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A Rearrangement Estimate for the Generalized Multilinear Anisotropic Fractional Integrals

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Abstract. In this paper, author studies $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_k}$ boundedness of the generalized multilinear anisotropic fractional integrals. We give a new proof of the Hardy-Littlewood-Sobolev multilinear anisotropic fractional integration theorem, based on a pointwise estimate of the rearrangement multilinear anisotropic fractional type integral.

Key Words and Phrases: Lebesgue space, multilinear anisotropic fractional integral.

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1. Introduction

Fractional maximal function and fractional integral is an important technical tool in harmonic analysis, real analysis and partial differential equations. Multilinear fractional maximal operator and multilinear fractional integral operator and related topics have been areas of research of many mathematicians such as R.Coifman and L. Grafakos [5], L. Grafakos [6, 7], L. Grafakos and N. Kalton [8], C.E. Kenig and E.M. Stein [12], Y. Ding and S. Lu [11] and others.

The purpose of this article is to describe several results about generalized multilinear anisotropic fractional integral operators. We study $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_k}$ boundedness of the generalized multilinear anisotropic fractional integrals. We give a new proof of the Hardy-Littlewood-Sobolev multilinear anisotropic fractional integration theorem, based on a pointwise estimate of the rearrangement generalized multilinear anisotropic fractional integral.

2. Rearrangements of functions

Let $\mathbb{R}^n$ is the n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$ with norms $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Let $\lambda > 0$, $a = (a_1, \ldots, a_n)$, $a_1 > 0, \ldots, a_n > 0$, $d = a_1 + \ldots + a_n$, $\delta_\lambda x = (\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n)$.

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Let $\rho(x)$ be a non-isotropic norm on $\mathbb{R}^n$ defined as the unique positive solution of the equation

$$\sum_{j=1}^{n} \frac{x_j^2}{\rho(x)^{2a_j}} = 1.$$

Note that $\rho(x)$ is equivalent to $\sum_{i=1}^{n} |x_i|^{1/a_i}$, i.e.,

$$c_1\rho(x) \le \sum_{i=1}^{n} |x_i|^{1/a_i} \le c_2\rho(x)$$

for certain positive c_1 and c_2 independent of x (see [2]).

It is immediate that $\rho(\delta_\lambda x) = \lambda\ \rho(x)$ for all $\lambda > 0$, $x \in \mathbb{R}^n$. With this norm, $\mathbb{R}^n$ is a space of homogeneous type in the sense of Coifman and Weiss [4] with homogeneous dimension $d = |a|$. In particular, there is a constant $c_0 \ge 1$ such that $\rho(x+y) \le c_0\ (\rho(x) + \rho(y))$ for all $x,\ y \in \mathbb{R}^n$.

One has the polar decomposition

$$x = \delta_\lambda\ \sigma \tag{1}$$

with $\sigma \in S^{n-1}$, $r = \rho(x)$ and $dx = r^{d-1} dr J(\sigma) d\sigma$, where $J(\sigma)$ is a smooth and nonnegative function of $\sigma \in S^{n-1}$ and is even in each of variables $\sigma_1, \ldots, \sigma_n$.

The isotropic and anisotropic balls of radius r and center x are defined

$$B(x, r) = \{y \in \mathbb{R}^n\ :\ |x - y| < r\},$$

$$\mathcal{E}(x, r) = \{y \in \mathbb{R}^n\ :\ \rho(x - y) < r\},$$

respectively.

For $1 \le p < \infty$ let $L_p(\mathbb{R}^n)$ be the space of all measurable functions g on $\mathbb{R}^n$ with finite norm

$$\|g\|_{L_p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |g(x)|^p dx \right)^{1/p}.$$

Let g be a measurable function on $\mathbb{R}^n$. The distribution function of g is defined by the equality

$$\lambda_g(t) = |\{x \in \mathbb{R}^n\ :\ |g(x)| > t\}|, \qquad t \ge 0.$$

We shall denote by $L_0(\mathbb{R}^n)$ the class of all measurable functions g on $\mathbb{R}^n$, which are finite almost everywhere and such that $\lambda_g(t) < \infty$ for all $t > 0$ (see [13]).

If a function g belongs to $L_0(\mathbb{R}^n)$, then its non-increasing rearrangement is defined to be the function g^* which is non-increasing on $]0, \infty[$ equimeasurable with $|g(x)|$:

$$|\{t > 0\ :\ g^*(t) > s\}| = \lambda_g(s)$$

for all $s \ge 0$.

Set

$$g^{**}(t) = \frac{1}{t}\int_0^t g^*(s)\ ds.$$

Moreover, by the Hardy-Littlewood theorem (see [3], p. 44), for every $f_1,\ f_2 \in L_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f_1(x)f_2(x)|\ dx \le \int_0^\infty f_1^*(t)f_2^*(t)\ dt.$$

It is well known that if $p>1$, then $\left(\int_0^\infty (g^{**}(t))^p\ dt\right)^{1/p}$ is comparable with the $L_p(\mathbb{R}^n)$ norm of g.

For $1 \le p < \infty$ the weak L_p space $WL_p(\mathbb{R}^n)$ is the set of all locally integrable functions g on $\mathbb{R}^n$ with finite norm

$$\|g\|_{WL_p(\mathbb{R}^n)} = \sup_{t>0}\ t\ \lambda_g(t)^{1/p}.$$

Equimeasurable rearrangements of functions play an important role in various fields of mathematics. Note some properties of the rearrangement (see, for example [3]):

1) if $0<t<t+s$, then

$$(g+h)^*(t+s) \le g^*(t)+h^*(s),$$

2) if $0<p<\infty$, then

$$\int_{\mathbb{R}^n} |g(x)|^p\ dx = \int_0^\infty (g^*(t))^p\ dt,$$

3) for any $t>0$

$$\sup_{|E|=t}\int_E |g(x)|\ dx = \int_0^t g^*(s)\ ds.$$

Let $k \ge 2$ be an integer and θ_j $(j=1,2,\cdots,k)$ be a fixed, distinct and nonzero real numbers.

Lemma 1. *[9] Let $f_1, f_2, \ldots, f_k \in L_0(\mathbb{R}^n)$, $k \ge 2$. Then for all $x \in \mathbb{R}^n$ and nonzero real numbers $\theta_1,\ldots,\theta_k$*

$$\int_{\mathbb{R}^n} |f_1(x-\theta_1 y)f_2(x-\theta_2 y)\cdots f_k(x-\theta_k y)|dy \le C_\theta \int_0^\infty f_1^*(t)f_2^*(t)\cdots f_k^*(t)dt, \qquad (2)$$

where $C_\theta = |\theta_1\ldots\theta_k|^{-n}$.

3. A rearrangement estimate for the generalized multilinear fractional integrals

By $\mathbf{f}$ we denote $(f_1, f_2, \cdots, f_k)$ and define

$$\mathbf{f}^*(t) = f_1^*(t)\ldots f_k^*(t),$$

$$\mathbf{f}^{**}(t) = \frac{1}{t}\int_0^t f_1^*(s)\dots f_k^*(s)\ ds, \qquad t>0.$$

Let $k \geq 2$ be an integer and θ_j $(j = 1, 2, \cdots, k)$ be fixed and nonzero real numbers. The analogy of O'Neil inequality (see, [14]) for k-linear integral operator by

$$(\mathbf{f}, g)(x) = \int_{\mathbb{R}^n} g(y) f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)\, dy,$$

is correct

Lemma 2. *[9] Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$. Then for all $0 < t < \infty$, the following inequality holds*

$$(\mathbf{f}, g)^{**}(t) \leq C_\theta \left(t\ \mathbf{f}^{**}(t)\ g^{**}(t) + \int_t^\infty \mathbf{f}^*(s)\ g^*(s)\ ds \right). \tag{3}$$

Lemma 3. *[9] Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$. Then for any $t > 0$*

$$(\mathbf{f}, g)^{**}(t) \leq C_\theta \int_t^\infty \mathbf{f}^{**}(t) g^{**}(t) dt. \tag{4}$$

In the following we define the k-sublinear anisotropic fractional maximal operator by

$$\mathcal{M}_{\Omega,\alpha}\mathbf{f}(x) = \sup_{r>0} \frac{1}{r^{d-\alpha}} \int_{\mathcal{E}(0,r)} |\Omega(y)||f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)|\ dy,$$

the k-linear anisotropic fractional integral operator by

$$R_{\Omega,\alpha}\mathbf{f}(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{\rho(y)^{d-\alpha}} f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)\, dy$$

and the generalized k-linear anisotropic fractional integral operator by

$$K_\alpha\mathbf{f}(x) = \int_{\mathbb{R}^n} K_\alpha(y) f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)\, dy,$$

where $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$.

Note that, if $K_\alpha(x) = \frac{\Omega(x)}{\rho(x)^{d-\alpha}}$, $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$, then $K_\alpha^*(t) = \left(\frac{A}{nt}\right)^{(d-\alpha)/d}$, $K_\alpha^{**}(t) = \frac{d}{\alpha}K_\alpha^*(t)$, where $A = \|\Omega\|^{d/(d-\alpha)}_{L_{d/(d-\alpha)}(S^{n-1})}$ and therefore $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$. And also, if $K_\alpha(x) = \frac{\Omega(x)}{\rho(x)^{d-\alpha}}$, $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$, then $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$.

The following lemma in the isotropic case was proved in [11]. In the anisotropic case it is proved analogously.

Lemma 4. *Suppose that $0 < \alpha < d$, $\Omega \in L_s(S^{n-1})$, $s \geq 1$. Then*

$$\mathcal{M}_{\Omega,\alpha}\mathbf{f}(x) \leq R_{|\Omega|,\alpha}(|\mathbf{f}|)(x), \tag{5}$$

where $|\mathbf{f}| = (|f_1|, \dots, |f_k|)$.

Proof. Indeed, for all $r > 0$, we have

$$R_{|\Omega|,\alpha}(|\mathbf{f}|)(x) \geq \int_{\mathcal{E}(0,r)} \frac{\Omega(y)}{\rho(y)^{d-\alpha}} f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)\,dy$$
$$\geq \frac{1}{r^{d-\alpha}} \int_{\mathcal{E}(0,r)} |\Omega(y)||f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)|\,dy,$$

where $\mathcal{E}(0,r)$ is the anisotropic ball centered at the origin of radius r. Taking supremum over all $r > 0$, we get (5).

For the generalized multilinear fractional integrals $K_\alpha \mathbf{f}$ the following theorem is valid:

Theorem 1. *Let $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$, $0 < \alpha < d$. Then*

$$(K_\alpha \mathbf{f})^*(t) \quad \leq \quad (K_\alpha \mathbf{f})^{**}(t) \quad \leq \quad C_1 \left(t^{\frac{\alpha}{d}-1} \int_0^t \mathbf{f}^*(s)\ ds + \int_t^\infty s^{\frac{\alpha}{d}-1}\ \mathbf{f}^*(s)\ ds \right), \qquad (6)$$

where $C_1 = \left(\frac{d}{\alpha}\right)^2\ C_\theta\ \|K_\alpha\|_{WL_{d/(d-\alpha)}}$.

Proof. Let $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$, then

$$K_\alpha^*(t) \leq \|K_\alpha\|_{WL_{d/(d-\alpha)}}\ t^{\frac{\alpha}{d}-1}, \qquad K_\alpha^{**}(t) \leq \frac{d}{\alpha}\ K_\alpha^*(t).$$

Taking into account inequality (3) we have (6).

Corollary 1. *Suppose that $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$. Then the following inequality*

$$(R_{\Omega,\alpha}\mathbf{f})^*(t) \quad \leq \quad (R_{\Omega,\alpha}\mathbf{f})^{**}(t) \quad \leq \quad C_2 \left(t^{\frac{\alpha}{d}-1} \int_0^t \mathbf{f}^*(s)\ ds + \int_t^\infty s^{\frac{\alpha}{d}-1}\ \mathbf{f}^*(s)\ ds \right),$$

holds, where $C_2 = \left(\frac{d}{\alpha}\right)\ C_\theta\ \left(\frac{A}{d}\right)^{(d-\alpha)/d}$, $A = \|\Omega\|^{d/(d-\alpha)}_{L_{d/(d-\alpha)}(S^{n-1})}$.

From Corollary 1 and Lemma 4 we get

Corollary 2. *Suppose that $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$. Then the following inequality*

$$(\mathcal{M}_{\Omega,\alpha}\mathbf{f})^*(t) \quad \leq \quad (\mathcal{M}_{\Omega,\alpha}\mathbf{f})^{**}(t) \quad \leq \quad C_2 \left(t^{\frac{\alpha}{d}-1} \int_0^t \mathbf{f}^*(s)\ ds + \int_t^\infty s^{\frac{\alpha}{d}-1}\ \mathbf{f}^*(s)\ ds \right),$$

holds.

Analogously we have

Theorem 2. *Let $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$, $0 < \alpha < d$. Then*

$$(K_\alpha \mathbf{f})^*(t) \leq (K_\alpha \mathbf{f})^{**}(t) \leq C_1 \int_t^\infty s^{\frac{\alpha}{d}-1}\ \mathbf{f}^{**}(s)\ ds. \qquad (7)$$

Corollary 3. *Suppose that* $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$. *Then the following inequality*

$$(R_{\Omega,\alpha}\mathbf{f})^*(t) \le (R_{\Omega,\alpha}\mathbf{f})^{**}(t) \le C_2 \int_t^\infty s^{\frac{\alpha}{d}-1}\mathbf{f}^{**}(s)\ ds$$

holds.

Corollary 4. *Suppose that* $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$. *Then the following inequality*

$$(\mathcal{M}_{\Omega,\alpha}\mathbf{f})^*(t) \le (\mathcal{M}_{\Omega,\alpha}\mathbf{f})^{**}(t) \le C_2 \int_t^\infty s^{\frac{\alpha}{d}-1}\mathbf{f}^{**}(s)\ ds$$

holds.

4. $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}$ boundedness of generalized multilinear fractional integral operators

In the sequel we shall use the following Lemma, which was proved in [1].

Lemma 5. *[1] Let* $0 < p \le 1$, $p \le q < \infty$ *and* k *be a non-negative measurable functions and* u, v *be weight functions on* $(0,\infty)$ *and*

$$T\varphi(t) = \int_0^\infty k(t,\tau)\varphi(\tau)d\tau.$$

Then the inequality

$$\left(\int_0^\infty (T\varphi(t))^q\, u(t)dt\right)^{1/q} \le C\left(\int_0^\infty \varphi(t)^p v(t)dt\right)^{1/p} \tag{8}$$

holds for all non-negative non-increasing functions φ *if and only if*

$$C_0 = \sup_{r>0}\left(\int_0^\infty\left(\int_0^r k(t,\tau)d\tau\right)^q u(t)dt\right)^{1/q}\left(\int_0^r v(t)dt\right)^{-1/p} < \infty.$$

The constant $C = C_0$ *is the best constant in* (8).

Corollary 5. *Let* $0 < p \le 1$, $p \le q < \infty$, $0 < \alpha < d$.

Then the inequality

$$\left(\int_0^\infty\left(\int_t^\infty \tau^{\frac{\alpha}{d}-1}\varphi(\tau)\ d\tau\right)^q\ dt\right)^{1/q} \le \mathcal{C}_0\left(\int_0^\infty \varphi(t)^p\ dt\right)^{1/p}$$

holds for all non-negative non-increasing functions φ *if and only if*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d}, \tag{9}$$

where $\mathcal{C}_0 = \left(\frac{d}{\alpha}\right)^{1+\frac{1}{q'}} B\left(\frac{d}{\alpha}, q+1\right)^{\frac{1}{q}}$, $B(s,r) = \int_0^1 (1-\tau)^{s-1}\tau^{r-1}d\tau$ *is the Beta function.*

It is said that p is the harmonic mean of $p_1, p_2, \ldots, p_k > 1$, if $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k}$. If $f_j \in L_{p_j}(\mathbb{R}^n)$, $j = 1, 2, \ldots, k$, then we say that $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$.

Theorem 3. *Suppose that $0 < \alpha < d, K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$. Let p be the harmonic mean of $p_1, p_2, \ldots, p_k > 1$ and q satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Then $K_\alpha \mathbf{f}$ is bounded operator from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $d/(d+\alpha) \leq p < d/\alpha$ (equivalently $1 \leq q < \infty$) and*

$$\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} \leq C \prod_{j=1}^{k} \|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C > 0$ independent of f.

Proof. Case I. $1 < p < \frac{d}{\alpha}$ (equivalently $\frac{d}{d-\alpha} < q < \infty$). Let us first prove Theorem 3 in this case.

Taking into account equality (2) and inequality (6) we have

$$\begin{aligned}
&\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} = \|(K_\alpha \mathbf{f})^*\|_{L_q(0,\infty)} \\
&\leq C_1 \left(\int_0^\infty t^{q(\alpha/d-1)} \left(\int_0^t \mathbf{f}^*(s)\ ds \right)^q dt \right)^{1/q} + C_1 \left(\int_0^\infty \left(\int_t^\infty s^{\alpha/d-1} \mathbf{f}^*(s)\ ds \right)^q dt \right)^{1/q},
\end{aligned}$$

where $C > 0$ independent of f.

Applying Hardy inequality we obtain, that for the validity of the following inequality

$$\left(\int_0^\infty t^{q(\alpha/d-1)} \left(\int_0^t \mathbf{f}^*(s) ds \right)^q dt \right)^{1/q} \leq C_3 \left(\int_0^\infty \mathbf{f}^*(s)^p ds \right)^{1/p}$$

it is necessary and sufficient that the following condition is satisfied

$$\begin{aligned}
&\sup_{t>0} \left(\int_t^\infty s^{q(\alpha/d-1)} ds \right)^{1/q} \left(\int_0^t ds \right)^{1/p'} \\
&\qquad = C_4 \sup_{t>0} t^{\frac{\alpha}{d} - \left(\frac{1}{p} - \frac{1}{q} \right)} < \infty \Leftrightarrow 1/p - 1/q = \alpha/d,
\end{aligned}$$

where $p' = \frac{p}{p-1}$.

For the validity of the following inequality

$$\left(\int_0^\infty \left(\int_t^\infty s^{\frac{\alpha-d}{d}} \mathbf{f}^*(s)\ ds \right)^q dt \right)^{1/q} \leq C_5 \left(\int_0^\infty \mathbf{f}^*(s)^p ds \right)^{1/p}$$

it is necessary and sufficient satisfying the following condition

$$\begin{aligned}
&\sup_{t>0} \left(\int_0^t ds \right)^{1/q} \left(\int_t^\infty s^{(\alpha/d-1)(1-p')} ds \right)^{1/p'} \\
&\qquad = C_6 \sup_{t>0} t^{\frac{\alpha}{d} - \left(\frac{1}{p} + \frac{1}{q} \right)} < \infty \Leftrightarrow 1/p - 1/q = \alpha/d.
\end{aligned}$$

Consequently applying equality (2) we obtain

$$\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} \le C_1(C_3+C_5)\,\|\mathbf{f}^*\|_{L_p(0,\infty)}$$
$$\le C_1(C_3+C_5)\prod_{j=1}^k \|f_j^*\|_{L_{p_j}(0,\infty)} = C_1(C_3+C_5)\prod_{j=1}^k \|f_j\|_{L_{p_j}(\mathbb{R}^n)}.$$

Case II. $\frac{d}{d+\alpha} \le p \le 1$ (equivalently $1 \le q \le \frac{d}{d-\alpha}$). Now let's prove Theorem 3 for this case.

Taking into account equality (2) and inequality (7) we have

$$\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} = \|(K_\alpha \mathbf{f})^*\|_{L_q(0,\infty)} \le \|(K_\alpha \mathbf{f})^{**}\|_{L_q(0,\infty)}$$
$$\le C_1\left(\int_0^\infty \left(\int_t^\infty s^{\alpha/d-1}\mathbf{f}^{**}(s)ds\right)^q dt\right)^{1/q}.$$

By virtue of Lemma 2 for the validity of the following inequality

$$\left(\int_0^\infty \left(\int_t^\infty s^{\alpha/d-1}\mathbf{f}^{**}(s)ds\right)^q dt\right)^{1/q} \le C_6\left(\int_0^\infty \mathbf{f}^{**}(s)^p ds\right)^{1/p}$$

it is necessary and sufficient satisfying the condition (9).

Consequently applying equality (2), Hardy inequality for monotonic functions and Holder inequality we obtain

$$\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} = \|(K_\alpha \mathbf{f})^*\|_{L_q(0,\infty)}$$
$$\le C_8\,\|\mathbf{f}^{**}\|_{L_p(0,\infty)} \le C_9\,\|\mathbf{f}^*\|_{L_p(0,\infty)}$$
$$\le C_9\prod_{j=1}^k \|f_j^*\|_{L_{p_j}(0,\infty)} = C_9\prod_{j=1}^k \|f_j\|_{L_{p_j}(\mathbb{R}^n)}.$$

Corollary 6. *Let $0<\alpha<d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$, p be the harmonic mean of $p_1, p_2, \dots, p_k > 1$ and q satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Then $R_{\Omega,\alpha}\mathbf{f}$ is a bounded operator from $L_{p_1}\times L_{p_2}\times\cdots\times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $d/(d+\alpha) \le p < d/\alpha$ (equivalently $1 \le q < \infty$) and*

$$\|R_{\Omega,\alpha}\ \mathbf{f}\|_{L_q(\mathbb{R}^n)} \le C\prod_{j=1}^k \|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C>0$ independent of f.

Corollary 7. *Let $0<\alpha<n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, p be the harmonic mean of $p_1, p_2, \dots, p_k > 1$ and q satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then the k-linear fractional integral operator*

$$I_{\Omega,\alpha}\mathbf{f}(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)\,dy$$

is a bounded operator from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $n/(n+\alpha) \le p < n/\alpha$ *(equivalently* $1 \le q < \infty$*) and*

$$\|I_{\Omega,\alpha}\ \mathbf{f}\|_{L_q(\mathbb{R}^n)} \le C \prod_{j=1}^{k} \|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C > 0$ *independent of* f*.*

Corollary 8. *Let* $0 < \alpha < d$*,* $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$*,* p *be the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$ *and* q *satisfy* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$*. Then* $\mathcal{M}_{\Omega,\alpha}\mathbf{f}$ *is a bounded operator from* $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $d/(d+\alpha) \le p < d/\alpha$ *(equivalently* $1 \le q < \infty$*) and*

$$\|\mathcal{M}_{\Omega,\alpha}\ \mathbf{f}\|_{L_q(\mathbb{R}^n)} \le C \prod_{j=1}^{k} \|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C > 0$ *independent of* f*.*

Remark 1. *Note that, Corollary 7 proved in [6], if* $\Omega \equiv 1$ *and in [11], if* $\Omega \in L_s(S^{n-1})$*,* $s > n/(n-\alpha)$ *and in [9, 10], if* $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$*.*

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Explicit Form of Laplace-Stieltjes Transform of Joint Distribution of the First Passage Time of Some Level "a" ($a > 0$) and Overshoots Across this Level by a Complex Semi-Markov Walk Process with Reflecting Screen at Zero

E.M. Neymanov

Abstract. In the paper, by the probability-statistical method we find explicit form of the Laplace-Stieltjes transform of joint distribution of the first passage time of some level "a" ($a > 0$) and overshoot across this level by a complex semi-Markov walk process with a reflecting screen at zero.

Key Words and Phrases: Laplace-Stieltjes transform, probability space, semi-Markov walk process.

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1. Introduction

In the paper [1, p. 61-63] asymptotic behavior of random walks in random medium with a delaying screen was considered. In [2, p. 160-165] random walk was studied in a strip. In the paper [3, p. 26-51] asymptotic expansion of distribution was found. In the paper [4, p. 61-63], various semi-Markov processes with a delaying screen and functional of these processes were studied. In [5, p. 77-84] the Laplace transform of distribution of the lower boundary functional of semi-Markov walk process with a delaying screen at zero was found. In [6, p. 49-60] the Laplace transform of ergodic distribution of semi-Markov walk process with a negative drift, non-negative jumps and a delaying screen at zero, was found.

In the present paper we study joint distribution of the first passage moment of some level "a" ($a > 0$) and the overshoot across this level by a complex semi-Markov walk process with a reflecting screen at zero.

2. Mathematical statement of the problem

Let on probability space $(\Omega, F, P(\cdot))$ be given the sequence $\left\{\xi_k^+, \eta_k^+, \xi_k^-, \eta_k^-\right\}_{k=1,\infty}$, where

$\xi_k^+, \eta_k^+, \xi_k^-, \eta_k^-$ are identically distributed between themselves positive random variables are identically.

