_2\to\infty} \frac{-\ln\left|a^{(k)}_{m_1,m_2}\right|}{\ln\left(m_1^{m_1} m_2^{m_2}\right)} = \frac{1}{\rho} = \lim_{m_1+m_2\to\infty} \frac{-\ln\left|a^{(k+1)}_{m_1,m_2}\right|}{\ln\left(m_1^{m_1} m_2^{m_2}\right)}, \quad \left(k = \overline{1, n-1}\right).$$

Hence,

$$\lim_{m_1+m_2\to\infty} \frac{-\ln\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|}{\ln\left(m_1^{m_1} m_2^{m_2}\right)} = 0 \qquad \left(k = \overline{1, n-1}\right),$$

or

$$\ln\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right| = o\ \left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \qquad \left(k = \overline{1, n-1}\right)$$

as $m_1 + m_2 \to \infty$.

Now let's prove the converse. Let the functions $f_k(z_1, z_2) \quad \left(k = \overline{1, (n-1)}\right)$ be of order $\rho_k \left(k = \overline{1, n}\right)$. Then

$$\frac{1}{\rho_k} - \frac{1}{\rho_{k+1}} = \lim_{m_1+m_2\to\infty} \frac{-\ln\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|}{\ln\left(m_1^{m_1} m_2^{m_2}\right)} = 0, \qquad \left(k = \overline{1, n-1}\right)$$

Consequently, $\rho_k = \rho_{k+1} \quad \left(k = \overline{1, (n-1)}\right)$. The theorem is proved.

Theorem 1.2. *Let the functions $\{f_k(z_1, z_2)\}_{k=1}^{n} \in B\left(C^2\right)$ be of regular growth. In order for the types of these functions to be the same, it is necessary and sufficient that the condition*

$$\ln\left\{\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right| = o\ (m_1 + m_2),\right.$$

be satisfied as $m_1 + m_2 \to \infty$.

Proof. $f_k(z_1, z_2) \quad \left(k = \overline{1, n}\right)$ are the regular functions, therefore

$$\varliminf_{m_1+m_2\to\infty} \frac{1}{e\rho}\left\{m_1^{m_1} m_2^{m_2}\left|a^{(k)}_{m_1,m_2}\right|^{\rho}\right\}^{\frac{1}{m_1+m_2}} = t_k = T_k =$$

$$= \varlimsup_{m_1+m_2\to\infty} \frac{1}{e\rho}\left\{m_1^{m_1} m_2^{m_2}\left|a^{(k)}_{m_1,m_2}\right|^{\rho}\right\}^{\frac{1}{m_1+m_2}}.$$

where $k = \overline{1, n}$.

Let the functions $\{f_k(z_1, z_2)\}$ be of the same type, i.e.

$$\lim_{m_1+m_2\to\infty} \frac{1}{e\rho}\left\{m_1^{m_1} m_2^{m_2}\left|a^{(k)}_{m_1+m_2}\right|^{\rho}\right\}^{\frac{1}{m_1+m_2}} = T = \lim_{m_1+m_2\to\infty} \frac{1}{e\rho}\left\{m_1^{m_1} m_2^{m_2}\left|a^{(k+1)}_{m_1+m_2}\right|^{\rho}\right\}^{\frac{1}{m_1+m_2}}.$$

Hence,

$$\lim_{m_1+m_2\to\infty}\frac{\rho}{m_1+m_2}\left\{\ln\left|a^{(k)}_{m_1,m_2}\right|-\ln\left|a^{(k+1)}_{m_1,m_2}\right|\right\}=0,$$

or

$$\ln\left\{\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|=o\left(m_1+m_2\right),\right.$$

as $m_1+m_2\to\infty$.

Let the functions $\{f_k(z_1,z_2)\}$ be of type T_k $\left(k=\overline{1,n}\right)$. Then

$$\ln T_k-\ln T_{k+1}=\frac{1}{\rho}\lim_{m_1,\,m_2\to\infty}\frac{1}{m_1+m_2}\ln\left|\frac{a^{(k)}_{m_1,\,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|=0.$$

Hence $T_k=T_{k+1}$ $\left(k=\overline{1,\,n}\right)$.

Theorem 1.3. *Let every function* $f_k(z_1,\,z_2)\in B\left(C^2\right)$ *in the system* (10) *be of order* ρ_k $\left(k=\overline{1,\,n}\right)$. *If*

$$\ln\left|a_{m_1,\,m_2}\right|^{-1}\sim\ln\prod_{k=1}^{n}\left|a^{(k)}_{m_1,\,m_2}\right|^{-1},\quad m_1+m_2\to\infty, \tag{11}$$

then the function (1) *is an entire function of order* ρ *such that*

$$\frac{1}{\rho}\geq\sum_{k=1}^{n}\frac{1}{\rho_k}. \tag{12}$$

Proof. First, let's prove that the function (1) is an entire function. By the condition of the theorem, the functions $f_k(z_1,\,z_2)\,,\ k=\overline{1,\,n}$, are entire functions. Therefore, by Lemma [2] we have

$$\lim_{m_1+m_2}\left|a^{(k)}_{m_1,\,m_2}\right|^{-\frac{1}{m_1+m_2}}=\infty,\ \ k=\overline{1,\,n}\,.$$

Hence for sufficiently large $R>0$ and sufficiently small $\varepsilon>0$, for $m_1+m_2>N_k$ and for fixed n we have

$$(R-\varepsilon)^{\frac{1}{n}}<\left|a^{(k)}_{m_1,\,m_2}\right|^{-\frac{1}{m_1+m_2}},\quad k=\overline{1,\,n}\,.$$

Taking logarithms of this last relation, we have

$$\frac{m_1+m_2}{n}\ln\left(R-\varepsilon\right)<\ln\left|a^{(k)}_{m_1,\,m_2}\right|^{-1},\ \ k=\overline{1,\,n}\,.$$

Assigning values 1, 2, ..., n to k and then summing up the resulting inequalities, we obtain

$$(m_1+m_2)\ln\left(R-\varepsilon\right)<\ln\prod_{k=1}^{n}\left|a^{(k)}_{m_1,\,m_2}\right|^{-1}.$$

Taking into account (11), we have

$$R-\varepsilon<|a_{m_1,\,m_2}|^{-\frac{1}{m_1+m_2}}$$

for $R>0$ and $\varepsilon>0$ as $m_1+m_2>N=\max(N_1,\,N_2,\,\ldots,\,N_n)$. This means that $f(z_1,\,z_2)$ is an entire function.
Hence, we have

$$\frac{1}{\rho_k}-\frac{\varepsilon}{n}<\frac{\ln\left|a^{(k)}_{m_1,\,m_2}\right|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}=\frac{1}{\rho_k},\quad \left(k=\overline{1,\,n}\right)$$

as $m_1+m_2>N_k$ $\left(k=\overline{1,\,n}\right)$ for any $\varepsilon>0$.

Assigning values 1, 2, …, n to k and summing up the resulting inequalities, we obtain

$$\sum_{k=1}^{n}\frac{1}{\rho_k}-\varepsilon<\frac{\ln\prod_{k=1}^{n}|a_{m_1,\,m_2}|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}.$$

Taking into account the condition (11), we have

$$\sum_{k=1}^{n}\frac{1}{\rho_k}-\varepsilon<\frac{\ln|a_{m_1,\,m_2}|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)},$$

as $m_1+m_2>N=\max(N_1,\,N_2,\,\ldots,\,N_n)$ for any $\varepsilon>0$. Passing to the limit as $m_1+m_2\to\infty$, we obtain

$$\sum_{k=1}^{n}\frac{1}{\rho_k}\le\frac{1}{\rho}=\varliminf_{m_1+m_2\to\infty}\frac{\ln|a_{m_1,m_2}|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}.$$

Hence, it follows that the inequality (12) is valid.

Theorem 1.4. *Let every function $f_k(z_1,\,z_2)\in B\left(C^2\right)$ in the system (10) be of order $\rho_k\left(k=\overline{1,\,n}\right)$. If*

$$\ln|a_{m_1,\,m_2}|^{-1}\sim\ln\prod_{k=1}^{n}\left(\ln\left|a^{(k)}_{m_1,\,m_2}\right|^{-1}\right)^{\alpha_k},\tag{13}$$

$0<\alpha_k<1,\ \sum_{k=1}^{n}\alpha_k=1$, *then the function (1) is an entire function of order ρ such that*

$$\rho\le\prod_{k=1}^{n}\rho_k^{\alpha_k}.\tag{14}$$

Proof. The entireness of the function $f_k(z_1,\,z_2)$ is easy to prove. By the condition of the theorem, $f_k(z_1,\,z_2)$, $\left(k=\overline{1,\,n}\right)$ are the entire functions. Then each of them is of order $\rho_k\,(0<\rho_k<\infty)$, $\left(k=\overline{1,\,n}\right)$. Therefore, we have

$$\left(\frac{1}{\rho_k}-\varepsilon\right)^{\alpha_k}<\left\{\frac{\ln\left|a^{(k)}_{m_1,\,m_2}\right|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}\right\}^{\alpha_k},\quad k=\overline{1,\,n},$$

for any $\varepsilon > 0$ and $\sum_{k=1}^{n} \alpha_k = 1$, $0 < \alpha_k < 1$ as $m_1 + m_2 > N_k$.

Assigning values 1, 2, ..., n to k in the last inequality and multiplying the resulting inequalities, we get

$$\prod_{k=1}^{n} \left(\frac{1}{\rho_k} - \varepsilon \right)^{\alpha_k} < \frac{\prod_{k=1}^{n} \left(\ln \left| a_{m_1, m_2}^{(k)} \right|^{-1} \right)^{\alpha_k}}{\ln \left(m_1^{m_1} m_2^{m_2} \right)}.$$

Taking into account the condition (13) of the theorem, we have

$$\prod_{k=1}^{n} \frac{1}{\rho_k^{\alpha_k}} \leq \frac{1}{\rho} \Rightarrow \rho \leq \prod_{k=1}^{n} \rho_k^{\alpha_k}.$$

Remark. *If $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \frac{1}{n}$, then the inequality (14) takes the form*

$$\rho \leq \sqrt[n]{\rho_1 \rho_2 \ldots \rho_n}.$$

Theorem 1.5. *Let every function $f_k(z_1, z_2) \in B\left(C^2\right)$ in the system (10) be of order ρ_k, $\left(k = \overline{1, n}\right)$. If*

$$n \left(\ln \left| a_{m_1, m_2} \right|^{-1} \right)^{-1} \sim \sum_{k=1}^{n} \left(\ln \left| a_{m_1, m_2}^{(k)} \right|^{-1} \right)^{-1}, \tag{15}$$

then the function (1) is an entire function of order ρ such that

$$\rho \leq \frac{1}{n} \sum_{k=1}^{n} \rho_k. \tag{16}$$

Furthermore, if λ_k $\left(k = \overline{1, n}\right)$ is the lower order of function $f_k(z_1, z_2)$, $\left(k = \overline{1, n}\right)$, then λ is the lower order of function (1) such that

$$\lambda \geq \frac{1}{n} \sum_{k=1}^{n} \lambda_k. \tag{17}$$

Proof. As $f_k(z_1, z_2)$, $\left(k = \overline{1, n}\right)$ are entire functions, we have

$$\frac{\ln \left(m_1^{m_1} m_2^{m_2} \right)}{\ln \left| a_{m_1, m_2}^{(k)} \right|^{-1}} < \rho_k + \varepsilon, \ \ m_1 + m_2 > N_k, \ \ k = \overline{1, n}, \ \ \varepsilon > 0.$$

For $k = 1, 2, \ldots, n$ summing the last inequalities we obtain

$$\sum_{k=1}^{n} \ln \left(m_1^{m_1} m_2^{m_2} \right) \left(\ln \left| a_{m_1, m_2}^{(k)} \right|^{-1} \right)^{-1} < \sum_{k=1}^{n} \rho_k + \varepsilon n,$$

or

$$\ln\left(m_1^{m_1} m_2^{m_2}\right) \sum_{k=1}^{n} \left(\ln\left|a_{m_1,\, m_2}^{(k)}\right|^{-1}\right)^{-1} < \sum_{k=1}^{n} \rho_k + \varepsilon n.$$

Taking into account the condition (15) of the theorem, we have

$$n \ln\left(m_1^{m_1} m_2^{m_2}\right) \left(\ln\left|a_{m_1,\, m_2}\right|^{-1}\right)^{-1} < \sum_{k=1}^{n} \rho_k + \varepsilon n.$$

Passing to the limit as $m_1 + m_2 \to \infty$

$$n\rho \leq \sum_{k=1}^{n} \rho_k \Rightarrow \rho \leq \frac{1}{n} \sum_{k=1}^{n} \rho_k,$$

we can easily prove the inequality (17).

Theorem 1.6. *Let every function $f_k(z_1,\, z_2) \in B\left(C^2\right)$ in the system (10) be of order ρ_k, of type $T_k\,(0 < T_k < \infty)$ and of lower type t_k $(0 < t_k < \infty)$, $k = \overline{1,\, n}$, and let*

$$n\left(\ln\left|a_{m_1,\, m_2}\right|^{-1}\right)^{-1} \sim \sum_{k=1}^{n} \left(\ln\left|a_{m_1,\, m_2}^{(k)}\right|^{-1}\right)^{-1}. \tag{18}$$

Then the function (1) is an entire function of order ρ such that

$$\rho = \frac{1}{n} \sum_{k=1}^{n} \rho_k. \tag{19}$$

Proof. According to (7), the type of an entire function is calculated by the formula

$$\overline{\underline{\lim}}_{m_1+m_2} \frac{1}{e\rho_k} \left\{m_1^{m_1} m_2^{m_2} \left|a_{m_1,\, m_2}^{(k)}\right|^{\rho_k}\right\}^{\frac{1}{m_1+m_2}} = {}^{T_k}_{t_k}, \quad k = \overline{1,\, n}\,.$$

Hence, for any $\varepsilon > 0$ and $m_1 + m_2 > N_k$, we have

$$\left(m_1^{m_1} m_2^{m_2}\right) \left|a_{m_1,\, m_2}^{(k)}\right|^{\rho_k} < \left[\left(T_k + \varepsilon\right) e\rho_k\right]^{m_1+m_2}, \quad k = \overline{1,\, n}\,.$$

Taking logarithms of this inequality, we have

$$\ln\left(m_1^{m_1} m_2^{m_2}\right) < \rho_k \ln\left|a_{m_1,m_2}^{(k)}\right| + \left(m_1 + m_2\right) \ln\left[\left(T_k + \varepsilon\right) e\rho_k\right].$$

Consequently

$$\frac{\ln\left(m_1^{m_1} m_2^{m_2}\right)}{\ln\left|a_{m_1,\, m_2}^{(k)}\right|^{-1}} < \rho_k + \frac{m_1 + m_2}{\ln\left|a_{m_1,\, m_2}^{(k)}\right|} \ln\left[\left(T_k + \varepsilon\right) e\rho_k\right],$$

or

$$\frac{\ln\left(m_1^{m_1} m_2^{m_2}\right)}{\ln\left|a_{m_1,\, m_2}^{(k)}\right|} < \rho_k + o\left(1\right), \quad k = \overline{1,\, n}\,.$$

Summing this inequality with respect to k, we obtain

$$\ln\left(m_1^{m_1} m_2^{m_2}\right) \sum_{k=1}^{n} \left(\ln\left|a_{m_1,\, m_2}^{(k)}\right|^{-1}\right)^{-1} < \sum_{k=1}^{n} \rho_k + o\left(1\right), \quad k = \overline{1,\, n}\,.$$

Taking into account here the condition (18) of the theorem and passing to the limit as $m_1 + m_2 \to \infty$, we get

$$n\rho = n \lim_{m_1+m_2} \frac{\ln\left(m_1^{m_1} m_2^{m_2}\right)}{\ln\left|a_{m_1,\, m_2}\right|} \leq \sum_{k=1}^{n} \rho_k,$$

or

$$\rho \leq \frac{1}{n} \sum_{k=1}^{n} \rho_k. \tag{20}$$

Similarly we can prove that

$$\rho > \lambda \geq \frac{1}{n} \sum_{k=1}^{n} \rho_k. \tag{21}$$

(20) and (21) imply (19).

Theorem 1.7. *Let every function $f_k\left(z_1,\, z_2\right) \in B\left(C^2\right)$ in the system (10) be of regular growth of order ρ_k $\left(k = \overline{1,\, n}\right)$. If*

$$\ln\left|a_{m_1,\, m_2}\right|^{-1} \sim \prod_{k=1}^{n} \left(\ln\left|a_{m_1,\, m_2}^{(k)}\right|^{-1}\right)^{\alpha_k}, \quad 0 < \alpha_k < 1, \tag{22}$$

$\left(k = \overline{1,\, n}\right)$, $\sum_{k=1}^{n} \alpha_k = 1$, *then the function* (1) *is an entire function of regular growth of order ρ such that*

$$\rho = \prod_{k=1}^{n} \rho_k^{\alpha_k}. \tag{23}$$

Proof. Using the definition of the order of an entire function, we have

$$\underline{\lim}_{m_1+m_2\to\infty} \frac{\ln\left|a_{m_1,\, m_2}^{(k)}\right|^{-1}}{\ln\left(m_1^{m_1} m_2^{m_2}\right)} = \frac{1}{\rho_k}, \quad k = \overline{1,\, n},$$

or

$$\underline{\lim}_{m_1+m_2\to\infty} \left(\frac{\ln\left|a_{m_1,\, m_2}^{(k)}\right|^{-1}}{\ln\left(m_1^{m_1} m_2^{m_2}\right)}\right)^{\alpha_k} = \frac{1}{\rho_k^{(\alpha_k)}}, \quad k = \overline{1,\, n}\,.$$

Assigning values 1, 2, $\ldots$, n to k and then multiplying the resulting equalities, we have

$$\underline{\lim}_{m_1+m_2\to\infty} \frac{\prod_{k=1}^{n} \left(\ln\left|a_{m_1,m_2}^{(k)}\right|^{-1}\right)^{\alpha_k}}{\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right)^{\sum_{k=1}^{n} \alpha_k}} = \prod_{k=1}^{n} \frac{1}{\rho_k^{(\alpha_k)}}. \tag{24}$$

Taking into account the condition (22) of the theorem in (24), we have

$$\frac{1}{\rho}=\underline{\lim}_{m_1+m_2\to\infty}\frac{\ln|a_{m_1m_2}|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}=\frac{1}{\prod_{k=1}^{n}\rho_k^{\alpha_k}},$$

or

$$\rho=\prod_{k=1}^{n}\rho_k^{\alpha_k},$$

which completes the proof.

Corollary. *If* $\alpha_1=\alpha_2=\ldots=\alpha_n=\frac{1}{n}$, *then the equality* (23) *takes the form* $\rho=\sqrt[n]{\rho_1\rho_2\ldots\rho_n}$.

Theorem 1.8. *Let every function* $f_k(z_1,z_2)\in B\left(C^2\right)$ *in the system* (10) *be of order* ρ_k *and of lower order* λ_k $(0<\lambda_k\le\rho_k<\infty)$, $k=\overline{1,n}$. *If*

$$n\left(\ln|a_{m_1,m_2}|^{-1}\right)^{-1}\sim\sum_{k=1}^{n}\left(\ln\left|a_{m_1,m_2}^{(k)}\right|^{-1}\right)^{-1}, \tag{25}$$

then $f(z_1,z_2)$ *is an entire function of order* ρ *and of lower order* λ *such that*

$$\sum_{k=1}^{n-1}\lambda_k\le(n\lambda-\lambda_n;\,n\rho-\rho_n)\le\sum_{k=1}^{n-1}\rho_k. \tag{26}$$

Proof. The entireness of the function $f(z_1,z_2)$ is easy to prove. According to (8), we have

$$\overline{\underline{\lim}}_{m_1+m_2\to\infty}\frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,m_2}^{(k)}\right|^{-1}}=\begin{matrix}\rho_k\\ \lambda_k\end{matrix},\quad k=\overline{1,n}.$$

Hence, for every $\varepsilon>0$ and $m_1+m_2>N_k$, we have

$$\frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,m_2}^{(k)}\right|^{-1}}<\rho_k+\varepsilon,\quad k=\overline{1,n}. \tag{27}$$

Summing up the inequalities (27) for $k=1,2,\ldots,n$, we obtain

$$\ln\left(m_1^{m_1}m_2^{m_2}\right)\sum_{k=1}^{n}\left(\ln\left|a_{m_1,m_2}^{(k)}\right|^{-1}\right)^{-1}<\sum_{k=1}^{n}\rho_k+\varepsilon n.$$

Taking into account the condition (25) of the theorem in last inequality, we obtain

$$n\frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,m_2}\right|^{-1}}<\sum_{k=1}^{n}\rho_k+\varepsilon n.$$

Passing to the limit as $m_1+m_2\to\infty$, we obtain

$$n\rho\le\sum_{k=1}^{n}\rho_k=\sum_{k=1}^{n-1}\rho_k+\rho_n, \tag{28}$$

or

$$n\rho - \rho_n \le \sum_{k=1}^{n-1} \rho_k.$$

We can easily prove that

$$n\lambda - \lambda_n \le \sum_{k=1}^{n-1} \rho_k, \quad n\lambda - \lambda_n \ge \sum_{k=1}^{n-1} \lambda_k, \quad n\rho - \rho_n \ge \sum_{k=1}^{n-1} \lambda_k. \tag{29}$$

(28) and (29) imply the validity of (26).