Denote

$$S_k = \left| S_{k-1} + ... + \eta^+_{\nu(\tau_{k-1})} + \eta^+_{\nu(\tau_k)} - \eta_k^- \right|, \tag{1}$$

where $S_0 = z$,

$$X^{\pm}(t) = \sum_{i=1}^{\nu^{\pm}(t)} \eta_i^{\pm}, \tag{2}$$

$$\tau_k^{\pm} = \sum_{i=1}^{k} \xi_k^{\pm}; k = 1, 2, ..; \ \tau_0^{\pm} = 0, \tag{3}$$

where

$$v^{\pm}(t) = \min \left\{ k : \sum_{i=1}^{k+1} \xi_k^{\pm} > t \right\}. \tag{4}$$

The process $X(t) = S_{k-1} + ... + \eta^+_{\nu(\tau_{k-1})} + \eta^+_{\nu(\tau_k)} - \eta_k^-$ if $\tau_{k-1}^{\pm} < t < \tau_k^{\pm}$ is called a complex semi-Markov walk process with a reflecting screen at zero. One of the realizations of the process $X(t)$ is of the form

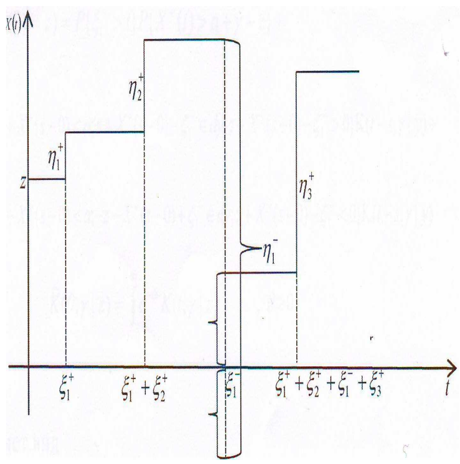

Fig. $v^{\pm}(t)$ **is the number of positive or negative jumps for time** t**.**

Our goal is to find the explicit form of the Laplace-Stieltjes transform of joint distribution of the first passage moment and overshoot of the level $a\,(a > 0)$.

Let τ_a be the first passage moment of the level $a\,(a > 0)$ and γ be an overshoot across this level.

We assume that ξ_1^+ has exponential distribution with the parameter λ_+.

Denote

$$\mathrm{K}(t, \gamma | X(0) = z) = P\{\tau_a < t, \gamma_a > a | X(0) = z\}$$

By total probability formula we have

$$\mathrm{K}(t, \gamma | X(0) = z) = P\{\tau_a < t, \gamma_a > \gamma; \xi_1^- > t | X(0) = z\} +$$

$$+\int_{s=0}^{t}\int_{y=0}^{a}P\{\xi_1^- \in ds; \sup_{0\le u\le s-0} X(u)<a; |X(s)|\in dy|X(0)=z\}\mathrm{K}(t-s,\gamma|y)=$$

$$=P\{\xi_1^->t; z+X^+(t)>a+\gamma\}+$$

$$+\int_{s=0}^{t}\int_{y=0}^{a}P\{\xi_1^-\in ds; z+X^+(s-0)<a; |z+X^+(s-0)-\zeta_1^-|\in dy\}\mathrm{K}(t-s,\gamma|y)$$

In view of $\{|u|<\varepsilon\}=\{-\varepsilon<u<\varepsilon\}$ we have

$$\mathrm{K}(t,\gamma|X(0)=z)=P\{\xi_1^->t\}P\{X^+(t)>a+\gamma-z\}+$$

$$+\int_{s=0}^{t}\int_{y=0}^{a}P\{\xi_1^-\in ds; z+X^+(s-0)<a; z+X^+(s-0)-$$

$$-\zeta_1^-\in dy; z+X^+(s-0)-\zeta_1^->0\}\mathrm{K}(t-s,\gamma|y)+$$

$$+\int_{s=0}^{t}\int_{y=0}^{a}P\{\xi_1^-\in ds; z+X^+(s-0)<a; -z-X^+(s-0)+$$

$$+\zeta_1^-\in dy; z+X^+(s-0)-\zeta_1^-<0\}\mathrm{K}(t-s,\gamma|y)$$

So, we get an integral equation for $\mathrm{K}(t,\gamma|X(0)=z)$.

$$\mathrm{K}(t,\gamma|X(0)=z)=P\{\xi_1^->t\}P\{X^+(t)>a+\gamma-z\}+$$

$$+\int_{s=0}^{t}\int_{y=0}^{a}P\{\xi_1^-\in ds; z+X^+(s-0)<a; z+X^+(s-0)-\zeta_1^-\in dy;$$

$$z+X^+(s-0)-\zeta_1^->0\}\mathrm{K}(t-s,\gamma|y)+$$

$$+\int_{s=0}^{t}\int_{y=0}^{a}P\{\xi_1^-\in ds; z+X^+(s-0)<a; -z-X^+(s-0)+\zeta_1^-\in dy;$$

$$z+X^+(s-0)-\zeta_1^-<0\}\mathrm{K}(t-s,\gamma|y). \tag{5}$$

Denote $\tilde{K}(\theta,\gamma|z)=\int_{t=0}^{\infty}e^{-\theta t}K(t,\gamma|z)$, $\theta>0$

Then (5) takes the form

$$\tilde{K}(\theta,\gamma|z)=\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t; X^+(t)>a+\gamma-z\}dt+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}d_yP\{X^+(t)<a-z;$$

$$z+X^+(t)-\zeta_1^-<y; z+X^+(t)-\zeta_1^->0\}dP\{\xi_1^-<t\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}d_yP\{X^+(t)<a-z; -z-X^+(t)+\zeta_1^-<y;$$

$$z+X^+(t)-\zeta_1^-<0\}dP\{\xi_1^-<t\} \tag{6}$$

Make a change of variables $X^{+}(t)=h$. Then (6) takes the form

$$\tilde{K}(\theta,\gamma|z)=\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^{-}>t\}dt-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^{-}>t\}P\{X^{+}(t)<a+\gamma-z\}dt+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{\zeta_1^{-}>z-y+h;\zeta_1^{-}<z+h\}d_t\times$$

$$\times P\{\xi_1^{-}<t\}d_hP\{X^{+}(t)<h\}+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\times$$

$$\times\int_{h=0}^{a-z}P\{\zeta_1^{-}<z+y+h;\zeta_1^{-}>z+h\}d_tP\{\xi_1^{-}<t\}d_hP\{X^{+}(t)<h\}=$$

$$=\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^{-}>t\}dt-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^{-}>t\}\times$$

$$\times\sum_{k=0}^{\infty}P\{\sum_{i=1}^{\infty}\zeta_i^{+}<a+\gamma-z\}P\{\nu^{+}(t)=k\}dt+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{z-y+h<\zeta_1^{-}<z+h\}d_t\times$$

$$\times P\{\xi_1^{-}<t\}d_hP\{X^{+}(t)<h\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{z+h<\zeta_1^{-}<z+h+y\}d_t\times$$

$$\times P\{\xi_1^{-}<t\}d_hP\{X^{+}(t)<h\}.$$

Taking into account $X^{+}(t)=\sum_{i=1}^{\nu^{+}(t)}\eta_i^{+}$, from the last equation we have

$$\tilde{K}(\theta,\gamma|z)=$$

$$=\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^{-}>t\}dt-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^{-}>t\}\sum_{k=0}^{\infty}\times$$

$$\times P\{\sum_{i=1}^{k}\zeta_i^{+}<a+\gamma-z\}P\{\nu^{+}(t)=k\}dt-$$

$$-\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{\zeta_1^{-}<z-y+h\}d_tP\{\xi_1^{-}<t\}d_hP\{X^{+}(t)<h\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{t=0}^{\infty}e^{-\theta t}d_y\int_{h=0}^{a-z}P\{\zeta_1^{-}<z+h+y\}d_tP\{\xi_1^{-}<t\}d_hP\{X^{+}(t)<h\}$$

From the fact that there should be $z-y+h>0$ or $h>\max(0,y-z)$, we have

$$\tilde{K}(\theta,\gamma|z)=$$

$$= \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} dt - \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} \sum_{k=0}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < a+\gamma-z\} P\{\nu^+(t)=k\} dt -$$

$$- \int_{y=0}^{z} \tilde{K}(\theta,\gamma|y) \int_{h=0}^{a-z} d_y P\{\zeta_1^- < -y+h+z\} \int_{t=0}^{\infty} e^{-\theta t} d_h \times$$

$$\times \sum_{k=0}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < h\} P\{\nu^+(t)=k\} d_t P\{\xi_1^- < t\} +$$

$$+ \int_{y=z}^{a} \tilde{K}(\theta,\gamma|y) \int_{h=y-z}^{a-z} d_y P\{\zeta_1^- < -y+h+z\} \int_{t=0}^{\infty} e^{-\theta t} d_h \times$$

$$\times \sum_{k=0}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < h\} P\{\nu^+(t)=k\} d_t P\{\xi_1^- < t\} +$$

$$+ \int_{y=0}^{a} \tilde{K}(\theta,\gamma|y) \int_{h=0}^{a-z} d_y P\{\zeta_1^- < y+h+z\} \int_{t=0}^{\infty} e^{-\theta t} d_h v \times$$

$$\times \sum_{k=0}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < h\} P\{\nu^+(t)=k\} d_t P\{\xi_1^- < t\}.$$

Simplify this equation. More exactly, taking into account

$$1 = \sum_{k=0}^{\infty} P\{\nu^+(t)=k\} = P\{\nu^+(t)=0\} + P\{\nu^+(t) \geq 1\}$$

the last equation takes the following form

$$\tilde{K}(\theta,\gamma|z) = \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} dt -$$

$$- \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} \varepsilon(a+\gamma-z) P\{\nu^+(t)=0\} dt -$$

$$- \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1^- > t\} \sum_{k=1}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < a+\gamma-z\} P\{\nu^+(t)=k\} dt -$$

$$- \int_{y=0}^{z} \tilde{K}(\theta,\gamma|y) \int_{h=0}^{a-z} d_y P\{\zeta_1^- < -y+h+z\} \int_{t=0}^{\infty} e^{-\theta t} d_h \varepsilon(h) P\{\nu^+(t)=0\} d_t P\{\xi_1^- < t\} -$$

$$- \int_{y=0}^{z} \tilde{K}(\theta,\gamma|y) \int_{h=0}^{a-z} d_y P\{\zeta_1^- < -y+h+z\} \times$$

$$\times \int_{t=0}^{\infty} e^{-\theta t} d_h \sum_{k=1}^{\infty} P\{\sum_{i=1}^{k} \zeta_i^+ < h\} P\{\nu^+(t)=k\} d_t P\{\xi_1^- < t\} - \qquad (7)$$

$$-\int_{y=z}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=y-z}^{a-z}d_yP\{\zeta_1^-<-y+h+z\}\int_{t=0}^{\infty}e^{-\theta t}d_h\varepsilon(h)P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}-$$

$$-\int_{y=z}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=y-z}^{a-z}d_yP\{\zeta_1^-<-y+h+z\}\times$$

$$\times\int_{t=0}^{\infty}e^{-\theta t}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_1^+<h\}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^-<y+h+z\}\int_{t=0}^{\infty}e^{-\theta t}d_h\varepsilon(h)P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^-<y+h+z\}\times$$

$$\times\int_{t=0}^{\infty}e^{-\theta t}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_1^+<h\}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}$$

By virtue of $\varepsilon(h)=\begin{cases}0, h<0\\ 1, h>0\end{cases}$ (7) takes the form

$$\tilde{K}(\theta,\gamma|z)=\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}dt-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}P\{\nu^+(t)=0\}dt\varepsilon(a+\gamma-z)-$$

$$-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_1^+<a+\gamma-z\}P\{\nu^+(t)=k\}dt-$$

$$-\int_{y=0}^{z}\tilde{K}(\theta,\gamma|y)d_yP\{\zeta_1^-<-y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}-$$

$$-\int_{y=o}^{z}\tilde{K}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^-<-y+h+z\}d_h\times$$

$$\times\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}-$$

$$-\int_{y=z}^{a}\tilde{K}(\theta,\gamma|y)d_yP\{\zeta_1^-<-y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}-$$

$$-\int_{y=z}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=y-z}^{a-z}d_yP\{\zeta_1^-<-y+h+z\}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\times$$

$$\times\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)d_yP\{\zeta_1^-<y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}+$$

$$+\int_{y=0}^{a}\tilde{K}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^-<y+h+z\}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\times$$
$$\times\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}. \tag{8}$$

Thus, when ξ_1^+, ξ_1^-, ζ_1^+, ζ_1^- have exponential distribution, we get integral equation (8). When ξ_1^+ has exponential distribution $\xi_1^-,\zeta_1^+,\zeta_1^-$ have Erlang distribution of any order, and one can get an integral equation of type (8). Solve equation (8) in the case when ξ_1^+, $\xi_1^-,\zeta_1^+,\zeta_1^-$ have Erlang distribution of first order.

Denote

$$\tilde{\tilde{K}}(\theta,\chi|z)=\int_{\gamma=0}^{\infty}e^{-\chi\gamma}d_\gamma\tilde{K}(\theta,\gamma|z),\ \ \chi>0.$$

Then (4) takes the form

$$\tilde{\tilde{K}}(\theta,\chi|z)=-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}P\{\nu^+(t)=0\}dt\int_{\gamma=0}^{\infty}d_\gamma\varepsilon(a+\gamma-z)-$$
$$-\int_{t=0}^{\infty}e^{-\theta t}P\{\xi_1^->t\}\sum_{k=1}^{\infty}P\{\nu^+(t)=k\}\int_{\gamma=0}^{\infty}e^{-\chi\gamma}d_\gamma P\{\sum_{i=1}^{k}\zeta_1^+<a+\gamma-z\}-$$
$$-\int_{y=0}^{z}\tilde{\tilde{K}}(\theta,\gamma|y)d_yP\{\zeta_1^-<-y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}-$$
$$-\int_{y=o}^{z}\tilde{\tilde{K}}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^-<-y+h+z\}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\times$$
$$\times\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}-$$
$$-\int_{y=z}^{a}\tilde{\tilde{K}}(\theta,\gamma|y)d_yP\{\zeta_1^-<-y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}-$$
$$-\int_{y=z}^{a}\tilde{\tilde{K}}(\theta,\gamma|y)\int_{h=y-z}^{a-z}d_yP\{\zeta_1^-<-y+h+z\}d_h\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\times$$
$$\times\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}+$$
$$+\int_{y=0}^{a}\tilde{\tilde{K}}(\theta,\gamma|y)d_yP\{\zeta_1^-<y+z\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=0\}d_tP\{\xi_1^-<t\}+$$
$$+\int_{y=0}^{a}\tilde{\tilde{K}}(\theta,\gamma|y)\int_{h=0}^{a-z}d_yP\{\zeta_1^-<y+h+z\}d_h\times$$
$$\times\sum_{k=1}^{\infty}P\{\sum_{i=1}^{k}\zeta_i^+<h\}\int_{t=0}^{\infty}e^{-\theta t}P\{\nu^+(t)=k\}d_tP\{\xi_1^-<t\}.$$

Now let

$$P\{\xi_1^{\pm} < t\} = \begin{cases} 0, t < 0 \\ 1 - e^{-\lambda_{\pm} t}, \lambda_{\pm} > 0, t > 0 \end{cases}$$

$$\zeta_1^{\pm} < x\} = \begin{cases} 0, x < 0 \\ 1 - e^{-\mu_{\pm} x}, x > 0, \mu_{\pm} > 0 \end{cases}$$

Then we get

$$\tilde{\tilde{K}}(\theta,\chi|z) = -\frac{e^{(a-z)\chi}}{\lambda_+ + \lambda_- + \theta} +$$
$$+\frac{\lambda_+\mu_+}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\chi+\mu_+)(\lambda_+ + \lambda_- + \theta)\right)} e^{-\frac{\mu_+(\lambda_- + \theta)(a-z)}{\lambda_+ + \lambda_- + \theta}} +$$
$$+\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta} e^{-\mu_- z} \int_{y=0}^{z} \tilde{\tilde{K}}(\theta,\chi|y) e^{\mu_- y} dy +$$
$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)\right)} \times$$
$$\times \left(e^{\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(a-z)\right)} - 1 \right) e^{-\mu_- z} \int_{y=0}^{z} \tilde{\tilde{K}}(\theta,\chi|y) e^{\mu_- y} dy +$$
$$+\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta} e^{-\mu_- z} \int_{y=z}^{a} \tilde{\tilde{K}}(\theta,\chi|y) e^{\mu_- y} dy +$$
$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)\right)} e^{-\mu_- z} \times$$
$$\times \int_{y=z}^{a} (e^{(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(a-z)} - e^{(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(y-z)}) \tilde{\tilde{K}}(\theta,\chi|y) e^{\mu_- y} dy +$$
$$+\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta} e^{-\mu_- z} \int_{y=0}^{a} \tilde{\tilde{K}}(\theta,\chi|y) e^{-\mu_- y} dy +$$
$$+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)\right)} \times$$
$$\times \left(e^{\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(a-z\right)} - 1 \right) e^{-\mu_- z} \int_{y=0}^{a} \tilde{\tilde{K}}(\theta,\chi|y) e^{-\mu_- y} dy. \tag{9}$$

Having multiplied the both sides by $e^{\mu_- z}$ and differentiated with respect to z, we get

$$e^{\mu_- z}\left[\mu_- \tilde{\tilde{K}}(\theta,\chi,z) + \tilde{\tilde{K}}'(\theta,\chi,z)\right] = -\frac{(\mu_- - \chi)e^{a\chi}}{\lambda_+ + \lambda_- + \theta} e^{(\mu_- - \chi)z} +$$
$$+\frac{\lambda_+\mu_+(\mu_-(\lambda_+ + \lambda_- + \theta) + \mu_+(\lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2\left(\lambda_+\mu_+ - (\chi+\mu_+)(\lambda_+ + \lambda_- + \theta)\right)} e^{-\frac{\mu_+(\lambda_- + \theta)a}{\lambda_+ + \lambda_- + \theta} + \left(\frac{\mu_+(\lambda_- + \theta)}{\lambda_+ + \lambda_- + \theta} + \mu_-\right)z} +$$
$$+\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta} \tilde{\tilde{K}}(\theta,\chi,z) e^{\mu_- z} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)\left(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)\right)} \times$$

$$\times\left[-\frac{\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)}{\lambda_+ + \lambda_- + \theta} e^{(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(a-z)} \times\right.$$

$$\left.\times \int_{y=0}^{z} \tilde{\tilde{K}}(\theta,\chi|y)e^{\mu_- y}dy + \left(e^{\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-\right)(a-z)} - 1\right)\tilde{\tilde{K}}(\theta,\chi,z)e^{\mu_+ z}\right] +$$

$$-\frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}\tilde{\tilde{K}}(\theta,\chi,z)e^{\mu_+ z} +$$

$$-\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta))}\left(e^{\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-\right)(a-z)} - 1\right)\times$$

$$\times\tilde{\tilde{K}}(\theta,\chi,z)e^{\mu_- z} - \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta))}\times$$

$$\times\frac{\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)}{\lambda_+ + \lambda_- + \theta} e^{-\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)z\right)}\times$$

$$\times\left[\int_{y=z}^{a} (e^{\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)a\right)} - e^{\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)y\right)})\tilde{\tilde{K}}(\theta,\chi|y)e^{\mu_- y}dy\right] -$$

$$-\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)(\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta))}\frac{\lambda_+\mu_+ - (\mu_+ + \mu_-)(\lambda_+ + \lambda_- + \theta)}{\lambda_+ + \lambda_- + \theta}\times$$

$$\times e^{-\left(\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_- + \theta} - \mu_+ - \mu_-)(a-z)\right)}\int_{y=0}^{a}\tilde{\tilde{K}}(\theta,\chi|y)e^{-\mu_- y}dy. \tag{10}$$

We differentiate the obtained equation by z. As a result, we get a second order inhomogeneous equation with constant coefficients

$$\tilde{\tilde{K}}''(\theta,\chi,z)+(\mu_- + \frac{\mu_+\lambda_+}{\lambda_+ + \lambda_- + \theta})\tilde{\tilde{K}}'(\theta,\chi,z)+\left[\frac{\lambda_+\mu_+\mu_-}{\lambda_+ + \lambda_- + \theta} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}\right]\tilde{\tilde{K}}(\theta,\chi,z) =$$

$$= \frac{(\mu_- - \chi)(\mu_+(\lambda_- + \theta) + \chi(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2}e^{(a-z)\chi}. \tag{11}$$

The roots of the appropriate characteristic equation are

$$k_{1;2}(\theta) = \frac{-(\mu_- + \frac{\mu_+\lambda_+}{\lambda_+ + \lambda_- + \theta}) \pm \sqrt{(\mu_- + \frac{\mu_+\lambda_+}{\lambda_+ + \lambda_- + \theta})^2 - 4\left[\frac{\lambda_+\mu_+\mu_-}{\lambda_+ + \lambda_- + \theta} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}\right]}}{2}$$

The solution of equation (11) is

$$\tilde{\tilde{K}}(\theta,\chi,z) = \frac{(\mu_- - \chi)((\mu_+(\lambda_- + \theta) + \chi(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2(\chi + k_1(\theta))(\chi + k_2(\theta))}e^{\chi(a-z)} +$$

$$+ C_1(\theta)e^{k_1(\theta)z} + C_2(\theta)e^{k_2(\theta)z}, \tag{12}$$

where $C_1(\theta)$ and $C_2(\theta)$ are constant with respect to z.