Theorem 1.9. *Let every function $f_k(z_1, z_2) \in B(C^2)$ in the system (10) be of order ρ_k and of lower order λ_k $(0 < \lambda_k \le \rho_k < \infty)$, $k = \overline{1, n}$, and let*

$$\ln |a_{m_1,m_2}|^{-1} \sim \prod_{k=1}^{n} \left(\ln \left| a_{m_1,m_2}^{(k)} \right|^{-1} \right)^{\alpha_k}, \tag{30}$$

where $0 < \alpha_k < 1$, $\sum_{k=1}^{n} \alpha_k = 1$. Then the function $f(z_1, z_2)$ is an entire function of order ρ and of lower order λ, such that

$$\prod_{k=1}^{n-1} \lambda_k^{\alpha_k} = \left\{ \frac{\lambda}{\lambda_n^{\alpha_n}}, \frac{\rho}{\rho_n^{\alpha_n}} \right\} \le \prod_{k=1}^{n-1} \rho_k^{\alpha_k}. \tag{31}$$

The proof is similar to that of Theorem 1.8.

Corollary. *If $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \frac{1}{n}$, then the relation (31) takes the form*

$$\sqrt[n-1]{\lambda_1 \lambda_2 \ldots \lambda_{n-1}} \le \frac{\lambda}{\sqrt[n]{\lambda_n}}, \ \frac{\rho}{\sqrt[n]{\rho_n}} \le \sqrt[n-1]{\rho_1 \rho_2 \ldots \rho_{n-1}}.$$

Theorem 1.10. *Let every function $f_k(z_1, z_2) \in B(C^2)$ in the system (10) be of order ρ_k and of lower order λ_k $(0 < \lambda_k < \infty)$. If*

$$\ln |a_{m_1,m_2}| \sim \ln \left(\prod_{k=1}^{n} \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} \right), \tag{32}$$

where $\alpha_k = const$ $\left(k = \overline{1, n}\right)$, then the function $f(z_1, z_2)$ is an entire function of order ρ and of lower order λ with

$$\sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k} \le \left\{ \frac{1}{\rho} - \frac{\alpha_n}{\rho_n}; \frac{1}{\lambda} - \frac{\alpha_n}{\lambda_n} \right\} \le \sum_{k=1}^{n-1} \frac{\alpha_k}{\lambda_k}. \tag{33}$$

Proof. The entireness of the function $f(z_1, z_2)$ is proved as in Theorem 1.3 using condition (32).

According to (8), we have

$$\ln \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} < -\frac{\alpha_k \ln \left(m_1^{m_1} m_2^{m_2}\right)}{\rho_k + \varepsilon}, \quad k = \overline{1, n}, \tag{34}$$

for any $\varepsilon > 0$ as $m_1 + m_2 > N_k$.

Summing this inequality for $k = 1, 2, \ldots, n$, we have

$$\ln \prod_{k=1}^{n} \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \sum_{k=1}^{n} \frac{\alpha_k}{\rho_1 + \varepsilon}.$$

Taking into account the condition (32) of the theorem, we get

$$\ln \left| a_{m_1,m_2} \right| < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \sum_{k=1}^{n} \frac{\alpha_k}{\rho_k + \varepsilon},$$

or

$$\frac{\ln\left(m_1^{m_1} m_2^{m_2}\right)}{\ln \left| a_{m_1,m_2} \right|^{-1}} < \frac{1}{\sum_{k=1}^{n} \frac{\alpha_k}{\rho_k + \varepsilon}}.$$

Passing to the limit as $m_1 + m_2 \to \infty$, we obtain

$$\sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k} \le \frac{1}{\rho} - \frac{\alpha_n}{\rho_n}. \tag{35}$$

From (34) we have

$$\ln \prod_{k=1}^{n-1} \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k + \varepsilon}. \tag{36}$$

For subsequence $\left\{ m_1 = m_1^{(i)},\, m_2 = m_2^{(i)} \right\}$, we have

$$\ln \left| a_{m_1,m_2}^{(n)} \right|^{\alpha_n} < -\frac{\alpha_n \ln\left(m_1^{m_1} m_2^{m_2}\right)}{\lambda_n + \varepsilon}. \tag{37}$$

Summing up the inequalities (36) and (37), we obtain

$$\ln \prod_{k=1}^{n-1} \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \left\{ \sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k + \varepsilon} + \frac{\alpha_n}{\lambda_n + \varepsilon} \right\}.$$

Taking into account the condition (32) of the theorem, we have

$$\ln \left| a_{m_1,m_2} \right| < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \left\{ \sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k + \varepsilon} + \frac{\alpha_n}{\lambda_n + \varepsilon} \right\}.$$

Passing to the limit as $m_1 + m_2 \to \infty$, we obtain

$$\sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k} \le \frac{1}{\lambda} - \frac{\alpha_n}{\lambda_n}. \tag{38}$$

It is easy to prove that

$$\frac{1}{\lambda} - \frac{\alpha_n}{\lambda_n} \le \sum_{k=1}^{n-1} \frac{\alpha_k}{\lambda_k}, \quad \frac{1}{\rho} - \frac{\alpha_n}{\rho_n} \le \sum_{k=1}^{n-1} \frac{\alpha_k}{\lambda_k}. \tag{39}$$

From (35), (38) and (39) we get the validity of (33).

2. On the type of the system of entire functions of several complex variables

Let there be given the functions $f(z_1, z_2) \in B(C^2)$,

$$f(z_1, z_2) = \sum_{m_1, m_2}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}, \tag{40}$$

$$\left\{ f_k(z_1, z_2) = \sum_{m_1, m_2}^{\infty} a_{m_1, m_2}^{(k)} z_1^{m_1} z_2^{m_2} \right\}_{k=1}^{n}, \tag{41}$$

It is known from [4] and [5] that

$$\overline{\lim_{r_1+r_2\to\infty}} \frac{\ln\ln M(r_1, r_2; f)}{\ln(r_1+r_2)} \overset{\rho}{\underset{\lambda}{=}} \overline{\lim_{m_1+m_2\to\infty}} \frac{\ln\left(m_1^{m_1} m_2^{m_2}\right)}{\ln|a_{m_1, m_2}|^{-1}}, \tag{42}$$

$$\overline{\lim_{r_1+r_2\to\infty}} \frac{\ln M(r_1, r_2; f)}{r_1^{\rho} + r_2^{\rho}} \overset{T}{\underset{t}{=}} \frac{1}{e\rho} \overline{\lim_{m_1+m_2\to\infty}} \left\{ m_1^{m_1} m_2^{m_2} |a_{m_1, m_2}|^{\rho} \right\}^{\frac{1}{m_1+m_2}}. \tag{43}$$

Theorem 2.1. *Let every function $f_k(z_1, z_2)$ in the system* (41) *be an entire function of order ρ_k and of type T_k $(0 < T_k < +\infty)$, $\left(k = \overline{1, n}\right)$. If*

$$|a_{m_1, m_2}| \sim \prod_{k=1}^{n} \left| a_{m_1, m_2}^{(k)} \right|, \tag{44}$$

then the function (40) *is an entire function of order ρ and of type T with*

$$\left(\frac{T}{\alpha}\right)^{\alpha} \le \prod_{k=1}^{n} \left(\frac{T_k}{\alpha_k}\right)^{\alpha_k}, \tag{45}$$

where $\alpha = \frac{1}{\rho}$, $\alpha_k = \frac{1}{\rho_k}$ and $\alpha = \sum_{k=1}^{n} \alpha_k$.

Proof. By the condition of the theorem, the functions in the system (41) are entire functions. Then

$$\lim_{m_1+m_2\to\infty} \left| a_n^{(k)} \right|^{-\frac{1}{m_1+m_2}} = +\infty, \quad k = \overline{1, n}.$$

Considering condition (44) of the theorem, we obtain

$$\lim_{m_1+m_2\to\infty} |a_{m_1, m_2}|^{-\frac{1}{m_1+m_2}} \ge \prod_{k=1}^{n} \lim_{m_1+m_2} \left| a_{m_1, m_2}^{(k)} \right|^{-\frac{1}{m_1+m_2}} = +\infty.$$

Hence, the function $f(z_1, z_2)$ is entire. Next, by virtue of condition (44) of the theorem, we have

$$\left(m_1^{m_1} m_2^{m_2} |a_{m_1, m_2}|^{\rho}\right)^{\frac{1}{\rho(m_1+m_2)}} = \left\{ m_1^{\frac{m_1}{m_1+m_2}} m_2^{\frac{m_2}{m_1+m_2}} \right\}^{\frac{1}{\rho}} |a_{m_1, m.2}|^{\frac{1}{m_1+m_2}} \sim$$

$$\sim \left(m_1^{\frac{m_1}{m_1+m_2}} m_2^{\frac{m_2}{m_1+m_2}} \right)^{\sum_{k=1}^{n} \frac{1}{\rho_k}} \left(\prod_{k=1}^{n} \left| a_{m_1,\, m_2}^{(k)} \right| \right)^{\frac{1}{m_1+m_2}} .$$

Hence, for any $\varepsilon > 0$ and $m_1 + m_2 > N = \max(N_1 \dots N_n)$, we have

$$\{m_1^{m_1} m_2^{m_2} |a_{m_1,\, m_2}|^{\rho}\}^{\frac{1}{\rho(m_1+m_2)}} < (1+\varepsilon) \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1, m_2}^{(1)} \right|^{\rho} \right\}^{\frac{1}{\rho(m_1+m_2)}} \times$$

$$\times \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(2)} \right|^{\rho_2} \right\}^{\frac{1}{\rho_2(m_1+m_2)}} \dots \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(n)} \right|^{\rho_n} \right\}^{\frac{1}{\rho_n(m_1+m_2)}} .$$

Passing to the limit as $m_1 + m_2 \to \infty$, we have

$$(\rho T)^{\frac{1}{\rho}} \le (\rho_1 T_1)^{\frac{1}{\rho_1}} \dots (\rho_n T_n)^{\frac{1}{\rho_n}} ,$$

which proves (45).

Theorem 2.2. *Let the functions in the system* (41) *be of order* ρ_k*, of type* T_k $(0 < T_k < +\infty)$ *and of lower type* t_k $(0 < t_k < +\infty)$, $k = \overline{1,\, n}$*. If*

$$|a_{m_1,\, m_2}| \sim \prod_{k=1}^{n} \left| a_{m_1,\, m_2}^{(k)} \right| , \tag{46}$$

then the function (40) *is an entire function of order* ρ*, of type* T *and of lower type* t *with*

$$(\rho t)^{\frac{1}{\rho}} \le \left\{ \begin{matrix} (\rho t_k)^{\frac{1}{\rho_k}} , \prod_{i=1, i\ne k}^{n} (\rho_i T_i)^{\frac{1}{\rho_i}} \\ (\rho T_k)^{\frac{1}{\rho_k}} , \prod_{i=1, i\ne k}^{n} (\rho_i t_i)^{\frac{1}{\rho_i}} \end{matrix} \right\} \le (\rho T)^{\frac{1}{\rho}} . \tag{47}$$

Proof. The proof of the entireness of function (40) is carried out as in Theorem 2.1. Let $\psi_k(x,\, y) \ge 0$, $k = \overline{1,\, n}$. Then

$$\underline{\lim} \prod_{k=1}^{n} \psi_k(x,\, y) \le \left\{ \begin{matrix} \underline{\lim} \psi_i(x,y),\ \overline{\lim} \prod_{k=1, k\ne i}^{n} \psi_k(x,y) \\ \overline{\lim} \psi_i(x,y),\ \underline{\lim} \prod_{k=1, k\ne n}^{n} \psi_k(x,y) \end{matrix} \right\} \le \overline{\lim} \prod_{k=1}^{n} \psi_k(x,\, y) . \tag{48}$$

From (43) it follows that

$$\left\{ m_1^{\frac{m_1}{m_1+m_2}} m_2^{\frac{m_2}{m_1+m_2}} \right\}^{\frac{1}{\rho}} |a_{m_1,\, m_2}|^{\frac{1}{m_1+m_2}} \sim \left\{ m_1^{\frac{m_1}{m_1+m_2}} m_2^{\frac{m_2}{m_1+m_2}} \right\}^{\sum_{k=1}^{n} \frac{1}{\rho_k}} \times$$

$$\times \left(\prod_{k=1}^{n} \left| a_{m_1,\, m_2}^{(k)} \right| \right)^{\frac{1}{m_1+m_2}} = \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(1)} \right|^{\rho_1} \right\}^{\frac{1}{\rho_1(m_1+m_2)}} \times$$

$$\times \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(2)} \right|^{\rho_2} \right\}^{\frac{1}{\rho_2(m_1+m_2)}} \dots \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(n)} \right|^{\rho_n} \right\}^{\frac{1}{\rho_n(m_1+m_2)}} =$$

$$= \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(k)} \right|^{\rho_k} \right\}^{\frac{1}{\rho_k (m_1+m_2)}} \times \left\{ m_1^{m_1} m_2^{m_2} \prod_{\substack{i=1 \\ i \neq k}}^{n} \left| a_{m_1,\, m_2}^{(i)} \right|^{\rho_i} \right\}^{\frac{1}{\rho_i (m_1+m_2)}} .$$

Using (49), we have

$$(\rho t)^{\frac{1}{\rho}} \leq \begin{Bmatrix} (\rho_k t_k)^{\frac{1}{\rho_k}} \ ,\prod_{i=1, i\neq k}^{n} (\rho_i T_i)^{\frac{1}{\rho_i}} \\ (\rho_k T_k)^{\frac{1}{\rho_k}} \ ,\prod_{i=1, i\neq k}^{n} (\rho_i t_i)^{\frac{1}{\rho_i}} \end{Bmatrix} \leq (\rho T)^{\frac{1}{\rho}} .$$

Note that in case of one variable Theorem 2.2 was proved in [7].

In particular, for two functions $f_1(z_1,\, z_2)$ and $f_2(z_1,\, z_2)$ we have a relation

$$(\rho t)^{\frac{1}{\rho}} \leq \begin{Bmatrix} (\rho_1 t_1)^{\frac{1}{\rho_1}} & , (\rho_2 T_2)^{\frac{1}{\rho_2}} \\ (\rho_1 T_1)^{\frac{1}{\rho_1}} & , (\rho_2 T_2)^{\frac{1}{\rho_2}} \end{Bmatrix} \leq (\rho T)^{\frac{1}{\rho}} .$$

In case of one variable this last relation was proved in [8].

Theorem 2.3. *Let every function $f_k(z_1,\, z_2)$ in the system* (41) *be of regular order ρ_k, $k=\overline{1,\, n}$, of type T_k and of lower type t_k. If*

$$\ln \left| a_{m_1,\, m_2} \right| \sim \left\{ \left| a_{m_1,\, m_2}^{(1)} \right|^{\alpha_1} \left| a_{m_1,\, m_2}^{(2)} \right|^{\alpha_2} \cdots \left| a_{m_1,\, m_2}^{(k)} \right|^{\alpha_k} \right\}, \tag{49}$$

where α_k is a constant $(k=1,\, 2,\, \ldots,\, n)$, *then the function* (40) *is an entire function of order ρ, of type T and of lower type t such that*

$$\prod_{k=1}^{n-1} (t_k \rho_k)^{\frac{\alpha_k}{\rho_k}} \leq \left\{ \frac{(\rho t)^{\frac{1}{\rho}}}{(\rho_n t_n)^{\frac{\alpha_n}{\rho_n}}},\ \frac{(\rho T)^{\frac{1}{\rho}}}{(\rho_n T_n)^{\frac{\alpha_n}{\rho_n}}} \right\} \leq \prod_{k=1}^{n-1} (T_k \rho_k)^{\frac{\alpha_k}{\rho_k}} . \tag{50}$$

Proof. According to (43), we have

$$\varlimsup_{m_1+m_2\to\infty} \frac{1}{e\rho_k} \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(k)} \right|^{\rho_k} \right\}^{\frac{1}{m_1+m_2}} =_{t_k}^{T_k}, \tag{51}$$

$$k = (1,\, 2,\, \ldots,\, n) .$$

From (51), for any $\varepsilon > 0$ and $m_1 + m_2 > N_k$, we have

$$\left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(k)} \right|^{\rho_k} \right\}^{\frac{1}{m_1+m_2}} < e\rho_k \left(T_k + \varepsilon \right), \ \ k = \overline{1,\, n},$$

and for the subsequence $\left\{ m_1 = m_1^{(i)},\, m_2 = m_2^{(i)} \right\}$

$$\left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(n)} \right|^{\rho_n} \right\}^{\frac{1}{m_1+m_2}} < e\rho_n \left(t_n + \varepsilon \right),$$

or

$$\left(m_1^{m_1} m_2^{m_2}\right)^{\frac{\alpha_k}{\rho_k}} \left|a_{m_1,\, m_2}^{(k)}\right|^{\alpha_k} < \left[\left(T_k + \varepsilon\right) e\rho_k\right]^{\frac{\alpha_k (m_1+m_2)}{\rho_k}}, \qquad (52)$$
$$k = (1,\, 2,\, \ldots,\, n)\,.$$

Taking logarithms of these inequalities and then summing them up for $k = 1,\, 2,\, \ldots,\, n-1$, we obtain

$$\ln \prod_{k=1}^{n} \left|a_{m_1,\, m_2}^{(k)}\right|^{\alpha_k} <$$
$$< \ln \left\{ \frac{\prod_{k=1}^{n-1} \left[\left(T_k + \varepsilon\right) e\rho_k\right]^{\frac{\alpha_k (m_1+m_2)}{\rho_k}}}{\left(m_1^{m_1} m_2^{m_2}\right)^{\frac{\alpha_k}{\rho_k}}} \times \frac{\left[\left(t_k + \varepsilon\right) e\rho_k\right]^{\frac{\alpha_n (m_1+m_2)}{\rho_n}}}{\left(m_1^{m_1} m_2^{m_2}\right)^{\frac{\alpha_n}{\rho_n}}} \right\}.$$

Taking into account the condition (49), we obtain

$$\ln \left|a_{m_1,\, m_2}\right| < \sum_{k=1}^{n-1} \frac{\alpha_k \left(m_1 + m_2\right)}{\rho_k} \ln \left[\left(T_k + \varepsilon\right) e\rho_k\right] +$$
$$+ \frac{\alpha_n \left(m_1 + m_2\right)}{\rho_n} \ln \left[\left(t_n + \varepsilon\right) e\rho_n\right] - \left(\frac{\alpha_1}{\rho_1} + \frac{\alpha_2}{\rho_2} + \ldots + \frac{\alpha_n}{\rho_n}\right) \ln \left(m_1^{m_1} m_2^{m_2}\right).$$

According to the Theorem 2.9.12 [3], we have

$$\frac{1}{\rho} = \frac{\alpha_1}{\rho_1} + \frac{\alpha_2}{\rho_2} + \ldots + \frac{\alpha_n}{\rho_n}.$$

Then we get

$$\frac{\ln \left(m_1^{m_1} m_2^{m_2}\right)^{\frac{1}{\rho}} + \ln \left|a_{m_1,\, m_2}\right|}{m_1 + m_2} < \sum_{k=1}^{n} \frac{\alpha_k}{\rho_k} \ln \left[\left(T_k + \varepsilon\right) e\rho_k\right] + \frac{\alpha_n}{\rho_n} \ln \left[\left(t_n + \varepsilon\right) e\rho_n\right].$$

Hence, we obtain

$$\ln \left\{m_1^{m_1} m_2^{m_2} \left|a_{m_1,\, m_2}\right|^{\rho}\right\}^{\frac{1}{\rho(m_1+m_2)}} < \ln \prod_{k=1}^{n-1} \left[\left(T_k + \varepsilon\right) e\rho_k\right]^{\frac{\alpha_k}{\rho_k}} \left[\left(t_n + \varepsilon\right) e\rho_n\right]^{\frac{\alpha_n}{\rho_n}}.$$

Passing to the limit as $m_1 + m_2 \to \infty$, we have

$$\left(te\rho\right)^{\frac{1}{\rho}} \leq \prod_{k=1}^{n-1} \left(T_k e\rho_k\right)^{\frac{\alpha_k}{\rho_k}} \left(t_n e\rho_n\right)^{\frac{\alpha_n}{\rho_n}},$$

or

$$\frac{\left(\rho t\right)^{\frac{1}{\rho}}}{\left(\rho_n t_n\right)^{\frac{\alpha_n}{\rho_n}}} \leq \prod_{k=1}^{n-1} \left(\rho_k T_k\right)^{\frac{\alpha_k}{\rho_k}}. \qquad (53)$$

Note that

$$\frac{(T\rho)^{\frac{1}{\rho}}}{(T_n\rho_n)^{\frac{\alpha_n}{\rho_n}}} < \prod_{k=1}^{n-1} (T_k\rho_k)^{\frac{\alpha_k}{\rho_k}}, \tag{54}$$

and

$$\frac{(\rho t)^{\frac{1}{\rho}}}{(\rho_n t_n)^{\frac{\alpha_n}{\rho_n}}} \geq \prod_{k=1}^{n-1} (\rho_k t_k)^{\frac{\alpha_k}{\rho_k}}, \quad \prod_{k=1}^{n} (t_k\rho_k)^{\frac{\alpha_k}{\rho_k}} \leq \frac{(T\rho)^{\frac{1}{\rho}}}{(T_n\rho_n)^{\frac{\alpha_n}{\rho_n}}}. \tag{55}$$

From (53), (54) and (55) we get the validity of the theorem.

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A Type of Shannon-McMillan Approximation Theorems for Second-Order Nonhomogeneous Markov Chains Indexed by a Double Rooted Tree

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Abstract. In this paper, a class of small deviation theorems for the relative entropy densities of arbitrary random field on a double rooted tree are discussed by comparing between the arbitrary measure μ and the second-order nonhomogeneous Markov measure μ_Q on the double rooted tree. As corollaries, some Shannon-McMillan theorems for the arbitrary random field, second-order Markov chain field and a limit property for the random conditional entropy of second-order homogeneous Markov chain on the double rooted tree are obtained. The existing result is extended.

Key Words and Phrases: Shannon-McMillan theorem, a double rooted tree, arbitrary random field, second-order nonhomogeneous Markov chain, relative entropy density.
2000 Mathematics Subject Classifications: 60F15

1. Introduction

A tree is a graph $S = \{T, E\}$ which is connected and contains no circuits. Given any two vertices $\sigma, t(\ \sigma \neq t \in T)$, let $\overline{\sigma t}$ be the unique path connecting σ and t. Define the graph distance $d(\sigma, t)$ to be the number of edges contained in the path $\overline{\sigma t}$.

Let T_o be an arbitrary infinite tree that is partially finite (i.e. it has infinite vertices, and each vertex connects with finite vertices) and has a root o. Meanwhile, we consider another kind of double root tree T, that is, it is formed with the root o of T_o connecting with an arbitrary point denoted by the root -1. For a better explanation of the double root tree T, we take Cayley tree $T_{C,N}$ for example. It's a special case of the tree T_o, the root o of Cayley tree has N neighbors and all the other vertices of it have $N + 1$ neighbors each. The double root tree $T^{'}_{C,N}$ (see Fig.1) is formed with root o of tree $T_{C,N}$ connecting with another root -1.

Let σ, t be vertices of the double root tree T. Write $t \leq \sigma$ $(\sigma, t \neq o, -1)$ if t is on the unique path connecting o to σ, and $|\sigma|$ for the number of edges on this path. For any

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two vertices σ, t $(\sigma, t \neq o, -1)$ of the tree T, denote by $\sigma \wedge t$ the vertex farthest from o satisfying $\sigma \wedge t \leq \sigma$ and $\sigma \wedge t \leq t$.

The set of all vertices with distance n from root o is called the n-th generation of T, which is denoted by L_n. We say that L_n is the set of all vertices on level n and especially root -1 is on the -1st level on tree T. We denote by $T^{(n)}$ the subtree of the tree T containing the vertices from level -1 (the root -1) to level n and denote by $T_o^{(n)}$ the subtree of the tree T_o containing the vertices from level 0 (the root o) to level n. Let $t(\neq o, -1)$ be a vertex of the tree T. We denote the first predecessor of t by 1_t, the second predecessor of t by 2_t, and denote by n_t the n-th predecessor of t. Let $X^A = \{X_t, t \in A\}$, and let x^A be a realization of X^A and denote by $|A|$ the number of vertices of A .

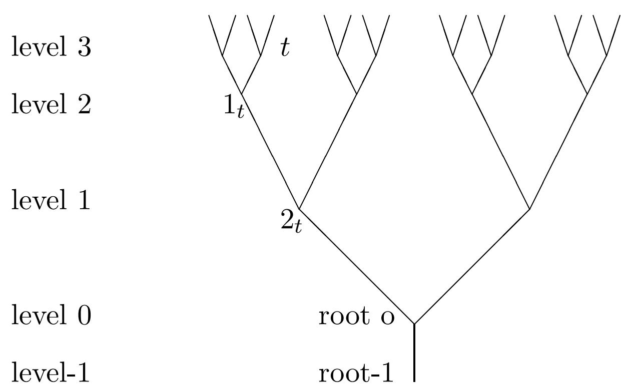

Fig.1 Double root tree $T_{C,2}^{'}$

Definition 1 *Let* $S = \{s_1, s_2, \cdots, s_M\}$ *and* $Q(z|y,x)$ *be a nonnegative function on* S^3. *Let*

$$Q = ((Q(z|y,x)), \quad Q(z|y,x) \geq 0, \quad x, y, z \in S.$$

If

$$\sum_{z \in S} Q(z|y,x) = 1,$$

then Q *is called a second-order transition matrix.*

Definition 2 *Let* T *be a double root tree and* $S = \{s_1, s_2, \cdots, s_M\}$ *be a finite state space, and* $\{X_t, t \in T\}$ *be a collection of* S*-valued random variables defined on the probability space* $(\Omega, \mathcal{F}, Q)$. *Let*

$$q = (q(x,y)), \quad x, y \in S \tag{1}$$

be a distribution on S^2, *and*

$$Q_t = (Q_t(z|y,x)), \quad x, y, z \in S, \quad t \in T \backslash \{o\}\{-1\} \tag{2}$$

be a collection of second-order transition matrices. For any vertex t ($t \neq o, -1$), if

$$Q(X_t = z|X_{1_t} = y, X_{2_t} = x, \text{ and } X_\sigma \text{ for } \sigma \wedge t \leq 2_t)$$

$$= Q(X_t = z|X_{1_t} = y, X_{2_t} = x) = Q_t(z|y,x) \quad \forall x, y, z \in S \tag{3}$$

and

$$Q(X_{-1} = x, X_o = y) = q(x,y), \quad x, y \in S, \tag{4}$$

then $\{X_t, t \in T\}$ is called a S-valued second-order nonhomogeneous Markov chain indexed by a tree T with the initial distribution (1) and second-order transition matrices (2), or called a T-indexed second-order nonhomogeneous Markov chain.

Definition 3. *Let $(Q_t = Q_t(z|x,y), t \in T^{(n)}\backslash\{o,-1\})$ and $q = (q(x,y))$ be defined as before, μ_Q be a second-order nonhomogeneous Markov measure on $(\Omega, \mathcal{F})$. If*

$$\mu_Q(x_0, x_{-1}) = q(x_0, x_{-1}) \tag{5}$$

$$\mu_Q(x^{T^{(n)}}) = q(x_0, x_{-1}) \prod_{t \in T^{(n)}\backslash\{o,-1\}} Q_t(x_t|x_{1_t}, x_{2_t}), \quad n \geq 1, \tag{6}$$

then μ_Q will be called a second-order Markov chains field on an infinite tree T determined by the stochastic matrices Q_t and the initial distribution q.

Let μ be an arbitrary probability measure defined on $(\Omega, \mathcal{F})$, *log* is the natural logarithmic. Denote

$$f_n(\omega) = -\frac{1}{|T^{(n)}|} \log \mu(X^{T^{(n)}}). \tag{7}$$

$f_n(\omega)$ is called the entropy density on subgraph $T^{(n)}$ with respect to the measure μ. If $\mu = \mu_Q$, then by (6), (7) we get

$$f_n(\omega) = -\frac{1}{|T^{(n)}|}[\log q(X_0, X_{-1}) + \sum_{t \in T^{(n)}\backslash\{o,-1\}} Q_t(X_t|X_{1_t}, X_{2_t})]. \tag{8}$$

The convergence of $f_n(\omega)$ in a sense (L_1 convergence, convergence in probability, or almost sure convergence) is called Shannon-McMillan theorem or the asymptotic equipartition property (AEP) in information theory. There have been some works on limit theorems for tree-indexed stochastic processes. Benjamini and Peres [1] have given the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye [2] have studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Ye and Berger (see [4],[5]), by using Pemantle's result [3] and a combinatorial approach, have studied the Shannon-McMillan theorem with convergence in probability for a PPS-invariant and ergodic random field on a homogeneous tree. Yang and Liu [8] have studied a strong law of large numbers for the frequency of occurrence of states for Markov chains field on a homogeneous tree (a particular case of tree-indexed Markov chains field and PPS-invariant random fields). Yang (see [6]) has studied the strong law of large numbers for frequency of occurrence of state and Shannon-McMillan theorem for homogeneous Markov chains indexed by a homogeneous tree. Recently, Yang (see [13]) has studied the strong law of large numbers and Shannon-McMillan theorem for nonhomogeneous Markov chains indexed by a homogeneous tree. Huang and Yang (see [11]) have also studied the strong law of large numbers for Markov chains indexed by an infinite tree with uniformly bounded degree. Peng and Yang have studied a class of small deviation theorems for functionals for arbitrary random field on a homogeneous trees (see[9]). Wang has also studied some Shannon-McMillan approximation theorems for arbitrary random field on the generalized Bethe tree (see[10]).

In this paper, we study a class of Shannon-McMillan random approximation theorems for arbitrary random fields on the double rooted tree by comparison the arbitrary measure with the second-order nonhomogeneous Markov measure and constructing a supermartingale on the double rooted tree. As corollaries, a class of Shannon-McMillan theorems for arbitrary random field and second-order Markov chains field on the double rooted tree are obtained. A limit property for the expectation of the random conditional entropy of second-order homogeneous Markov chain indexed by the double rooted tree is studied. Yang and Ye's result (see[13]) is extended.

2. Main result and its proof

Lemma 1 (see [8]). *Let μ_1 and μ_2 be two probability measures defined on $(\Omega, \mathcal{F})$, $D \in \mathcal{F}$, $\{\tau_n, n \geq 0\}$ be a sequence of positive-valued random variables such that*

$$\liminf_{n} \frac{\tau_n}{|T^{(n)}|} > 0. \quad \mu_1 - \quad a.s. \quad D. \tag{10}$$

Then

$$\limsup_{n\to\infty} \frac{1}{\tau_n} \log \frac{\mu_2(X^{T^{(n)}})}{\mu_1(X^{T^{(n)}})} \leq 0. \quad \mu_1 - \quad a.s. \quad D. \tag{11}$$

In particular, let $\tau_n = |T^{(n)}|$, then

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu_2(X^{T^{(n)}})}{\mu_1(X^{T^{(n)}})} \leq 0. \qquad \mu_1 - \quad a.s. \tag{12}$$

Proof . See reference [8].

Let

$$\varphi(\mu|\mu_Q) = \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu(X^{T^{(n)}})}{\mu_Q(X^{T^{(n)}})}, \tag{13}$$

$\varphi(\mu|\mu_Q)$ is called the sample relative entropy rate of $X^{T^{(n)}}$ with respect to μ and μ_Q. $\varphi(\mu|\mu_Q)$ is also called asymptotic logarithmic likelihood ratio. By (12) and (13)

$$\varphi(\mu|\mu_Q) \geq \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu(X^{T^{(n)}})}{\mu_Q(X^{T^{(n)}})} \geq 0. \quad \mu - a.s. \tag{14}$$

Hence $\varphi(\mu|\mu_Q)$ can be looked on as a type of a measure of the deviation between the arbitrary random field and the second-order nonhomogeneous Markov chain fields on the double rooted tree.

Although $\varphi(\mu|\mu_Q)$ is not a proper metric between two probability measures, we nevertheless think of it as a measure of "dissimilarity" between their joint distribution μ and second-order Markov distribution μ_Q. Obviously, $\varphi(\mu|\mu_Q) = 0$ if and only if $\mu = \mu_Q$. It has been shown in (14) that $\varphi(\mu|\mu_Q) \geq 0$, a.s. in any case. Hence, $\varphi(\mu|\mu_Q)$ can be used as a random measure of the deviation between the true joint distribution $\mu(x^{T^{(n)}})$ and the second-order Markov distribution $\mu_Q(x^{T^{(n)}})$. Roughly speaking, this deviation may be regarded as the one between coordinate stochastic process $x^{T^{(n)}}$ and the Markov case. The smaller $\varphi(\mu|\mu_Q)$ is, the smaller the deviation is.

Theorem 1. *Let $X = \{X_t, t \in T\}$ be an arbitrary random field on a double rooted tree. $f_n(\omega)$ and $\varphi(\mu|\mu_Q)$ are respectively defined as (7) and (13). Denote by $H_t^Q(X_t|X_{1_t}, X_{2_t})$ the random conditional entropy of X_t relative to X_{1_t}, X_{2_t} on the measure μ_Q, that is*

$$H_t^Q(X_t|X_{1_t}, X_{2_t}) = -\sum_{x_t\in S} Q_t(x_t|X_{1_t}, X_{2_t}) \log Q_t(x_t|X_{1_t}, X_{2_t}), \qquad t \in T^{(n)}\backslash\{o, -1\}. \tag{15}$$

Let

$$D(c) = \{\omega : \varphi(\mu|\mu_Q) \leq c\} \tag{16}$$

$$\alpha(c) = \min\left\{\frac{2xe^{-2}}{(1-x)^2} + \frac{c}{x}, \quad 0 < x < 1\right\}, \quad c > 0; \quad \alpha(0) = 0. \tag{17}$$

$$\beta(c) = \max\left\{\frac{2xe^{-2}}{(1+x)^2} + \frac{c}{x}, \quad -1 < x < 0\right\}, \quad c > 0; \quad \beta(0) = 0. \tag{18}$$

Then

$$\limsup_{n\to\infty}[f_n(\omega) - \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\backslash\{o,-1\}} H_t^Q(X_t|X_{1_t}, X_{2_t})] \leq \alpha(c)M, \quad \mu - a.s. \ \omega \in D(c) \tag{19}$$

$$\liminf_{n\to\infty}[f_n(\omega) - \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\backslash\{o,-1\}} H_t^Q(X_t|X_{1_t}, X_{2_t})] \geq \beta(c)M - c. \quad \mu - a.s. \ \omega \in D(c) \tag{20}$$

Proof. Consider the probability space $(\Omega, \mathcal{F}, \mu)$, let $\lambda > 0$ be a constant, $\delta_j(\cdot)$ be Kronecker function. Denote $g_t(j) = -\log Q_t(j|X_{1_t}, X_{2_t})$, we construct the following product distribution:

$$\mu_Q(x^{T^{(n)}};\lambda) = q(x_0, x_{-1}) \prod_{t\in T^{(n)}\setminus\{o,-1\}} \exp\{\lambda g_t(j)\delta_j(x_t)\}[\frac{Q_t(x_t|x_{1_t}, x_{2_t})}{1+(e^{\lambda g_t(j)}-1)Q_t(j|x_{1_t}, x_{2_t})}]. \tag{22}$$

By (22) we can write

$$\begin{aligned}
&\sum_{x^{L_n}\in S^{L_n}} \mu_Q(x^{T^{(n)}};\lambda)\\
&= \sum_{x^{L_n}\in S^{L_n}} q(x_0, x_{-1}) \prod_{t\in T^{(n)}\setminus\{o,-1\}} \exp\{\lambda g_t(j)\delta_j(x_t)\}[\frac{Q_t(x_t|x_{1_t}, x_{2_t})}{1+(e^{\lambda g_t(j)}-1)Q_t(j|x_{1_t}, x_{2_t})}]\\
&= \mu_Q(x^{T^{(n-1)}};\lambda) \sum_{x^{L_n}\in S^{L_n}} \prod_{t\in L_n} \exp\{\lambda g_t(j)\delta_j(x_t)\}[\frac{Q_t(x_t|x_{1_t}, x_{2_t})}{1+(e^{\lambda g_t(j)}-1)Q_t(j|x_{1_t}, x_{2_t})}]\\
&= \mu_Q(x^{T^{(n-1)}};\lambda) \prod_{t\in L_n} \sum_{x_t\in S} \exp\{\lambda g_t(j)\delta_j(x_t)\}[\frac{Q_t(x_t|x_{1_t}, x_{2_t})}{1+(e^{\lambda g_t(j)}-1)Q_t(j|x_{1_t}, x_{2_t})}]\\
&= \mu_Q(x^{T^{(n-1)}};\lambda) \prod_{t\in L_{n-1}} \frac{1}{1+(e^{\lambda g_t(j)}-1)Q_t(j|x_{1_t}, x_{2_t})}[\sum_{x_t=j} + \sum_{x_t\neq j}]\\
&= \mu_Q(x^{T^{(n-1)}};\lambda) \prod_{t\in L_{n-1}} \frac{e^{\lambda g_t(j)}Q_t(j|x_{1_t}, x_{2_t}) + 1 - Q_t(j|x_{1_t}, x_{2_t})}{1+(e^{\lambda g_t(j)}-1)Q_t(j|x_{1_t}, x_{2_t})}\\
&= \mu_Q(x^{T^{(n-1)}};\lambda)
\end{aligned} \tag{23}$$

Therefore $\mu_Q(x^{T^{(n)}};\lambda)$, $n = 1, 2, \cdots$ are a class of consistent distributions on $S^{T^{(n)}}$. Let

$$U_n(\lambda,\omega) = \frac{\mu_Q(X^{T^{(n)}};\lambda)}{\mu(X^{T^{(n)}})} \tag{24}$$

By (22) and (24) we attain

$$U_n(\lambda,\omega) = \exp\{\sum_{t\in T^{(n)}\setminus\{o,-1\}} \lambda g_t(j)\delta_j(X_t)\} \prod_{t\in T^{(n)}\setminus\{o,-1\}} [\frac{1}{1+(e^{\lambda g_t(j)}-1)Q_t(j|X_{1_t}, X_{2_t})}]$$

$$\cdot q(X_0) \prod_{t\in T^{(n)}\setminus\{o,-1\}} Q_t(X_t|X_{1_t}, X_{2_t}) \Big/ \mu(X^{T^{(n)}}). \tag{25}$$

It is easy to see that $U_n(\lambda,\omega)$ is a nonnegative sup-martingale from Doob's martingale convergence theorem (see [12]) since μ and μ_Q are two probability measures. Moreover,

$$\lim_{n\to\infty} U_n(\lambda,\omega) = U_\infty(\lambda,\omega) < \infty. \quad \mu-a.s. \tag{26}$$

By (12) and (24) we have

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log U_n(\lambda,\omega) \le 0. \qquad \mu - a.s. \tag{27}$$

According to (6), (25), we can rewrite (27) as

$$\limsup_{n\to\infty}\{\frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \lambda g_t(j)\delta_j(X_t)$$
$$-\frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \log[1+(e^{\lambda g_t(j)}-1)Q_t(j|X_{1_t},X_{2_t})] + \frac{1}{|T^{(n)}|}\log\frac{\mu_Q(X^{T^{(n)}})}{\mu(X^{T^{(n)}})}\}$$
$$\le 0 \qquad \mu - a.s. \tag{28}$$

Letting $\lambda = 0$ in (28), we have

$$\varphi(\mu|\mu_Q) \ge \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \log\frac{\mu(X^{T^{(n)}})}{\mu_Q(X^{T^{(n)}})} \ge 0. \qquad \mu - a.s. \qquad \omega\in D(c). \tag{29}$$

By use of (16) and (28) we obtain

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \{\lambda g_t(j)\delta_j(X_t) - \log[1+(e^{\lambda g_t(j)}-1)Q_t(j|X_{1_t},X_{2_t})\}$$
$$\le \varphi(\mu|\mu_Q) \le c. \qquad \mu - a.s. \qquad \omega\in D(c). \tag{30}$$

By virtue of (30), the properties of super limit and the inequalities $1-1/x \le \ln x \le x-1, (x>0)$, $e^x - 1 - x \le (1/2)x^2 e^{|x|}$, we can write

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \lambda\{g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t},X_{2_t})\}$$
$$\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \{\log[1+(e^{\lambda g_t(j)}-1)Q_t(j|X_{1_t},X_{2_t})] - \lambda g_t(j)Q_t(j|X_{1_t},X_{2_t})\}+c$$
$$\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} Q_t(j|X_{1_t},X_{2_t})[e^{\lambda g_t(j)} - 1 - \lambda g_t(j)] + c$$
$$\le (\lambda^2/2) \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} Q_t(j|X_{1_t},X_{2_t}) g_t^2(j) e^{|\lambda g_t(j)|} + c$$
$$= (\lambda^2/2) \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} Q_t(j|X_{1_t},X_{2_t}) \log^2 Q_t(j|X_{1_t},X_{2_t}) e^{-|\lambda|\log Q_t(j|X_{1_t},X_{2_t})} + c$$

$$= (\lambda^2/2) \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \log^2 Q_t(j|X_{1_t}, X_{2_t}) \cdot Q_t(j|X_{1_t}, X_{2_t})^{1-|\lambda|} + c.$$

$$\mu - a.s. \qquad \omega \in D(c) \tag{31}$$

In the case $0 < \lambda < 1$, dividing two sides of (31) by λ, we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} [g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t}, X_{2_t})] \\ \leq\ & \frac{\lambda}{2} \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \log^2 Q_t(j|X_{1_t}, X_{2_t}) \cdot Q_t(j|X_{1_t}, X_{2_t})^{1-\lambda} + \frac{c}{\lambda} \end{aligned}$$

$$\mu - a.s. \qquad \omega \in D(c) \tag{32}$$

Consider the function

$$\phi(x) = (\log x)^2 x^{1-\lambda}, \quad 0 < x \leq 1, \quad 0 < \lambda < 1. \quad (set \ \ \phi(0) = 0) \tag{33}$$

It can be concluded that on the internal $[0, 1]$,

$$\max\{\phi(x), 0 \leq x \leq 1\} = \phi(e^{2/(\lambda-1)}) = (\frac{2}{\lambda - 1})^2 e^{-2}. \tag{34}$$

By (32) and (34) we have that when $0 < \lambda < 1$,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} [g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t}, X_{2_t})] \\ \leq\ & \frac{\lambda}{2} \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} (\frac{2}{\lambda - 1})^2 e^{-2} + \frac{c}{\lambda} \\ =\ & \frac{2\lambda e^{-2}}{(1-\lambda)^2} \limsup_{n\to\infty} \frac{|T^{(n)}| - 2}{|T^{(n)}|} + \frac{c}{\lambda} \leq \frac{2\lambda e^{-2}}{(1-\lambda)^2} + \frac{c}{\lambda}. \qquad \mu - a.s. \qquad \omega \in D(c) \end{aligned} \tag{35}$$

When $c > 0$, $h(\lambda) = (2\lambda e^{-2})/(1 - \lambda)^2$+c/$\lambda$ attains its smallest value $\alpha(c)$ at $\lambda_o \in (0, 1)$. Hence letting $\lambda = \lambda_o$ in (35), we attain from (17) that

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} [g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t}, X_{2_t})] \leq \alpha(c).$$

$$\mu - a.s. \qquad \omega \in D(c). \tag{36}$$

By (7), (6), (15), (29) and (36), noticing $g_t(j) = -\log Q_t(j|X_{1_t}, X_{2_t})$, we can deduce

$$\limsup_{n\to\infty} [f_n(\omega) - \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} H_t^Q(X_t|X_{1_t}, X_{2_t})]$$

$$= \limsup_{n\to\infty}\{-\frac{1}{|T^{(n)}|}\log\mu(X^{T^{(n)}}) - \frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}} H_t^Q(X_t|X_{1_t})\}$$

$$\leq \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}\{-\log Q_t(X_t|X_{1_t},X_{2_t}) - H_t^Q(X_t|X_{1_t},X_{2_t})\}$$

$$+\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}[\sum_{t\in T^{(n)}\setminus\{o,-1\}}\log Q_t(X_t|X_{1_t},X_{2_t}) - \log\mu(X^{T^{(n)}})] = \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}$$

$$\sum_{t\in T^{(n)}\setminus\{o,-1\}}\sum_{j\in S}\{-\delta_j(X_t)\log Q_t(j|X_{1_t},X_{2_t}) + Q_t(j|X_{1_t},X_{2_t})\log Q_t(j|X_{1_t},X_{2_t})\}$$

$$+\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\log\frac{\mu_Q(X^{T^{(n)}})}{\mu(X^{T^{(n)}})}$$

$$= \limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}\sum_{j=s_1}^{s_M}[g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t},X_{2_t})]$$

$$+\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\log\frac{\mu_Q(X^{T^{(n)}})}{\mu(X^{T^{(n)}})}$$

$$le\sum_{j=s_1}^{s_M}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}[g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t},X_{2_t})]$$

$$\leq \alpha(c)M \qquad \mu-a.s. \qquad \omega\in D(c), \tag{37}$$

thus in the case $c>0$, (19) follows from (37).