Find $C_1(\theta)$ and $C_2(\theta)$.
In (9), having substituted $z=a$, we get an equation with respect to $C_1(\theta)$ and $C_2(\theta)$

$$C_1(\theta)\left[e^{k_1 a}-\frac{\lambda_-\mu_-}{\lambda_++\lambda_-+\theta}e^{-\mu_- a}\left[\frac{1}{k_1(\theta)+\mu_-}\left(e^{(k_1(\theta)+\mu_-)a}-1\right)+\right.\right.$$

$$\left.\left.+\frac{1}{k_1(\theta)-\mu_-}\left(e^{(k_1(\theta)-\mu_-)a}-1\right)\right]\right]+$$

$$+C_2(\theta)\left[e^{k_2 a}-\frac{\lambda_-\mu_-}{\lambda_++\lambda_-+\theta}e^{-\mu_- a}\left[\frac{1}{k_2(\theta)+\mu_-}\left(e^{(k_2(\theta)+\mu_-)a}-1\right)+\right.\right.$$

$$\left.\left.+\frac{1}{k_2(\theta)-\mu_-}\left(e^{(k_{21}(\theta)-\mu_-)a}-1\right)\right]\right]=$$

$$=-\frac{(\mu_--\chi)((\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2(\chi+k_1(\theta))(\chi+k_2(\theta))}-$$

$$-\frac{\mu_++\chi}{\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta)}+\frac{\lambda_-\mu_-}{\lambda_++\lambda_-+\theta}\times$$

$$\times\frac{(\mu_--\chi)(\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2(\chi+k_1(\theta))(\chi+k_2(\theta))}\times$$

$$\times\left[\frac{1}{\mu_--\chi}e^{(\mu_--\chi)a}-\frac{1}{\mu_-+\chi}e^{-(\mu_-+\chi)a}-\frac{2\chi}{(\mu_-+\chi)(\mu_--\chi)}\right]e^{(\chi-\mu_-)a}.$$

In (10), having substituted $z=a$, we get an equation with respect to $C_1(\theta)$ and $C_2(\theta)$

$$C_1(\theta)\left[(\mu_-+k_1(\theta))e^{k_1(\theta)a}+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\lambda_-+\theta)^2}e^{-\mu_- a}\times\right.$$

$$\left.\times\left[\frac{1}{k_1(\theta)+\mu_-}\left(e^{(k_1(\theta)+\mu_-)a}-1\right)+\frac{1}{k_1(\theta)-\mu_-}\left(e^{(k_1(\theta)-\mu_-)a}-1\right)\right]\right]+$$

$$+C_2(\theta)\left[\left(\mu_-+k_2(\theta)\right)e^{k_2(\theta)a}+\frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_++\lambda_-+\theta)^2}e^{-\mu_- a}\times\right.$$

$$\left.\times\left[\frac{1}{k_2(\theta)+\mu_-}\left(e^{(k_2(\theta)+\mu_-)a}-1\right)+\frac{1}{k_2(\theta)-\mu_-}\left(e^{(k_2(\theta)-\mu_-)a}-1\right)\right]\right]=$$

$$=-\frac{(\mu_--\chi)^2((\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2(\chi+k_1(\theta))(\chi+k_2(\theta))}-\frac{\mu_--\chi}{\lambda_++\lambda_-+\theta}-$$

$$-\frac{\lambda_+\mu_+(\mu_+(\lambda_-+\theta)+\mu_-(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^2((\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}-$$

$$-\frac{\lambda_+\lambda_-\mu_+\mu_-(\mu_--\chi)(\mu_+(\lambda_-+\theta)+\chi(\lambda_++\lambda_-+\theta))}{(\lambda_++\lambda_-+\theta)^4(\chi+k_1(\theta))(\chi+k_2(\theta))}\times$$

$$\times\left[\frac{1}{\mu_--\chi}e^{(\mu_--\chi)a}-\frac{1}{\mu_-+\chi}e^{-(\mu_-+\chi)a}-\frac{2\chi}{(\mu_-+\chi)(\mu_--\chi)}\right]e^{(\chi-\mu_-)a}.$$

Thus, we get a system of linear algebraic equations with respect to $C_1(\theta)$and $C_2(\theta)$. Denote

$$S_1 = e^{-\mu_- a}\left[\frac{1}{k_1(\theta)+\mu_-}\left(e^{(k_1(\theta)+\mu_-)a}-1\right)+\frac{1}{k_1(\theta)-\mu_-}\left(e^{(k_1(\theta)-\mu_-)a}-1\right)\right],$$

$$S_2 = e^{-\mu_- a}\left[\frac{1}{k_2(\theta)+\mu_-}\left(e^{(k_2(\theta)+\mu_-)a}-1\right)+\frac{1}{k_2(\theta)-\mu_-}\left(e^{(k_2(\theta)-\mu_-)a}-1\right)\right],$$

$$A = -\frac{(\mu_- - \chi)((\mu_+(\lambda_- + \theta)+\chi(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2(\chi + k_1(\theta))(\chi + k_2(\theta))} -$$

$$-\frac{\mu_+ + \chi}{\mu_+(\lambda_- + \theta)+\chi(\lambda_+ + \lambda_- + \theta)} + \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}\times$$

$$\times\frac{(\mu_- - \chi)(\mu_+(\lambda_- + \theta)+\chi(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2(\chi + k_1(\theta))(\chi + k_2(\theta))}\times$$

$$\times\left[\frac{1}{\mu_- - \chi}e^{(\mu_- - \chi)a} - \frac{1}{\mu_- + \chi}e^{-(\mu_- + \chi)a} - \frac{2\chi}{(\mu_- + \chi)(\mu_- - \chi)}\right]e^{(\chi - \mu_-)a},$$

$$B = -\frac{(\mu_- - \chi)^2((\mu_+(\lambda_- + \theta)+\chi(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2(\chi + k_1(\theta))(\chi + k_2(\theta))} - \frac{\mu_- - \chi}{\lambda_+ + \lambda_- + \theta} -$$

$$-\frac{\lambda_+\mu_+(\mu_+(\lambda_- + \theta)+\mu_-(\lambda_+ + \lambda_- + \theta))}{(\lambda_+ + \lambda_- + \theta)^2((\mu_+(\lambda_- + \theta)+\chi(\lambda_+ + \lambda_- + \theta))},$$

$$C_1(\theta) =$$

$$= \frac{A[(\mu_- + k_2(\theta))e^{k_2(\theta)a} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}S_2] - B[e^{k_2 a} - \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}S_2]}{(k_2 - k_1)e^{(k_1+k_2)a} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}(S_2 e^{k_1 a} - S_1 e^{k_2 a}) + \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}(S_2(\mu_- + k_1(\theta))e^{k_1 a} - S_1(\mu_- + k_2(\theta))e^{k_2 a})},$$

$$C_2(\theta) =$$

$$= \frac{B[e^{k_1 a} - \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}S_1] - A[(\mu_- + k_1(\theta))e^{k_1(\theta)a} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}S_1]}{(k_2 - k_1)e^{(k_1+k_2)a} + \frac{\lambda_+\lambda_-\mu_+\mu_-}{(\lambda_+ + \lambda_- + \theta)^2}(S_2 e^{k_1 a} - S_1 e^{k_2 a}) + \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_- + \theta}[S_2(\mu_- + k_1(\theta))e^{k_1 a} - S_1(\mu_- + k_2(\theta))e^{k_2 a}]}$$

Finally, we find the solution of equation (8).

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Interpolation Theorems for Lizorkin-Triebel-Morrey type Spaces with Many Groups Variables

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Abstract. In this paper, we introduce a new function space $F^{l^{\varrho}}_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau}(G,s)$ with the parameters of many groups of variables of type Lizorkin-Triebel-Morrey. In view of interpolation theorems we study some properties of functions, which are belonging to intersection of these spaces.
Key Words and Phrases: intersection of spaces Lizorkin-Triebel-Morrey type, many groups of variables, integral representation, interpolation theorems.

1. Introduction

In this paper we study interpolation theorems for space

$$F^{l}_{p,\theta,a,\varkappa,\tau}(G,s), \tag{1}$$

that is, with help of theory embedding we study some characterization of function which are belonging to intersection of space $F^{l^{\varrho}}_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau}(G,s)$ $(\varrho = 1, 2, \ldots, N)$, that is, the space Lizorkin-Triebel-Morrey type with many group variables.

Let $G \subset R^n$ be a domain and $1 \le s \le n$; s, n be naturals, in addition $e_n = \{1, 2, ..., n\}$, $n_1 + ... + n_s = n$. Hence we suppose the sufficient smooth function f(x), where the points $x = (x_1, ..., x_s) \in R^n$ have coordinates $x_k = (x_{k.1}; ...; x_{k,n_k}) \in R^{n_k}$ $(k \in e_s = \{1, ..., s\})$. Consequently, $R^n = R^{n_1} \times R^{n_2} \times \cdots \times R^{n_s}$.

Let $l = (l_1, ..., l_s)$ be a given positive vector such that, $l_k = (l_{k.1}; ...; l_{k,n_k})$, $(k \subset e_s)$, that is, $l_{k,j} > 0$, $(j = 1, ..., n_k)$ for every $k \in e_s$ and we shall denote by Q the set of vectors $i = (i_1, ..., i_s)$, where $i_k = 1, 2, ..., n_k$ for all $k \in e_s$. The number of the set Q is equal to: $|Q| = \prod_{k=1}^{s}(1 + n_k)$.

Therefore, to the vector $i = (i_1, ..., i_s) \in Q$, we let correspond the vector $l^i = \left(l_1^{i_1}; ...; l_s^{i_s}\right)$, where vectors $l^i = \left(l_1^{i_1}; ...; l_s^{i_s}\right)$ are coordinates of $l = (l_1, ..., l_s)$ and $l^0 = (0, 0, ..., 0)$, $l_k^1 = (l_{k,1}, 0, ..., 0), ..., l_k^{i_k} = (0, 0, .., l_{k,n_k})$ for all $k \in e_s$. And the the vectors e^i, we correspond the vector $\overline{l}^i = \left(\overline{l}_1^{i_1}, \overline{l}_2^{i_2}, ..., \ \overline{l}_s^{i_s}\right)$, where $\overline{l}_k^{i_k} = \left(\overline{l}_{k,1}^{i_1}, \overline{l}_{k,2}^{i_2}, ..., \overline{l}_{k,n_k}^{i_k}\right)$ $(k \in e_s)$, and

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the largest number $\bar{l}^{i_k}_{k,j}$ is less than $l^{i_k}_{k,j}$ for every $l^{i_k}_{k,j}$ >0,when $l^{i_k}_{k,j}$=0 then we assume that $\bar{l}^{i_k}_{k,j}=0$ for each $k\in e_s$.

Let $R^{|e^i|}=R^{|e^{i_1}|}\times R^{|e^{i_2}|}\times\cdots\times R^{|e^{i_s}|}$, where $R^{|e^{i_k}|}=R^{\left|e^{i_{k,1}}\right|}\times R^{\left|e^{i_{k,2}}\right|}\times\cdots\times R^{\left|e^{i_{k,n_k}}\right|}$.

Further for every $k\in e_s$, $R^{|e^{i_k}|}=\{t_k=(t_{k,1},...,t_{k,n_k})\in R^{n_k}, t_{k,j}\in R^{k,n_k}, t_{k,j}=0, \forall j\notin e^{i_k}=\sup p\bar{l}^{i_k}\ ,\ k\in e_s\}$.

Definition 1. *We denote by $F^{<l>}_{p,\theta,a,\varkappa,\tau}(s,G)$ normed Lizorkin-Triebel-Morrey space of function f on G, with many groups variables, with finite norm*

$$\|f\|_{F^{<l>}_{p,\theta,a,\varkappa,\tau}}(G,s)=\sum_{i\in Q}\|f\|_{L^{<l^i>}_{p,\theta,a,\varkappa,\tau}(G)}, \tag{2}$$

$$\|f\|_{L^{<l^i>}_{p,\theta,a,\varkappa,\tau}(G)}=\left\|\left\{\int_0^{t^i_{0,1}}\int_0^{t^i_{0,s}}\left[\frac{\Delta^{2\omega}(t,G)\,D^{\bar{l}^i}f}{\prod_{k\in e^i}t_k^{\left|\beta^{i_k}_k\right|}}\right]^{\theta}\prod_{k\in e^i}\frac{dt_k}{t_k}\right\}^{1/\theta}\right\|_{p,a,\varkappa,\tau}, \tag{3}$$

and

$$\|f\|_{p,a,\varkappa,\tau:\ G}=sup_{x\in G}\left\{\int_0^{\infty}\cdots\int_0^{\infty}\left[\prod_{k\in e_s}[t_k]_1^{\frac{-|\varkappa_k|a}{p}}\|f\|_{p,G_{t^\varkappa}(x)}\right]^{\tau}\prod_{k\in e_s}\frac{dt_k}{t_k}\right\}^{1/\tau}, \tag{4}$$

Further it means that, $D^{\bar{l}^i}f=D_1^{\bar{l}^{i_1}_1}\cdots D_s^{\bar{l}^{i_s}_s}f$, $D_s^{\bar{l}^{i_k}_k}f=D_{k,1}^{\bar{l}^{k}_k}\cdots D_{k,n_k}^{\bar{l}^{i_k}_k}f$; $G_{t^\varkappa}(x)=G\cap I_{t^\varkappa}(x)$; $I_{t^\varkappa}(x)=I_{t_1^{\varkappa_1}}(x_1)\times I_{t_2^{\varkappa_2}}(x_2)\times\cdots\times I_{t_s^{\varkappa_s}}(x_s)$; $I_{t_k^{\varkappa_k}}(x_k)=$

$\left\{y_k:|y_k-x_k|<\frac{1}{2}t_k^{|\varkappa_k|},\ k\in e_s\right\}$, $|\beta_k|=\sum_{j=1}^{n_k}\beta^{i_k}_{k,j}$; $\frac{dt_k}{t_k}=\prod_{j\in e^i_k}\frac{dt_{k,j}}{t_{k,j}}$, where $0<\beta^{i_k}_{k,j}=l^{i_k}_{k,j}-\bar{l}^{i_k}_{k,j}\le 1$ for $l^{i_k}_k>0$, but when $l^{i_k}_{k,j}=0$, $\beta^{i_k}_{k,j}=0$; $t=(t_1,...,t_s)$, $t_k=(t_{k,1},...,t_{k,n_k})$, $\omega=(\omega_1,...,\omega_s)$, $\omega_k=(\omega_{k,\,1},...,\omega_{k,n_k})$ and in addition $\omega_{k,j}=1$ or $\omega_{k,\,j}=0$, $k\in e_s$, $e^i=\sup p\bar{l}^i=\sup p\omega$, $1<\theta<\infty$; $(1\le p<\infty)$; $t_0=(t_{0,1},...,t_{0,s})$, $t_{0,k}=(t_{0,k,1},...,t_{0,k,n_k})$ be a fixed vector and $\varkappa\in(0,\infty)^n$, $a\in[0,1]$, $\tau\in[1,\ \infty]$, $[t_k]_1=\min\{1,t_k\}$, $k\in e_s$.

When $s=1$ then space (1) is equivalent to the space Lizorkin-Triebel-Morrey type $F^{<l>}_{p,\theta,a,\varkappa,\tau}(G)$, which was investigated in [1, 4, 9], when s=n then the space (1) is equivalent to the space Lizorkin-Triebel-Morrey type with mixed derivatives, $S^{<l>}_{p,\theta,a,\varkappa,\tau}F(G)$ which was studied in [5, 6], when $a=0,\tau=\infty$ $s=1$, $N=1$, then this space is equivalent to the space $F^l_{p,\theta}(G)$, which was developed in [2, 13, 14].

Similarly results for the Morrey spaces was investigated in [3, 12, 13].

It is clear, that $V(\sigma)\subset I_{T^\sigma}, U-$ is an open set, which belonging to the domain G and $U+V\subset G$. Here it is said that, the subdomain $U\subset G\subset R^n$ calls domain satisfying the condition " $\sigma-semi-horn''$, if the vector $\sigma=(\sigma_1,...,\sigma_s)$ is such that, $x+V(\sigma)\subset G$ for all $x\subset U$. It is said that, the domain $G\subset E_n$ satisfying the condition "$\sigma-semi-horn''$, that is, $G \subset A(T^\sigma)$, if we have finite sub domains $G_1, ..., G_N \subset G$, satisfying the condition "$\sigma - semi - horn$" and surfacing the domain G, that is,

$$G = \bigcup_{j=1}^{N} G_j. \tag{5}$$

But we suppose $G \in A_\epsilon\ (T^\sigma)\ (\epsilon > 0)$, if we substitute the condition $G = \bigcup_{j=1}^{N} G_{j,\epsilon}$ in the condition (5). Note that $G_{j,\epsilon} = \{x : x \in G_j :\ \rho(x, G/G_j) > \epsilon\}$.

2. Preliminaries

Let $\Psi_i \in C_0^\infty(R^n)$ be such, that their carries belonging to $I_1 = \left\{x : |x_j| < \frac{1}{2}; j = 1, .., n_k\right\}$. Then we put

$$V(\sigma) = \bigcup_{\substack{0 < t_j \le T_j;\\ j \in e_n}} \left\{ y : \left(\frac{y}{t^\sigma}\right) \in S(\Psi_i) \right\},$$

where $0 < T_j \le 1$, $j \in e_n$. U is an open set which belonging to the domain G. Furthermore we assume that $U + V \subset G$, for $T = (T_1, ..., T_s)$, $T_k = (T_{k,1}, ..., T_{k,n_k})$, $0 < T_{k,j} \le 1$, $k \in e_s, j = 1, .., n_k$, $(t^\sigma + T^\sigma)^i = t_k^{\sigma_k}$, $(k \in e^i)$; $(t^\sigma + T^\sigma)^i = T^\sigma$, $(k \in e_s/e^i)$, $\sigma = (\sigma_1, ..., \sigma_s)$, $\sigma_j > 0$, $j = 1, .., n_k$. Let $G_{(t^\sigma+T^\sigma)^i}(U) = \left(U + I_{(t^\sigma+T^\sigma)^i}(x)\right) \bigcap G = Z$, $p_\varrho = (p_{\varrho 1}, ..., p_{\varrho n})$, $q_\varrho = (q_{\varrho 1}, ..., q_{\varrho n})$, $\alpha_\varrho \ge 0$, $\sum_{\varrho=1}^{N} \alpha_\varrho = 1$, $\frac{1}{p} = \sum_{\varrho=1}^{N} \frac{\alpha_\varrho}{p_\varrho}$, $\frac{1}{q} = \sum_{\varrho=1}^{N} \frac{\alpha_\varrho}{q_\varrho}$, $\frac{1}{\theta} = \sum_{\varrho=1}^{N} \frac{\alpha_\varrho}{\theta_\varrho}$, $l = \sum_{\varrho=1}^{N} l^\varrho \alpha_\varrho$.

Lemma 1. *Let* $1 \le p_\varrho \le q_\varrho \le r_\varrho \le \infty; \varrho = 1, 2, .., N;$ $0 < |\varkappa_k| < |\sigma_k|; 0 \le \eta_{k,j} \le T_{k,j} \le 1; \eta = (\eta_1, ..., \eta_n), 0 < \eta_{k,j} \cdot t_{k,j} \le T_{k,j} \le 1;$ $(k \in e_s,\ j = 1, 2, .., n_k)$, $1 \le \tau \le \infty$; $v = (v_1, ..., v_s)$, $v_{k,j} \ge 0$ *are integers,* $0 < \rho_{k,j} < \infty$; $j = 1, ..., n_k; k \in e_s$; *and* $\Delta^{2\omega}(t) D^{\bar{l}^i} f \in L_{p_\varrho, a, \varkappa, \tau}(G)$,

$$\mu_{k,i_k} = \sum_{\varrho=1}^{N} l_{k,i_k}^{\varrho} \alpha_\varrho \sigma_k - (v_k, \sigma_k) - (|\sigma_k| - |\varkappa_k| a)\left(\frac{1}{p} - \frac{1}{q}\right),$$

$$(v_k, \sigma_k) = \sum_{j=1}^{n_k} \sigma_{k,j} v_{k,j}, \quad |\sigma_k| = \sum_{j=1}^{n_k} \sigma_{k,j}, |\varkappa_k| = \sum_{j=1}^{n_k} \varkappa_{k,j},$$

$$F_\eta^i(x) = \prod_{k \in e_s/e^i} T_k^{-|\sigma_k| + \sigma_{k,i_k} \bar{l}_{k,i_k} - (v_k, \sigma_k)} \int_0^{\eta^i} \cdots \int_0^{\eta^i} \varphi_i(x, t,\ T)$$

$$\times \prod_{k \in e^i} \frac{dt_k}{t_k^{1 + |\sigma_k| - \sigma_{k,i_k} \bar{l}_{k,i_k} + (v_k, \sigma_k)}}, \tag{6}$$

$$F^i_{\eta T}(x) = \prod_{k\in e_s/e^i} T_k^{-|\sigma_k|+\sigma_{ki_k},\bar{l}_{k,i_k}-(v_k,\sigma_k)} \int_{\eta^i}^{T^i} \cdots \int_{\eta^i}^{T^i} \varphi_i(x,t,T)$$

$$\times \prod_{k\in e^i} \frac{dt_k}{t_k^{1+|\sigma_k|-\sigma_{k,i_k}\bar{l}_{k,i_k}+(v_k,\sigma_k)}} \tag{7}$$

Here $\left|\beta_k^{\varrho}\right| = \sum_{j=1}^{n_k} \beta_{k,j}^{i_k,\ \varrho}$, $(v_k,\sigma_k) = \sum_{j=1}^{n_k} \sigma_{k,j} v_{k,j}$, $|\sigma_k| = \sum_{j=1}^{n_k} \sigma_{k,j}$, $|\varkappa_k| = \sum_{j=1}^{n_k} \varkappa_{k,j}$,

$$\varphi_i(x,t,T)$$

$$= \int_{R^{|e^i|}} \int_{R^n} \left\{ \Delta^{2\omega}(u)\, D^{\bar{l}^i} f(x+y)\, \Psi_i^{(v)} \left(\frac{y}{(t^\sigma+T^\sigma)^i},\ \frac{u}{(t^\sigma+T^\sigma)^i} \right) \right\} dydu, \tag{8}$$

where $\Psi_i \in C^\infty(R^n \times R^n)$, and $\Psi_i(\cdot,z) \in C_0^\infty$.

Then the following inequalities hold:

$$\sup_{\overline{x}\in U} \left\|F^i_{\eta}\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})} \le C_1 \prod_{\varrho=1}^{N} \left\{ \left\| \prod_{k\epsilon e^i} t_k^{-|\beta_k{}^{\varrho}|} \Delta^{2\omega}(t)\, D^{\bar{l}^{i,\varrho}} f \right\|_{p_\varrho,a,\varkappa,\tau} \right\}^{\alpha_\varrho}$$

$$\times \prod_{k\in e_s} [\rho_k]_1^{\frac{|\varkappa_k|a}{p}} \prod_{k\in e_s/e^i} T_k^{\mu_{k,i_k}} \prod_{k\epsilon e^i} t_k^{\mu_{k,i_k}}; (\mu_{k,i_k} > 0), \tag{9}$$

$$\sup_{\overline{x}\in U} \left\|F^i_{\eta T}\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})} \le C_2 \prod_{\varrho=1}^{N} \left(\left\| \prod_{k\epsilon e^i} t_k^{-|\beta_k{}^{\varrho}|} \Delta^{2\omega}(t)\, D^{\bar{l}^{i,\varrho}} f \right\|_{p_\varrho,a,\ \varkappa,\ \tau} \right)^{\alpha_\varrho}$$

$$\times \begin{cases} \prod_{k\epsilon e^i} T_k^{\mu_{k,i_k}}; & \mu_{k,i_k} > 0, \\ \prod_{k\in e^i} ln \frac{T_k}{\eta_k}; & \mu_{k,i_k} = 0, \\ \prod_{k\epsilon e^i} \eta_k^{\mu_{k,i_k}}; & \mu_{k,i_k} < 0, \end{cases} \times \prod_{k\in e_s} [\rho_k]_1^{\frac{|\varkappa_k|a}{p}}. \tag{10}$$

Where C_1, and C_2 are constants independent of f, ρ, η and T.

Proof. Using Minkowski's inequality for any $\overline{x} \in U$, we have:

$$\sup_{\overline{x}\in U} \left\|F^i_{\eta}\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})} \le\ C \prod_{k\in e_s/e^i} T_k^{-|\sigma_k|+\sigma_{k,i_k}\bar{l}_{k,i_k}-(v_k,\sigma_k)}$$

$$\times \int_{0^i}^{\eta^i} \|\varphi_i(\cdot;t;T)\|_{q,U_{\rho^{\varkappa}}(\overline{x})} \prod_{k\in e^i} t_k^{-1+|\sigma_k|+\sigma_{k,i_k}\bar{l}_{k,i_k}-(v_k,\sigma_k)} dt_k \tag{11}$$

We must estimate $\|\varphi_i(\cdot,t,\ T)\|_{q,U_{\rho^{\varkappa}(\overline{x})}}$ from the Holder's inequality $(q \le r)$ we get:

$$\|\varphi_i(\cdot,t,T)\|_{q,U_{\rho^{\varkappa}}(\overline{x})} \le C_1 \left(\int_{U_{\rho^{\varkappa}}(\overline{x})} \prod_{\varrho=1}^{N} \{|\varphi_i(x\cdot,t,T)|\}^{\alpha_\varrho q} dx \right)^{1/q}.$$

Using the Holder's inequality into right part with the indication $\lambda_\varrho = \frac{q_\varrho}{q\alpha_\varrho}$, $\varrho = 1, 2, \ldots, N$, $\left(\sum_{\varrho=1}^{N} \frac{1}{o_\varrho} = q \sum_{\varrho=1}^{N} \frac{\alpha_\varrho}{q_\varrho} = 1\right)$. Then we have

$$\|\varphi_i (\cdot, t)\|_{q_\varrho, U_{\rho^\varkappa}(\overline{x})} \leq C_2 \prod_{\varrho=1}^{N} \left\{ \|\varphi_i (\cdot, t,\ T)\|_{r_\varrho, U_{\rho^\varkappa}(\overline{x})} \right\}^{\alpha_\varrho}. \tag{12}$$

Once again, using Holder's inequality ($q_\varrho \leq r_\varrho$) we have

$$\|\varphi_i (\cdot, t)\|_{q_\varrho, U_{\rho^\varkappa}(\overline{x})} \leq \|\varphi_i (\cdot, t, T)\|_{r_\varrho, U_{\rho^\varkappa}(\overline{x})} \times \prod_{j\in e_s} \rho_j^{|\varkappa_j|\left(\frac{1}{q_\varrho} - \frac{1}{r_\varrho}\right)}. \tag{13}$$

Let X be a characterization function of the set $S(\Psi_i)$. Noting that, $1 \leq p_\varrho \leq r_\varrho \leq \infty$; $s_\varrho \leq r_\varrho$ $\left(\frac{1}{s_\varrho} = 1 - \frac{1}{p_\varrho} + \frac{1}{r_\varrho}\right)$ we get

$$\left|\Delta^{2\omega} D^{\overline{l}^{i,\varrho}} f \Psi_i\right| = \left(\left|\Delta^{2\omega} D^{\overline{l}^{i,\varrho}} f\right|^{p_\varrho} |\Psi_i|^{s_\varrho}\right)^{\frac{1}{r_\varrho}} \left(\left|\Delta^{2\omega} D^{\overline{l}^{i,\varrho}} f\right|^{p_\varrho} X\right)^{\frac{1}{p_\varrho} - \frac{1}{r_\varrho}} \left(|\Psi_i|^{s_\varrho}\right)^{\frac{1}{r_\varrho}}$$

and using for $|\varphi_i|$ Holder's inequality $\left(\frac{1}{r_\varrho} + \left(\frac{1}{p_\varrho} - \frac{1}{r_\varrho}\right) + \left(\frac{1}{s_\varrho} + \frac{1}{r_\varrho}\right) = 1\right)$, then we have

$$\|\varphi_i (\cdot, t, T)\|_{r_\varrho, U_{\rho^\varkappa}(\overline{x})} \leq \sup_{x \in U_{\rho^\varkappa}(\overline{x})}$$

$$\left(\int_{R^{|e^i|}} \int_{R^n} \left|\Delta^{2\omega} D^{\overline{l}^{i,\varrho}} f(x+y)\right|^{p_\varrho} X\left(\frac{y}{(t^\sigma + T^\sigma)^i}\right) dudy\right)^{\frac{1}{p_\varrho} - \frac{1}{r_\varrho}}$$

$$\times \sup_{x \in V} \left(\int_{R^{|e^i|}} \int_{R^n} \left|\Delta^{2\omega}(u) D^{\overline{l}^{i,\varrho}} f(x+y)\right|^{p_\varrho} dudy\right)^{\frac{1}{r_\varrho}}$$

$$\times \left(\int_{R^{|e^i|}} \int_{R^n} \left|\Psi_i\left(\frac{y}{(t^\sigma + T^\sigma)^i}, \frac{u}{(t^\sigma + T^\sigma)^i}\right)\right|^{s_\varrho} dudy\right)^{\frac{1}{s_\varrho}}. \tag{14}$$

Because of $U + V \subset Z$, and $Z_{(t^\sigma + T^\sigma)^i}(x) \subset Z_{(t^\varkappa + T^\varkappa)^i}(x)$, for all $x \in U$ and $0 < t_j \leq T_j \leq 1$, $|\varkappa_k| \leq |\sigma_k|$, $k \in e_n$ we find:

$$\int_{R^n} \left|\int_{R^{|e^i|}} \Delta_u^{2\omega}(u) D^{\overline{l}^{i,\varrho}} f(x+y)\, du\right|^{p_\varrho} X\left(\frac{y}{(t^\sigma + T^\sigma)^i}\right) dy$$

$$\leq \int\limits_{Z_{(t^\sigma + T^\sigma)^i}(\overline{x})} \left|\int_{R^{|e^i|}} \Delta^{2\omega}(u) D^{\overline{l}^{i,\varrho}} f(x+y)\right|^{p_\varrho} dudy$$

$$\times \left\| \prod_{k\in e^i} t_k^{\left|\beta_k^\varrho\right|} \Delta^{2\omega}(t) D^{\overline{l}^{i,\varrho}} f \right\|_{p_\varrho,a,\varkappa}^{p_\varrho} \prod_{k\in e^i} t_k^{|\varkappa_k|a} \prod_{k\in e_s/e^i} T_k^{|\varkappa_k|a}. \tag{15}$$

Next for $y \in V$

$$\int_{U_{\rho^\varkappa}(\overline{x})} \left| \int_{R^{|e^i|}} \Delta^{2\omega}(u) D^{\overline{l}^{i,\varrho}} f(x+y)\, du \right|^{p_\varrho} dx$$

$$\leq \int_{Z_{\rho^\varkappa}(\overline{x}+y)} \left| \int_{R^{|e^i|}} \Delta^{2\omega}(u) D^{\overline{l}^{i,\varrho}} f(x)\, du \right|^{p_\varrho} dx$$

$$\leq \left\| \prod_{k\in e^i} t_k^{-\left|\beta_k^\varrho\right|} \Delta^{2\omega}(t) D^{\overline{l}^{i,\varrho}} f \right\|_{p_\varrho,a,\varkappa}^{p_\varrho}$$

$$\times \prod_{k\in e^i} t_k^{\left|\beta_k^\varrho\right| p_\varrho} \prod_{k\in e_s} [\rho_k]_1^{|\varkappa_k|a}, \tag{16}$$

$$\int_{R^{|e^i|}} \int_{R^n} \left| \Psi_i \left(\frac{y}{(t^\sigma + T^\sigma)^i}, \frac{u}{(t^\sigma + T^\sigma)^i} \right) \right|^s dudy$$

$$= \prod_{k\in e^i} t_k^{|\sigma_k|} \prod_{k\in e_s/e^i} T_k^{|\sigma_k|} \|\Psi_i\|_{s_\varrho}^{s_\varrho}. \tag{17}$$

From (12)-(17) we get

$$\|\varphi_i(\cdot,t,T)\|_{q,U_{\rho^\varkappa}(\overline{x})} \leq C \prod_{\varrho=1}^{N} \left\{ \left\| \prod_{k\epsilon e^i} t_k^{-|\beta_k{}^\varrho|} \Delta^{2\omega}(t) D^{\overline{l}^{i,\varrho}} f \right\|_{p_\varrho,a,\ \varkappa} \right\}^{\alpha_\varrho}$$

$$\times \prod_{k\in e_s/e^i} T^{|\sigma_k| - (|\sigma_k| - |\varkappa_k|a)\left(\frac{1}{p} - \frac{1}{q}\right)}{}_k \prod_{k\epsilon e^i} t_k^{|\sigma_k| - (|\sigma_k| - |\varkappa_k|a)\left(\frac{1}{p} - \frac{1}{q}\right)}$$

$$\times \prod_{k\in e_n} [\rho_k]_1^{\frac{|\varkappa_k|a}{r}} \prod_{k\in e_n} \rho_k^{|\varkappa_k|\left(\frac{1}{q} - \frac{1}{r}\right)}. \tag{18}$$

Taking consideration $\|\cdot\|_{p,a,\varkappa} \leq \|\cdot\|_{p,a,\varkappa,\tau}$ for $1 \leq \tau \leq \infty$ and putting (18) into (11) for $r = q$, then we get the inequality (9). Similarly, we can prove the inequality (10). ◄

Lemma 2. *Let* $1 \leq p_\varrho \leq q_\varrho < \infty; \varrho = 1,2,..,N;\ 0 < \ |\varkappa_k| \leq |\sigma_k|; 0 \leq T_k \leq 1; (k \in e_s,$ $j = 1,2,..,n_k),\ \ 1 \leq \tau_1 \leq \tau_2 \leq \infty;\ \ \mu_{k,i_k} > 0$ *and* $\Delta^{2\omega}(u) D^{\overline{l}^i} \in L_{p_\varrho,a,\varkappa,\tau}$

$$\mu_{k,i_k,0} = \sigma_{k,i_k} \sum_{\varrho=1}^{N} l_{k,i_k}^\varrho \alpha_{\varrho} - (v_k, \sigma_k) - (|\sigma_k| - |\varkappa_k|\, a)\, \frac{1}{p}.$$

Then the following inequality holds for the function $B_{\eta}^{i}(x)$:

$$\left\|F_{\eta}^{i}\right\|_{q,b,\varkappa,\tau_2;U} \leq$$

$$\times C^{1}\prod_{\varrho=1}^{N}\left\{\left\|\prod_{k\epsilon e^{i}} t_{k}^{-\left|\beta_{k}{}^{\varrho}\right|}\Delta^{2\omega}(t)\, D^{\bar{l}^{i},\ \varrho} f\right\|_{p_{\varrho},a,\ \varkappa,\tau_1}\right\}^{\alpha_{\varrho}}, \tag{19}$$

where b, is an arbitrary number satisfying the following condition:

$$\begin{gathered} 0 \leq b \leq 1,\ \text{ if}\, \mu_{k,i_k,0} > 0, \\ 0 \leq b < 1,\ \text{if}\, \mu_{k,i_k,0} = 0, \\ 0 \leq b < 1 + \frac{\mu_{k,i_k,0}\, q\,(1-a)}{|\sigma_k| - |\varkappa_k|\, a},\ \text{if}\, \mu_{k,i_k,0} < 0. \end{gathered} \tag{20}$$

The proof of this lemma is similarly 1.

Using these facts, we can show the general theorems, which give us the structure of such space $F_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau_1}^{l^{\varrho}}(G,\ s)\ (\varrho = 1,2,..,N)$.

3. Embedding theorems

Using these facts, we can show the general theorems, which give us the structure of such space $F_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau_1}^{l^{\varrho}}(G,\ s)\ (\varrho = 1,2,..,N)$.

Theorem 1. *Let $G \in A(T^{\sigma})$ be a domain , $1 \leq p_{\varrho} \leq q_{\varrho} \leq \infty$, $(\varrho = 1,2,..,N)$; $v = (v_1,\ldots,v_n)$; $v_j \geq 0$ are integers, (j=1,2,.., n) and in addition*

1) $v_{k,j} \geq l_{k,j}^{0}$ $(j = 1,2,..,n_k; k \in e_s)$;

2) $v_{k,j} \geq l_{k,j}^{i_k} + 1$, $v_{k,i_k} < l_{k,i_k}^{i_k} + 1$, $0 < \varkappa_k < \sigma_k$ $(k \in e_s)$; $1 \leq \tau_1 \leq \tau_2 \leq \infty$, $f \in \bigcap_{\varrho=1}^{N} F_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau_1}^{<l^{\varrho}>}(G,s)$ and let $\mu_{k,i_k} > 0$, $(i_k = 1,2,...,n_k, k \in e_s)$.

Then following inequality holds:

$$\|D^{v} f\|_{q,G} \leq C^{1} B_1(T) \prod_{\varrho=1}^{N}\left\{\|f\|_{F_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau_1}^{<l^{\varrho}>}(G,s)}\right\}^{\alpha_{\varrho}}, \tag{21}$$

$$\begin{gathered} \|D^{v} f\|_{p,b,\varkappa,\tau_2;G} \leq C^{2} \prod_{\varrho=1}^{N}\left\{\|f\|_{F_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau_1}^{<l^{\varrho}>}(G,\ s)}\right\}^{\alpha_{\varrho}}, \\ (p_{\varrho,j} \leq q_{\varrho,j} < \infty, j \in e_n). \end{gathered} \tag{22}$$

where $B_1(T) = \sum_{i\in Q}\prod_{k\in e_s} T_k^{\mu_{k,i_k}}$.

Particular, if $\mu_{k,i_k,0} > 0$, $(i_k = 1,2,...,n_k,\ k \in e_s)$ then the function $D^{v} f$ is continuous on G and

$$\sup\nolimits_{x\in G}|D^{v} f| \leq C^{3} {B_1}^{0}(T) \prod_{\varrho=1}^{N}\left\{\|f\|_{F_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau_1}^{<l^{\varrho}>}(G,\ s)}\right\}^{\alpha_{\varrho}}, \tag{23}$$

where

$$B_1{}^0(T) = \sum_{i=(i_1,...,i_s)\in Q} \prod_{j\in e_s} T_j^{\mu_{k,i_k,0}},$$

and $T_k \in (0, \min(1, T_{0,k})]$, $(k \in e_s)$, $T_0 = (T_{0,1}, ..., T_{0,k})$ is a fixed positive vector, b is an arbitrary number satisfying condition (**??**), C^1 and C^2 C^1,C^3 are constants independent of f, and C^1 dependent of the vector T.

The proof of Theorem 1. Obviously, in this case for $f \in F^{<l^\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau}(G,s)$ generalized derivatives D^{vf} exit. It means that, if $\mu_{k,i_k} > 0(k \in e_s)$, because of $p_\varrho \le q_\varrho, |\varkappa_k| < |\sigma_k|\,(k \in e_s)$, $a \in [0,1]^n$, $f \in F^{<l^\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau}(G,s) \to F^{<l^\varrho>}_{p_\varrho,\theta_\varrho}(G,s)$, $\varrho = 1,2,..,N)$

It means that, for almost every point of $x \in G$, there exits generalized derivatives D^{vf} with the same carries [3]:

$$D^v f(x) = \sum_{i=(i_1,...,i_s)\in Q} (-1)^{\left|\bar{l}^i - v\right|} C_i$$

$$\prod_{k\in e_s/e^i} T_k^{-|\sigma_k| + \sigma_{k,i_k}\bar{l}_{k,i_k} - (v_k,\sigma_k)}$$

$$\times \int_0^{T_1^i} \cdots \int_0^{T_n^i} \prod_{k\in e^i} t_k^{-1-|\sigma_k| + \sigma_{k,i_k}\bar{l}_{k,i_k} - (v_k,\sigma_k)} dt_k$$

$$\times \int_{R^{|e^i|}} \int_{R^n} \{\Delta^{2\omega}(u)\, D^{\bar{l}^{i,\varrho}} f(x+y)$$

$$\times \Psi_i^{(v)}\left(\frac{y}{(t^\sigma + T^\sigma)^i}, \frac{u}{(t^\sigma + T^\sigma)^i}\right)\}\; dydu. \tag{24}$$

Using the Minkowski's inequality, then we have:

$$\|D^v f\|_{q,\,G} \le C_1 \sum_{i=(i_1,...,i_s)\in Q} \left\|F_T^i\right\|_{q;G}. \tag{25}$$

From (10) for $U = G$, $\eta = T$, $\varrho \to \infty$ we get

$$\left\|F_T^i\right\|_{q;G} \le$$

$$\times C_2 \prod_{k\in e_s} T_k^{\mu_{k,i_k}} \prod_{\varrho=1}^{N} \left\{\left\|\prod_{k\in e^i} t_k^{-|\beta_k{}^\varrho|} \Delta^{2\omega}(t)\, D^{\bar{l}^i,\varrho} f\right\|_{p_\varrho,a,\varkappa}\right\}^{\alpha_\varrho}.$$

Using it for (25), and taking consideration $p_\varrho \le \theta_\varrho$ and $1 < \theta_\varrho < \infty$, $\varrho = 1,2,..,N$, we get (21).

Using (19) we can proof (22).

Next we suppose $\mu_{k,i_k0} > 0$, $k \in e_s$. We must show that, the function D^{vf} is continuous on G. From (24) and (25) for $q_j \equiv \infty, j \in e_n$, $\mu_{k,i_k} = \mu_{k,i_k,0}$, $k \in e_s$ we have:

$$\|D^v f - D^v f_{T^\sigma}\|_{\infty,G} \leq \sum_{i \in Q} \prod_{k \in e_s/e^i} T_k^{\mu_{k,i_k}}$$

$$\times \prod_{\varrho=1}^{N} \left(\left\| \int_0^{h_0^i} \cdots \int_0^{h_{0n}^i} \left(\left(\prod_{k \in e^i} t_k^{-|\beta^\varrho{}_k|} \Delta^{2\omega}(\cdot) D^{\bar{l}^i} f \right)^{\theta_\varrho} \prod_{k \in e^i} \frac{dt_k}{t_k} \right)^{1/\theta_\varrho} \right\|_{p_\varrho,a,\varkappa,\tau} \right)^{\alpha_\varrho}.$$

$\lim_{T\to 0}\|D^v f - D^v f_{T^\sigma}\|_{\infty,G} = 0$. Because of $D^v f_{T^\sigma}$ is continuous on G, then convergence of $L_\infty(G)$ coincides with the absolutely convergence. Consequently, it is continuous on G. This completes the proof.

Let γ be a n dimensional vector.

Theorem 2. *Let all conditions of Theorem 1 be satisfied.In addition, $G \in A_\in (T^\sigma)$. Then for $\mu_{k,i_k} > 0$, $(i_k = 1, 2, ..., n_k,\ k \in e_s)$ the derivative D^{vf} satisfies condition the Holder on the domain G, for metric L_q with indication ε. More precisely,*

$$\|\Delta(\gamma, G) D^v f\ \|_{q,G} \leq C \prod_{\varrho=1}^{N} \left\{ \|f\|_{F^{<l^\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau_1}(G,s)} \right\}^{\alpha_\varrho}$$

$$\times \prod_{k \in e^i} |\gamma_k|^{\varepsilon_k}, \tag{26}$$

where $\varepsilon = (\varepsilon_1, ..., \varepsilon_s)$, $\varepsilon_k = (\varepsilon_{k,1}, ..., \varepsilon_{k,n_k})$, and ε_k is an arbitrary number satisfying the condition:

$$0 < \varepsilon_k \leq 1,\ \mathit{if}\ \frac{\mu_{k,i_k}}{\sigma_0} > 1,$$

$$0 < \varepsilon_k < 1,\ \mathit{if}\ \frac{\mu_{k,i_k}}{\sigma_0} = 1,$$

$$0 < \varepsilon_k \leq \frac{\mu_{k,i_k}}{\sigma_0},\ \mathit{if}\ \frac{\mu_{k,i_k}}{\sigma_0} < 1. \tag{27}$$

where $\mu_k = \min\mu_{k,i_k}$, $\sigma_0 = \max|\sigma_k|$ $(i_k = 1, 2, \ldots, n_k,\ k \in e_s)$. If $\mu_{k,i_k,0} > 0$, $(i_k = 1, 2, \ldots, n_k,\ k \in e_s)$ then

$$\sup_{x\in G} |\Delta(\gamma,\ G) D^v f(x)| \leq C \prod_{\varrho=1}^{N} \left\{ \|f\|_{F^{<l^\varrho>}_{p_\varrho,\theta_\varrho,a,\tau_1}(G,s)} \right\}^{\alpha_\varrho} \prod_{k \in e^i} |\gamma_k|^{\varepsilon_k{}^0}, \tag{28}$$

where ε_k^0 satisfies the same condition, but we must substitute $\mu_{k,i_k,0}$ into μ_k and C is a constant independent of f and γ.

The proof of this theorem is similarly 1.

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Multilinear Rough Fractional Integral on Product Morrey Spaces

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Abstract. We will study the boundedness of multilinear fractional integral operator $I_{\Omega,\alpha,m}$ with rough kernels $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s \le \infty$ on product Morrey spaces. We find for the operator $I_{\Omega,\alpha,m}$ necessary and sufficient conditions on the parameters of the boundedness on product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to Morrey spaces $L^{q,\lambda}(\mathbb{R}^n)$.

Key Words and Phrases: product Morrey spaces, multilinear rough fractional integral.

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1. Introduction

The classical Morrey spaces, introduced by Morrey [9] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations. The boundedness of fractional integral operators on the classical Morrey spaces was studied by Adams [1], Chiarenza and Frasca *et al.* [2].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. Also for $\overrightarrow{x} = (x_1,\ldots,x_m) \in \mathbb{R}^{mn}$ and $r > 0$, we denote by $B(\overrightarrow{x},r)$ the open ball centered at $\overrightarrow{x} \in \mathbb{R}^{mn}$ of radius r, and $B(\overrightarrow{x},r)$ We denote by $\overrightarrow{f}$ the m-tuple $(f_1,f_2,\ldots,f_m)$, $\overrightarrow{y} = (y_1,\ldots,y_m)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

Definition 1. *Let $1 \le p < \infty$, $0 \le \lambda \le n$, $[t]_1 = \min\{1,t\}$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, and by $WL_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms*

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}, \qquad \|f\|_{WL_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))}$$

respectively.

In 1999, Kenig and Stein [8] studied the following multilinear fractional integral

$$I_{\alpha,m}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{nm-\alpha}} dy_1 dy_2 \dots dy_m,$$

and showed that $I_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p_1 + \dots + 1/p_m - \beta/n > 0$ for each $p_i > 1 (i = 1, \dots, m)$. If some $p_i = 1$, then $I_{\alpha,m}$ is bounded $L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n)$ to $L_{q,\infty}(\mathbb{R}^n)$. Obviously, the multilinear fractional integral $I_{\alpha,m}$ is a natural generalization of the classical fractional integral $I_\alpha \equiv I_{\alpha,1}$.

Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$. The multi-sublinear fractional maximal operator $\mathcal{M}_{\alpha,m}$ with rough kernels Ω is defined by

$$\mathcal{M}_{\alpha,m}(\overrightarrow{f})(x) = \sup_{r>0} \frac{1}{r^{nm-\alpha}} \int_{B(\overrightarrow{0},r)} |\Omega(\overrightarrow{y})| \prod_{j=1}^{m} |f_i(x-y_i)| d\overrightarrow{y}, \qquad 0 \le \alpha < nm.$$

If $m = 1$, then $M_{\Omega,\alpha} \equiv \mathcal{M}_{\Omega,\alpha,1}$ is the fractional maximal operator with rough kernel Ω. When $m = 1$ and $\Omega \equiv 1$, then $M_\alpha \equiv \mathcal{M}_{1,\alpha,1}$ is the classical fractional maximal operator.

In [7] we proved the boundedness of the multi-sublinear fractional maximal operator with rough kernels $\mathcal{M}_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, if $p > s'$, $1 < p_1, \dots, p_m < \infty$, $1/q = 1/p_1 + \dots + 1/p_m - \alpha/(mn - \lambda)$ and from the space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $WL^{q,\lambda}(\mathbb{R}^n)$, if $p = s'$, $1 \le p_1, \dots, p_m < \infty$ and $1/q = 1/p_1 + \dots + 1/p_m - \alpha/(n - \lambda)$ and at least one exponent p_i, $1 \le i \le m$ equals one.

In this work, we prove the boundedness of the multilinear fractional integral operator with rough kernels $I_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, if $p > s'$, $1 < p_1, \dots, p_m < \infty$, $1/q = 1/p_1 + \dots + 1/p_m - \alpha/(mn - \lambda)$ and from the space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $WL^{q,\lambda}(\mathbb{R}^n)$, if $p = s'$, $1 \le p_1, \dots, p_m < \infty$ and $1/q = 1/p_1 + \dots + 1/p_m - \alpha/(n - \lambda)$ and at least one exponent p_i, $1 \le i \le m$ equals one.

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Boundedness of multilinear fractional integral operator $\mathcal{M}_{\Omega,\alpha,m}$ on product Morrey spaces

In this part, we investigate the boundedness of multilinear fractional integral operator $I_{\Omega,\alpha,m}$ on product Morrey spaces.

Spanne and Adams obtained two remarkable results on Morrey spaces (see Definition 1.1 of the Morrey spaces in Section 1) for I_α. Their results can be summarized as follows.

Theorem 1. *[5, 10] (Spanne, but published by Peetre) Let $0 < \alpha < n$, $0 \le \lambda < n - \alpha p$, $1/q = 1/p - \alpha/n$ and $\mu/q = \lambda/p$. Then for $p > 1$, the operator I_α are bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ and for $p = 1$, I_α is bounded from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\mu}(\mathbb{R}^n)$.*

Theorem 2. *[1, 4] Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n - \alpha p$.*

(i) If $p > 1$, then condition $1/p - 1/q = \alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of the operator I_α from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.

(ii) If $p = 1$, then condition $1 - 1/q = \alpha/(n-\lambda)$ s necessary and sufficient for the boundedness of the operator I_α from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$.

If $\lambda = 0$, then the statement of Theorems 1 and 2 reduces to the well known Hardy-Littlewood-Sobolev inequality.

When $m \ge 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, in [6] was find out $\mathcal{M}_{\Omega,m}$ also have the same properties by providing the following multi-version result of the Chiarenza and Frasca [2].

Theorem 3. *[6] Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \dots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \quad for \quad 0 \le \lambda_j < n. \tag{1}$$

(i) If $p > s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{L^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{WL^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

Lemma 1. *[11] Let $0 < \alpha < mn$, $1 \le s' < mn/\alpha$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$ and $f \in L^{p_1}(\mathbb{R}^n) \times \ldots \times L^{p_m}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ for any $x \in \mathbb{R}^n$*

$$\left| I_{\Omega,\alpha,m}\boldsymbol{f}(x) \right| \le C \Big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\boldsymbol{f}(x) \Big]^{\frac{1}{2}} \Big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\boldsymbol{f}(x) \Big]^{\frac{1}{2}}. \tag{2}$$

Lemma 2. *[7] Let $0 < \alpha < mn$, $1 \le s' < mn/\alpha$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \dots, p_m > 1$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \times \ldots \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$*

$$\mathcal{M}_{\Omega,\alpha,m}\boldsymbol{f}(x) \le C_0 \prod_{j=1}^{m} \Big[M_{\frac{\alpha s'}{m}}(f_j^{s'})(x) \Big]^{\frac{1}{s'}} \le C_0 \prod_{j=1}^{m} \Big[M_{\frac{\alpha s' p_j}{mp}}(f_j^{\frac{s' p_j}{p}})(x) \Big]^{\frac{p}{s' p_j}}, \tag{3}$$

where $C_0 = \frac{\|\Omega\|_{L_s(\mathbb{S}^{mn-1})}}{(mn)^{\frac{1}{s}}}$.

When $m \geq 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, we find out $I_{\Omega,\alpha,m}$ also have the same properties by providing the following multi-version of the Theorem 2.

Theorem 4. *Let* $0 < \alpha < mn$, $1 < s \leq \infty$ *and* $\Omega \in L^s(\mathbb{S}^{mn-1})$. *Let also* $\sum_{j=1}^m \frac{\lambda_j}{p_j} = \frac{\lambda}{p}$, $\frac{1}{p_j} - \frac{1}{q_j} = \frac{\alpha}{m(n-\lambda_j)}$ *and* $0 \leq \lambda_j < n - \frac{\alpha p_j}{m}$, $j = 1, \ldots, m$.