In the case $-1<\lambda<0$, by (31) we have

$$\liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}[g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t},X_{2_t})]$$

$$\geq \frac{\lambda}{2}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}\log^2 Q_t(j|X_{1_t},X_{2_t})\cdot Q_t(j|X_{1_t},X_{2_t})^{1+\lambda} + \frac{c}{\lambda}.$$

$$\mu-a.s. \qquad \omega\in D(c). \tag{38}$$

By (34) and (38), we gain

$$\liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}[g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t},X_{2_t})]$$

$$\geq \frac{\lambda}{2}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}(\frac{2}{1+\lambda})^2 e^{-2} + \frac{c}{\lambda}$$

$$\geq \frac{2\lambda e^{-2}}{(1+\lambda)^2}+\frac{\mathrm{c}}{\lambda}.$$

$$\mu - a.s. \qquad \omega \in D(c). \tag{39}$$

In the case $c > 0$, the function $u(\lambda) = (2\lambda e^{-2})/(1+\lambda)^2 + c/\lambda$ attains the largest value $\beta(c)$ at $\lambda^o \in (-1, 0)$. Thereby letting $\lambda = \lambda^o$ in (39), we have

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} [g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t}, X_{2_t})] \geq \beta(c).$$

$$\mu - a.s. \qquad \omega \in D(c). \tag{40}$$

By (7), (6), (13), (16) and (40), noticing that $g_t(j) = -\log Q_t(j|X_{1_t}, X_{2_t})$, we can write

$$\begin{aligned}
&\liminf_{n\to\infty}[f_n(\omega) - \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} H_t^Q(X_t|X_{1_t}, X_{2_t})\\
&\geq \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \{-\log Q_t(X_t|X_{1_t}, X_{2_t}) - H_t^Q(X_t|X_{1_t}, X_{2_t})\}\\
&\quad + \liminf_{n\to\infty} \frac{1}{|T^{(n)}|}[\sum_{t\in T^{(n)}\setminus\{o,-1\}} \log Q_t(X_t|X_{1_t}, X_{2_t}) - \log\mu(X^{T^{(n)}})]\\
&\geq \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \sum_{j=s_1}^{s_M} [g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t}, X_{2_t})]\\
&\quad - \limsup_{n\to\infty} \frac{1}{|T^{(n)}|}[\log\mu(X^{T^{(n)}}) - \sum_{t\in T^{(n)}\setminus\{o\}} \log Q_t(X_t|X_{1_t})]\\
&\geq \sum_{j=s_1}^{s_M} \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} [g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t}, X_{2_t})] - \varphi(\mu|\mu_Q)\\
&\geq \beta(c)M - c. \qquad \mu - a.s. \qquad \omega \in D(c).
\end{aligned} \tag{41}$$

In accordance with (41), we see that (20) also holds in the case $c > 0$. When $c = 0$, take $0 < \lambda_i < 1, (i = 1, 2, \cdots)$ such that $\lambda_i \to 0 \ (i \to \infty)$, by (35) we acquire

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} [g_t(j)\delta_j(X_t) - g_t(j)Q_t(j|X_{1_t}, X_{2_t})] \leq 0.$$

$$\mu - a.s. \qquad \omega \in D(0). \tag{42}$$

Imitating the proof of (37), we have by (17) and (42)

$$\limsup_{n\to\infty}[f_n(\omega) - \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} H_t^Q(X_t|X_{1_t}, X_{2_t})] \leq 0. \ \mu - a.s. \qquad \omega \in D(0). \tag{43}$$

Since $\alpha(0)=0$, we know that (19) also holds in the case $c=0$ from (37). By the similar means, we can obtain that (20) holds in the case $c=0$.

Corollary 1. *Under the assumption of Theorem 1, we have that in the case* $0\le c<1$,

$$\limsup_{n\to\infty}[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}H_t^Q(X_t|X_{1_t},X_{2_t})]\le M[\frac{2e^{-2}}{(1-\sqrt{c})^2}+1]\sqrt{c},$$

$$\mu-a.s. \qquad \omega\in D(c) \tag{44}$$

$$\liminf_{n\to\infty}[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}H_t^Q(X_t|X_{1_t},X_{2_t})]\ge -M[\frac{2e^{-2}}{(1-\sqrt{c})^2}+1]\sqrt{c}-c.$$

$$\mu-a.s. \qquad \omega\in D(c). \tag{45}$$

Proof. Letting $x=\sqrt{c}$ in (19) and (20), we have

$$[2e^{-2}/(1-\sqrt{c})^2+1]\sqrt{c}\ge\alpha(c), \quad -[2e^{-2}/(1-\sqrt{c})^2+1]\sqrt{c}\le\beta(c).$$

Therefore (44), (45) follow from (19) and (20), respectively.

Corollary 2. *Under the assumption of Theorem 1, we have*

$$\lim_{n\to\infty}[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}H_t^Q(X_t|X_{1_t},X_{2_t})]=0. \quad \mu-a.s. \qquad \omega\in D(0). \tag{46}$$

Proof. Letting $c=0$ in Corollary 1, (46) follows from (44) and (45).

Corollary 3. *Let* $X=\{X_t,t\in T\}$ *be the second-order nonhomogeneous Markov chains field indexed by the double rooted tree with the initial distribution (5) and the joint distribution (6),* $f_n(\omega)$ *be defined as (8). Then*

$$\lim_{n\to\infty}[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}H_t^Q(X_t|X_{1_t},X_{2_t})]=0. \qquad \mu_Q-a.s. \tag{47}$$

Proof. Let $\mu\equiv\mu_Q$ in Theorem 1, then $\varphi(\mu|\mu_Q)\equiv 0$. Thereby $D(0)=\Omega$. (47) follows from (46) correspondingly.

Remark. When the second-order nonhomogeneous Markov chain indexed by the tree degenerates into the first-order nonhomogeneous Markov chain indexed by a tree, we can see easily that $Q_t(X_t|X_{1_t},X_{2_t})=Q_t(X_t|X_{1_t})$, $H_t^Q(X_t|X_{1_t},X_{2_t})=H_t^Q(X_t|X_{1_t})$. At the moment, (47) is changed into

$$\lim_{n\to\infty}[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o,-1\}}H_t^Q(X_t|X_{1_t})]=0. \qquad \mu_Q-a.s.$$

This is a main result of Yang and Ye (see [13]).

3. Shannon-McMillan theorem for homogeneous Markov chains fields on a double rooted tree

Let $X=\{X_t, t\in T\}$ be another second-order homogeneous Markov chain indexed by a double rooted tree with the initial distribution and the joint distribution on the measure μ_P as follows:

$$\mu_P(x_0,x_{-1})=p(x_0,x_{-1}) \tag{48}$$

$$\mu_P(x^{T^{(n)}})=p(x_0,x_{-1})\prod_{t\in T^{(n)}\backslash\{o,-1\}}P(x_t|x_{1_t},x_{2_t}),\quad n\geq 1, \tag{49}$$

where $P=P(z|x,y),\ x,y,z\in S$ is a strictly positive stochastic matrix on S^3, $p=(p(x,y))$ is a strictly positive distribution on S^2. Thereby the relative entropy density of $X=\{X_t,t\in T\}$ on the measure μ_P is

$$f_n(\omega)=-\frac{1}{|T^{(n)}|}[\log p(X_0,X_{-1})+\sum_{t\in T^{(n)}\backslash\{o,-1\}}\log P(x_t|x_{1_t},x_{2_t})]. \tag{50}$$

Let a be a real number, denote $[a]^+=\max\{a,0\}$. We have the following result:

Theorem 2. *Let $X=\{X_t,t\in T\}$ be the second-order homogeneous Markov chains field with the initial distribution (48) and joint distribution (49) under the measure μ_P. $f_n(\omega)$ is defined by (50). Let $Q=Q(z|x,y),\ x,y,z\in S$ be defined by definition 1, $\alpha=\min\{Q(j|i,k),\ i,k,j\in S\}>0$. Denote*

$$H_Q(X_t|X_{1_t},X_{2_t})=-\sum_{x_t\in S}Q(x_t|X_{1_t},X_{2_t})\log Q(x_t|X_{1_t},X_{2_t}),\quad t\in T^{(n)}\backslash\{o,-1\}.$$

If

$$\sum_{i\in S}\sum_{k\in S}\sum_{j\in S}[P(j|i,k)-Q(j|i,k)]^+\leq \alpha\cdot c, \tag{51}$$

then

$$\limsup_{n\to\infty}[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\backslash\{o,-1\}}H_Q(X_t|X_{1_t},X_{2_t})]\leq \alpha(c)M.\qquad \mu_P-a.s. \tag{52}$$

$$\liminf_{n\to\infty}[f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\backslash\{o,-1\}}H_Q(X_t|X_{1_t},X_{2_t})]\geq \beta(c)M-c.\qquad \mu_P-a.s. \tag{53}$$

Proof. Let $\mu=\mu_P,\ Q_t(z|x,y)\equiv Q(z|x,y),\ x,y,z\in S,\ t\in T^{(n)}\backslash\{o,-1\}$, we obtain that $H_t^Q(X_t|X_{1_t},X_{2_t})=H_Q(X_t|X_{1_t},X_{2_t}),\ t\in T^{(n)}\backslash\{o,-1\}$, thus (50) follows from (7) and (49). By the inequalities $\log x \le x-1$ $(x>0)$, $a \le [a]^+$ and (51), noticing that $\alpha = \min\{Q(j|i,k),\ i,k,j \in S\} > 0$, we can conclude

$$\begin{aligned}
\varphi(\mu_P|\mu_Q) &= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu_P(X^{T^{(n)}})}{\mu_Q(X^{T^{(n)}})} \\
&= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{p(X_0,X_{-1}) \prod_{t\in T^{(n)}\setminus\{o,-1\}} P(X_t|X_{1_t},X_{2_t})}{q(X_0,X_{-1}) \prod_{t\in T^{(n)}\setminus\{o,-1\}} Q(X_t|X_{1_t},X_{2_t})} \\
&\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{p(X_0,X_{-1})}{q(X_0,X_{-1})} + \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \log \frac{P(X_t|X_{1_t},X_{2_t})}{Q(X_t|X_{1_t},X_{2_t})} \\
&\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \sum_{i\in S}\sum_{k\in S}\sum_{j\in S} \delta_j(X_t)\delta_i(X_{1_t})\delta_k(X_{2_t}) \log \frac{P(j|i,k)}{Q(j|i,k)} \\
&\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \sum_{i\in S}\sum_{k\in S}\sum_{j\in S} \delta_j(X_t)\delta_i(X_{1_t})\delta_k(X_{2_t}) \frac{P(j|i,k)-Q(j|i,k)}{Q(j|i,k)} \\
&\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \sum_{i\in S}\sum_{k\in S}\sum_{j\in S} \frac{P(j|i,k)-Q(j|i,k)}{Q(j|i,k)} \\
&\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \sum_{i\in S}\sum_{k\in S}\sum_{j\in S} \frac{[P(j|i,k)-Q(j|i,k)]^+}{\alpha} \\
&\le \sum_{i\in S}\sum_{k\in S}\sum_{j\in S} \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \frac{[P(j|i,k)-Q(j|i,k)]^+}{\alpha} \\
&\le \sum_{i\in S}\sum_{k\in S}\sum_{j\in S} \limsup_{n\to\infty} \frac{|T^{(n)}|-2}{|T^{(n)}|} \frac{[P(j|i,k)-Q(j|i,k)]^+}{\alpha} \\
&= \frac{1}{\alpha} \sum_{i\in S}\sum_{k\in S}\sum_{j\in S} [P(j|i,k)-Q(j|i,k)]^+.
\end{aligned} \tag{54}$$

By (51) and (54) we have

$$\varphi(\mu_P|\mu_Q) \le c. \qquad a.s. \tag{55}$$

By (16) and (55) we know $D(c)=\Omega$. Hence (52), (53) follow from (19), (20), respectively.

Theorem 3. *Let $X=\{X_t, t\in T\}$ be a second-order homogeneous Markov chains field indexed by the double rooted tree with the initial distribution (1) and the transition matrix $Q=Q(z|x,y),\ x,y,z\in S$. Then*

$$\lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} E_Q[H_Q(X_t|X_{1_t},X_{2_t})] = -\sum_{i\in S}\sum_{j\in S}\sum_{k\in S} q(i,j)Q(k|i,j)\log Q(k|i,j),$$

$$\mu_Q - a.s., \tag{56}$$

where E_Q represents the expectation under the measure μ_Q.

Proof. By the definition of $H_Q(X_t|X_{1_t}, X_{2_t})$ in Theorem 2 and the property of the conditional expectation, we can write

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} E_Q[H_Q(X_t|X_{1_t}, X_{2_t})] \\
= & \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} E_Q[E_Q(-\log Q(X_t|X_{1_t}, X_{2_t})|X_{1_t}, X_{2_t})] \\
= & \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} E_Q(-\log Q(X_t|X_{1_t}, X_{2_t})) \\
= & \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \sum_{x_{1_t}\in S} \sum_{x_{2_t}\in S} \sum_{x_t\in S} [-q(x_{1_t}, x_{2_t}, x_t) \log Q(x_t|x_{1_t}, x_{2_t})] \\
= & \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \sum_{i\in S} \sum_{j\in S} \sum_{k\in S} [-q(i,j,k) \log Q(k|i,j)] \\
= & \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o,-1\}} \sum_{i\in S} \sum_{j\in S} \sum_{k\in S} [-q(i,j)Q(k|i,j) \log Q(k|i,j)] \\
= & \sum_{i\in S} \sum_{j\in S} \sum_{k\in S} [-q(i,j)Q(k|i,j) \log Q(k|i,j)] \cdot \lim_{n\to\infty} \frac{|T^{(n)}|-2}{|T^{(n)}|} \\
= & \sum_{i\in S} \sum_{j\in S} \sum_{k\in S} [-q(i,j)Q(k|i,j) \log Q(k|i,j)].
\end{aligned} \tag{57}$$

Therefore, (56) follows from (57) directly.

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Analysis of the Shoaling of the Aral Sea by Effective Albedo

Z.J.Zabidov

Abstract. This article discusses the current situation in the shoaled areas of the Aral Sea based on the photos made from Terra and Aqua satellites.

Key Words and Phrases: satellite data, optical parameters, effective albedo, desertification

2000 Mathematics Subject Classifications: 97K80, 62M15

1. Introduction

The availability of scientific methods for exploring the shoaled and flooded areas with the use of special measurement systems installed on geophysical satellites is of great importance today. The data provided by the satellites at different times is playing an important role in controlling the dynamics of river course changes.

First studies of water-level in the Aral Sea date back to the 19th Century. Until the mid-20th Century, there were almost no dramatic changes in the water-level in the Aral Sea. At those times, the Aral Sea was one of the four largest lakes in the world with an area of 68,000 square kilometres. With a length of 426 km and a width of 284 km, it had a maximal depth of 68 m. Amu Darya and Syr Darya were the rivers that fed it. Since the mid-20th Century, those rivers have been used for the irrigation of cotton and rice plantations in Central Asia, and this significantly reduced the water volume of the Aral Sea. The total volume of precipitation in this region is incomparably less than the amount of vaporizable water. At the end, the Aral's sea level started to drop since 1961.

In this work, we analyze the situation around the shoaled areas of the Aral Sea using spectrometric data provided by the satellites over the past years. Our approach is based on the calculation of digital values of the spectral effective albedo of the atmosphere-underlying surface system which is the main optical parameter used in the exploration of Earth surface via satellite.

2. Methodology and The Analysis of Obtained Results

Photographing the Earth's surface from the space is closely related to the two-dimensional distributions of the flux of radiation reflected by the atmosphere-underlying surface system. The main optical parameter that characterizes the radiation properties of the environment (or the object) is effective albedo. On the upper boundary of the atmosphere, the albedo is defined by

$$\bar{A}(z=\infty) = \frac{F^{\uparrow}(z=\infty)}{\pi F_0 \cos\theta_0},$$

where $\bar{A}(z=\infty)$ is the relationship between the extra-atmospheric stream of sunlight $\pi F_0 \cos\theta_0$ and the energy reflected by the Earth surface. Spectral albedo is determined by the reflected part of the flux of spectral radiation. Photometric data related to the Earth surface is usually more comprehensive on the visible part of spectre. The maximal value of albedo is reached on the solar wave of length λ=0,55 mkm. The range of albedo on the surface strongly depends on the humidity of that surface.

Photographs taken by satellite make the exploration of Earth surface more comprehensive and more reliable. A lot of special applications have been designed for the processing of digital images. Image Processor is one of them.

Figure 1 is a collection of photographs of the Aral Sea taken at different times. All those photos are made by Terra and Aqua satellites equipped with MODIS technology. Figure 1 reveals the shoaling dynamics in the Aral Sea. Space observations of the last 10 years testifies to the significant growth of shoaled areas.

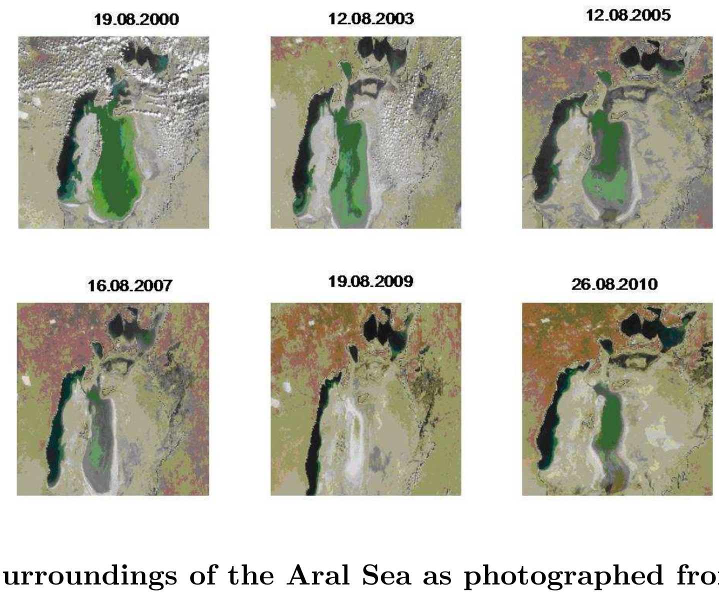

Figure 1. Surroundings of the Aral Sea as photographed from satellite

Since 1989, the Aral Sea has splitted into two different water basins – the North Aral Sea and the South Aral Sea. In 2003, the South Aral Sea, in turn, splitted into the western and eastern basins as a result of shoaling. Also, water level decline was the reason the Vozrozhdenie island turned into a peninsula in 2001. From 1960 to 2003, the surface area and the water volume of the Aral Sea shrinked to 25% and 10% of their original sizes, respectively, as a result of shoaling.