(i) If $p > s'$ *and* $\sum_{j=1}^m \frac{\lambda_j}{q_j} = \frac{\lambda}{q}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* $I_{\Omega,\alpha,m}$ *from product Morrey space* $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$. *Moreover, there exists a positive constant* C *such that for all* $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|I_{\Omega,\alpha,m}\boldsymbol{f}\|_{L^{q,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = s'$ *and* $\lambda \sum_{j=1}^m \frac{1}{p_j q_j} = \sum_{j=1}^m \frac{\lambda_j}{p_j q_j}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* $I_{\Omega,\alpha,m}$ *from product Morrey space* $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to the weak Morrey space* $WL^{q,\lambda}(\mathbb{R}^n)$. *Moreover, there exists a positive constant* C *such that for all* $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|I_{\Omega,\alpha,m}\boldsymbol{f}\|_{WL^{q,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

Proof.

(i) *Sufficiency.* Following the method used in [3], we choose a small positive number ε with $0 < \varepsilon < \min\{\alpha, \frac{m(n-\lambda_j)}{p_j} - \alpha, \frac{n-\lambda}{p} - \alpha\}$. One can then see from the condition of Theorem 4 that $1 \leq s' < p_j < \frac{m(n-\lambda_j)}{\alpha+\varepsilon}$ and $1 \leq s' < p_j < \frac{m(n-\lambda_j)}{\alpha-\varepsilon}$, and we let

$$\frac{1}{\widetilde{q}_1} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - \frac{\alpha+\varepsilon}{n-\lambda} = \frac{1}{p} - \frac{\alpha+\varepsilon}{n-\lambda},$$

and

$$\frac{1}{\widetilde{q}_2} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - \frac{\alpha-\varepsilon}{n-\lambda} = \frac{1}{p} - \frac{\alpha-\varepsilon}{n-\lambda}.$$

Now if each $p_j > s'$, then from [7], Theorem 1.1(i) implies that

$$\|\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}\|_{L^{q,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}, \quad \|\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}\|_{L^{q,\lambda}} \quad \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

A simple calculation yields $\frac{q}{2\widetilde{q}_1} + \frac{q}{2\widetilde{q}_2} = 1$. Hence, using Lemma 1, the Holder inequality and the above inequalities, we have

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda} \int_{B(x,t)} |I_{\Omega,\alpha,m} f(y)|^q dy\right)^{1/q}$$

$$\leq C \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{B(x,t)}\Big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\Big]^{\frac{q}{2}}\Big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\Big]^{\frac{q}{2}}dy\Big)^{\frac{1}{q}}$$
$$\leq C \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{B(x,t)}\Big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\Big]^{\widetilde{q}_1}dy\Big)^{\frac{1}{2\widetilde{q}_1}} \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\Big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\Big]^{\widetilde{q}_2}dy\Big)^{\frac{1}{2\widetilde{q}_1}}$$
$$\leq C\|\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}\|^{1/2}_{L^{\widetilde{q}_1,\lambda}}\|\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}\|^{1/2}_{L^{\widetilde{q}_2,\lambda}} = C\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}},$$

Necessity. Suppose that $I_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L_{q,\lambda}(\mathbb{R}^n)$. Define $\mathbf{f}_\varepsilon(x) = \big(f_1(\varepsilon x),\ldots,f_m(\varepsilon x)\big)$ for $\varepsilon > 0$. Then it is easy to show that

$$I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y) = \varepsilon^{-\alpha}I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y). \tag{4}$$

Thus

$$\begin{aligned}\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{L^{q,\lambda}} &= \varepsilon^{-\alpha}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{B(x,t)}|I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)|^q dy\Big)^{1/q}\\ &= \varepsilon^{-\alpha-n/q}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{B(\varepsilon x,\varepsilon t)}|I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{1/q}\\ &= \varepsilon^{-\alpha-n/q+\lambda/q}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{(\varepsilon t)^\lambda}\int_{B(\varepsilon x,\varepsilon t)}|I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{1/q}\\ &= \varepsilon^{-\alpha-(n-\lambda)/q}\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}}.\end{aligned}$$

Since $I_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}\times\ldots\times L^{p_m,\lambda_m}$ to $L^{q,\lambda}$, we have

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = \varepsilon^{\alpha+(n-\lambda)/q}\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{L^{q,\lambda}} \leq C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m\|f_j(\varepsilon\cdot)\|_{L^{p_j,\lambda_j}}$$
$$= C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^{\lambda_j}}\int_{B(x,t)}|f_j(\varepsilon y)|^{p_j}dy\Big)^{1/p_j}$$

$$= C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m\varepsilon^{-n/p_j}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)}|f_j(y)|^{p_j}dy\Big)^{1/p_j}$$
$$= C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m\varepsilon^{(\lambda_j-n)/p_j}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{(\varepsilon t)^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)}|f_j(y)|^{p_j}dy\Big)^{1/p_j}$$
$$= C\varepsilon^{\alpha+(n-\lambda)/q-(n-\lambda)/p}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}},$$

where C is independent of ε.

If $(n-\lambda)/p < (n-\lambda)/q + \alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = 0$ as $\varepsilon \to 0$.

If $(n-\lambda)/p > (n-\lambda)/q + \alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = 0$ as $\varepsilon \to \infty$.

Therefore we get $(n-\lambda)/p = (n-\lambda)/q + \alpha$.

(ii) *Sufficiency.* If $p_i = s'$ for some i, we take $\eta^2 = \beta^{2-\frac{q}{\widetilde{q}_2}}\Big(\prod\limits_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\Big)^{\frac{q}{\widetilde{q}_2}-1}$ for any $\beta > 0$, then applying Lemma 1 and Theorem 4 in [7], we get

$$\begin{aligned}
&\Big|\big\{y \in B(x,t) : \big|I_{\Omega,\alpha,m}\mathbf{f}(y)\big| > \beta\big\}\Big| \\
&\le C\Big|\big\{y \in B(x,t) : C\big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\big]^{\frac{1}{2}}\big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\big]^{\frac{1}{2}} > \beta\big\}\Big| \\
&\le C\Big|\big\{y \in B(x,t) : \sqrt{C}\big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\big]^{\frac{1}{2}} > \eta\big\}\Big| \\
&+ \Big|\big\{y \in B(x,t) : \sqrt{C}\big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\big]^{\frac{1}{2}} > \beta/\eta\big\}\Big| \\
&\le C\Big|\big\{y \in B(x,t) : \mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y) > C\eta^2\big\}\Big| + \Big|\big\{y \in B(x,t) : \mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y) > C\beta^2/\eta^2\big\}\Big| \\
&= Ct^{\lambda}\Big[\Big(\frac{1}{\eta^2}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\Big)^{\widetilde{q}_1} + \Big(\frac{\eta^2}{\beta^2}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\Big)^{\widetilde{q}_2}\Big] \\
&= Ct^{\lambda}\left(\frac{1}{\beta}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\right)^{q}.
\end{aligned}$$

Hence, we obtain the following inequality

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} &= \sup_{\beta>0}\beta \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^{\lambda}}\Big|\Big\{y \in B(x,t) : |I_{\Omega,\alpha,m}\mathbf{f}(y)| > \beta\Big\}\Big|\Big)^{\frac{1}{p}} \\
&\le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.
\end{aligned}$$

This is the conclusion (ii) of Theorem 4.

Necessity. Suppose that $I_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $WL_{q,\lambda}(\mathbb{R}^n)$. From equality (4) we get

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}_{\varepsilon}\|_{WL^{q,\lambda}} &= \sup_{\tau>0}\tau \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^{\lambda}}\int_{\big\{y\in B(x,t):I_{\Omega,\alpha,m}\mathbf{f}_{\varepsilon}(y)>\tau\big\}} dy\Big)^{1/q} \\
&= \sup_{\tau>0}\tau \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^{\lambda}}\int_{\big\{y\in B(x,t):I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^{\alpha}\big\}} dy\Big)^{1/q}
\end{aligned}$$

$$= \varepsilon^{-\frac{n}{q}} \sup_{\tau>0} \tau \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^\lambda} \int_{\{y\in B(x,\varepsilon t): I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\}} dy \Big)^{1/q}$$
$$= \varepsilon^{-\alpha-\frac{n}{q}+\frac{\lambda}{q}} \sup_{\tau>0} \tau\varepsilon^\alpha \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{(\varepsilon t)^\lambda} \int_{\{y\in B(x,\varepsilon t): I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\}} dy \Big)^{1/q}$$
$$= \varepsilon^{-\alpha-(n-\lambda)/q} \|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}}.$$

By the boundedness of the operator $I_{\Omega,\alpha,m}$ from $L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$ to $WL^{q,\lambda}$, we have

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} &= \varepsilon^{\alpha+(n-\lambda)/q} \|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{WL^{q,\lambda}} \\
&\le C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \|f_j(\varepsilon\cdot)\|_{L^{p_j,\lambda_j}} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^{\lambda_j}} \int_{B(x,t)} |f_j(\varepsilon y)|^{p_j} dy \Big)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \varepsilon^{-n/p_j} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^{\lambda_j}} \int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy \Big)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \varepsilon^{(\lambda_j-n)/p_j} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{(\varepsilon t)^{\lambda_j}} \int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy \Big)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q-(n-\lambda)/p} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}},
\end{aligned}$$

where C is independent of ε.

If $(n-\lambda)/p < (n-\lambda)/q + \alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} = 0$ as $\varepsilon \to 0$.

If $(n-\lambda)/p > (n-\lambda)/q + \alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} = 0$ as $\varepsilon \to \infty$.

Therefore we get $(n-\lambda)/p = (n-\lambda)/q + \alpha$.

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On Morrey type Spaces and Some Properties

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Abstract. Subspace $M_\rho^{p,\alpha}$ of the weighted Morrey -type space $L_\rho^{p,\alpha}$ is defined, it is proved that infinitely differentiable functions are dense in it. An approximation properties of the Poisson kernel is studied in $M_\rho^{p,\alpha}$. A sufficient condition for belonging of the product to the space $M_\rho^{p,\alpha}$ is obtained. It is proved that $M_\rho^{p,\alpha}$ is an invariant subspace of a singular integral operator.

Key Words and Phrases: Morrey-type classes, Minkowski inequality, Poisson kernel.

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1. Introduction

The concept of Morrey space was introduced by C. Morrey [1] in 1938 in the study of qualitative properties of the solutions of elliptic type equations with BMO (Bounded Mean Oscillations) coefficients (see also [2, 3]). This space provides a large class of weak solutions to the Navier-Stokes system [4]. In the context of fluid dynamics, Morrey-type spaces have been used to model the fluid flow in case where the vorticity is a singular measure supported on some sets in R^n [5]. There appeared lately a large number of research works which considered many problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc in Morrey-type spaces (for more details see [2-26]). It should be noted that the matter of approximation in Morrey-type spaces has only started to be studied recently (see, e.g., [11, 12, 16, 17]), and many problems in this field are still unsolved. This work is just dedicated to this field. Subspace $M_\rho^{p,\alpha}$ of the weighted Morrey -type space $L_\rho^{p,\alpha}$ is defined, it is proved that infinitely differentiable functions are dense in it. An approximation properties of the Poisson kernel is studied in $M_\rho^{p,\alpha}$. A sufficient condition for belonging of the product to the space $M_\rho^{p,\alpha}$ is obtained. It is proved that $M_\rho^{p,\alpha}$ is an invariant subspace of a singular integral operator .

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2. Needful Information

We will need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$. All the constants throughout this paper (can be different in different places) will be denoted by c.

The expression $f(x) \sim g(x)$, $x \in M$, means

$$\exists \delta > 0 : \delta \leq \left| \frac{f(x)}{g(x)} \right| \leq \delta^{-1}, \ \forall x \in M.$$

Similar meaning is intended by the expression $f(x) \sim g(x)$, $x \to a$.

By Morrey-Lebesgue space $L^{p,\,\alpha}(\Gamma)$, $0 < \alpha \leq 1$, $p \geq 1$, we mean the normed space of all measurable functions $f(\cdot)$ on Γ with the finite norm

$$\|f\|_{L^{p,\,\alpha}(\Gamma)} = \sup_B \left(\left| B \bigcap \Gamma \right|_\Gamma^{\alpha-1} \int_{B \bigcap \Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty.$$

$L^{p,\,\alpha}(\Gamma)$ is a Banach space with $L^{p,\,1}(\Gamma) = L_p(\Gamma)$, $L^{p,\,0}(\Gamma) = L_\infty(\Gamma)$. Similarly we define the weighted Morrey-Lebesgue space $L_\mu^{p,\,\alpha}(\Gamma)$ with the weight function $\mu(\cdot)$ on Γ equipped with the norm

$$\|f\|_{L_\mu^{p,\,\alpha}(\Gamma)} = \|f\mu\|_{L^{p,\,\alpha}(\Gamma)}, f \in L_\mu^{p,\,\alpha}(\Gamma).$$

The inclusion $L^{p,\,\alpha_1}(\Gamma) \subset L^{p,\,\alpha_2}(\Gamma)$ is valid for $0 < \alpha_1 \leq \alpha_2 \leq 1$. Thus, $L^{p,\,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall \alpha \in (0,\,1]$, $\forall p \geq 1$. For $\Gamma = [-\pi,\,\pi]$ we will use the notation $L^{p,\,\alpha}(-\pi,\,\pi) = L^{p,\,\alpha}$.

More details on Morrey-type spaces can be found in [2-26].

In the sequel, we will need some auxiliary facts. Recall Minkowski's (integral) inequality.

Let $(X;\, \mathrm{A}_x;\, \nu)$ and $(Y;\, \mathrm{A}_y;\, \mu)$ be measurable spaces with $\sigma-$finite measures ν and μ, respectively. If $F(x;\, y)$ is $\nu \times \mu-$measurable, then we have

$$\left\| \int_X F(\cdot;\, y)\, d\nu(x) \right\|_{L^p(d\mu)} \leq \int_X \|F(x;\, \cdot)\|_{L^p(d\mu)} \, d\nu(x), 1 \leq p < +\infty,$$

where

$$\|g(y)\|_{L^p(d\mu)} = \left(\int_Y |g(y)|^p \, d\mu \right)^{1/p}.$$

Now let $Y \equiv R$ and $\mu(\cdot)$ be a Borel measure on R. We have

$$\left(\int_I \left| \int_X F(x;\, y)\, d\nu(x) \right|^p d\mu(y) \right)^{1/p} \leq$$

$$\leq \int_X \left(\int_I |F(x;\, y)|^p \, d\mu(y) \right)^{1/p} d\nu(x) .$$

Thus

$$\left(\frac{1}{|I|^{1-\alpha}} \int_I \left| \int_X F(x;\, y)\, d\nu(x) \right|^p d\mu(y) \right)^{1/p} \leq$$

$$\leq \int_X \left(\frac{1}{|I|^{1-\alpha}} \int_I |F(x;\, y)|^p \, d\mu(y) \right)^{1/p} d\nu(x) \leq$$

$$\leq \int_X \|F(x;\, y)\|_{L^{p,\,\alpha}(d\mu)} \, d\nu(x) \,, \forall I \subset R.$$

Taking sup over $I \subset R$, we get

$$\left\| \int_X F(x;\, y)\, d\nu(x) \right\|_{L^{p,\,\alpha}(d\mu)} \leq \int_X \|F(x;\, y)\|_{L^{p,\,\alpha}(d\mu)} \, d\nu(x) . \tag{1}$$

So the Minkowski inequality (1) holds in the Morrey-type space $L^{p,\,\alpha}(d\mu)$.

Thus, the following Minkowski's inequality regarding Morrey type spaces is true.

Statement 1. *Let* $(X;\, \mathrm{A}_x;\, \nu)$ *be a measurable space with a* $\sigma-$*finite measure* ν *and* $(R;\, \mathrm{B};;\, \mu\,)$ *be a measurable space with a Borel measure* μ *on* $\sigma-$*algebra of Borel sets* B *of* R. *Then, for* $\nu \times \mu-$ *measurable function* $F(x;\, y)$, *the following analog of Minkowski's inequality is valid*

$$\left\| \int_X F(x;\, \cdot)\, d\nu(x) \right\|_{L^{p,\,\alpha}(d\mu)} \leq \int_X \|F(x;\, \cdot)\|_{L^{p,\,\alpha}(d\mu)} \, d\nu(x) .$$

By S_Γ we denote the following singular integral operator

$$(S_\Gamma f)(\tau) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)\, d\zeta}{\zeta - \tau}, \quad \tau \in \Gamma,$$

where $\Gamma \subset C$ is some rectifiable curve on complex plane C. Let $\omega = \{z \in C : |z| < 1\}$ be the unit disk on C and $\partial\omega = \gamma$ be its boundary. Define the Morrey-Hardy space $H_+^{p,\,\alpha}$ of analytic functions $f(z)$ inside ω equipped with the following norm

$$\|f\|_{H_+^{p,\,\alpha}} = \sup_{0<r<1} \left\| f\left(re^{it}\right) \right\|_{L^{p,\,\alpha}} .$$

In what follows, we assume that the function $f(\cdot)$ periodically continued on the whole axis R .

We will also use the following concepts. Let $\Gamma \subset C$ be some bounded rectifiable curve, $t = t(\sigma)$, $0 \leq \sigma \leq 1$, be its parametric representation with respect to the arc length σ, and l be the length of Γ. Let $d\mu(t) = d\sigma$, i.e. let $\mu(\cdot)$ be a linear measure on Γ. Let

$$\Gamma_t(r) = \{\tau \in \Gamma : |\tau - t| < r\} \,, \Gamma_{t(s)}(r) = \{\tau(\sigma) \in \Gamma : |\sigma - s| < r\} .$$

It is absolutely clear that $\Gamma_{t(s)}(r) \subset \Gamma_t(r)$.

Definition 1. *Curve Γ is said to be Carleson if $\exists c > 0$:*

$$\sup_{t\in\Gamma} \mu\left(\Gamma_t\left(r\right)\right) \le cr, \forall r > 0.$$

Curve Γ is said to satisfy the chord-arc condition at the point $t_0 = t\left(s_0\right) \in \Gamma$ if there exists a constant $m > 0$ independent of t such that $|s - s_0| \le m\left|t\left(s\right) - t\left(s_0\right)\right|$, $\forall t\left(s\right) \in \Gamma$. Γ satisfies a chord-arc condition uniformly on Γ if $\exists m > 0 : |s - \sigma| \le m\left|t\left(s\right) - t\left(\sigma\right)\right|$, $\forall t\left(s\right), t\left(\sigma\right) \in \Gamma$.

Let's state the following lemma of [10] which is of independent interest.

Lemma 1. *[10] Let Γ be a bounded rectifiable curve. If the power function $|t - t_0|^\gamma$, $t_0 \in \Gamma$, belongs to the space $L^{p,\alpha}\left(\Gamma\right)$, $1 \le p < \infty$, $0 < \alpha < 1$, then the inequality $\gamma \ge -\frac{\alpha}{p}$ holds. If Γ is a Carleson curve, then this condition is also sufficient.*

We will extensively use the following theorem of N.Samko [10].

Theorem 1. *[10] Let the curve Γ satisfy the chord-arc condition and the weight $\rho\left(\cdot\right)$ be defined by*

$$\rho\left(t\right) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}\, ;\, \{t_k\}_1^m \subset \Gamma, t_i \ne t_j, i \ne j. \tag{2}$$

A singular operator S_Γ is bounded in the weighted space $L_\rho^{p,\alpha}\left(\Gamma\right)$, $1 < p < +\infty$, $0 < \alpha \le 1$, if the following inequalities are satisfied

$$-\frac{\alpha}{p} < \alpha_k < -\frac{\alpha}{p} + 1, k = \overline{1, m}. \tag{3}$$

Moreover, if Γ is smooth in some neighborhoods of the points t_k, $k = \overline{1, m}$, then the validity of inequalities (3) *is necessary for the boundedness of the operator S_Γ in $L_\rho^{p,\alpha}\left(\Gamma\right)$.*

In what follows, as Γ we will consider a boundary of unit disk: $\gamma = \partial\omega$. Consider the weighted space $L_\rho^{p,\alpha}\left(\gamma\right) =: L_\rho^{p,\alpha}$ with the weight $\rho\left(\,\cdot\,\right)$. In an absolutely similar way to the non-weighted case, we define the space $M_\rho^{p,\alpha}$ with the weight $\rho\left(\cdot\right)$. Denote by $\tilde{M}_\rho^{p,\alpha}$ the set of functions whose shifts are continuous in $L_\rho^{p,\alpha}$, i.e.

$$\|S_\delta f - f\|_{p,\alpha;\rho} = \|f\left(\cdot + \delta\right) - f\left(\cdot\right)\|_{p,\alpha;\rho} \to 0, \delta \to 0,$$

where S_δ is a shift operator: $\left(S_\delta f\right)\left(x\right) = f\left(x + \delta\right)$ and we will consider that the function $f\left(\cdot\right)$ (in sequel also) periodically continued to the whole real axis R. It is not difficult to see that $\tilde{M}_\rho^{p,\alpha}$ is a linear subspace of $L_\rho^{p,\alpha}$. Denote the closure of $\tilde{M}_\rho^{p,\alpha}$ in $L_\rho^{p,\alpha}$ by $M_\rho^{p,\alpha}$.

Consider the following class

$$\tilde{L}_\rho^{p,\alpha} =: \left\{f \in L_\rho^{p,\alpha} : \|T_\delta f - f\|_{p,\alpha;\rho} \to 0,\, \delta \to 0\right\}.$$

It is evident that the class $\tilde{L}_\rho^{p,\alpha}$ is a linear subspace of $L_\rho^{p,\alpha}$. Let us denote by $M_\rho^{p,\alpha}$ the closure of $\tilde{L}_\rho^{p,\alpha}$ in $L_\rho^{p,\alpha}$.

Let us remember the following properties of Poisson kernel $P_r(\varphi)$:

$$P_r(\varphi) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\varphi + r^2}\ ,\ 0 < r < 1.$$

(a) $\sup\limits_{|t|<\delta} P_r(t) \to 0$ as $|r| \to 1$;

(b) $\int_{|t|>\delta} P_r(t)\,dt \to 0$ as $|r| \to 1$ for $\forall \delta > 0$.

These properties directly follows from the expression for $P_r(\varphi)$. We have

$$\|(P_r * f)(\cdot) - f(\cdot)\|_{p,\alpha;\rho} = \left\| \frac{1}{2\pi} \int_{-\pi} P_r(t) f(t-s)\,dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(s)\,dt \right\|_{p,\alpha;\rho} \le$$

$$\le \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt =$$

$$= \frac{1}{2\pi} \left[\int_{|t|>\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt + \int_{|t|<\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt \right].$$

Regarding the second integral in the right-hand side, we have

$$\frac{1}{2\pi} \int_{|t|<\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt \le$$

$$\le \sup_{|t|<\delta} \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,,\quad \text{as}\quad \delta \to 0.$$

To estimate the first integral, consider

$$\|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho} \le \|f\|_{p,\alpha;\rho} + \|f(t-\cdot)\|_{p,\alpha;\rho} = 2\|f\|_{p,\alpha;\rho}\,.$$

We have

$$\|f(t-\cdot)\|^p_{p,\alpha;\rho} = \sup_B \frac{1}{|B\bigcap\gamma|^{1-\alpha}} \int_{B\bigcap\gamma} |f(t-s)|^p\,ds =$$

$$= \sup_B \frac{1}{|B\bigcap\gamma|^{1-\alpha}} \int_{(B\bigcap\gamma)_t} |f(s)|^p\,ds,$$

where $(B\bigcap\gamma)_t \equiv \{s : t-s \in B\bigcap\gamma\}$. It is clear that

$$\left|B\bigcap\gamma\right| = \left|\left(B\bigcap\gamma\right)_t\right|,$$

holds. Therefore

$$\|f(t-\cdot)\|_{p,\alpha;\rho} = \|f\|_{p,\alpha;\rho}\,.$$

As a result

$$\int_{|t|>\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt \le$$

$$\le 2\|f\|_{p,\alpha;\rho} \int_{|t|>\delta} P_r(t)\,dt \to 0 \text{ as } |r| \to 1.$$

So we have proved the following

Theorem 2. *If* $f \in M_\rho^{p,\alpha}$, $1 \le p < +\infty \wedge 0 \le \alpha \le 1$, *then* $\|P_r * f - f\|_{p,\alpha;\rho} \to 0$ *as* $|r| \to 1$.

From Theorem 2 we immediately get the validity of the following

Theorem 3. *Let* $f \in M_\rho^{p,\alpha}$, $0 < \alpha \le 1$, $1 < p < +\infty$. *Then it holds*

$$\left\|(Kf)(r\xi) - f^+(\xi)\right\|_{p,\alpha;\rho} \to 0,\ r \to 1-0.$$

Similar assertion is true in case of $f^-(\xi)$ when $r \to 1+0$.