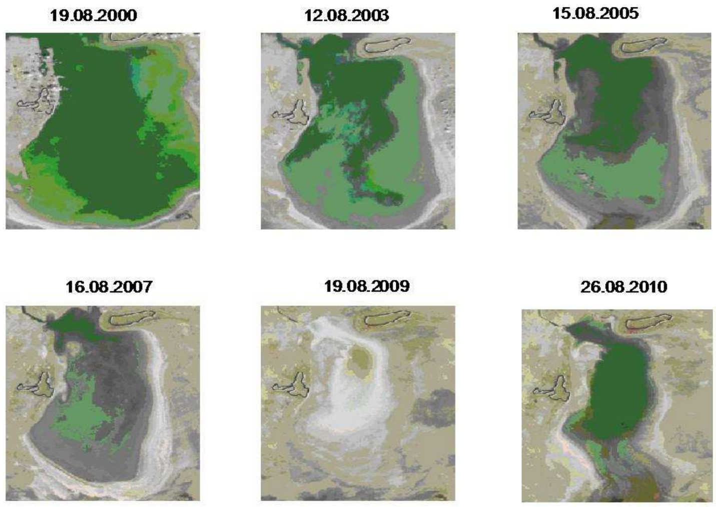

Figure 2. The Eastern Aral Sea as photographed from satellite

Figure 3 shows the two-dimensional distributions of the effective albedo of the eastern parts of the Large and Small Aral Seas. To get the images in this figure, distributions of reflected luminous flux have been estimated using Image Processor, with the upward luminous flux and the inclination of luminous flux taken in accordance with the solar constant and the upper culmination of the Sun, respectively, at the height of the aerial photography.

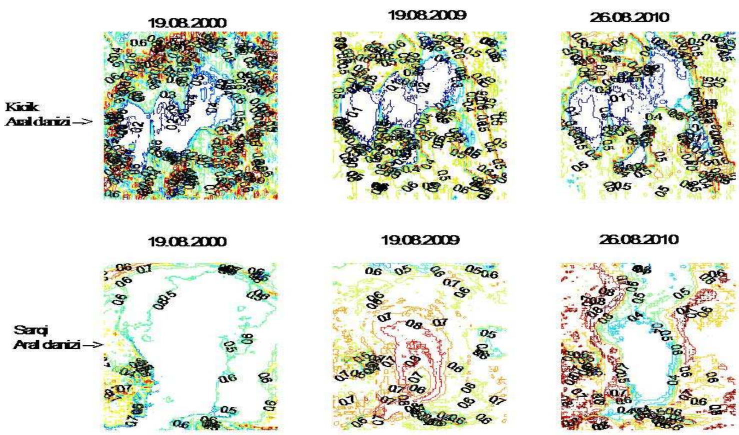

Figure 3.Distribution of the Isolines of Effective Albedo in Surroundings of the Aral Sea

Table 1 shows the variations of water surface area of the Aral Sea between 1960 and 2010. We can see that the surface area of the Aral Sea has drastically shrinked over the past 50 years. The variations of effective avbedo of some part of the surface are closely related to the characteristics (humidity, shoaling) of that part. Figure 3 shows the distribution of the isolines of effective albedo in different parts of the Aral Sea based on the satellite-made photos taken at different times. We can also see that the variations of albedo in the Small Aral Sea are less dynamic than those in the Eastern Aral Sea. As a result of shoaling, the percentage of the parts of the Eastern Aral Sea with albedo varying between 0.5$\le A \le$0.85 has sharply grown in 2000-2010, while the percentage of those with albedo varying between 0.2$\le A \le$0.35 has sharply reduced during the same period.

Table 1. Aral Sea's water resources

	1960	1990	2003	2004	2007	2008	2009	2010
Water level , m	53,40	38,24	31,0					
Water volume, km	1083	323	112,8		75			
Water surface area, thousand km	68,90	36,8	18,24	17,2	14,183.	10,579	11,8[]	13,9

3. Conclusion

Two-dimensional distributions of the effective albedo of the shoaled areas of the Aral Sea are determined in this work using satellite-made images. The following results are obtained:

1. shoaling process in the Eastern Aral Sea is more dynamic than in the North and Western Aral Seas;
2. effective albedo of the shoaled areas varies between 0.5$\le A \le$0.85.

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Hardy-Littlewood-Stein-Weiss Inequality in the Generalized Morrey Spaces with Variable Exponent

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Abstract. We consider generalized weighted Morrey spaces $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^\gamma}(\Omega)$ with variable exponent $p(x)$ and a general function $\omega(x,r)$ defining the Morrey-type norm. In case of bounded sets $\Omega \subset \mathbb{R}^n$ we prove the boundedness of the Hardy-Littlewood maximal operator and Calderon-Zygmund singular operators with standard kernel, in such spaces. We also prove a Sobolev-Adams type $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^\gamma}(\Omega) \to \mathcal{M}^{q(\cdot),\omega,|x-x_0|^\gamma}(\Omega)$-theorem for the potential operators $I^{\alpha(\cdot)}$, also of variable order. In all the cases the conditions for the boundedness are given it terms of Zygmund-type integral inequalities on $\omega(x,r)$, which do not assume any assumption on monotonicity of $\omega(x,r)$ in r.

Key Words and Phrases: Maximal operator, fractional maximal operator, Riesz potential, singular integral operators, generalized Morrey space, Hardy-Littlewood-Sobolev-Morrey type estimate, BMO space.
2000 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction.

Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [45]). They are defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))},$$

where $0 \le \lambda < n$, $1 \le p < \infty$. In the theory of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$ play an important role. Later, Morrey spaces found important applications to Navier-Stokes ([44], [66]) and Schrödinger ([50], [52], [53], [64], [65]) equations, elliptic problems with discontinuous coefficients ([10], [18]), and potential theory ([1], [2]). An exposition of the Morrey spaces can be found in the books [19] and [42].

As is known, over the last two decades there has been an increasing interest in the study of variable exponent spaces and operators with variable parameters in such spaces, we refer the readers to the surveying papers [16], [29], [38], [59], on the progress in this field, including topics of Harmonic Analysis and Operator Theory (see also references therein).

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [3] and [47] in the Euclidean setting and in [30] in the setting of metric measure spaces, in case of bounded sets. In [3] there was proved the boundedness of the maximal operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$ and $\lambda(\cdot)$, and for potential operators, under the same log-condition and the assumptions $\inf_{x\in\Omega}\alpha(x)>0$, $\sup_{x\in\Omega}[\lambda(x)+\alpha(x)p(x)]<n$, there was proved a Sobolev type $\mathcal{L}^{p(\cdot),\lambda(\cdot)}\to\mathcal{L}^{q(\cdot),\lambda(\cdot)}$– theorem. In the case of constant α, there was also proved a boundedness theorem in the limiting case $p(x)=\frac{n-\lambda(x)}{\alpha}$, when the potential operator I^α acts from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ into BMO. In [47] the maximal operator and potential operators were considered in a somewhat more general space, but under more restrictive conditions on $p(x)$. P. Hästö in [26] used his new "local-to-global" approach to extend the result of [3] on the maximal operator to the case of the whole space $\mathbb{R}^n$. In [30] there was proved the boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces.

Generalized Morrey spaces of such a kind in the case of constant p were studied in [5], [17], [43], [46], [48], [49]. In [22] there was proved the boundedness of the maximal operator, singular integral operator and the potential operators in generalized variable exponent Morrey spaces $\mathcal{M}^{p(\cdot),\omega}(\Omega)$.

In the case of constant p and λ, the results on the boundedness of potential operators and classical Calderon-Zygmund singular operators date back to [1] and [51], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [11]; for further results in the case of constant p and λ see, e.g., [6]– [9].

We introduce the generalized variable exponent weighted Morrey spaces $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^\gamma}(\Omega)$ over an open set $\Omega\subseteq\mathbb{R}^n$. Within the frameworks of the spaces $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^\gamma}(\Omega)$, over bounded sets $\Omega\subseteq\mathbb{R}^n$ we consider the Hardy-Littlewood maximal operator

$$Mf(x)=\sup_{r>0}|B(x,r)|^{-1}\int_{\widetilde{B}(x,r)}|f(y)|dy,$$

potential type operators

$$I^{\alpha(x)}f(x)=\int_\Omega|x-y|^{\alpha(x)-n}f(y)dy,\ 0<\alpha(x)<n,$$

the fractional maximal operator

$$M^{\alpha(x)}f(x)=\sup_{r>0}|B(x,r)|^{\frac{\alpha(x)}{n}-1}\int_{\widetilde{B}(x,r)}|f(y)|dy,\ 0\le\alpha(x)<n$$

of variable order $\alpha(x)$ and Calderon-Zygmund type singular operator

$$Tf(x)=\int_\Omega K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n} \quad \text{for all} \quad x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|x-\xi|.$$

We find the condition on the function $\omega(x,r)$ for the boundedness of the maximal operator M and the singular integral operators T in generalized weighted Morrey space $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^{\gamma}}(\Omega)$ with variable $p(x)$ under the log-condition on $p(\cdot)$. For potential operators, under the same log-condition and the assumptions

$$\inf_{x\in\Omega} \alpha(x) > 0, \ \sup_{x\in\Omega} \alpha(x)p(x) < n$$

we also find the condition on $\omega(x,r)$ for the validity of a Sobolev-Adams type $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^{\gamma}}(\Omega) \to \mathcal{M}^{q(\cdot),\omega,,|x-x_0|^{\mu}}(\Omega)$-theorem, which recovers the known result for the case of the classical weighted Morrey spaces with variable exponents, when $\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)}$.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent Lebesgue and Morrey spaces. In Section 3 we introduce the generalized Morrey spaces with variable exponents and recall some facts about generalized Morrey spaces with constant p. In Section 4 we deal with the maximal operator, while potential operators are studied in Section 5. In Section 6 we treat Calderon-Zygmund singular operators.

The main results are given in Theorems 17, 18, 19, 20, 24, 25. We emphasize that the results we obtain for generalized weighted Morrey spaces are new even in the case when $p(x)$ is constant, because we do not impose any monotonicity type condition on $\omega(x,r)$.

N o t a t i o n :
$\mathbb{R}^n$ is the n-dimensional Euclidean space,
$\Omega \subseteq \mathbb{R}^n$ is an open set, $\ell = \text{diam } \Omega$;
$0 \in \overline{\Omega}$;
$\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$;
$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\})$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$;
by c,C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line.

2. Preliminaries on variable exponent Lebesgue and Morrey spaces

Let $p(\cdot)$ be a measurable function on Ω with values in $[1,\infty)$. An open set Ω is assumed to be bounded throughout the whole paper. We suppose that

$$1 < p_- \le p(x) \le p_+ < \infty, \tag{1}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1, \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty.$

By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 :\ I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent. The Hölder inequality is valid in the form

$$\int_\Omega |f(x)||g(x)|dx \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

For the basics on variable exponent Lebesgue spaces we refer to [63], [41].

The weighted Lebesgue space $L^{p(\cdot),\omega}(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot),\omega}(\Omega)} = \inf\left\{\eta > 0 :\ \int_\Omega \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \omega(x)dx \le 1\right\}.$$

Definition 1. *By $WL(\Omega)$ (weak Lipschitz) we denote the class of functions defined on Ω satisfying the log-condition*

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2} \ \ x, y \in \overline{\Omega}, \tag{2}$$

where $A = A(p) > 0$ does not depend on x, y.

Theorem 1. *([14]) Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p \in WL(\Omega)$ satisfy condition* (1)*. Then the maximal operator M is bounded in $L^{p(\cdot)}(\Omega)$.*

Theorem 2. *([36]) Let Ω be bounded and $p \in WL(\Omega)$ satisfy condition* (1), (9)*, $x_0 \in \overline{\Omega}$ and let*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{p'(x_0)}. \tag{3}$$

Then the weighted maximal operator M_β is bounded in $L^{p(\cdot)}(\Omega)$.

The following theorem for bounded sets Ω, but for variable $\alpha(x)$, was proved in [59].

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $p, \alpha \in WL(\Omega)$ satisfy assumption* (1)*, $x_0 \in \overline{\Omega}$ and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0, \ \sup_{x\in\Omega} \alpha(x)p(x) < n, \tag{4}$$

$$\alpha(x_0)p(x_0) - n < \gamma < n(p(x_0) - 1), \tag{5}$$

$$\mu = \frac{q(x_0)\gamma}{p(x_0)}. \tag{6}$$

Then the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot),|x-x_0|^\gamma}(\Omega)$ to $L^{q(\cdot),|x-x_0|^\mu}(\Omega)$ with

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}. \tag{7}$$

Singular operators within the framework of the spaces with variable exponents were studied in [15]. From Theorem 4.8 and Remark 4.6 of [15] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition on $p(x)$ at infinity.

Theorem 4. *([15]) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $p \in WL(\Omega)$ satisfy condition* (1)*. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

We will also make use of the estimate provided by the following lemma (see [57], Corollary to Lemma 3.22).

Lemma 1. *Let Ω be a bounded domain and p satisfy the assumption $1 \le p_- \le p(x) \le p_+ < \infty$ and condition* (2)*. Let also $\beta \in WL(\Omega)$ and $\sup_{x\in\Omega}[n+\nu(x)p(x)] < 0$, $\sup_{x\in\Omega}[n+\nu(x)p(x) + \beta(x)] < 0$. Then*

$$\| |x-y|^{\nu(x)} \chi_{B(x,r)}(y) \|_{L^{p(\cdot),|\cdot|^{\beta(x)}}} \le C r^{\nu(x)+\frac{n}{p(x)}} (r + |x|)^{\frac{\beta(x)}{p(x)}}, \quad x \in \Omega,\ 0 < r < \ell, \tag{8}$$

where C does not depend on x and r.

Remark 1. *It can be shown that the constant C in* (8) *may be estimated as $C = C_0 \ell^{n\left(\frac{1}{p_-} - \frac{1}{p_+}\right)}$, where C_0 does not depend on Ω.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ and variable weighted Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}(\Omega)$ are defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f \chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^{\gamma}}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}} \||\cdot|^{\frac{\gamma}{p(\cdot)}} f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

respectively. Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}|dy,$$

where $f_{\widetilde{B}(x,t)}(x) = |\widetilde{B}(x,t)|^{-1} \int_{\widetilde{B}(x,t)} f(y)dy$.

Definition 2. *We define the $BMO_{|\cdot|^\beta}(\Omega)$ space as the set of all locally integrable functions f with the finite norm*

$$\|f\|_{BMO_{|\cdot|^\beta}} = \sup_{x\in\Omega} |x|^\beta M^\sharp f(x) = \|M^\sharp f\|_{L_{\infty,|\cdot|^\beta}}.$$

The following statements are known.

Theorem 5. *([3]) Let Ω be bounded and $p \in WL(\Omega)$ satisfy condition (1) and let a measurable function λ satisfy the conditions*

$$0 \le \lambda(x), \qquad \sup_{x\in\Omega} \lambda(x) < n. \tag{9}$$

Then the maximal operator M is bounded in $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

Theorem 5 was extended to unbounded domains in [26].

Note that the boundedness of the maximal operator in Morrey spaces with variable $p(x)$ was studied in [30] in the more general setting of quasimetric measure spaces.

Theorem 6. *([3]) Let Ω be bounded, $p, \alpha, \lambda \in WL(\Omega)$ and p satisfy condition (1). Let also $\lambda(x) \ge 0$ and the conditions (4), (7) be fulfilled. Then the operator $I^{\alpha(\cdot)}$ is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $\mathcal{L}^{q(\cdot),\mu(\cdot)}(\Omega)$, where*

$$\frac{\mu(x)}{q(x)} = \frac{\lambda(x)}{p(x)}. \tag{10}$$

Theorem 7. *([3]) Let Ω be bounded, $p, \alpha, \lambda \in WL(\Omega)$ and p satisfy condition (1). Let also $\lambda(x) \ge 0$ and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0, \ \sup_{x\in\Omega} [\lambda(x) + \alpha(x)p(x)] < n \tag{11}$$

hold. Then the operator $I^{\alpha(\cdot)}$ is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $\mathcal{L}^{q(\cdot),\lambda(\cdot)}(\Omega)$, where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n-\lambda(x)}. \tag{12}$$

Theorem 8. *([3]) Let Ω be bounded and $p,\alpha,\lambda \in WL(\Omega)$ satisfy conditions* (1) *and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0,\ \ \lambda(x) + \alpha(x)p(x) = n$$

hold. Then the operator $M^{\alpha(\cdot)}$ is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^\infty(\Omega)$.

Theorem 9. *([3]) Let Ω be bounded and $p,\lambda \in WL(\Omega)$ satisfy conditions* (1) *and let $0 < \alpha < n$, $0 \le \lambda(x)$, $\sup \lambda(x) < n - \alpha$,*

$$p(x) = \frac{n - \lambda(x)}{\alpha}.$$

Then the operator I^α is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $BMO(\Omega)$.

3. Variable exponent generalized Morrey spaces

Throughout this paper the functions $\omega(x,r)$, $\omega_1(x,r)$ and $\omega_2(x,r)$ are non-negative measurable functions on $\Omega \times (0,\ell)$, $\ell = diam\,\Omega$.

We find it convenient to define the generalized Morrey spaces as follows.

Definition 3. *Let $1 \le p < \infty$. The generalized Morrey space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ is defined by the norms*

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},$$

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega,|\cdot|^\gamma}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))}.$$

According to this definition, we recover the space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$:

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega) = \mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)\Big|_{\omega(x,r)=r^{\frac{\lambda(x)-n}{p(x)}}}.$$

In the sequel we assume that

$$\inf_{x\in\Omega, r>0} \omega(x,r) > 0 \tag{13}$$

which makes the space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ nontrivial. Note that when p is constant, in the case of $w(x,r) \equiv const > 0$, we have the space $L^\infty(\Omega)$.

3.1. Preliminaries on Morrey spaces with constant exponents p

In [20], [21], [46] and [48] sufficient conditions on weights ω_1 and ω_2 for the boundedness of the singular operator T from $\mathcal{M}^{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{p,\omega_2}(\mathbb{R}^n)$ were obtained. In [48] the following condition was imposed on $w(x,r)$:

$$c^{-1}\omega(x,r) \le \omega(x,t) \le c\,\omega(x,r) \tag{14}$$

whenever $r \le t \le 2r$, where $c(\ge 1)$ does not depend on t,r and $x \in \mathbb{R}^n$, with

$$\int_r^\infty \omega(x,t)^p \frac{dt}{t} \le C\,\omega(x,r)^p \tag{15}$$

for tmaximal or singular operator and

$$\int_r^\infty t^{\alpha p}\omega(x,t)^p \frac{dt}{t} \le C\,r^{\alpha p}\omega(x,r)^p \tag{16}$$

for potential or fractional maximal operator, where $C(>0)$ does not depend on r and $x \in \mathbb{R}^n$.

Remark 2. *The left-hand side inequality in* (14) *is satisfied for any non-negative function $w(x,r)$ such that there exists a number $a \in \mathbb{R}^1$ such that the function $r^a w(x,r)$ is almost increasing in r uniformly in x:*

$$t^a w(x,t) \le c r^a w(x,r) \qquad \textit{for all} \qquad 0 < t \le r < \infty$$

where $c \ge 1$ does not depend on x,r,t.

Note that integral conditions of type (15) after the paper [4] of 1956 are often referred to as Bary-Stechkin or Zygmund-Bary-Stechkin conditions, see also [25]. The classes of almost monotonic functions satisfying such integral conditions were later studied in a number of papers (see [28], [54], [55] and references therein), where the characterization of integral inequalities of such a kind was given in terms of certain lower and upper indices known as Matuszewska-Orlicz indices. Note that in the cited papers the integral inequalities were studied as $r \to 0$. Such inequalities are also of interest when they allow to impose different conditions as $r \to 0$ and $r \to \infty$; such a case was dealt with in [56], [40].

In [48] the following statements were proved.

Theorem 10. *[48] Let $1 < p < \infty$ and $\omega(x,r)$ satisfy conditions* (14)-(15)*. Then the operators M and T are bounded in $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$.*

Theorem 11. *[48] Let $1 < p < \infty, 0 < \alpha < \frac{n}{p}$, and $\omega(x,t)$ satisfy conditions* (14) *and* (16)*. Then the operators M^α and I^α are bounded from $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\omega}(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.*

The following statement, containing the results in [46], [48], was proved in [20] (see also [21]). Note that Theorems 12 and 13 do not impose condition (14).

Theorem 12. *[20] Let $1 < p < \infty$ and $\omega_1(x,r)$, $\omega_2(x,r)$ be positive measurable functions satisfying the condition*

$$\int_r^\infty \omega_1(x,t)\frac{dt}{t} \le c_1\,\omega_2(x,r) \tag{17}$$

with $c_1 > 0$ not depending on $x \in \mathbb{R}^n$ and $t > 0$. Then the operators M and T are bounded from $\mathcal{M}^{p,\omega_1(\cdot)}(\mathbb{R}^n)$ to $\mathcal{M}^{p,\omega_2(\cdot)}(\mathbb{R}^n)$.

Theorem 13. *[20] Let $0 < \alpha < n$, $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\omega_1(x,r)$, $\omega_2(x,r)$ be positive measurable functions satisfying the condition*

$$\int_r^\infty t^\alpha \omega_1(x,t)\frac{dt}{t} \le c_1\,\omega_2(x,r). \tag{18}$$

Then the operators M^α and I^α are bounded from $\mathcal{M}^{p,\omega_1(\cdot)}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\omega_2(\cdot)}(\mathbb{R}^n)$.

Theorem 14. *[22] Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p \in WL(\Omega)$ satisfy assumption (1) and the function $\omega_1(x,r)$ and $\omega_2(x,r)$ satisfy the condition*

$$\int_r^\ell \omega_1(x,t)\frac{dt}{t} \le C\,\omega_2(x,r), \tag{19}$$

where C does not depend on x and t. Then the maximal operators M and T are bounded from the space $\mathcal{M}^{p(\cdot),\omega_1}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\omega_2}(\Omega)$.

Theorem 15. *[22] Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p, q \in WL(\Omega)$ satisfy assumption (1), $\alpha(x), q(x)$ satisfy the conditions (4), (7) and the functions $\omega_1(x,r)$ and $\omega_2(x,r)$ fulfill the condition*

$$\int_r^\ell t^{\alpha(x)}\omega_1(x,t)\frac{dt}{t} \le C\,\omega_2(x,r), \tag{20}$$

where C does not depend on x and r. Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathcal{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega_2(\cdot)}(\Omega)$.

Theorem 16. *[22] Let $p \in WL(\Omega)$ satisfy assumption (1), $\alpha(x)$ fulfill the condition (4) and let $\omega(x,t)$ satisfy condition (19) and the conditions*

$$\omega(x,r) \le \frac{C}{r^{\frac{\alpha(x)}{1-\frac{p(x)}{q(x)}}}}, \tag{21}$$

$$\int_r^\ell t^{\alpha(x)-1}\,\omega(x,t)dt \le C\omega(x,r)^{\frac{p(x)}{q(x)}}, \tag{22}$$

where $q(x) \ge p(x)$ and C does not depend on $x \in \Omega$ and $r \in (0,\ell]$. Suppose also that for almost every $x \in \Omega$, the function $w(x,r)$ fulfills the condition

there exist an $a = a(x) > 0$ such that $\omega(x,\cdot) : [0,\ell] \to [a,\infty)$ is surjective. (23)

Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega(\cdot)}(\Omega)$.

4. The maximal operator in the spaces $\mathcal{M}^{p(\cdot),\omega(\cdot),|\cdot|^\gamma}(\Omega)$

Theorem 17. *Let Ω be bounded and $p \in WL(\Omega)$ satisfy condition (1), $x_0 \in \overline{\Omega}$ and (3). Then*

$$\|Mf\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))}$$
$$\leq Ct^{\frac{n}{p(x)}}(t+|x-x_0|)^{\frac{\beta}{p(x)}} \int_t^\ell s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}} \|f\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,s))} ds, \ 0<t<\frac{\ell}{2}, \quad (24)$$

for every $f \in L^{p(\cdot),|x-x_0|^\beta}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\Omega\backslash\widetilde{B}(x,2t)}(y), \quad t>0. \quad (25)$$

Then

$$\|Mf\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))} \leq \|Mf_1\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))} + \|Mf_2\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))}.$$

By the Theorem 2 we obtain

$$\begin{aligned}\|Mf_1\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))} &\leq \|Mf_1\|_{L^{p(\cdot),|x-x_0|^\beta}(\Omega)} \\ &\leq C\|f_1\|_{L^{p(\cdot),|x-x_0|^\beta}(\Omega)} = C\|f\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,2t))}, \end{aligned} \quad (26)$$

where C does not depend on f. We assume for simplicity that $x_0 = 0$. From (26) we obtain

$$\|Mf_1\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,t))} \leq Ct^{\frac{n}{p(x)}}(t+|x-x_0|)^{\frac{\beta}{p(x)}} \int_{2t}^\ell s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))} ds$$
$$\leq Ct^{\frac{n}{p(x)}}(t+|x-x_0|)^{\frac{\beta}{p(x)}} \int_t^\ell s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))} ds \quad (27)$$

easily obtained from the fact that $\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,2t))}$ is non-decreasing in t, so that $\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,2t))}$ on the right-hand side of (26) is dominated by the right-hand side of (27). Note that this "complication" of estimate in comparison with (26) is done because the term Mf_2 will be estimated below in a similar way (see (29)).

To estimate Mf_2, we first prove the following auxiliary inequality

$$\int_{\Omega\backslash\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \leq C\int_t^\ell s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))} ds, \quad 0<t<\ell. \quad (28)$$

To this end, we choose $\delta > \frac{n}{p_-}$ and proceed as follows:

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \le \delta \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n+\delta}|f(y)|dy \int_{|x-y|}^{\ell} s^{-\delta-1}ds$$
$$= \delta \int_t^{\ell} s^{-\delta-1}ds \int_{\{y\in\Omega:2t\le|x-y|\le s\}} |x-y|^{-n+\delta}|f(y)|dy$$
$$\le C\int_t^{\ell} s^{-\delta-1}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}\||x-y|^{-n+\delta}\|_{L^{p'(\cdot),|\cdot|^{\beta/(1-p(x))}}(\widetilde{B}(x,s))}ds.$$

We then make use of Lemma 1 and obtain (28).

For $z \in \widetilde{B}(x,t)$ we get

$$Mf_2(z) = \sup_{r>0}|B(z,r)|^{-1}\int_{\widetilde{B}(z,r)} |f_2(y)|dy$$
$$\le C\sup_{r\ge 2t}\int_{(\Omega\setminus\widetilde{B}(x,2t))\cap\widetilde{B}(z,r)} |y-z|^{-n}|f(y)|dy$$
$$\le C\sup_{r\ge 2t}\int_{(\Omega\setminus\widetilde{B}(x,2t))\cap\widetilde{B}(z,r)} |x-y|^{-n}|f(y)|dy$$
$$\le C\int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n}|f(y)|dy.$$

Then by (28)

$$Mf_2(z) \le C\int_{2t}^{\ell} s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}ds,$$
$$\le C\int_{t}^{\ell} s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}ds,$$

where C does not depend on x, r. Thus, the function $Mf_2(z)$, with fixed x and t, is dominated by the expression not depending on z. Then

$$\|Mf_2\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,t))} \le C\int_{t}^{\ell} s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}ds\,\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|\cdot|^\beta}(\Omega)}. \tag{29}$$

Since $\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|\cdot|^\beta}(\Omega)} \le Ct^{\frac{n}{p(x)}}(t+|x|)^{\frac{\beta}{p(x)}}$ by Lemma 1, we then obtain (24) from (27) and (29).

The following theorem extends Theorem 15 to the case of generalized weighted Morrey spaces $\mathcal{M}^{p(\cdot),\omega,|\cdot|^\beta}(\Omega)$.

Theorem 18. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p \in WL(\Omega)$ satisfy assumption* (1)*, $x_0 \in \overline{\Omega}$,* (3) *and the function $\omega_1(x,r)$ and $\omega_2(x,r)$ satisfy the condition*

$$\int_t^{\ell}(r+|x-x_0|)^{-\frac{\beta}{p(x)}}\omega_1(x,r)\frac{dr}{r} \le C\,(t+|x-x_0|)^{-\frac{\beta}{p(x)}}\omega_2(x,t). \tag{30}$$

Then the maximal operator M is bounded from the space $\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\beta}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\beta}(\Omega)$. We have

$$\|Mf\|_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)} = \sup_{x\in\Omega,\, t\in(0,\ell)} \omega_2^{-1}(x,t)t^{-\frac{n}{p(x)}}\|Mf\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))}.$$

The estimation is obvious for $\frac{\ell}{2} \le t \le \ell$ in view of (13). For

$$\|Mf\|^{\sim}_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)} = \sup_{x\in\Omega,\, t\in\left(0,\frac{\ell}{2}\right)} \omega_2^{-1}(x,t)t^{-\frac{n}{p(x)}}\|Mf\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))}$$

by Theorem 17 we obtain

$$\|Mf\|^{\sim}_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)} \le C \sup_{x\in\Omega,\, 0<t\le\ell} \omega_2^{-1}(x,t)(t+|x-x_0|)^{\frac{\beta}{p(x)}}$$
$$\int_t^\ell r^{-\frac{n}{p(x)}-1}(r+|x-x_0|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,r))}dr.$$

Hence

$$\|Mf\|^{\sim}_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)} \le$$
$$\le C\|f\|_{\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\beta}(\Omega)} \sup_{x\in\Omega,\, t\in(0,\ell)} \omega_2^{-1}(x,t)(t+|x-x_0|)^{\frac{\beta}{p(x)}}$$
$$\int_t^\ell \omega_1(x,r)(r+|x-x_0|)^{-\frac{\beta}{p(x)}}\frac{dr}{r} \le C\|f\|_{\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\beta}(\Omega)},$$

by (30), which completes the proof.

5. Riesz potential operator in the spaces $\mathcal{M}^{p(\cdot),\omega(\cdot),|\cdot|^\gamma}(\Omega)$

5.1. Spanne type result

Theorem 19. *Let $p \in WL(\Omega)$ satisfy conditions* (1) *and let* (3), $x_0 \in \overline{\Omega}$, (5), (6), $\alpha(x), q(x)$ *satisfy the conditions* (4) *and* (7). *Then*

$$\|I^{\alpha(\cdot)}f\|_{L^{q(\cdot),|x-x_0|^\mu}(\widetilde{B}(x,t))}$$
$$\le Ct^{\frac{n}{q(x)}}(t+|x-x_0|)^{\frac{\gamma}{p(x)}}\int_t^l s^{-\frac{n}{q(x)}-1}(s+|x-x_0|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,s))}ds, \quad 0<t\le\frac{\ell}{2} \tag{31}$$

where t is an arbitrary number in $\left(0,\frac{\ell}{2}\right)$ and C does not depend on f, x and t.

Proof. As in the proof of Theorem 17, we represent function f in form (25) and have

$$I^{\alpha(\cdot)}f(x) = I^{\alpha(\cdot)}f_1(x) + I^{\alpha(\cdot)}f_2(x).$$

By Theorem 3 we obtain

$$\begin{aligned}\|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot),|x-x_0|^{\mu}}(\widetilde{B}(x,t))} &\le \|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot),|x-x_0|^{\mu}}(\Omega)} \\ &\le C\|f_1\|_{L^{p(\cdot),|x-x_0|^{\gamma}}(\Omega)} = C\|f\|_{L^{p(\cdot),|x-x_0|^{\gamma}}(\widetilde{B}(x,2t))}.\end{aligned}$$

Then

$$\|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot),|x-x_0|^{\mu}}(\widetilde{B}(x,t))} \le C\|f\|_{L^{p(\cdot),|x-x_0|^{\gamma}}(\widetilde{B}(x,2t))},$$

where the constant C is independent of f.

We assume for simplicity that $x_0 = 0$. Taking into account that

$$\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,2t))} \le Ct^{\frac{n}{q(x)}}(t+|x|)^{\frac{\gamma}{p(x)}}\int_t^l s^{-\frac{n}{q(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}ds,$$

we get

$$\|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot),|\cdot|^{\mu}}(\widetilde{B}(x,t))} \le Ct^{\frac{n}{q(x)}}(t+|x|)^{\frac{\gamma}{p(x)}}\int_t^l s^{-\frac{n}{q(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}ds. \tag{32}$$

When $|x-z| \le t$, $|z-y| \ge 2t$, we have $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$\begin{aligned}|I^{\alpha(\cdot)}f_2(x)| &\le \int_{\Omega\setminus\widetilde{B}(x,2t)} |z-y|^{\alpha(y)-n}|f(y)|dy \\ &\le C\int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{\alpha(x)-n}|f(y)|dy.\end{aligned}$$

We choose $\beta > \frac{n}{q(x)}$ and obtain

$$\int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{\alpha(x)-n}|f(y)|dy$$

$$\begin{aligned}&= \beta\int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{\alpha(x)-n+\beta}|f(y)|\left(\int_{|x-y|}^l s^{-\beta-1}ds\right)dy \\ &= \beta\int_{2t}^l s^{-\beta-1}\left(\int_{\{y\in\Omega:2t\le|x-y|\le s\}} |x-y|^{\alpha(x)-n+\beta}|f(y)|dy\right)ds \\ &\le C\int_{2t}^l s^{-\beta-1}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}\||x-y|^{\alpha(x)-n+\beta}\|_{L^{p'(\cdot),|\cdot|^{\gamma/(1-p(x))}}(\widetilde{B}(x,s))}ds \\ &\le C\int_{2t}^l s^{\alpha(x)-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}ds.\end{aligned}$$

Hence

$$\|I^{\alpha(\cdot)} f_2\|_{L^{q(\cdot),|\cdot|^\mu}(\widetilde{B}(x,t))} \le C \int_{2t}^{l} s^{-\frac{n}{q(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,s))} ds \|\chi_{\widetilde{B}(x,t)}\|_{L^{q(\cdot),|\cdot|^\mu}(\Omega)}.$$

Therefore

$$\|I^{\alpha(\cdot)} f_2\|_{L^{q(\cdot),|\cdot|^\mu}(\widetilde{B}(x,t))} \le C t^{\frac{n}{q(x)}} (t+|x|)^{\frac{\gamma}{p(x)}} \int_{2t}^{l} s^{-\frac{n}{q(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,s))} ds \tag{33}$$

which together with (32) yields (31).

Theorem 20. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p, q \in WL(\Omega)$ satisfy assumptions* (1), (5), (6), (3), *$x_0 \in \overline{\Omega}$, $\alpha(x), q(x)$ satisfy the conditions* (4), (7) *and the functions $\omega_1(x,r)$ and $\omega_2(x,r)$ fulfill the condition*

$$\int_r^\ell t^{\alpha(x)}(t+|x-x_0|)^{-\frac{\gamma}{p(x)}} \omega_1(x,t) \frac{dt}{t} \le C\,(r+|x-x_0|)^{-\frac{\gamma}{p(x)}} \omega_2(x,r). \tag{34}$$

Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathcal{M}^{p(\cdot),\omega_1(\cdot),|x-x_0|^\gamma}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega_2(\cdot),|x-x_0|^\mu}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega,|x-x_0|^\gamma}(\Omega)$. As usual, when estimating the norm

$$\|I^{\alpha(\cdot)} f\|_{\mathcal{M}^{q(\cdot),\omega_2,|x-x_0|^\mu}(\Omega)} = \sup_{x\in\Omega,\, t>0} \frac{t^{-\frac{n}{q(x)}}}{\omega_2(x,t)} \|I^{\alpha(\cdot)} f \chi_{\widetilde{B}(x,t)}\|_{L^{q(\cdot),|x-x_0|^\mu}(\Omega)}, \tag{35}$$

it suffices to consider only the values $t \in \left(0, \frac{\ell}{2}\right)$, thanks to condition (13). We estimate $\|I^{\alpha(\cdot)} f \chi_{\widetilde{B}(x,t)}\|_{L^{q(\cdot),|x-x_0|^\mu}(\Omega)}$ in (35) by means of Theorem 19 and obtain

$$\begin{aligned}
&\|I^{\alpha(\cdot)} f\|_{\mathcal{M}^{q(\cdot),\omega_2,|x-x_0|^\mu}(\Omega)} \\
&\le C \sup_{x\in\Omega,\, t>0} \frac{(t+|x-x_0|)^{\frac{\gamma}{p(x)}}}{\omega_2(x,t)} \int_t^\ell r^{-\frac{n}{q(x)}-1}(r+|x-x_0|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,r))} dr \\
&\le C \|f\|_{\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\gamma}(\Omega)} \sup_{x\in\Omega,\, t>0} \frac{(t+|x-x_0|)^{\frac{\gamma}{p(x)}}}{\omega_2(x,t)} \int_t^\ell \frac{r^{\alpha(x)}(r+|x-x_0|)^{-\frac{\gamma}{p(x)}} \omega_1(x,r)}{r} dr.
\end{aligned}$$

It remains to make use of condition (34).

Theorem 21. *Let $p \in WL(\Omega)$ satisfy assumption* (1), *$x_0 \in \overline{\Omega}$, γ, μ satisfy conditions*

$$0 \le \gamma < \frac{n}{p'(x_0)}, \quad \mu = \frac{\gamma}{p(x_0)}, \tag{36}$$

$\inf\limits_{x\in\Omega} \alpha(x) > 0$ and let $\omega(x,t)$ satisfy condition

$$r^{\alpha(x)} \omega(x,r) \le C. \tag{37}$$

Then the operator $M^{\alpha(\cdot)}$ is bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot),|x-x_0|^\gamma}(\Omega)$ to $L^{\infty,|x-x_0|^\mu}(\Omega)$.

Proof. Let $x \in \Omega$ and $r > 0$. We assume for simplicity that $x_0 = 0$. By the Hölder inequality we get successively

$$
\begin{aligned}
& r^{\alpha(x)-n} \int_{\widetilde{B}(x,r)} |f(y)| dy \\
& \le C r^{\alpha(x)-n} r^{\frac{n}{p(x)}} \omega(x,r) r^{-\frac{n}{p(x)}} \omega^{-1}(x,r) \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))} \|\chi_{\widetilde{B}(x,r)}\|_{L^{p'(\cdot),|\cdot|^{\gamma/(1-p(\cdot))}}} \\
& \le C r^{\alpha(x)} \omega(x,r) |x|^{-\frac{\gamma}{p(x)}} \|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} \le C |x|^{-\frac{\gamma}{p(x)}} \|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.
\end{aligned}
$$

We again refer to the logarithmic condition for $p(x)$ which provides the equivalence

$$|x|^{\frac{\gamma}{p(x)}} \sim |x|^{\frac{\gamma}{p(0)}}.$$

Theorem 22. *Let $p \in WL(\Omega)$ satisfy assumption* (1)*, $x_0 \in \overline{\Omega}$, γ, μ satisfy condition* (36)*, $0 < \alpha < n$ and let $\omega(x,t)$ satisfy condition* (37)*.*
Then the operator I^α is bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot),|x-x_0|^\gamma}(\Omega)$ to $BMO_{|x-x_0|^\mu}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega(\cdot),|x-x_0|^\gamma}(\Omega)$. In [1] it was proved that

$$M^\sharp(I^\alpha f)(x) \le C M^\alpha f(x). \tag{38}$$

The proof of Theorem 22 follows from the Theorem 21 and inequality (38).

6. Singular operators in the spaces $\mathcal{M}^{p(\cdot),\omega(\cdot),|\cdot|^\gamma}(\Omega)$

Theorem 23. *[33] Let Ω be bounded, $p \in WL(\Omega)$ and p satisfy conditions* (1) *and* (3)*, $x_0 \in \overline{\Omega}$. Then the operators T and T^* are bounded in the space $L^{p(\cdot),|x-x_0|^\gamma}(\Omega)$.*

Theorem 24. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $p \in WL(\Omega)$ satisfy conditions* (1)*,* (3)*, $x_0 \in \overline{\Omega}$ and $f \in L^{p(\cdot),|x-x_0|^\gamma}(\Omega)$. Then*

$$
\begin{aligned}
& \|Tf\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))} \\
& \le C t^{\frac{n}{p(x)}} (t+|x-x_0|)^{\frac{\gamma}{p(x)}} \int_t^\ell r^{-\frac{n}{p(x)}-1} (r+|x-x_0|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,r))} dr, \quad 0 < t \le \frac{\ell}{2},
\end{aligned}
\tag{39}
$$

where C does not depend on f and t.

Proof. We represent function f as in (25) and have

$$\|Tf\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))} \le \|Tf_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))} + \|Tf_2\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))}.$$

By the Theorem 23 we obtain

$$\|Tf_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))} \le \|Tf_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\Omega)} \le C\|f_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\Omega)},$$

so that

$$\|Tf_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))} \le C\|f\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,2t))}.$$

We assume for simplicity that $x_0 = 0$. Taking into account the inequality

$$\|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))} \le$$

$$Ct^{\frac{n}{p(x)}}(t+|x|)^{\frac{\gamma}{p(x)}} \int_{2t}^{\ell} r^{-\frac{n}{p(x)}-1}(r+|x|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))} dr, \quad 0 < t \le \frac{\ell}{2},$$

we get

$$\|Tf_1\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))} \le Ct^{\frac{n}{p(x)}}(t+|x|)^{\frac{\gamma}{p(x)}} \int_{t}^{\ell} r^{-\frac{n}{p(x)}-1}(r+|x|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))} dr. \tag{40}$$

To estimate $\|Tf_2\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))}$, we observe that

$$|Tf_2(z)| \le C \int\limits_{\Omega\setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$|Tf_2(z)| \le C \int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n}|f(y)|dy.$$

Hence by estimate (8) (with $\nu(x) \equiv 0$) and inequality (28), we get

$$\|Tf_2\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))} \le C \int_t^{\ell} r^{-\frac{n}{p(x)}-1}(r+|x|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))} dr \, \|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|\cdot|^\gamma}(\Omega)}. \tag{41}$$

From (40) and (41) we arrive at (39).