3. Subspace $M_\rho^{p,\alpha}$

Let $\rho : [-\pi, \pi] \to (0, +\infty)$ be some weight function and consider the space $M_\rho^{p,\alpha}$. It is easy to see that if $\rho \in L^{p,\alpha}$, then $C[-\pi, \pi] \subset M_\rho^{p,\alpha}$ is true. Indeed, let $f \in C[-\pi, \pi]$. Without loss of generality, we assume that the function f periodically continued on the whole axis. We have

$$|f(x+\delta) - f(x)| \le \|f(\cdot+\delta) - f(\cdot)\|_\infty \to 0, \delta \to 0.$$

Consequently

$$\|f(\cdot+\delta) - f(\cdot)\|_{p,\alpha;\rho} = \|(f(\cdot+\delta) - f(\cdot))\rho(\cdot)\|_{p,\alpha} \le$$

$$\le \|f(\cdot+\delta) - f(\cdot)\|_\infty \|\rho(\cdot)\|_{p,\alpha} \to 0, \delta \to 0.$$

Hence, we have $f \in M_\rho^{p,\alpha}$.

Let us show that the set of infinitely differentiable functions is dense in $M_\rho^{p,\alpha}$. Consider the following averaged function

$$\omega_\varepsilon(t) = \begin{cases} c_\varepsilon \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |t|^2}\right), & |t| < \varepsilon, \\ 0, & |t| \ge \varepsilon, \end{cases}$$

where

$$c_\varepsilon \int_{-\infty}^{+\infty} \omega_\varepsilon(t)\, dt = 1.$$

Take $\forall f \in M_\rho^{p,\alpha}$ and consider the convolution $f * g$:

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(t-s)\, g(s)\, ds,$$

and let

$$f_\varepsilon(t) = (\omega_\varepsilon * f)(t) = (f * \omega_\varepsilon)(t).$$

It is clear that f_ε is infinitely differentiable on $[-\pi,\, \pi]$. We have

$$\|f_\varepsilon - f\|_{p,\,\alpha;\,\rho} = \left\| \int_{-\infty}^{+\infty} \omega_\varepsilon(s)\, f(\cdot - s)\, ds - f(\cdot) \right\|_{p,\,\alpha;\,\rho} =$$

$$= \left\| \int_{-\infty}^{+\infty} \omega_\varepsilon(s) \left[f(\cdot - s) - f(\cdot) \right] ds \right\|_{p,\,\alpha;\,\rho}.$$

Applying Minkowski inequality (1) to this expression, we obtain

$$\|f_\varepsilon - f\|_{p,\,\alpha;\,\rho} \le \int_{-\infty}^{+\infty} \omega_\varepsilon(s) \|f(\cdot - s) - f(\cdot)\|_{p,\,\alpha;\,\rho}\, ds =$$

$$= \int_{-\varepsilon}^{\varepsilon} \omega_\varepsilon(s) \|f(\cdot - s) - f(\cdot)\|_{p,\,\alpha;\,\rho}\, ds \le$$

$$= \sup_{|s| \le \varepsilon} \|f(\cdot - s) - f(\cdot)\|_{p,\,\alpha;\,\rho} \to 0, \varepsilon \to 0.$$

Thus, the following theorem is true.

Theorem 4. *Let $\rho \in L^{p,\,\alpha}$, $1 < p < +\infty$, $0 < \alpha \le 1$. Then infinitely differentiable functions are dense in $M_\rho^{p,\,\alpha}$.*

Consider the singular operator $S(\cdot)$:

$$Sf(t) = \frac{1}{\pi} \int_\gamma \frac{f(\tau)\, d\tau}{\tau - t}, t \in \gamma.$$

Applying Theorem 1 [10] to the operator S we obtain the following result.

Theorem 5. *Let the weight $\rho(\cdot)$ be defined by the expression (2), where $\Gamma = \gamma$. Then the operator S is bounded in $L_\rho^{p,\,\alpha}$,$1 < p < +\infty$, $0 < \alpha \le 1$, i.e. the following inequality holds*

$$\|Sf\|_{p,\,\alpha;\,\rho} \le c\, \|f\|_{p,\,\alpha;\,\rho}, \forall f \in L_\rho^{p,\,\alpha},$$

if and only if the following inequalities are fulfilled

$$-\frac{\alpha}{p} < \alpha_k < -\frac{\alpha}{p} + 1, k = \overline{1,\, m}. \tag{4}$$

Let us show that the subspace $M_\rho^{p,\,\alpha}$ is an invariant with respect to the operator S. It is sufficient to prove that the shift operator S is continuous in $M_\rho^{p,\,\alpha}$. So, let $f \in M_\rho^{p,\,\alpha}$ and $\delta \in R$. Consider the shift operator S:

$$(Sf)\left(te^{i\delta}\right) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau)\, d\tau}{\tau - te^{i\delta}}, t \in \gamma.$$

We have

$$(Sf)\left(te^{i\delta}\right) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau)\, d\tau e^{-i\delta}}{\tau e^{-i\delta} - t} =$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f\left(\tau e^{-i\delta}e^{i\delta}\right)d\tau e^{-i\delta}}{\tau e^{-i\delta}-t} = \frac{1}{2\pi i}\int_\gamma \frac{f\left(\xi e^{i\delta}\right)d\xi}{\xi - t}.$$

Consequently

$$(Sf)\left(te^{i\delta}\right) - (Sf)(t) = \frac{1}{2\pi i}\int_\gamma \frac{f\left(\xi e^{i\delta}\right) - f(\xi)}{\xi - t}d\xi =$$

$$= \left(S\left(f\left(\cdot e^{i\delta}\right) - f(\cdot)\right)\right)(t).$$

Paying attention to Theorem 5, hence we immediately obtain

$$\left\|(Sf)\left(te^{i\delta}\right) - (Sf)(t)\right\|_{p,\,\alpha;\,\rho} \leq$$

$$\leq c\left\|f\left(e^{i\delta}\right) - f(\cdot)\right\|_{p,\,\alpha;\,\rho} \to 0, \delta \to 0,$$

as $f \in M_\rho^{p,\,\alpha}$. Thus, the following theorem is true.

Theorem 6. *Let the weight ρ be defined by the expression*

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}, t \in \gamma, \tag{5}$$

where $\{t_k\}_{k=\overline{1,\,m}} \subset \gamma-$are different points. If the inequalities (4) hold, then the operator S boundedly acts in $M_\rho^{p,\,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$.

Remark 1. *In previous statements and in their proofs the spaces $L_\rho^{p,\,\alpha}$, $M_\rho^{p,\,\alpha}(-\pi,\,\pi)$ and $L_\rho^{p,\,\alpha}$, $M_\rho^{p,\,\alpha}$, are naturally identified, respectively, i.e. $L_\rho^{p,\,\alpha} = L_\rho^{p,\,\alpha}(-\pi,\pi) = L_\rho^{p,\,\alpha}(\gamma)$ & $M_\rho^{p,\,\alpha} = M_\rho^{p,\,\alpha}(-\pi,\pi) = M_\rho^{p,\,\alpha}(\gamma)$.*

Consider the following Cauchy integral

$$(Kf)(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi - z}, z \notin \gamma.$$

Let $f \in L_\rho^{p,\,\alpha}$, where the weight $\rho(\cdot)$ is defined by the expression (5). Applying Holder's inequality we obtain

$$\|f\|_{L_1} = \left\|f\rho\rho^{-1}\right\|_{L_1} \leq c\,\|f\rho\|_{p,\,\alpha}\left\|\rho^{-1}\right\|_{q,\,\alpha} =$$

$$= c\,\|f\|_{p,\,\alpha;\,\rho}\left\|\rho^{-1}\right\|_{q;\,\alpha}. \tag{6}$$

Suppose that the following inequalities are fulfilled

$$-\frac{\alpha}{p} < \alpha_k < \frac{\alpha}{q}, k = \overline{1,\,m}, \frac{1}{p} + \frac{1}{q} = 1. \tag{7}$$

Then from (6) it follows that $f \in L_1$. As a result, according to the classical facts, the following Sokhotskii-Plemelj is true

$$f^{\pm}(\xi) = \pm\frac{1}{2}f(\xi) + (Sf)(\xi), \xi \in \gamma, \tag{8}$$

where $f^{+}(\xi)$ (respectively, $f^{-}(\xi)$) boundary values of a Cauchy integral $(Kf)(z)$ on γ inside ω (outside ω). Paying attention to Theorem 6, form (8) we obtain that if $f \in M_{\rho}^{p,\alpha}$, then $f^{\pm} \in M_{\rho}^{p,\alpha}$ and the following inequality holds.

$$\left\|f^{\pm}\right\|_{p,\alpha;\rho} \leq c\left\|f\right\|_{p,\alpha;\rho}, \forall f \in M_{\rho}^{p,\alpha}. \tag{9}$$

Assume
$K_z(s) = \frac{e^{is}}{e^{is}-z}$ −Cauchy kernel; $P_z(s) = Re\frac{e^{is}+z}{e^{is}-z}$ − Poisson kernel;
$Q_z(s) = Im\frac{e^{is}+z}{e^{is}-z}$ −is the conjugate Poisson kernel,
(Re−is a real part, Im−is an imaginary part). We have

$$K_z(s) = \frac{1}{2} + \frac{1}{2}\left(P_z(s) + iQ_z(s)\right) = \frac{1}{2} + \frac{1}{2}\frac{e^{is}+z}{e^{is}-z}, z \in \omega. \tag{10}$$

Let $F(z) = u(z) + i\vartheta(z)$ be an analytic function in ω. It is clear that $F \in H_{\rho}^{p,\alpha}$ if and only if $u;\vartheta \in h_{\rho}^{p,\alpha}$. Paying attention to the relation (10) we arrive at the conclusion that many of the properties of functions from $h_{\rho}^{p,\alpha}$ transferred to the function from $H_{\rho}^{p,\alpha}$. For example, $\forall F \in H_{\rho}^{p,\alpha}$ has a.e. on γ the nontangential boundary values F^{+}, since, $\lim\limits_{r\to 1-0} F\left(re^{it}\right) = F^{+}\left(e^{it}\right)$, a.e. $t \in [-\pi, \pi]$. We have

$$F^{+}(\tau) = u^{+}(\tau) + i\vartheta^{+}(\tau), \tau \in \gamma.$$

Let all the conditions of Theorem 2 be fulfilled. Then the following representation is true.

$$u\left(re^{i\theta}\right) = \left(P_r * u^{+}\right)(\theta), \vartheta\left(re^{i\theta}\right) = \left(P_r * u^{+}\right)(\theta),$$

and, as a result

$$F\left(re^{i\theta}\right) = \left(P_r * F^{+}\right)(\theta).$$

Paying attention to Theorem 2, we obtain the following result.

Theorem 7. *Let the weight $\rho(\cdot)$ is defined by the expression* (5) *and the inequality* (7) *holds. Then the Sokhotskii-Plemelj formula* (8) *is valid and for the boundary values the inequality* (9) *holds .*

Let us prove the following

Theorem 8. *Let $f(\cdot) \in L_{\infty} \bigcap M_{\rho}^{p,1} \wedge g(\cdot) \in M_{\rho}^{p,\alpha}$ and let the weight function ρ satisfies the following condition*

$$\exists\delta_0 > 0 : \frac{1}{|I|^{1-\alpha}}\int_I \rho^p(t)\,dt \leq c\,|I|^{\delta_0}, \forall I \in [-\pi, \pi].$$

Then $f(\cdot)\,g(\cdot) \in M_{\rho}^{p,\alpha}$ when $0 < \alpha \leq 1$ and $p \geq 1$.

Proof. Let $f(\cdot) \in L_\infty \bigcap M_\rho^{p,1}$ and $g(\cdot) \in M_\rho^{p,\alpha}$, $0 < \alpha \le 1$, $p \ge 1$. For $\alpha = 1$ it is evident that $M_\rho^{p,1} = L_{p,\rho}$ and the following estimation is true

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|^p \rho^p(t)\, dt \le c \int_{-\pi}^{\pi} |g(t)|^p \rho^p(t)\, dt = c\, \|g\|_{p,\rho}^p < +\infty.$$

So, $f(\cdot)\, g(\cdot) \in M_\rho^{p,\alpha}$ when $\alpha = 1$.

Consider the case $0 < \alpha < 1$. We have

$$\sup_I \left(\frac{1}{|I|^{1-\alpha}} \int_I |f(t)\, g(t)|^p \rho^p(t)\, dt \right)^{\frac{1}{p}} \le$$

$$\le c \sup_I \left(\frac{1}{|I|^{1-\alpha}} \int_I |g(t)|^p \rho^p(t)\, dt \right)^{\frac{1}{p}} = c\, \|g\|_{p,\alpha;\rho} < +\infty.$$

Let us consider

$$\Delta_\delta = \|f(\cdot+\delta)\, g(\cdot+\delta) - f(\cdot)\, g(\cdot)\|_{p,\alpha;\rho}.$$

For any $\varepsilon > 0$ and $m > 0$ there is a $\varphi(\cdot) \in C[-\pi;\pi]$ such that $\|g(\cdot) - \varphi(\cdot)\|_{p,\alpha;\rho} < \frac{\varepsilon}{m}$, as $g \in M_\rho^{p,\alpha}$. We have

$$\Delta_\delta = \|f(\cdot+\delta)\,[g(\cdot+\delta) - \varphi(\cdot+\delta) + \varphi(\cdot+\delta)] - f(\cdot)\,[g(\cdot) - \varphi(\cdot) + \varphi(\cdot)]\|_{p,\alpha;\rho} \le$$

$$\le c_f\, \|g(\cdot+\delta) - \varphi(\cdot+\delta)\|_{p,\alpha;\rho} + \|f(\cdot+\delta)\,\varphi(\cdot+\delta) - f(\cdot)\,\varphi(\cdot)\|_{p,\alpha;\rho} + c_f\, \|g(\cdot) - \varphi(\cdot)\|_{p,\alpha;\rho},$$

where $c_f = \|f(\cdot)\|_{L_\infty}$. From $\|g(\cdot) - \varphi(\cdot)\|_{p,\alpha;\rho} < \frac{\varepsilon}{m}$ it follows

$$\|g(\cdot+\delta) - \varphi(\cdot+\delta)\|_{p,\alpha;\rho} \le \|g(\cdot+\delta) - g(\cdot)\|_{p,\alpha;\rho} +$$

$$+ \|g(\cdot) - \varphi(\cdot)\|_{p,\alpha;\rho} + \|\varphi(\cdot) - \varphi(\cdot+\delta)\|_{p,\alpha;\rho}.$$

It is obvious that $\|g(\cdot+\delta) - g(\cdot)\|_{p,\alpha;\rho} \to 0$, $\delta \to 0$.

Let the weight function ρ satisfies the following condition

$$\exists \delta_0 > 0 : \frac{1}{|I|^{1-\alpha}} \int_I \rho^p(t)\, dt \le c\, |I|^{\delta_0}, \forall I \in [-\pi;\pi],$$

where $c > 0$ is some constant.

It follows from uniformly continuity that for $\forall m > 0$, $\varepsilon > 0$ there exists $\delta_1 > 0 : \forall \delta \in (-\delta_1, \delta_1)$:

$$\|\varphi(\cdot) - \varphi(\cdot+\delta)\|_{p,\alpha;\rho} = \sup_I \left(\frac{1}{|I|^{1-\alpha}} \int_I |\varphi(t) - \varphi(t+\delta)|^p \rho^p(t) \right)^{\frac{1}{p}} <$$

$$< \frac{\varepsilon}{m} \sup_I \left(\frac{1}{|I|^{1-\alpha}} \int_I \rho^p(t) \right)^{\frac{1}{p}} < \frac{\varepsilon}{m} \sup_I \left(c\, |I|^{\delta_0} \right)^{\frac{1}{p}} = c \frac{\varepsilon}{m} (2\pi)^{\frac{\delta_0}{p}}.$$

Then the previous inequality implies

$$\Delta_\delta \leq c_f \frac{\varepsilon}{m}\left(2 + c\,(2\pi)^{\frac{\delta_0}{p}}\right) + \|f\,(\cdot+\delta)\,\varphi\,(\cdot+\delta) - f\,(\cdot)\,\varphi\,(\cdot)\|_{p,\,\alpha;\,\rho}\,.$$

Thus, it is suffices to prove that for $\varphi\,(\cdot) \in C\,[-\pi;\,\pi]$ it is true

$$\lim_{\delta\to 0} \|f\,(\cdot+\delta)\,\varphi\,(\cdot+\delta) - f\,(\cdot)\,\varphi\,(\cdot)\|_{p,\,\alpha;\,\rho} = 0.$$

We have

$$\|f\,(\cdot+\delta)\,\varphi\,(\cdot+\delta) - f\,(\cdot)\,\varphi\,(\cdot)\|_{p,\,\alpha;\,\rho} \leq \|f\,(\cdot+\delta)\,[\varphi\,(\cdot+\delta) - \varphi\,(\cdot)]\|_{p,\,\alpha;\,\rho} +$$

$$+\,\|[f\,(\cdot+\delta) - f\,(\cdot)]\,\varphi\,(\cdot)\|_{p,\,\alpha;\,\rho} \leq c_f\,\|\varphi\,(\cdot+\delta) - \varphi\,(\cdot)\|_{p,\,\alpha;\,\rho} + c_\varphi\,\|f\,(\cdot+\delta) - f\,(\cdot)\|_{p,\,\alpha;\,\rho}\,.$$

where $c_\varphi = \|\varphi\,(\cdot)\|_{L_\infty}$. Let us take

$$\Delta_\delta\,(f) = \|f\,(\cdot+\delta) - f\,(\cdot)\|_{p,\,\alpha;\,\rho}\,.$$

Let $\vartheta > 0$ be an arbitrary number. We have

$$\Delta_\delta\,(f) = \max\left\{\Delta_\delta^{(1)}\,(f)\,,\,\Delta_\delta^{(1)}\,(f)\right\},$$

where

$$\Delta_\delta^{(1)}\,(f) = \sup_{I:|I|\leq\vartheta}\left(\frac{1}{|I|^{1-\alpha}}\int_I |f\,(t+\delta) - f\,(t)|^p\,\rho^p\,(t)\,dt\right)^{\frac{1}{p}},$$

$$\Delta_\delta^{(21)}\,(f) = \sup_{I:|I|\leq\vartheta}\left(\frac{1}{|I|^{1-\alpha}}\int_I |f\,(t+\delta) - f\,(t)|^p\,\rho^p\,(t)\,dt\right)^{\frac{1}{p}}.$$

Regarding $\Delta_\delta^{(1)}\,(f)$, we have

$$\Delta_\delta^{(1)}\,(f) \leq 2c_f \sup_{|I|\leq\vartheta}\left(\frac{1}{|I|^{1-\alpha}}\int_I \rho^p\,(t)\,dt\right)^{\frac{1}{p}} \leq$$

$$\leq 2c_f \sup_{|I|\leq\vartheta}\,(|I|)^{\frac{\delta_0}{p}} = \tilde{c}\vartheta^{\frac{\delta_0}{p}}\,.$$

Regarding $\Delta_\delta^{(2)}\,(f)$, we have

$$\Delta_\delta^{(2)}\,(f) \leq \vartheta_\varepsilon^{\frac{\alpha-1}{p}} \sup_{|I|\geq\vartheta}\left(\int_I |f\,(t+\delta) - f\,(t)|^p\,\rho^p\,(t)\,dt\right)^{\frac{1}{p}} \leq$$

$$\leq \vartheta_\varepsilon^{\frac{\alpha-1}{p}}\left(\int_{-\pi}^{\pi} |f\,(t+\delta) - f\,(t)|^p\,\rho^p\,(t)\,dt\right)^{\frac{1}{p}} \leq$$

$$\vartheta_\varepsilon^{\frac{\alpha-1}{p}} \left\| f\left(\cdot+\delta\right)-f\left(\cdot\right)\right\|_{p,\rho}.$$

where $\vartheta=\left(\frac{\varepsilon}{\tilde{c}}\right)^{\frac{p}{\delta_0}}:=\vartheta_\varepsilon$. It is clear that $\exists\delta_2>0$:

$$\left\| f\left(\cdot+\delta\right)-f\left(\cdot\right)\right\|_{p,\rho}<\vartheta_\varepsilon^{\frac{1}{p}}, \forall\delta\in\left(-\delta_2,\,\delta_2\right),$$

where we can choose $\varepsilon_1=\vartheta_\varepsilon^{\frac{1}{p}}$ for any $\varepsilon_1>0$. Hence we get $\Delta_\delta^{(2)}\left(f\right)\leq\left(\frac{\varepsilon}{\tilde{c}}\right)^{\frac{\alpha}{\delta_0}}$.
Now let us take $\Delta_\delta\left(f\right)\leq\max\left\{\varepsilon_2,\,\left(\frac{\varepsilon_2}{\tilde{c}}\right)^{\frac{\alpha}{\delta_0}}\right\}$, where $\varepsilon_2=\frac{\varepsilon}{m}$ for any $m>0$.
Consequently

$$\Delta_\delta\leq\frac{\varepsilon}{m}\left(c_f\left(2+2c\left(2\pi\right)\right)^{\frac{\delta_0}{p}}+c_\varphi\right), \forall\delta\in\left(-\delta_3,\,\delta_3\right),$$

where $\delta_3=\min\left\{\delta_1,\,\delta_2\right\}$. By taking $m=\left(c_f\left(2+2c\left(2\pi\right)\right)^{\frac{\delta_0}{p}}+c_\varphi\right)$, we get $\Delta_\delta\leq\varepsilon$, $\forall\delta\in\left(-\delta_3,\delta_3\right)$. It follows that $\Delta_\delta\to0$ as $\delta\to0$. ◀

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On the Completeness of Double System of Exponents in the Weighted Lebesgue Space

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Abstract. This paper considers double system of exponentials with linear phase in the weighted space $L_{p,\rho}$ with power weight $\rho(\cdot)$ on the segment $[\pi, \pi]$. Under certain conditions on the weight function $\rho(\cdot)$ and on the perturbation parameters, the completeness of this system in $L_{p,\rho}$ is proved. An explicit expression for the biorthogonal system in the case of minimality is derived. The obtained results generalize all previously known results in this direction.

Key Words and Phrases: exponential system, completeness, weighted space.

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1. Introduction

Investigation of many partial differential equation by the application of Fourier method reduces to perturbed trigonometric system of sines (or cosines) of the form

$$\{\sin(nt + \alpha(t))\}_{n \in N}, \tag{1}$$

where $\alpha : [0, \pi] \to R$ is some function (N is a set of natural numbers). Similar problems were studied, for example, in the papers [4, 5, 6, 7, 8, 9, 11]. To justify the Fourier method it is necessary to study the basicity properties (completeness, minimality, basis property, etc.) of these systems in different functional spaces. Complex versions of these systems are perturbed system of exponents of the form

$$\left\{e^{i(nt + \beta(t) sign\, n)}\right\}_{n \in Z}, \tag{2}$$

where $\beta : [-\pi, \pi] \to R$ is some function (Z is the set of integers). Basis properties of the systems (1) and (2) in corresponding spaces are closely linked, in Lebesgue spaces L_p they are well studied by various mathematicians (see, for example [4, 5, 6, 12, 13, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 10]). The case $L_\infty = C[-\pi, \pi]$ is treated in [38]. In connection with application to solution of differential equations, the interest in Lebesgues spaces $L_{p(\cdot)}$ with

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variable summability power $p(\cdot)$ and in Morrey spaces $L^{p,\alpha}$ greatly increased in recent years. Problems of approximation in these spaces have also begun to be studied and basicity problems of the systems (1), (2) in $L_{p(\cdot)}$ are studied in [29, 30], but basicity of the classical system of exponents with linear phase in Morrey spaces are studied in [33, 34]. Note that study of basicity properties of the systems (1), (2) in weighted spaces $L_{p,\rho}$ is equivalent to the study of analogous properties of the systems (1), (2) with corresponding degenerate coefficients in the spaces L_p. For this reason, it can be assumed that the study of the basicity of trigonometric systems in weighted Lebesgue spaces takes its origin from the paper of K.Babenko [18]. Later this area was developed in the works [14, 15, 16, 28, 31, 32, 35, 36]. The problem of basicity of the exponential system in the weighted space $L_{p,\rho} \equiv L_{p,\rho}(-\pi,\pi)$, $1 < p < +\infty$, is solved in the paper [37]. Such a condition is a Muckenhaupt condition with respect to the weight function $\rho(\cdot)$:

$$\sup_I \left(\frac{1}{|I|} \int_I \rho(t)\, dt \right) \left(\frac{1}{|I|} \int_I \rho^{-\frac{1}{p-1}} dt \right)^{p-1} < \infty, \tag{3}$$

where sup is taken over all intervals $I \subset [-\pi,\pi]$ and $|I|$ is the length of the interval I (see e.g. [3]).