Theorem 25. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $p \in WL(\Omega)$ satisfy condition* (1), (3), *$x_0 \in \overline{\Omega}$ and $\omega_1(x,t)$ and $\omega_2(x,r)$ fulfill conditions* (30). *Then the singular integral operator T is bounded from the space $\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\gamma}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\gamma}(\Omega)$.*

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\gamma}(\Omega)$. As usual, when estimating the norm

$$\|Tf\|_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\gamma}(\Omega)} = \sup_{x\in\Omega,\, t>0} \frac{t^{-\frac{n}{p(x)}}}{\omega_2(x,t)} \|Tf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|x-x_0|^\gamma}(\Omega)}, \tag{42}$$

it suffices to consider only the values $t \in \left(0, \frac{\ell}{2}\right)$, thanks to condition (13). We estimate $\|Tf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|x-x_0|^\gamma}(\Omega)}$ in (42) by means of Theorem 24 and obtain

$$\begin{aligned}
&\|Tf\|_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\gamma}(\Omega)} \\
&\leq C \sup_{x\in\Omega,\, t>0} \frac{(t+|x-x_0|)^{\frac{\gamma}{p(x)}}}{\omega_2(x,t)} \int_t^\ell r^{-\frac{n}{p(x)}-1}(r+|x-x_0|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,r))} dr \\
&\leq C\|f\|_{\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\gamma}(\Omega)} \sup_{x\in\Omega,\, t>0} \frac{(t+|x-x_0|)^{\frac{\gamma}{p(x)}}}{\omega_2(x,t)} \int_t^\ell \frac{(r+|x-x_0|)^{-\frac{\gamma}{p(x)}}\omega_1(x,r)}{r} dr.
\end{aligned}$$

It remains to make use of condition (30).

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A New Proof of Laguerre's Theorem

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Abstract. Using Möbius transformation and its characterstics we have obtained a different proof of well known Laguerre's theorem on the zeros of the polar derivative of a polynomial.

Key Words and Phrases: polar derivative of a polynomial, zeros, circular region, Laguerre's theorem, Möbius transformation

2000 Mathematics Subject Classifications: Primary 30C15, Secondary 30C10

1. Introduction

Concerning relative location of zeros of a polynomial and its polar derivative we have the following well known interesting theorem due to Laguerre ([2], [3, p. 49]).

Laguerre's theorem. *If all the zeros z_j of the n^{th} degree polynomial $f(z)$ lie in a circular region C and if Z is any zero of*

$$f_1(z) = nf(z) + (\zeta - z)f'(z),$$

the polar derivative of $f(z)$, then not both points Z and ζ may lie outside of C. Furthermore, if $f(Z) \neq 0$, then any circle K through Z and ζ either passes through all the zeros of $f(z)$ or separates these zeros.

In the literature there exists certain other proof [1] of Laguerre's theorem. In this paper we have used Möbius transformation and its characteristics ([4, chapter 10], [5, chapter 5]) to obtain a different proof of Laguerre's theorem.

2. Lemmas

For the proof of the Laguerre's theorem we require the following lemmas.

Lemma 1. *If each complex number $w_j, j = 1, 2, \ldots, p$, has the properties that $w_j \neq 0$ and*

$$\gamma \leq \ Arg\ w_j < \gamma + \pi,\ \ j = 1, 2, \ldots, p,$$

where γ is a real constant, then their sum $w = \sum_{j=1}^{p} w_j$ can not vanish.

This lemma is due to Marden [3, Theorem (1,1)].

Lemma 2. *If each complex number $w_j, j = 1, 2, \ldots, p$, has the properties that $w_j \neq 0$ and*

$$\gamma < \ Arg\ w_j < \gamma + \pi,\ \textit{for at least one } j, j = 1, 2, \ldots, p, \tag{1}$$

with

$$\gamma \leq \ Arg\ w_j \leq \gamma + \pi,\ \textit{for remaining } j\ 's, j = 1, 2, \ldots, p, \tag{2}$$

where γ is a real constant, then their sum $w = \sum_{j=1}^{p} w_j$ can not vanish.

Proof of Lemma 2. We begin with the case

$$\gamma = 0.$$

Then

$$Im\ w_j > 0,\ \text{for at least one } j, j = 1, 2, \ldots, p,\ \text{(by (1))},$$

with

$$Im\ w_j \geq 0,\ \text{for remaining } j\ 's, j = 1, 2, \ldots, p,\ \text{(by (2))}.$$

Therefore

$$Im\ w > 0$$

and accordingly

$$w \neq 0.$$

In the case that

$$\gamma \neq 0,$$

we may consider the quantities

$$w'_j = w_j e^{-\gamma i}, j = 1, 2, \ldots, p.$$

These satisfy inequalities (1) and (2) with

$$\gamma = 0$$

and consequently their sum w' does not vanish. As

$$w' = e^{-i\gamma} w,$$

it follows that

$$w \neq 0.$$

This completes the proof of Lemma 2.

3. Proof of Laguerre's theorem

To prove the first part of the theroem we can assume that

$$f(Z) \neq 0 \tag{3}$$

(as for the possibility

$$f(Z) = 0,$$

proof is trivial). As

$$0 = f_1(Z) = (\zeta - Z)f'(Z) + nf(Z), \tag{4}$$

we get by using (3) that

$$\zeta \neq Z \tag{5}$$

and as

$$f(z) = \prod_{j=1}^{p}(z - z_j)^{m_j}, \sum_{j=1}^{p} m_j = n, \tag{6}$$

we get by using (3), (4) and (5) that

$$0 = \frac{f_1(Z)}{f(Z)(\zeta - Z)} = \frac{f'(Z)}{f(Z)} + \frac{n}{\zeta - Z},$$

i.e.

$$\frac{n}{Z - \zeta} = \sum_{j=1}^{p} \frac{m_j}{Z - z_j} \text{ (by (6))}. \tag{7}$$

By using the symbols

$$w = \frac{1}{Z - \zeta} \tag{8}$$

$$w_j = \frac{1}{Z - z_j}, j = 1, 2, \ldots, p, \tag{9}$$

and (6), (7) can be rewritten as

$$\sum_{j=1}^{p} m_j(w_j - w) = 0. \tag{10}$$

For proving first part of the theorem we have to show that not both points Z and ζ may lie outside of C. On the contrary, we assume that both points Z and ζ are outside of C. We now consider Möbius transformation

$$\tau = g(z) = \frac{1}{Z - z}. \tag{11}$$

Let γ be the boundary of the circular region C and let Γ be the image of γ under the transformation (11). As γ is a straight line or a circle, Γ will also be a straight line or a circle. Accordingly we think of two possibilities:
(i) Γ is a circle. Therefore

$$Z \notin \gamma$$

and as

$$g(Z) = \infty \in \text{ domain (known as exterior of } \Gamma \text{ and represented by the symbol } E(\Gamma)), \tag{12}$$

we can say that

$$g(C) = \text{ domain (known as interior of } \Gamma \text{ and represented by the symbol } I(\Gamma)) \tag{13}$$

or

$$g(C) = \overline{I(\Gamma)} \tag{14}$$

which imply that

$$w_j \in I(\Gamma), j = 1, 2, \ldots, p, \text{ (by (6), (9) and (11))} \tag{15}$$

or

$$w_j \in \overline{I(\Gamma)}, j = 1, 2, \ldots, p, \text{ (by (6), (9) and (11))} \tag{16}$$

respectively. Further by (12) and by our assumption that both points Z and ζ lie outside of C, we can say that

$$w \in E(\Gamma), \text{ (by (8) and (11))} \tag{17}$$

or

$$w \in \Gamma, \text{ (by (8) and (11)).} \tag{18}$$

(Please note that w can belong to Γ only when (15) happens but (16) does not happen.) (19)

Now by (15), (16), (17), (18) and (19) we can say that there will definitely exist a real number η such that

$$\eta < Arg\ (w_j - w) < \eta + \pi, \ j = 1, 2, \ldots, p \tag{20}$$

and therefore by Lemma 1 we can say that

$$\sum_{j=1}^{p} m_j (w_j - w) \neq 0$$

which contradicts the fact represented by (10). Hence our assumption that both points Z and ζ are outside of C should be wrong and we can conclude for the possibility under consideration that not both points Z and ζ may lie outside of C.
(ii) Γ *is a straight line.* Therefore

$$Z \in \gamma \tag{21}$$

and

$$g(C) = \text{domain (known as an open half plane with boundary } \Gamma) . \tag{22}$$

Now (22) helps us to say that

$$\left.\begin{array}{l} w_j \in \text{ half plane } g(C) \text{ with boundary } \Gamma, j = 1, 2, \ldots, p, \\ \text{with} \\ w_j \notin \Gamma,\ j = 1, 2, \ldots, p, \end{array}\right\}, \text{(by (6), (9) \& (11))}. \tag{23}$$

Further by (21) and by our assumption that both points Z and ζ lie outside of C, we can say that
$w \in$ domain (known as second half plane with boundary Γ and different from half plane $g(C)$),

$$\text{(by (8) \& (11))} \tag{24}$$

or

$$w \in \Gamma,\ \text{(by (8) and (11))}. \tag{25}$$

By (23), (24) and (25) we can say that there will definitely exist a real number δ such that

$$\delta < \ Arg\ (w_j - w) < \delta + \pi,\ j = 1, 2, \ldots, p$$

and now the proof of the first part of the theorem for the present possibility can be completed similar to the proof of the first part of the theorem for the possibility (i) after expression (20). This completes the proof of the first part of the theorem.

To prove the second part of the theorem we are given that

$$f(Z) \neq 0$$

and therefore (10) is still true. We now assume that a circle K through Z and ζ has at least one z_j in its interior, no z_j in its exterior and the remaining z_j $'s$ on its circumference. Under Möbius transformation (11), K will be transformed onto a straight line Γ_0, with

$$g(I(K)) = \text{ an open half plane with boundary } \Gamma_0. \tag{26}$$

(26) and our assumption help us to say that

$$w_j \ \in \ \text{ open half plane } g(I(K)) \text{ with boundary } \Gamma_0, \text{ for}$$

$$\text{at least one } j, 1 \leq j \leq p, \text{ (by (6),(9) \& (11))}, \tag{27}$$

with

$$w_j \in \Gamma_0 \text{ for remaining } j\ 's, 1 \leq j \leq p, \text{ (by (6),(9) \& (11))}. \tag{28}$$

Further as

$$\zeta \in K,$$

we have

$$w \in \Gamma_0, \text{ (by (8) and (11))}. \tag{29}$$

Now by (27), (28) and (29) we can say that there will exist a real number α such that

$$\alpha < \ Arg\ (w_j - w) < \alpha + \pi, \text{ for at least one } j, 1 \leq j \leq p,$$

with

$$\alpha \leq \ Arg\ (w_j - w) \leq \alpha + \pi, \text{ for remaining } j\ 's, 1 \leq j \leq p.$$

(It should be noted here that $(w_j - w)$ may vanish for certain $j\ 's, (1 \leq j \leq p)$). Therefore by Lemma 2 we can say that

$$\sum_{j=1}^{p} m_j(w - w_j) \neq 0$$

which contradicts the fact represented by (10). Hence our assumption that a circle K through Z and ζ has at least one z_j in its interior, no z_j in its exterior and the remaining $z_j\ 's$ on its circumference, should be wrong. One can similarly show that the assumption that a circle K through Z and ζ has at least one z_j in its exterior, no z_j in its interior and the remaining $z_j\ 's$ on its circumference will be wrong. Therefore we conclude that any circle K through Z and ζ must separate $z_j\ 's$ unless it passes through all of them. This completes the proof of the second part of the theorem, thereby completing the proof of the theorem also.

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A Generalized Contraction Mapping Principle

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Abstract. We introduce a generalized contraction mapping principle in fuzzy metric spaces with the help of two real functions. The methodology is different from other similar existing results.

Key Words and Phrases: Fixed point, complete fuzzy metric space, contraction mapping, Cauchy sequence.

1. Introduction

In this paper we introduce a new contraction mapping in fuzzy metric spaces which entails the spirit of generalization by Geraghty of the Banach's contraction mapping principle [7]. For this purpose we use two functions one of which has been recently considered by Shen et al in [17] proving a contraction mapping theorem in fuzzy metric spaces and the other by Geraghty [7]. We obtain our result in the fuzzy metric space defined by George and Veeramani [6]. The fuzzy fixed point theory has developed largely based on this space. One of the reasons for such successful development of fuzzy fixed point theory is that the space has a Hausdorff topology, a feature which has been widely utilized in this domain of study. Some recent references on the aforesaid topic are [4, 5, 9, 13, 14]. There are also other definitions of fuzzy metric spaces as, for instance, Kaleva and Seikkala defined a fuzzy metric with the help of fuzzy numbers [11]. A recent fixed point result on this space is deduced in [18].

In the following we state some concepts essential for the discussion in the rest of the paper.

Definition 1.1[10, 16] A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a continuous t-norm if the following properties are satisfied:

(i) $*$ is associative and commutative,

(ii) $a * 1 = a$ for all $a \in [0,1]$,

(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$

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(iv) $*$ is continuous.

Some examples of continuous t-norm are $a *_1 b = \min\{a, b\}$, $a *_2 b = \frac{ab}{\max\{a,b,\lambda\}}$ for $0 < \lambda < 1$, $a *_3 b = ab$ and $a *_4 b = \max\{a + b - 1, 0\}$.
George and Veeramani in their paper [6] introduced the following definition of fuzzy metric space. We will be concerned only with this definition of fuzzy metric space.

Definition 1.2[6] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, t) > 0$,

(ii) $M(x, y, t) = 1$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and

(v) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by

$$B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, t, r) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology and is called the topology on X induced by the fuzzy metric M. This topology is metrizable as we indicated above.

Example 1.3[6] Let X be the set of all real numbers and d be any metric on X. Let $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. For each $t > 0$, $x, y \in X$, let

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a fuzzy metric space.

Example 1.4 Let (X, d) be a metric space and ψ be an increasing and continuous function of $(0, \infty)$ into $(0, 1)$ such that $\lim_{t\to\infty} \psi(t) = 1$. Three typical examples of these functions are $\psi(t) = \dfrac{t}{t+1}$, $\psi(t) = \sin(\dfrac{\pi t}{2t+1})$ and $\psi(t) = 1 - e^{-t}$. Let $*$ be any continuous t-norm. For each $t > 0$, $x, y \in X$, let

$$M(x, y, t) = \psi(t)^{d(x,y)}.$$

Then $(X, M, *)$ is a fuzzy metric space.

Definition 1.5[6] Let $(X, M, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

The following lemma was proved by Grabiec [8] for fuzzy metric spaces defined by Kramosil et al [12]. The proof is also applicable to the fuzzy metric space given in Definition 1.2.

Lemma 1.6[8] Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is nondecreasing for all $x, y \in X$.

Lemma 1.7[15] M is a continuous function on $X^2 \times (0, \infty)$.

Definition 1.8 [17] Let $\psi : [0, 1] \to [0, 1]$ be a function that satisfies the following conditions:

$(P1)$ ψ is strictly decreasing left continuous,

$(P2)$ $\psi(\lambda) = 0$ if and only if $\lambda = 1$.

Obviously, we obtain that $\lim_{\lambda\to 1} \psi(\lambda) = \psi(1) = 0$.

Definition 1.9 [1, 2, 3, 7] Let S be the class of functions $\beta : R^+ \to [0, 1)$ with

(i) $R^+ = \{t \in R/t > 0\}$,

(ii) $\beta(t_n) \to 1$ implies $t_n \to 0$. (1.1)

2. Main Result

Theorem 2.1 Let $(X, M, *)$ be a complete fuzzy metric space. Let $T : X \to X$ be a mapping, $\psi : [0, 1] \to [0, 1]$ is as in Definition 1.8 and β satisfies the Definition 1.9. If the mapping T satisfies the condition

$$\psi(M(Tx, Ty, t)) \leq \beta(\psi((M(x, y, t))).\psi((M(x, y, t)), \tag{2.1}$$

for all $x, y \in X$, $t > 0$ and $x \neq y$, then T has a unique fixed point in X.

Proof. Starting with x_0 in X, we define the sequence $\{x_n\}$ in X as follows:

$$x_{n+1} = Tx_n. \tag{2.2}$$

Let for all $t > 0$, $n \geq 0$,

$$\delta_n(t) = M(x_n, x_{n+1}, t). \tag{2.3}$$

Now from (2.1), for every $t > 0$, we have

$$\psi(\delta_n(t)) = \psi(M(x_n, x_{n+1}, t))$$

$$\begin{aligned}
&= \psi(M(Tx_{n-1}, Tx_n, t)) \\
&\leq \beta(\psi((M(x_{n-1}, x_n, t))).\psi(M(x_{n-1}, x_n, t)) \\
&= \beta(\psi((M(x_{n-1}, x_n, t))).\psi(\delta_{n-1}(t)) \qquad (2.4)\\
&< \psi(\delta_{n-1}(t))
\end{aligned}$$

$$\psi(\delta_n(t)) < \psi(\delta_{n-1}(t)).$$

Since $\{\psi(\delta_n(t))\}$ is strictly decreasing for every t, there exists $\delta(t) \geq 0$ such that $\lim_{n\to\infty} \psi(\delta_n(t)) = \delta(t)$.

$$\text{Let } \delta(t) > 0 \text{ for some } t. \tag{2.5}$$

From (2.4), we have

$$\frac{\psi(\delta_n(t))}{\psi(\delta_{n-1}(t))} \leq \beta(\psi((M(x_{n-1}, x_n, t))) < 1.$$

Taking $n \to \infty$ in the above inequality and using (2.5), we have

$$\lim_{n\to\infty} \beta(\psi((M(x_{n-1}, x_n, t))) = 1.$$

Using the property of (1.1), we have

$$\delta(t) = \lim_{n\to\infty} \psi(\delta_n(t)) = \lim_{n\to\infty} \psi((M(x_{n-1}, x_n, t)) = 0.$$

So we arrive at a contradiction.

Therefore $\delta(t) > 0$ for all t, and we have

$$\lim_{n\to\infty} M(x_{n-1}, x_n, t) = 1. \tag{2.6}$$

We now prove that the sequence $\{x_n\}$ is a Cauchy sequence. If not, then there exist $0 < \epsilon < 1$ and two sequences $\{m(k)\}$ and $\{n(k)\}$, where $m(k) > n(k) > k$ for every $n \geq 0$ and $t > 0$, such that

$$M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \epsilon.$$

$$\text{and } M(x_{m(k)-1}, x_{n(k)}, t) > 1 - \epsilon. \tag{2.7}$$

Then, $M(x_{m(k)-1}, x_{n(k-1)}, t) \geq M(x_{m(k)-1}, x_{n(k)}, \frac{t}{2}) * M(x_{n(k)}, x_{n(k)-1}, \frac{t}{2})$

$$\geq (1-\epsilon) * M(x_{n(k)}, x_{n(k)-1}, \tfrac{t}{2}). \tag{2.8}$$

Since M is continuous we can find $\eta > 0$ such that

$$1 - \epsilon \geq M(x_{m(k)}, x_{n(k)}, t) \geq$$

$$M(x_{m(k)-1}, x_{n(k)}, \frac{\eta}{2}) * M(x_{m(k)-1}, x_{n(k)-1}, t-\eta) * M(x_{n(k)-1}, x_{n(k)}, \frac{\eta}{2}). \tag{2.9}$$

Taking $k \to \infty$ in the above two inequalities (2.8) and (2.9), using (2.7) and the fact that $*$ is continuous, we have

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t) \geq (1-\epsilon) \geq \lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t-\eta).$$

Since M is continuous and η is arbitrary, we have

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t) = 1 - \epsilon. \tag{2.10}$$

Now by (2.1), we have

$$\psi(1-\epsilon) \leq \psi(M(x_{m(k)}, x_{n(k)}, t)) \leq \beta(\psi(M(x_{m(k)-1}, x_{n(k)-1}, t))).\psi(M(x_{m(k)-1}, x_{n(k)-1}, t)).$$

Taking $k \to \infty$, we have

$$\psi(1-\epsilon) \leq \lim_{k\to\infty} \beta(\psi(M(x_{m(k)-1}, x_{n(k)-1}, t))).\psi(1-\epsilon).$$

Using definition 1.7, the last inequality implies $\lim_{k\to\infty} \beta(\psi(M(x_{m(k)-1}, x_{n(k)-1}, t))) = 1.$
Since $\beta \in S$, we have $\lim_{k\to\infty} \psi(M(x_{m(k)-1}, x_{n(k)-1}, t)) = 0,$
which implies $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t) = 1.$
This fact and (2.10) give us $\epsilon = 0$, which is a contradiction.
Hence the sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

$$\begin{aligned}\psi(M(x_{n+1}, Tx, t)) &= \psi(M(Tx_n, Tx, t)) \\ &\leq \beta(\psi(M(x_n, x, t)).\psi(M(x_n, x, t)) \\ &< \psi(M(x_n, x, t)).\end{aligned}$$

Taking $n \to \infty$ on the both sides of the above inequality, we have

$$\begin{aligned}\psi(M(x, Tx, t)) &\leq \psi(M(x, x, t)), \\ &= \psi(1), \\ &= 0.\end{aligned}$$

Since $\psi(M(x, Tx, , t)) = 0$, by the property of $(P2)$, we have $M(x, Tx, t) = 1$,
that is, $x = Tx$,
that is, x is a fixed point of T.
To show uniqueness, let $y \neq x$ be another fixed point of T.
$\psi(M(x, y, t)) = \psi(M(Tx, Ty, t)) \leq \beta(\psi(M(x, y, t))).\psi(M(x, y, t)) < \psi(M(x, y, t)),$
which is a contradiction.
The proof is completed.