In the papers [2, 15] the system (2) is considered in the case when $\beta(t) = \beta t$, where $\beta \in R$ is some real parameter and its basicity in $L_{p,\rho}$, $1 < p < +\infty$, is studied when $\rho(\cdot)$ has the following form

$$\rho(t) = \prod_{k=-r}^{r} |t - t_k|^{\alpha_k} ,$$

where $-\pi = t_{-r} < t_{-r+1} < ... < t_r = \pi$.

The class of weights, satisfying the condition (3), is denoted by A_p. It is easy to see that

$$\rho \in A_p \Leftrightarrow -1 < \alpha_k < p-1, \;\; k = \overline{-r,r}.$$

It is additionally required in [2] that the condition $\alpha_{-r} = \alpha_r$ holds, which means that degeneration must be present at both ends of the segment $[-\pi,\pi]$. This effect does not take place in the paper [15].

In this paper the completeness of the exponential system

$$\left\{ e^{i\left(n+\frac{\beta}{2} sign\, n\right)t} \right\}_{n\in Z} , \tag{4}$$

in the weighted space $L_{p,\rho}$, $1 < p < +\infty$, where $\beta \in C$ is a complex parameter, is studied. Under certain conditions on the parameter β and the weight function $\rho(\cdot)$, the completeness of the system (4) is established in the space $L_{p,\rho}$.

2. Preliminaries. Main lemma

Consider the following double system of exponents

$$\left\{ e^{i[(n+\beta_1)t+\gamma]}; e^{-i[(k+\beta_2)t+\gamma_2]} \right\}_{n\in Z_+; k\in N} , \tag{5}$$

where $\beta_k = Re\beta_k + iIm\beta_k$, $\gamma_k = Re\,\gamma_k + iIm\,\gamma_k$, $k = 1,2$, are complex parameters, $Z_+ = \{0\} \bigcup N$. We assume that the weight function $\rho(\cdot)$ is of the following power form

$$\rho(t) = \prod_{k=-r}^{r} |t - t_k|^{\alpha_k},$$

where $-\pi = t_{-r} < t_{-r+1} < ... < t_0 = 0 < ... < t_r = \pi$, $\{\alpha_k\}_{k=-\overline{r,r}} \subset R$ are some numbers. We consider the weighted space $L_{p,\rho}$, $1 < p < +\infty$, with the norm $\|\cdot\|_{p,\rho}$:

$$\|f\|_{p,\rho} = \left(\int_{-\pi}^{\pi} |f(t)|^p \rho(t)\, dt \right)^{1/p}.$$

It is easy to see that basicity properties of the system (5) in $L_{p,\rho}$ are equivalent to basicity properties of the system

$$\left\{ e^{i(n+\beta_1)t}; e^{-i(k+\beta_2)t} \right\}_{n \in Z_+; k \in N}, \tag{6}$$

in $L_{p,\rho}$. We put $g(t) = e^{\frac{i}{2}(\beta_2 - \beta_1)t}$. It is evident that $\exists \delta > 0$:

$$0 < \delta \le |g(t)| \le \delta^{-1} < +\infty, \;\; \forall t \in [-\pi, \pi].$$

Multiplying the system (6) to the function $g(t)$, we immediately obtain from here that the basicity properties of the system (6) on $L_{p,\rho}$ are equivalent to the basicity properties of the system (4) on $L_{p,\rho}$, $\beta = \beta_1 + \beta_2$. Thus, the study of basicity properties of the system (5) on $L_{p,\rho}$ is reduced to the investigation of corresponding properties with respect to the system (4) on $L_{p,\rho}$.

Let $\beta \in C$ be some complex number. We will assume throughout the paper that $(1+z)^{\beta}$ is some fixed branch of multivalued analytic function $(1+z)^{\beta}$ on the complex plane with the cut along the semiline $(-\infty, -1) \subset R$ on the real axis and take

$$(1+z)^{-\beta} = \frac{1}{(1+z)^{\beta}}.$$

Analogously, we define a branch z^{β} of a multivalued function z^{β} on C with the cut along $(-\infty, 0) \subset R$ and $z^{-\beta} = \frac{1}{z^{\beta}}$.

We will essentially use the following main lemma in the proof of main results.

Lemma 1. *[38] Let $Re\beta > -1$. Then the following Cauchy integral formulas hold*

$$J_m^-(z) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i(\beta+m)\theta} \left(1 + e^{i\theta}\right)^{\beta}}{e^{i\theta} - z} d\theta \equiv \begin{cases} 0, & |z| < 1, \\ -z^{-m-\beta-1} (1+z)^{\beta}, & |z| > 1, \end{cases}$$

$$J_m^+(z) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i(m+1)\theta} \left(1 + e^{i\theta}\right)^{\beta}}{e^{i\theta} - z} d\theta \equiv \begin{cases} 0, & |z| > 1, \\ z^m (1+z)^{\beta}, & |z| < 1, \end{cases}$$

$$\forall m \in Z_+.$$

Consider the following system of functions

$$\vartheta_n^+(t) = \frac{e^{-i\frac{\beta}{2}t}}{2\pi}\left(1+e^{it}\right)^\beta \sum_{k=0}^{n} C_{-\beta}^{n-k} e^{-ikt}, \;\; n \in Z_+;$$
$$\vartheta_m^-(t) = -\frac{e^{-i\frac{\beta}{2}t}}{2\pi}\left(1+e^{it}\right)^\beta \sum_{k=1}^{m} C_{-\beta}^{m-k} e^{ikt}, \;\; m \in N;$$

where

$$C_{-\gamma}^k = \frac{\gamma(\gamma-1)\ldots(\gamma-k+1)}{k!},$$

is a binomial coefficient. Accordingly, we denote

$$e_n^+(t) \equiv e^{i\left(n+\frac{\beta}{2}\right)t}, \;\; n \in Z_+; \;\; e_k^-(t) \equiv e^{-i\left(n+\frac{\beta}{2}\right)t}, \;\; k \in N.$$

Assume that $Re\beta > -1$. The expansion in powers of z of the function $(1+z)^{-\beta} J_m^+(z)$ that is analytic on $|z|<1$ is

$$(1+z)^{-\beta} J_m^+(z) = \sum_{n=0}^{\infty} a_{n;m}^+ z^n,$$

where

$$a_{n;m}^+ = \int_{-\pi}^{\pi} e^{i\left(m+\frac{\beta}{2}\right)t} \vartheta_n^+(t)\, dt.$$

On the other hand, it follows from Lemma 1 that

$$(1+z)^{-\beta} J_m^+(z) \equiv z^m, \quad |z|<1.$$

Comparing the corresponding coefficients, we arrive at the following equalities

$$\int_{-\pi}^{\pi} e_m^+(t)\, \vartheta_n^+(t)\, dt = \delta_{nm}, \quad \forall n,m \in Z_+.$$

Expanding the function $(1+z)^{-\beta} J_m^+(z)$ at infinity in powers of z^{-1}, we obtain

$$(1+z)^{-\beta} J_m^+(z) = \sum_{n=1}^{\infty} b_{n;m}^+ z^{-n}, \quad |z|>1,$$

where

$$b_{n;m}^+ = \int_{-\pi}^{\pi} e^{i\left(m+\frac{\beta}{2}\right)t} \vartheta_n^-(t)\, dt, \quad m \in Z_+, \quad n \in N.$$

It is easy to see that

$$\lim_{|z|\to\infty} (1+z)^{-\beta} J_m^+(z) = 0.$$

On the other hand, again, as follows from Lemma 1, we have

$$(1+z)^{-\beta} J_m^+(z) \equiv 0, \quad |z|>1.$$

These two expansions imply

$$\int_{-\pi}^{\pi} e^{i\left(m+\frac{\beta}{2}\right)t}\vartheta_n^-(t)\,dt = 0, \quad \forall m \in Z_+, \quad \forall n \in N.$$

The relations

$$\begin{array}{l} \int_{-\pi}^{\pi} e_m^-(t)\,\vartheta_n^+(t)\,dt = 0, \quad m \in N, \quad n \in Z_+; \\ \int_{-\pi}^{\pi} e_m^-(t)\,\vartheta_n^-(t)\,dt = \delta_{nm}, \qquad \forall n, m \in N \end{array}$$

can be proved analogously.

As a result, we obtain the validity of the following statement.

Proposition 1. *Let $Re\beta > -1$. Then for all admissible values of indices n and m the following relations*

$$\int_{-\pi}^{\pi} e_n^{\pm}(t)\,\vartheta_m^{\pm}(t)\,dt = \delta_{nm}, \qquad \int_{-\pi}^{\pi} e_n^{\pm}(t)\,\vartheta_m^{\mp}(t)\,dt = 0,$$

hold.

Define the following system of functions

$$h_n^{\pm}(t) = \rho^{-1}(t)\,\overline{\vartheta_n^{\pm}(t)}\,.$$

3. Completeness in $L_{p,\rho}$

The following lemma on the uniform convergence plays an important role in the study of the completeness of the exponential system (4) in $L_{p,\rho}$.

Lemma 2. *Let $-1 < Re\beta < 0$, or $\beta = 0$. If $\psi(\cdot)$ is an arbitrary Hlder function on $[-\pi,\pi] : e^{i\beta\pi}\psi(-\pi) = \psi(\pi) = 0$, then the series*

$$\sum_{n=0}^{\infty} a_n^+ e_n^+(t) + \sum_{n=1}^{\infty} a_n^- e_n^-(t)\,,$$

uniformly converges to $\psi(\cdot)$ on $[-\pi,\pi]$, where $a_n^{\pm} = \int_{-\pi}^{\pi}\psi(t)\,h_n^{\pm}(t)\,\rho(t)\,dt$.

Proof. Consider the following conjugate problem: find a piecewise analytic function $F(z)$ inside and outside of the unit circle, which the boundary values on the unit circle satisfy the following condition

$$F^+\left(e^{it}\right) + e^{-i\beta t}F^-\left(e^{it}\right) = e^{-i\frac{\beta}{2}t}\psi(t)\,, \quad t \in (-\pi,\pi]\,. \tag{7}$$

We will solve this problem by the method developed in the monograph F.D. Gakhov [1] (see page 427). Consider the following multi-valued analytic function in the complex plane

$$\omega(z) = (z+1)^{\gamma}\,.$$

We carry out cut on the plane z from zero to infinity $(-\infty)$ along the negative real axis. On the plane that have cut in this way, this function will be unique and the incision for it will be line of the rupture . Denote this branch by

$$\omega_{-1}(z) = (z+1)_{-1}^{\gamma}.$$

Let us define

$$\gamma = \frac{1}{2\pi i}\ln e^{-i2\beta\pi} \Rightarrow Re\gamma = -Re\beta.$$

A solution of problem (7) is the following Cauchy type integral

$$F^{+}(z) = (z+1)_{-1}^{\gamma} X_1^{+}(z)\,\Psi^{+}(z),$$

$$F^{-}(z) = \left(\frac{z+1}{z}\right)_{-1}^{\gamma} X_1^{-}(z)\,\Psi^{-}(z),$$

where

$$X_1(z) = \exp\left[\Gamma(z)\right],$$

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\ln\left[\tau^{-\gamma}G(\arg\tau)\right]}{\tau - z}d\tau,$$

$$\Psi(z) = \frac{1}{2\pi i}\int_L \frac{(\tau+1)_{-1}^{-\gamma}\varphi(\arg\tau)}{X_1^{+}(\tau)(\tau - z)}d\tau,$$

$$G(t) = e^{-i\beta t};\ \varphi(t) = e^{-i\frac{\beta}{2}t}\psi(t),$$

$L-$ is a unit circle, which goes around from the point $e^{-i\pi}$ to the point $e^{i\pi}$ in the positive direction.

The fact that $F(z)$ satisfies the boundary condition (7), follows directly from the Sokhotskii-Plemelj formulas. Let $0 < Re\gamma < 1$. It is clear that the function $G(t)$ satisfies the Holder condition on the interval $[-\pi,\pi]$. Moreover, it is easy to verify that the function $\tau^{-\gamma}G(\arg\tau)$ is continuous at a point $\tau = -1$, and as a result it satisfies a certain Holder condition on the unit circle. Then according to the results of the monograph F.D.Gakhov [1] (see page 55) the function $X_1^{\pm}(\tau)$ satisfies the Holder condition on L. Denote

$$L_{-\pi} = \left\{z = e^{it} : t \in \left[-\pi, -\frac{\pi}{2}\right]\right\}.$$

Assume

$$\varphi^{*}(\tau) = \frac{\psi(\arg\tau)}{X^{+}(\tau)\,\tau^{\frac{\beta}{2}}}, \qquad \tau \in L.$$

Let $\left[(z+1)^{-\gamma}\right]^{*}$ be a branch, holomorphic function $(z+1)^{-\gamma}$ in the cut along a plane $L_{-\pi}$ that takes values $(t+1)_{-1}^{-\gamma}$ on the left side $L_{-\pi}$. So

$\left[(t+1)^{-\gamma}\right]^{*} = (t+1)_{-1}^{-\gamma}$ on $L_{\pi} = \left\{z = e^{it} : t \in \left[\frac{\pi}{2}, \pi\right]\right\}$,

then using the results from the monograph of [1] (see page 74), the function $\Psi(z)$ near the point $z = -1$ on the contour L can be represented as

$$\Psi(t)=\left[\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0)-\frac{ctg\,\gamma\pi}{2i}\varphi^*(-1-0)\right]\frac{1}{[(t+1)^{\gamma}]^*}+\Phi(t),\quad \text{for } t\in L_{\pi};\quad (8)$$

$$\Psi(t)=\left[\frac{ctg\,\gamma\pi}{2i}\varphi^*(-1+0)-\frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1-0)\right]\frac{1}{[(t+1)^{\gamma}]^*}+\Phi(t),\quad \text{for } t\in L_{-\pi},\quad (9)$$

where under $\left[(t+1)^{-\gamma}\right]^*$ we mean the limit of the function $\left[(z+1)^{-\gamma}\right]^*$, when z tends to t on the left of $L_{-\pi}\bigcup L_{\pi}$, and moreover

$$\Phi(t)=\frac{\Phi^*(t)}{|t+1|^{\gamma_0}},\ \ \gamma_0<Re\,\gamma, \qquad (10)$$

and the function $\Phi^*(t)$ belongs to the Holder class at the neighborhood of the point $z=-1$.

Applying the Sokhotskii-Plemelj formula from these representations near the point $z=-1$ we have

$$F^+(t)=(t+1)^{\gamma}_{-1}X_1^+(t)\left[\frac{1}{2}(t+1)^{-\gamma}_{-1}\varphi^*(t)+\frac{1}{2\pi i}\int_L\frac{\varphi^*(\tau)}{(\tau+1)^{\gamma}_{-1}(\tau-t)}d\tau\right]=X_1^+(t)\left[\frac{1}{2}\varphi^*(t)+\right.$$

$$\left.+\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0)-\frac{ctg\,\gamma\pi}{2i}\varphi^*(-1-0)\right]+(t+1)^{\gamma}_{-1}X_1^+(t)\,\Phi(t).$$

Passing to the limit as $t\to-1-0$, and taking into account the relation (10), we obtain

$$F^+(-1-0)=X_1^+(-1)\left[\frac{1}{2}\varphi^*(-1-0)+\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\ \varphi^*(-1+0)-\frac{ctg\,\gamma\pi}{2i}\varphi^*(-1-0)\right]=$$

$$=X_1^+(-1)\left[\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0)-\frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1-0)\right].$$

Similarly, from the expressions (8) and (9) we obtain

$$F^+(-1+0)=X_1^+(-1)\left[\frac{1}{2}\varphi^*(-1+0)+\frac{ctg\gamma\pi}{2i}\varphi^*(-1+0)-\frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1-0)\right]=$$

$$=X_1^+(-1)\left[\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0)-\frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1-0)\right].$$

Thus, $F^+(-1-0)=F^+(-1+0)$, i.e. $F^+(t)$ is continuous at the point $z=-1$, and as a result, it satisfies a certain Holder condition on L. Expanding $F^+(z)$ on z at zero, we obtain

$$F^+(z)=\sum_{n=0}^{\infty}a_n^+z^n,$$

where

$$a_n^+ = \int_{-\pi}^{\pi} \psi(t)\, h_n^+(t)\, \rho(t)\, dt, \quad n \in Z_+.$$

We have

$$\frac{1}{2\pi i} \int_{|z|=r<1} F^+(z)\, z^{-n-1} dz = \begin{cases} a_n^+, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Passing to the limit as $r \to 1-0$, hence we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F^+\left(e^{it}\right) e^{-int} dt = \begin{cases} a_n^+, & n \geq 0\,, \\ 0, & n < 0. \end{cases}$$

As, the function $F^+\left(e^{it}\right)$ satisfies Holder condition on L, then its Fourier series on classical system of exponents $\left\{e^{int}\right\}_{n\in Z}$ uniformly converges to it on $[-\pi, \pi]$, and consequently

$$F^+\left(e^{it}\right) = \sum_{n=0}^{\infty} a_n^+ e^{int}, \quad t \in [-\pi, \pi]\,.$$

Now, we investigate the boundary properties of the function $F^-(z)$. Similarly to the case $F^+(z)$, using the representation (8) - (10), and the Sohotskogo- Plemelj formula, we obtain

$$F^-(t) = t_{-1}^{-\gamma} (1+t)_{-1}^{\gamma} X_1^-(t)\, \Psi^-(t) = t_{-1}^{-\gamma} (1+t)_{-1}^{\gamma} X_1^-(t)$$

$$\left\{-\frac{1}{2}\varphi^*(t)\,(1+t)_{-1}^{-\gamma} + \left[\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0) - \frac{ctg\gamma\pi}{2i}\varphi^*(-1-0)\right](1+t)_{-1}^{-\gamma} + \Phi(t)\right\} =$$

$$= t_{-1}^{-\gamma} X_1^-(t) \left[-\frac{\varphi^*(t)}{2} + \frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0) - \frac{ctg\gamma\pi}{2i}\varphi^*(-1-0) + (1+t)_{-1}^{\gamma}\, \Phi(t)\right].$$

Passing to the limit as $t \to 1-0$, we have

$$F^-(-1-0) = e^{-i\gamma\pi} X_1^-(-1) \left[-\frac{\varphi^*(-1-0)}{2} + \frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0) - \frac{e^{i\gamma\pi}}{2i\,\sin\gamma\pi}\varphi^*(-1-0)\right] =$$

$$= e^{-i\gamma\pi} X_1^-(-1) \frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\left[\varphi^*(-1+0) - \varphi^*(-1-0)\right] = 0.$$

$$F^-(-1+0) = e^{i\gamma\pi} X_1^-(-1) \left[-\frac{\varphi^*(-1+0)}{2} + \frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}\varphi^*(-1+0) - \frac{ctg\gamma\pi}{2i}\varphi^*(-1-0)\right] =$$

$$= e^{i\gamma\pi} X_1^-(-1) \frac{ctg\gamma\pi}{2i}\left[\varphi^*(-1+0) - \varphi^*(-1-0)\right] = 0.$$

Thus, $F^-(-1-0) = F^-(-1+0) = 0$, and as a result, $F^-(t)$ satisfies Holder condition on L. Similarly to the case $F^+(t)$, it is proved that the series

$$\sum_{n=1}^{\infty} a_n^- e^{-int},$$

uniformly converges to $F^{-}\left(e^{it}\right)$ on $[-\pi,\pi]$. Then from the boundary condition (7) it follows that the biorthogonal series

$$\sum_{n=0}^{\infty} a_n^{+} e_n^{+}(t) + \sum_{n=1}^{\infty} a_n^{-} e_n^{-}(t),$$

uniformly converges to $\psi(t)$ on $[-\pi,\pi]$. ◀

Using the representation (8), (9) and the expression of the functions $F^{\pm}(z)$ we establish the validity of the following lemma.

Lemma 3. *Let $0 < Re\beta < 1$ and $\psi(\cdot)$ be an arbitrary Holder function on $[-\pi,\pi]$. Then the series*

$$\sum_{n=0}^{\infty} a_n^{+} e_n^{+}(t) + \sum_{n=1}^{\infty} a_n^{-} e_n^{-}(t),$$

where

$$a_n^{\pm} = \int_{-\pi}^{\pi} \psi(t)\, h_n^{\pm}(t)\, \rho(t)\, dt,$$

uniformly converges to $\psi(\cdot)$ on every compact $G \subset (-\pi,\pi)$, and if $\left|1+e^{it}\right|^{-Re\beta} \in L_{p,\rho}$ converges to it in $L_{p,\rho}$, and the following inequality holds

$$-1 < \alpha_k < \frac{p}{q},\ k = -\overline{r,r}. \tag{11}$$

Indeed, the first part follows from the fact that in this case the functions $F^{\pm}\left(e^{it}\right)$ are Holder functions on each compact $G \subset (-\pi,\pi)$. And under fulfilling the inequality (11) the system of exponents $\left\{e^{int}\right\}_{n\in Z}$ forms a basis for $L_{p,\rho}$ and from the inclusion $\left|1+e^{it}\right|^{-Re\beta} \in L_{p,\rho}$follows that $F^{\pm}\left(e^{it}\right) \in L_{p,\rho}$.

The following theorem follows directly from this lemma.

Theorem 1. *Let $\rho \in L_1$ and the parameter β satisfy one of the following conditions:*
i)$-Re\beta \in \bigcup_{k=0}^{\infty}(k,k+1)$;
ii) $-\beta \in Z_+$;
iii) $\left|1+e^{it}\right|^{-Re\beta} \in L_{p,\rho}$ and the following inequalities hold

$$-1 < \alpha_k < \frac{p}{q}, k = -\overline{r,r}.$$

Then the system (4) is complete in $L_{p,\rho}$, for $\forall p \geq 1$, if $\rho \in L_1$.

Indeed, let us consider the case *i*), the case *ii*) proves similarly. Let, for example, $-Re\beta \in (1,2)$, i.e. $-2 < Re\beta < -1 \Rightarrow Re\tilde{\beta} < 0$, where $\tilde{\beta} = \beta + 1$. Consider the system

$$\left\{e^{i\left(n+\frac{\beta}{2}\right)t}; e^{-i\left(n+\frac{\beta}{2}\right)t}\right\}_{n\in N}. \tag{12}$$

Presenting this system in the form of

$$\left\{e^{i\left(n-1+1+\frac{\beta}{2}\right)t}; e^{-i\left(n+\frac{\beta}{2}\right)t}\right\}_{n\in N},$$

and multiplying it by $e^{-i\frac{\beta}{2}t}$, we get the following system

$$\left\{e^{i\left(n+\frac{\tilde{\beta}}{2}\right)t}\right\}_{n\in Z}, \tag{13}$$

where $\tilde{\beta} = \beta + 1$, as a result, as it follows from Lemma 2, the corresponding biorthogonal series of an arbitrary Holder function $f \in C^{\alpha}[-\alpha, \pi] : f(-\pi) = f(\pi) = 0$ uniformly converges to it on $[-\pi, \pi]$. Denote the partial sums of this series by $S_m(f)$, $m \in N$. Consequently

$$\|f - S_m(f)\|_{p,\rho}^p = \int_{-\pi}^{\pi} |f(t) - S_m(f)(t)|^p \rho(t)\, dt \le$$

$$\le \int_{-\pi}^{\pi} \rho(t)\, dt \max_{[-\pi,\pi]} |f(t) - S_m(f)(t)|^p \to 0,\ m \to \infty.$$

Since the set of such functions is dense in $L_{p,\rho}$, hence, we obtain the completeness of the system (13), and at the same time of the system (12) in $L_{p,\rho}$. From the completeness of the system (12) follows the completeness of the system (4) in $L_{p,\rho}$. The remaining cases are proved by mathematical induction. Case *iii*) directly follows from the Lemma 3.

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A Remark on the Levelling Algorithm for the Approximation by Sums of Two Compositions

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Abstract. Let X be compact subset of the d-dimensional Euclidean space and $C(X)$ be the space of continuous functions on X. In [6], the second author, under suitable conditions, showed that the Diliberto-Straus levelling algorithm holds for a subspace of $C(X)$ consisting of sums of two compositions. In the proof, he substantially used the theory of bolts and bolt functionals. In the current paper, we prove the result differently, by implementing Golomb's and also Light and Cheney's ideas.