Particularly, taking $\beta(t) = k$, $0 < k < 1$ we obtain the following corollary.

Corollary 2.2 Let $(X, M, *)$ be a complete fuzzy metric space. Let $T : X \to X$ be a mapping, $\psi : [0, 1] \to [0, 1]$ is as in Definition 1.7 and β satisfies the Definition 1.8. If the mapping T satisfies the condition

$$\psi(M(Tx, Ty, t)) \leq k.\psi((M(x, y, t)),$$

for all $x, y \in X$, $t > 0$ and $x \neq y$, then T has a unique fixed point in X.

Conclusion: The corollary can be viewed as a version of the contraction mapping principle in fuzzy metric spaces. With this consideration, Theorem 2.1 is an extension of the fuzzy contractions mapping principle.

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Future prospects of Azerbaijan in the context of increasing international role of the Caspian region

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Abstract. This work presents an analysis of current geopolitical situation in the Caspian region as the severe confrontation for Caspian hydrocarbons is growing on. Azeri prospects in these conditions are discussed.
Key Words and Phrases: Oil production, gas and chemical industry, Azeri-Russian relations, the Caspian region.

Azerbaijan has some huge deposits of hydrocarbons and good perspectives for exploiting them.

Economically, this is quite feasible when we talk about oil. Indeed, oil reserves in the Caspian are mostly easy to extract and, consequently, highly remunerative. In general, Caspian region countries do not face problems like those faced by Western powers which have to exploit new hard-to-reach fields of energy, such as shale gas and shale oil in the United States, oil sands in Canada, underwater mining complexes in Norway, etc [1].

Besides, there is another factor which is very important in global competition for resources: Caspian oil is mostly "light", i.e. structurally very close (unlike Russian oil brand Urals produced in Tyumen) to the "reference pattern" produced in the Persian Gulf countries.

As for the gas resources, just three Central Asian countries located at the Caspian seashore – Kazakstan, Turkmenistan and neighboring Uzbekistan – account for 12 percent of the world's total gas reserves. In quantitative terms, it makes about 22 trillion cubic meters, very close to the Russian reserves of 24 trillion cubic meters [2].

Also, Azerbaijan has vast reserves of gas easy to extract and to transport. In 1999, the discovery of the giant Shah Deniz gas field instantly made this country one of the largest natural gas proven reserve holder in the world. That was soon followed by the signing production sharing agreements with world giants Statoil and Royal Dutch/Shell [3]. Moreover, the launch of second gas platform in Shah Deniz will make country's total volume of gas reserves exceed 1 trillion cubic meters. And, let's recall what Azeri President

Ilham Aliyev said during his visit to Brussels in July 2013: "Hopefully Azerbaijan's gas reserves will not be restricted to those produced in Shah Deniz. We have more than 3 trillion cubic meters of proven reserves" [4, p. 151].

The indisputable advantage of Azerbaijan as an international player in the energy sector is that the Azeri part of the Caspian seashore, which comprises about 79.000 square kilometers (20 percent of all Caspian water area), is currently best explored [5]. In our opinion, this factor makes Azerbaijan much more attractive for potential strategic investors in the field of exploitation of oil and gas fields.

Taking into account the fact that almost all the gas reserves of Russia are located far away from the Caspian (mostly in Yamalo-Nenets Autonomous Okrug and other northern territories which account for from 85 to 90 percent of total Russian gas production), we can conclude that in the near future the major players in international oil trade in the Caspian region will be such post-Soviet countries as Kazakstan, Azerbaijan, Turkmenistan and Uzbekistan, and, of course, Iran. Oil profits will allow these countries to shape their own economic and foreign policies.

Russia, in turn, due to a number of political factors, will have to fight to maintain its positions in global energy markets. Not only now, but for the years ahead, too. The reason is not only the severe economic competition but also the counteraction by the leading Western powers. On the other hand, we have to mention the following fact: as the demand for oil and gas grows continuously all over the world, oil production drops in many places. For example, *Economist Intelligence* reports that the average oil production decrease in well-known oil fields is about 2-6 percent a year (including Western Siberia, Russia's old-times oil region, where the oil production drops continuously since 2006), while the global oil demand grows by 2% every year [6]. This gives us the reason to suppose that the leading oil producing countries and their oil companies are going to concentrate on the discovery and exploitation of completely new fields, including those in the Caspian.

As a natural consequence of the above-mentioned, we can assert that those countries with a powerful economic and political potential which are going to enter a phase of ever-growing need for hydrocarbons will be quite naturally most active and most involved in the Caspian region states. In this aspect, two major oil consumers meeting the above criteria come into one's mind: European Union and China.

It should be also noted that the countries which have their own hydrocarbon resources (we mean China, the United Kingdom and Norway) are currently on the brink of depleting them. Very soon oil and gas production in most of these countries will start to decrease, and, with current exploitation rates, in the next 12-15 years all their reserves will be fully exhausted.

Now let's recall that the leading world powers first showed interest in the oil and gas potential of the Caspian region in the early 1990s, i.e. right after the collapse of the former Soviet Union. For the leading Western powers, oil and gas resources of the Caspian are not only the alternative source of energy today, but also the profit potential for the future when, as mentioned above, all the other raw material sources will be exhausted. Moreover, we should mention another important factor which makes the Caspian resources so attractive for the West: the ever-growing influence of China.

In the early 1990s, Chinese company Sinopec with 88% of shares in Petrochina, has entered the severe battle for global hydrocarbon resources. Widely backed by the government of China, this company started its successful activities in more than 25 countries of the world including CIS countries. Sinopec, inter alia, was allowed to take part in oil filed exploitation in Azerbaijan and Kazakstan.

Today, this trend is getting more and more distinctive. And the reasons of this are obvious. Stressing the growing global influence of China, Azeri economist F.Murshudli claims that China is becoming the major stimulator for all emerging markets as the financial and economical crisis in Europe goes on [7].

In our opinion, the above arguments are also important in the context of the rivalry between Chinese and West European oil and gas companies. The rivalry which extends on the Caspian region countries, too. Today, the biggest companies in the West, Saudi Arabia and China consider these countries as the prospective independent hydrocarbon producers (Norway, Mexico, Canada and some others may also be added to the list)[8]. A big fight is going on to make a friendship with these countries, to get an allowance to work in these countries, to construct new pipelines or to use the older ones there. And, let's recall once again that the Caspian oil is mostly "light", i.e. not only much easier to produce from financial point of view as compared to other oil regions, but also much easier to process.

As this severe fighting for exploitation of Caspian hydrocarbons is going on, Azerbaijan enjoys good prospects mainly due to its balanced foreign policy. A.Vlasov, Director General of Post-Soviet Studies Center at the Moscow State University, says that the foreign policy pursued by the Azeri government aims at bringing the interests of foreign players in the Caspian into conformance with the national interests of Azerbaijan [9]. It should be taken into account that the geographical location of this country as well as its cultural and historical legacy also predetermines the character of its foreign policy. The country is located at the crossroads of Islamic and European civilizations, and, quite naturally, has a full membership in such international organizations as the Council of Europe and the Organization of Islamic Cooperation.

The Republic of Azerbaijan is shaping its multilateral and bilateral relations with both leading World powers and CIS countries on the basis of above-stated principles. Diversification of Azerbaijan's foreign policy is reflected, in particular, in its membership in GUAM, the politico-economic organization created in 1997 by four CIS countries – Georgia (officially left CIS in August 2009), Ukraine, Azerbaijan and Moldova. From 1999 to May 2005, when Uzbekistan was another member of this organization, the acronym was GUUAM.

Due to objective reasons, the relationship with the Russian Federation is most important for the Republic of Azerbaijan as compared to its bilateral relations with other CIS countries. As a testimony to the rapidly growing relationship between two countries, we can mention the fact that the joint businesses in both countries are mostly intended for a long period of time. Of course, this is the result of both countries reluctance to create agreeable working conditions for each other. Besides, it should not be forgotten that the friendly relationship between the Republic of Azerbaijan and the Russian Federation is based on the principles of good neighbor policy in the Caspian.

That is why Azeri politicians with a realistic view of Russia's role in global and regional processes are absolutely right when stressing the special importance of Azeri-Russian relations for the peace and prosperity in the Caspian region. "Over the past 10 years, taking full responsibility for the region, Baku and Moscow repeatedly showed their willingness to compromise which helped to avoid any situation where a possible conflict of interest could arise in the region... Both countries have enough potential to contribute to the enhancement of the global importance of our region," wrote in his recent book Ramiz Mehdiyev, head of the Presidential Administration of Azerbaijan.[4, p.173].

Hence, we are now able to shape the principles that determine the basis of modern Azeri-Russian relations. These are:

1. mutual respect for the principles of national sovereignty in the context of foreign policy;
2. equal rights for both countries when signing any kind of political or economical agreement;
3. full freedom for either country for political and economical relations with a third party if these are not harmful for other party;
4. commitment to coordinate with each other in the Caspian region for the sake of neighborliness and cooperation on a long term basis.

It seems like long-term political and economical relations based on the above-stated principles will be mutually beneficial for both countries in the context of current realities in the Caspian region.

Today's objective reality is that Azerbaijan is not only reliable geopolitical and economical partner for the Russian Federation, but also the most influential political player in Southern Caucasus. The world is interested in Azerbaijan, and this is proved by the fact that the special session of Davos World Economic Forum (WEF) entitled Strategic Dialog on the Future of Southern Caucasus and Central Asia has been held in April 2013 in Baku. 140 world-renowned experts, political scientists and economists from 40 countries including Klaus Schwab, founder and executive chairman of the WEF, have attended this session. The choice of Azerbaijan as the venue for such a big event shows how much the global financial and economic elites appreciate this country and how much importance they attach to this country in the settlement of prospective regional conflicts in the Caspian region and in the Caucasus as a whole.

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Some Results About Common Fixed Point Theorems for Multi-Valued Mappings

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Abstract. V. Popa has proved common fixed point theorems for multi-valued mappings which verify rational inequalities, which contain the Hausdorff metric in their expressions. Recently, A. Petcu in [1, 2, 3] has proved other common fixed point theorems for two or more multi-valued mappings without using the Hausdorff metric. In this paper by providing some different conditions we shall study existence of common fixed points for multi-valued mappings.

Key Words and Phrases: Complete metric space, Common fixed point, Multi-valued mappings.

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1. Introduction

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis. Its core subject is concerned with the conditions for the existence of one or more fixed points of a mapping or multi-valued mapping T from a topological space X into itself, that is, we can find $x \in X$ such that $Tx = x$ (for mapping) or $x \in Tx$ (for multi-valued mapping).
In [4] V. Popa has proved common fixed point theorems for multi-valued mappings which verify rational inequalities, which contain the Hausdorff metric in their expressions.
In [1] A. Petcu has proved other common fixed point theorems for two or more multi-valued mappings without using the Hausdorff metric.

In this paper by providing some different conditions we study existence of common fixed points for multi-valued mappings.
Let X be a nonempty set, $P(X)$ the set of all nonempty subsets of X, T a multifunction of X into $P(X)$, $F(T)$ the fixed points set of T, that is $F(T) = \{x \in X : x \in Tx\}$. Throughout the paper, for a topological space X we denote the set of all nonempty closed subsets of X by $P_{cl}(X)$ and the set of all nonempty closed and bounded subsets of X by $P_{b,cl}(X)$ when X is a metric space.

Let (X,d) be a metric space, for $x \in X$ and $A, B \subseteq X$, set $D(x,A) = \inf_{y\in A} d(x,y)$ and

$$H(A,B) = \max\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\}.$$

We also denote

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}.$$

It is known that, H is a metric on closed bounded subsets of X which is called the Hausdorff metric.

2. Main results

Let $\mathcal{F}$ be all multi-valued mappings of X in to $P_{b,cl}(X)$.
Define the following equivalence relation for the elements of $\mathcal{F}$:

$$F \sim G \quad if\ and\ only\ if \quad fixF = fixG \qquad (F, G \in \mathcal{F}).$$

Denote the equivalence class of $\mathcal{F}$ by $\tilde{\mathcal{F}}$ and define it as follows:

$$\tilde{\mathcal{F}} = \frac{\mathcal{F}}{\sim} = \{\tilde{F} : F \in \mathcal{F}\}.$$

Also define $\tilde{d}$ *on* $\tilde{\mathcal{F}}$ such that

$$\tilde{d}(\tilde{F}, \tilde{G}) = H(fixF, fixG).$$

It is easy to see that $(\tilde{\mathcal{F}}, \tilde{d})$ is a metric space.

Lemma 1. *Let (X,d) be a metric space and $S, T : X \to P_{b,cl}(X)$ be two multi-valued mappings such that $\forall x \in X, \forall y \in Sx$ (or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sy$) with*

$$d^{3m}(x,y) - \frac{3}{4\sqrt[3]{4}} c^2 d^{2m}(y,z) d(x,y) - \frac{c^3}{8} d^{3m}(y,z) \geq 0, \tag{2.1}$$

where $m \geq 1, c > 1$ and $F(S) \neq \phi$. Then $F(T) \neq \phi$ and $\tilde{S} = \tilde{T}$.

Proof. Let $u \in F(S)$, that is $u \in Su$. Then there exists $z \in Tu$ and (2.1) becomes

$$d^3(u,u) - \frac{3}{4\sqrt[3]{4}} c^2 d^2(u,z) d(u,u) - \frac{c^3}{8} d^3(u,z) \geq 0,$$

from where we get $-\frac{c^3}{8} d^3(u,z) \geq 0, c > 1$, that is $d(u,z) = 0$. Then $z = u$ and therefore $u \in Tu$ which implies $F(S) \subset F(T)$.

Analogously we prove that $F(T) \subset F(S)$, therefore $F(S) = F(T)$ and hence $\tilde{S} = \tilde{T}$.

Let $V : X \to P_{b,cl}(X)$ with (X, d) a metric space. The following property will be used further:

$$for\ any\ convergent\ sequence\ (x_n)_{n\geq 0}\ \ from\ \ X\ \ with\ \ \lim_{n\to\infty} x_n = x,$$

$$x_{2n-1} \in Vx_{2n-2}(or\ x_{2n} \in Vx_{2n-1}), it\ results\ \ x \in Vx. \qquad (a)$$

Theorem 1. *Let (X, d) be a complete metric space and $S, T : X \to p_{b,cl}(X)$ be two multi-valued mappings such that $\forall x \in X, \forall y \in Sx$ (or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sy$) with inequality (2.1) holding, where $m \geq 1$, $c > 1$.*
If one of the multi-valued mappings S, T verifies condition (a), then $\tilde{S} = \tilde{T}$.

Proof. Let $x_0 \in X$ be arbitrarily fixed and $x_1 \in Sx_0$. Then there exists $x_2 \in Tx_1$ such that

$$d^{3m}(x_0, x_1) - \frac{3}{4\sqrt[3]{4}}c^2 d^{2m}(x_1, x_2)d(x_0, x_1) - \frac{c^3}{8}d^{3m}(x_1, x_2) \geq 0.$$

Then there exists $x_3 \in Sx_2$ such that

$$d^{3m}d(x_1, x_2) - \frac{3}{4\sqrt[3]{4}}c^2 d^{2m}(x_2, x_3)d(x_1, x_2) - \frac{c^3}{8}d^{3m}(x_2, x_3) \geq 0.$$

Continuing this reasoning we obtain a sequence
$x_0, x_1, x_3, \ldots, x_{n-1}, x_n \ldots$ with $x_{2n-1} \in Sx_{2n-2}$, $x_{2n} \in Tx_{2n-1}$ which verifies the inequality

$$d^{3m}(x_n, x_{n-1}) - \frac{3c^2}{4\sqrt[3]{4}}d^{2m}(x_n, x_{n+1})d(x_n, x_{n-1}) - \frac{c^3}{8}d^{3m}(x_n, x_{n+1}) \geq 0, \qquad (2.2)$$

for all $n \geq 1$. The first member in the inequality (2.2) is a third degree trinomial in the variable $d^m(x_n, x_{n-1})$ with the discriminant

$$\Delta = 4(\frac{-3}{4\sqrt[3]{4}}c^2 d^{2m}(x_n, x_{n+1}))^3 + 27(\frac{-c^3}{8}d^{3m}(x_n, x_{n+1}))^2 = 0.$$

Inequality (2.2) holds if

$$d^m(x_n, x_{n-1}) \geq -2\sqrt[3]{\frac{c^3}{8}d^{3m}(x_n, x_{n+1})} = cd^m(x_n, x_{n+1}).$$

We denote $k^m = \frac{1}{c}$. Then we have $k < 1$ and

$$0 \leq d^m(x_n, x_{n+1}) < k^m d^m(x_n, x_{n-1}),$$

that is

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n), \quad \forall n \geq 1,$$

from where we deduce

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \quad \forall n \geq 1.$$

A routine calculation leads to

$$d(x_n, x_{n+p}) \le \frac{k^n}{1-k} d(x_0, x_1), \quad n, p \in \mathbb{N},$$

which shows that $(x_n)_{n\ge 0}$ is a Cauchy sequence and since the space X is complete it results that $(x_n)_{n\ge 0}$ is convergent. Let $u = \lim_{n\to\infty} x_n,\ u \in X$. We have
$x_{2n-1} \in Sx_{2n-2}$ and assuming that S verifies (a) it results that $u \in Su$. With lemma (1) we deduce that $u \in Tu$ and $F(S) = F(T)$ and so $\tilde{S} = \tilde{T}$.

Lemma 2. *[5] If $A, B \in B(X)$ and $k \in \mathbb{R}, k > 1$, then for any $a \in A$ there exists $b \in B$ such that $d(a, b) \le kH(A, B)$.*

According to the above lemma the following lemma is true.

Lemma 3. *Let $k > 1$ and the multi-valued mappings $S, T : X \to P_{cl,b}(X)$ be given. Then for any $y \in Sx$(or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sy$) such that*

$$d(y, z) \le kH(Sx, Ty).$$

Theorem 2. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{b,cl}(X)$ be two multi-valued mappings such that*

$$H^m(T_1x, T_2y) \le \frac{8d^{3m}(x, T_1x)}{c^2\delta^{2m}(y, T_2y) + 6c\delta^m(y, T_2y)\delta^m(x, T_1x) + 8\delta^{2m}(x, T_1x)}, \tag{2.3}$$

and for any x, y from X

$$c^2\delta^{2m}(y, T_2y) + 6c\delta^m(y, T_2y)\delta^m(x, T_1x) + 8\delta^{2m}(x, T_1x) \ne 0,$$

where $m \ge 1, c > 1$. Then $\tilde{T_1} = \tilde{T_2}$.

Proof. Eliminating the denominator, (2.3) becomes

$$\begin{aligned} H^m(T_1x, T_2y)(c^2\delta^{2m}(y, T_2y) + 6c\delta^m(y, T_2y)\delta^m(x, T_1x) + 8\delta^{2m}(x, T_1x)) \\ \le 8d^{3m}(x, T_1x). \end{aligned} \tag{2.4}$$

Inequality (2.4) is valid for any x, y from X and in particular for $y \in T_1x$.
Let $1 < c < k^m$. For $x \in X,\ y \in T_1x$ with lemma (3) it results that there exists $z \in T_2y$ such that $d(y, z) \le kH(T_1x, T_2y)$, and from here we have

$$cd^m(y, z)(c^2d^{2m}(y, z) + \frac{6c}{\sqrt[3]{4}}d^m(y, z)d^m(x, y) \le 8d^{3m}(x, y).$$

Consequently, $\forall x \in X, \forall y \in T_1x$, there exists $z \in T_2y$ such that

$$d^{3m}(x, y) - \frac{3}{4\sqrt[3]{4}}cd^m(y, z)d^m(x, y) - \frac{c^3}{8}d^{3m}(y, z) \ge 0,$$

where $m \geq 1, 1 < c < k^m$, condition which has the form of inequality (2.1). We prove now that T_1 verifies condition (a). Let $(x_n)_{n\geq 0}$ be a convergent sequence from X with $\lim_{n\to\infty} x_n = x \in X$ and $x_{2n-1} \in T_1 x_{2n-2},\ x_{2n} \in T_2 x_{2n-1}$.
We have

$$d(T_1x, x_{2n}) \leq H(T_1x, T_2x_{2n-1}),$$

from where with (2.4) we obtain

$$cd^m(T_1x, x_{2n})(c^2d^{2m}(x_{2n-1}, x_{2n}) + 6cd^m(x_{2n-1}, x_{2n})d^m(x_{2n}, T_1x) + 8d(x_{2n}, T_1x))$$
$$\leq 8d^{3m}(x_{2n}, T_1x),$$

from where, for $n \to \infty$, it results

$$d(x, T_1x) \leq \frac{1}{c}d(x, T_1x),$$

that is $d(T_1x, x) = 0$. Because T_1x is a closed set we deduce $x \in T_1x$ and by previous theorem and lemma we obtain $F(T_1) = F(T_2)$, therefore $\tilde{T_1} = \tilde{T_2}$.

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