Key Words and Phrases: uniform approximation, levelling algorithm, best approximation operator, central proximity map.
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1. Introduction

Assume E is a Banach space and X and Y are closed subspaces of E. In addition, assume A and B are best approximation operators acting from E onto X and Y respectively. There are many papers devoted to methods of computing the distance to a given element $z \in E$ from $X + Y$. In this paper, we consider a method called the *levelling algorithm*. This method can be described as follows: Starting with $z_1 = z$ compute $z_2 = z_1 - Az_1$, $z_3 = z_2 - Bz_2$, $z_4 = z_3 - Az_3$, and so forth. Obviously, $z - z_n \in X + Y$ and the sequence $\{\|z_n\|\}_{n=1}^{\infty}$ is nonincreasing. J. von Neumann [17] was the first to prove that in Hilbert space setting the sequence $\{\|z_n\|\}_{n=1}^{\infty}$ converges to the error of approximation from $X+Y$. But for other Banach spaces, the convergence of the algorithm depends on certain additional conditions. The general result of Golomb [5] (see also Light and Cheney [13, p.57]) states that in the above Banach space setting the sequence $\{\|z_n\|\}_{n=1}^{\infty}$ converges in norm to the error of approximation from $X + Y$ provided that the sum $X + Y$ is closed and the equalities

$$\|z - Az + x\| = \|z - Az - x\|, \quad \|z - Bz + y\| = \|z - Bz - y\|, \tag{1}$$

hold for all $z \in E$, $x \in X$ and $y \in Y$. Note that best approximation operators with the property (1) are called central proximity maps (see [13]).

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In 1951, Diliberto and Straus [3] considered the levelling algorithm in the space of continuous functions. They proved that for the problem of uniform approximation of a bivariate function defined on a unit square by sums of univariate functions, the sequence produced by the levelling algorithm converges to the desired quantity. In this paper, we generalize Diliberto-Straus's result to linear superpositions consisting of two summands. More precisely, we consider the levelling algorithm in the problem of approximating from the set of sums of superpositions, which contains functions of the form $f(s(x))+g(p(x))$,where $s(x)$ and $p(x)$ are fixed continuous mappings and f and g are variable univariate continuous functions on the images of s and p, respectively. Under suitable assumptions, we prove that the sequence produced by the levelling algorithm converges to the error of approximation. It should be noted that using the idea of *bolts* (for this terminology see [2, 4, 7, 9, 10, 11, 12]) and methods of Functional Analysis, the second author [6] proved the convergence of the algorithm in the setting considered in this paper. The method of the proof presented here is different and quite short. It is mainly based on the above result of Golomb [5] and ideas of Light and Cheney [13].

2. Levelling algorithm for the sum of two compositions

Let Q be a compact subset of the space $\mathbb{R}^d$. Fix two continuous maps $s : Q \longrightarrow \mathbb{R}$, $p : Q \longrightarrow \mathbb{R}$ and consider the following spaces

$$\begin{aligned} D_1 &= \{f(s(x)) : \ f \in C(\mathbb{R})\}, \\ D_2 &= \{g(p(x)) : \ g \in C(\mathbb{R})\}, \\ D &= D_1 + D_2. \end{aligned}$$

Note that the space D, in particular cases, turn into sums of univariate functions, sums of two ridge functions, sums of two radial functions, etc. The literature abounds with the use of ridge functions (see, e.g., [2, 7, 8, 15, 18, 20]) and radial functions (see, e.g., [4, 14, 16] and a great deal of references therein). Ridge functions and radial functions are defined as multivariate functions of the form $g(\mathbf{a}\cdot\mathbf{x})$ and $g(\|\mathbf{x}-\mathbf{a}\|)$ respectively, where $\mathbf{a}\in\mathbb{R}^d$ is a fixed vector, $\mathbf{x}\in\mathbb{R}^d$ is the variable, $\mathbf{a}\cdot\mathbf{x}$ is the usual inner product, $\|\cdot\|$ is the norm induced by this inner product and g is a univariate function.

We are going to deal with the problem of approximating a continuous function $h : Q \to \mathbb{R}$ using functions from the space D. By $s(Q)$ and $p(Q)$ we will denote the images of Q under the mappings s and p respectively. Define the following operators

$$F : C(Q) \to D_1, \quad (Fh)(a) = \frac{1}{2}\left(\max_{\substack{x\in Q\\ s(x)=a}} h(x) + \min_{\substack{x\in Q\\ s(x)=a}} h(x)\right), \quad \text{for all } a \in s(Q),$$

and

$$G : C(Q) \to D_2, \quad (Gh)(b) = \frac{1}{2}\left(\max_{\substack{x\in Q\\ p(x)=b}} h(x) + \min_{\substack{x\in Q\\ p(x)=b}} h(x)\right), \quad \text{for all } b \in p(Q).$$

In the sequel, we need that the above max and min functions be continuous. For this reason, we will chose the functions $s(x)$ and $p(x)$ from the certain class of functions defined below.

Definition 1. *We say that a function $f \in C(Q)$ belongs to the class $\mathcal{M}(Q)$, if for any two points x and y with $f(x) = f(y)$ and any sequence $\{x_n\}_{n=1}^{\infty}$ tending to x, there exist a sequence $\{y_k\}_{k=1}^{\infty}$ tending to y and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $f(y_k) = f(x_{n_k})$, for all $k = 1, 2, ...$*

Note that the class $\mathcal{M}(Q)$ strictly depends on the considered set Q. That is, a continuous function $f : Q \to \mathbb{R}$ may be in $\mathcal{M}(Q)$, but for many subsets $P \subset Q$, it may happen that the restriction of f to P is not in $\mathcal{M}(P)$. For example, let K be the unit square in $\mathbb{R}^2$ and $K_1 = [0, 1] \times [0, \frac{1}{2}] \cup [0, \frac{1}{2}] \times [0, 1]$. Clearly, the coordinate function $f(x, y) = x$ is in $\mathcal{M}(K)$, but not in $\mathcal{M}(K_1)$. Indeed, for the sequence $\{(\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2})\}_{n=1}^{\infty} \subset K_1$, which tends to $(\frac{1}{2}, \frac{1}{2})$, we cannot find a sequence $\{(x_k, y_k)\}_{k=1}^{\infty} \subset K_1$ tending to $(\frac{1}{2}, 1)$ such that $\{x_k\}_{k=1}^{\infty}$ is a subsequence of $\{\frac{1}{2} + \frac{1}{n+1}\}_{n=1}^{\infty}$.

Let f be a fixed continuous function on a compact set $Q \subset \mathbb{R}^d$. For each continuous function $h : Q \to \mathbb{R}$ consider the following max and min functions

$$r(a) = \max_{\substack{x \in Q \\ f(x)=a}} h(x) \text{ and } u(a) = \min_{\substack{x \in Q \\ f(x)=a}} h(x),\ a \in f(Q). \tag{2}$$

When these functions inherit continuity properties of the given f? It turns out that if $f \in \mathcal{M}(Q)$, then the functions in (2) are continuous for all $h \in C(Q)$.

Lemma 1. *Let Q be a compact set in $\mathbb{R}^d$ and $f \in \mathcal{M}(Q)$. Then the functions $r(a)$ and $u(a)$ are continuous for each function $h \in C(Q)$.*

Proof. Suppose the contrary. Suppose that $f \in \mathcal{M}(Q)$, but one of the functions $r(a)$ and $u(a)$ is not continuous. Without loss of generality we may assume that $r(a)$ is not continuous on the image of f. Let $r(a)$ be discontinuous at a point $a_0 \in f(Q)$. Then there exists a number $\varepsilon > 0$ and a sequence $\{a_n\}_{n=1}^{\infty} \subset f(Q)$ tending to a_0, such that

$$|r(a_n) - r(a_0)| > \varepsilon, \tag{3}$$

for all $n = 1, 2, ...$ Since the function h is continuous on Q, there exist points $x_k \in Q$, $k = 0, 1, 2, ...,$ such that $h(x_k) = r(a_k)$, $f(x_k) = a_k$, for $k = 0, 1, 2, ...$ Thus the inequality (3) can be written as

$$|h(x_n) - h(x_0)| > \varepsilon, \tag{4}$$

for all $n = 1, 2, ...$ Since Q is compact, the sequence $\{x_n\}_{n=1}^{\infty}$ has a converging subsequence. Without loss of generality assume that $\{x_n\}_{n=1}^{\infty}$ itself converges to a point $y_0 \in Q$. Then

$f(x_n) \to f(y_0)$, as $n \to \infty$. But by the assumption, we also have $f(x_n) \to f(x_0)$, as $n \to \infty$. Therefore, $f(y_0) = f(x_0) = a_0$. Note that x_0 and y_0 cannot be the same point, for the equality $x_0 = y_0$ violates the condition (4). By the definition of the class $\mathcal{M}(Q)$, we must have a subsequence $x_{n_k} \to y_0$ and a sequence $z_k \to x_0$ such that

$$f(x_{n_k}) = f(z_k),$$

for all $k = 1, 2, ...$ Since $f(x_{n_k}) = a_{n_k}$, $k = 1, 2, ...$, and on each level set $\{x \in Q : f(x) = a_{n_k}\}$, the function h takes its maximum value at x_{n_k}, we obtain that

$$h(z_k) \leq h(x_{n_k}), \ k = 1, 2, ...$$

Taking the limit in the last inequality as $k \to \infty$, gives us the new inequality

$$h(x_0) \leq h(y_0). \tag{5}$$

Recall that on the level set $\{x \in Q : f(x) = a_0\}$, the function h takes its maximum at x_0. Thus from (5) we conclude that $h(x_0) = h(y_0)$. This last equality contradicts the choice of the positive ε in (4), since $h(x_n) \to h(y_0)$, as $n \to \infty$. Hence the function r is continuous on $f(Q)$. By the same way one can prove that u is continuous on $f(Q)$. ◀

The following theorem plays a key role in the proof of our main result (Theorem 2).

Theorem 1. *Let the continuous mappings $s : Q \longrightarrow \mathbb{R}$, $p : Q \longrightarrow \mathbb{R}$ be in the class $\mathcal{M}(Q)$. Then the operators F and G are best approximation operators onto the spaces D_1 and D_2 respectively, both enjoying the properties of centrality and non-expansiveness.*

Proof. We prove this theorem for the operator F. A proof for G can be carried out by the same way.

Clearly, on the level set $s(x) = a$, the constant $(Fh)(a)$ is a best approximation to h, among all constants. Varying over $a \in s(Q)$, we obtain a best approximating function $Fh : s(Q) \to \mathbb{R}$, which is, due to Lemma 1, in the space D_1.

Now let us prove that the operator F is central. In other words, we must prove that for any functions $h(x) \in C(Q)$ and $f(s(x)) \in D_1$,

$$\|h - Fh - f\| = \|h - Fh + f\|. \tag{6}$$

Put $u = h - Fh$. There exists a point $x_0 \in Q$ such that

$$\|u + f\| = |u(x_0) + f(s(x_0))|.$$

First assume that $|u(x_0) + f(s(x_0))| = u(x_0) + f(s(x_0))$. Note that $Fu = 0$. This means that

$$\max_{\substack{x \in Q \\ s(x)=a}} u(x) = -\min_{\substack{x \in Q \\ s(x)=a}} u(x), \text{ for all } a \in s(Q). \tag{7}$$

Let

$$\min_{\substack{x\in Q \\ s(x)=s(x_0)}} u(x) = u(x_1). \tag{8}$$

From (7) and (8) it follows that

$$-u(x_1) \geq u(x_0).$$

Taking the last inequality and the equality $s(x_1) = s(x_0)$ into account we may write

$$\|u - f\| \geq f(s(x_1)) - u(x_1) \geq f(s(x_0)) + u(x_0) = \|u + f\|. \tag{9}$$

Changing in (9) the function f to $-f$ gives the reverse inequality $\|u + f\| \geq \|u - f\|$. Thus (6) holds.

Note that if $|u(x_0) + f(s(x_0))| = -(u(x_0) + f(s(x_0)))$, then by replacing Eq (8) by

$$\max_{\substack{x\in Q \\ s(x)=s(x_0)}} u(x) = u(x_1). \tag{10}$$

we will derive from (7) and (10) that $u(x_1) \geq -u(x_0)$. This inequality is then used to obtain the estimation

$$\|u - f\| \geq -(f(s(x_1) - u(x_1)) \geq -(f(s(x_0)) + u(x_0)) = \|u + f\|,$$

which in turn yields (6). The centrality has been proven.

Now we prove that the operator F is non-expansive. First note that it is nondecreasing. That is, if $h_1(x) \leq h_2(x)$, then $Fh_1(s(x)) \leq Fh_2(s(x))$ for all $x \in Q$. Besides, $F(h + c) = Fh + c$, for any real number c. Put $c = \|h_1 - h_2\|$. Then for any $x \in Q$, we can write

$$h_2(x) - c \leq h_1(x) \leq h_2(x) + c$$

and further

$$Fh_2(s(x)) - c \leq Fh_1(s(x)) \leq Fh_2(s(x)) + c.$$

From the last inequality we obtain that

$$\|Fh_1(s(x)) - Fh_2(s(x))\| \leq c = \|h_1 - h_2\|.$$

Thus we see that F is non-expansive. ◄

Consider the iteration

$$h_1(x) = h(x), h_{2n} = h_{2n-1} - Fh_{2n-1}, h_{2n+1} = h_{2n} - Gh_{2n}, n = 1, 2, ...$$

Our main result is the following theorem.

Theorem 2. *Assume $s, p \in \mathcal{M}(Q)$ and D is closed in $C(Q)$. Then $\|h_n\|$ converges to the error of approximation $E(h)$.*

Proof of Theorem 2 easily follows from Theorem 1 and the result of Golomb [5]: Let E be a Banach space and X and Y be closed subspaces of E. In addition, let the sum $X + Y$ be closed in E. If A and B are central proximity maps (see Introduction), then for an element $z \in E$ the sequence produced by the levelling algorithm $z_1 = z$, $z_2 = z_1 - Az_1$, $z_3 = z_2 - Bz_2$, $z_4 = z_3 - Az_3, ...$, converges in norm to the distance $dist(z, X + Y)$.

Remark 1. Theorem 2 in a more general form involving any compact Hausdorff space X and closed subalgebras of $C(X)$ will appear in [1].

Remark 2. A version of Theorem 2 was proved by the second author differently in [6]. He did not consider the classes $\mathcal{M}(Q)$ and assumed directly that the functions $r(a)$ and $u(a)$ are continuous for each function $h \in C(Q)$.

Remark 3. We do not yet know if the Diliberto and Straus algorithm converges without the closedness assumption on the subspace D. Note that this problem was posed in various settings in several works (see, e.g., [18, 19, 20]).

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On the Equivalence of Completeness of a System of Powers and Trivial Solvability of Homogeneous Riemann Problem

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Abstract. Double system of powers with degenerate coefficients is considered in this work. Some weighted Smirnov classes are introduced and conjugation problem for them is formulated. Equivalence of the completeness of a double system of powers in a weighted Lebesgue space and the trivial solvability of the corresponding homogeneous conjugation problem in weighted Smirnov classes is proved.

Key Words and Phrases: system of powers, completeness, weighted space, Smirnov classes.

2010 Mathematics Subject Classifications: 42C40, 42C15

1. Introduction

Consider the following system of powers:

$$\left\{A^{+}(t)\,\omega^{+}(t)\,\varphi^{n}(t)\,;\;A^{-}(t)\,\omega^{-}(t)\,\bar{\varphi}^{n}(t)\right\}_{n\geq 0}, \tag{1}$$

where $A^{\pm}(t) \equiv |A^{\pm}(t)|\, e^{i a^{\pm}(t)}$ and $\varphi(t)$ are complex-valued functions on the interval $[a,b]$ with the degenerate coefficients $\omega^{\pm}(\cdot)$:

$$\omega^{\pm}(t) \equiv \prod_{i=1}^{l^{\pm}} \left|t - t_i^{\pm}\right|^{\beta_i^{\pm}},$$

where $\left\{t_i^{\pm}\right\} \subset (a,b)$, $\left\{\beta_i^{\pm}\right\} \subset R$ are some sets (R is the real axis).

Very special cases of the system (1) arise when considering spectral problems of the theory of differential operators. As a typical example, we can mention so-called Kostyuchenko system $\left\{e^{iant}\sin nt\right\}_{n\geq 1}$, where $a \in C$ is a complex parameter (C is the complex plane). Many researches have been dedicated to the basis properties of this system (see, e.g., [1-6]). Final results on the basis properties of this system (completeness, minimality, basicity)

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have been obtained in [5]. Theoretical foundations for the study of basis properties of the systems like (1) have been laid by J.L. Walsh [7]. [8] and [9] also treated the above mentioned problems. A special case of the system (1) with $\varphi(t) \equiv e^{it}$ was considered in [10,11], where basicity criteria for the exponential system with degenerate coefficients in L_p have been obtained.

In the present work, we study the completeness of the system (1) in the weighted space $L_{p,\rho} \equiv L_{p,\rho}(a,b)$, $1 < p < +\infty$, with the weight $\rho : [a,b] \to (0,+\infty)$.

2. Needful Information

Before stating our main result, we make the following assumptions.

1) $|A^{\pm}(t)|; |\varphi'(t)|$ are measurable on (a,b) and the following condition holds:

$$\sup_{(a,b)} vrai \left\{ \left|A^{+}(t)\right|^{\pm 1}; \left|A^{-}(t)\right|^{\pm 1}; \left|\varphi'(t)\right|^{\pm 1} \right\} < +\infty.$$

2) $\Gamma = \varphi\{[a,b]\}$ is a simple closed ($\varphi(a) = \varphi(b)$) rectifiable Jordan curve. Γ is either a Radon curve (i.e. the angle $\theta_0(\varphi(t))$ between the tangent line to Γ at the point $\varphi = \varphi(t)$ and the real axis is a function of bounded variation on $[a,b]$), or a piecewise Lyapunov curve.

For definiteness, we will assume that when the point $\varphi = \varphi(t)$ runs across the curve Γ as t increases, the internal domain $int\Gamma$ stays on the left side.

To state our theorem, we have to introduce weighted Smirnov classes of analytic functions.

Let $D \equiv int\Gamma$, and $E_1(D)$ be a usual Smirnov class of analytic functions in D. Let $\omega(\tau)$ be some weight function on Γ and $L_{p,\omega}(\Gamma)$ be a weighted Lebesgue class of p-summable functions on Γ:

$$L_{p,\omega}(\Gamma) \overset{def}{\equiv} \left\{ f : \int_{\Gamma} |f(\tau)|^{p} \omega(\tau) \, |d\tau| < +\infty \right\}.$$

By $f^{+}(\tau)$ we denote the non-tangential boundary values of the function $f(z) \in E_1(D)$. Introduce

$$E_{p,\omega}(D) \overset{def}{\equiv} \left\{ f \in E_1(D) : \; f^{+}(\tau) \in L_{p,\omega}(\Gamma) \right\}.$$

Let's consider the following conjugation problem in the classes $E_{p^{\pm},\rho^{\pm}}(D)$:

$$F_1^{+}(\tau) + G(\tau)\overline{F_2^{+}(\tau)} = g(\tau), \; \tau \in \Gamma, \tag{2}$$

where $\overline{F_2^{+}(\tau)}$ is a complex conjugation, $g(\tau) \in L_{p,\omega}(\Gamma)$ is some function, and $G(\tau)$ is a given function. $g(\tau)$ and $G(\tau)$ are called the free term and the coefficient of the problem (2), respectively. By the solution of the problem (2) we mean a pair of analytic functions $F_1(z)$ and $F_2(z)$ in D, which belong to the classes $E_{p^{+},\rho^{+}}(D)$ and $E_{p^{-},\rho^{-}}(D)$, respectively, and whose boundary values $F_1^{+}(\tau)$ and $F_2^{+}(\tau)$ satisfy the equality (2) almost everywhere on Γ.

Further, denote by $t = \psi(\varphi)$ the inverse of the function $\varphi = \varphi(t)$ defined on $\Gamma \backslash \{\varphi(a) = \varphi(b)\}$. The point $\varphi_0 = \varphi(a) = \varphi(b)$ is considered as two different "stuck-together" endpoints $\varphi_0^+ = \varphi(a)$ and $\varphi_0^- = \varphi(b)$. Then, it is quite natural to assume that $\psi(\varphi_0^+) = a$ and $\psi(\varphi_0^-) = b$.

3. t-Besselian systems

Consider the following homogeneous conjugation problem:

$$F_1^+(\tau) - G(\tau)\overline{F_1^+(\tau)} = 0 \quad \text{a.e. on } \Gamma, \tag{3}$$

where the coefficient $G(\tau)$ is defined by the formula

$$G(\tau) = \frac{A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\,\bar{\varphi}'(\psi(\varphi))}{A^-(\psi(\varphi))\,\omega^-(\psi(\varphi))\,\varphi'(\psi(\varphi))}.$$

The following theorem is true.

Theorem 1. *Let $\rho : [a,b] \to (0,+\infty)$ be some weight function, the coefficients $A^{\pm}(t)$ satisfy the conditions 1), 2,) and $\omega^{\pm} \in L_{p,\rho}(a,b)$, where $p \in (1,+\infty)$ is some number. Then the system* (1) *is complete in $L_{p,\rho}(a,b)$ only when the homogeneous conjugation problem* (3) *has only the trivial solution in the classes $E_{q,\rho^{\pm}}(D)$, $\frac{1}{p}+\frac{1}{q}=1$, where*

$$\rho^{\pm}(\varphi) = \left|\omega^{\pm}(\psi(\varphi))\right|^{-q}\rho^{1-q}(\psi(\varphi)), \quad \varphi \in .$$

Proof. The completeness of the system (1) in $L_{p,\rho}(a,b)$ is equivalent to saying that every function $f(t) \in L_{q,\rho}(a,b)$, $\frac{1}{p}+\frac{1}{q}=1$, is equal to zero almost everywhere with

$$\left.\begin{array}{l} \int_a^b A^+(t)\,\omega^+(t)\,\varphi^n(t)\,\bar{f}(t)\,\rho(t)\,dt = 0\,, \\ \int_a^b A^-(t)\,\omega^-(t)\,\bar{\varphi}^n(t)\,\bar{f}(t)\,\rho(t)\,dt = 0\,,\ n \geq 0. \end{array}\right\} \tag{4}$$

From the first of (4) we have

$$\int_a^b A^+(t)\,\omega^+(t)\,\bar{f}(t)\,\varphi^n(t)\,\rho(t)\,dt = \int_\Gamma A^+(\psi(\varphi))\,\omega^+(\psi(\varphi)) \times$$

$$\times \bar{f}(\psi(\varphi))\left[\varphi'(\psi(\varphi))\right]^{-1}\rho(\psi(\varphi))\,\varphi^n d\varphi = \int_\Gamma F_1(\varphi)\,\varphi^n d\varphi = 0, \tag{5}$$

where

$$F_1(\varphi) = A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\left[\varphi'(\psi(\varphi))\right]^{-1}\bar{f}(\psi(\varphi))\,\rho(\psi(\varphi))\,.$$

It is not difficult to conclude from the conditions of the theorem that $F_1(\varphi) \in L_1(\Gamma)$. Then, due to the results of [12], the equalities (5) are equivalent to the existence of the function $F_1 \in E_1(D)$ such that $F_1^+(\varphi) = F_1(\varphi)$ a.e. on Γ.

It is not difficult to see that $F_1(\varphi) \in L_{q,\rho^+}(\Gamma)$, where $\rho^+ \equiv |\omega^+|^{-q}\rho^{1-q}$. Consequently, by definition, the function $F_1(z)$ belongs to the class $E_{q,\rho^+}(D)$.

Similarly, from the second of (4) we have

$$\int_a^b \overline{A^-(t)}\omega^-(t)\, f(t)\,\varphi^n(t)\,\rho(t)\,dt = \int_\Gamma \overline{A^-(\psi(\varphi))}\omega^-(\psi(\varphi))\times$$

$$\times f(\psi(\varphi))\left[\varphi'(\psi(\varphi))\right]^{-1}\rho(\psi(\varphi))\,\varphi^n d\varphi = \int_\Gamma F_2(\varphi)\,\varphi^n d\varphi = 0\,,\ n\geq 0,$$

where

$$F_2(\varphi) = \overline{A^-(\psi(\varphi))}\omega^-(\psi(\varphi))\left[\varphi'(\psi(\varphi))\right]^{-1} f(\psi(\varphi))\,\rho(\psi(\varphi))\,.$$

Proceeding as above, we arrive at the conclusion that there exists the function $F_2(z)\in E_1(D)$ such that $F_2^+(\varphi)=F_2(\varphi)$ a.e. on Γ, where $F_2^+(\varphi)$ is a non-tangential boundary value of $F_2(z)$ on Γ. From $F_2(\varphi)\in L_{q,\rho^-}(\Gamma)$, $\rho^-\equiv|\omega^-|^{-q}\rho^{1-q}$, it follows that the function $F_2(z)$ belongs to the class $E_{q,\rho^-}(D)$. Expressing the function $f(t)$ in terms of $F_1(\varphi)$ and $F_2(\varphi)$, we have

$$F_1^+(\varphi) = G(\varphi)\,\overline{F_2^+(\varphi)}\,, \varphi\in\Gamma,$$

where

$$G(\varphi) = \frac{A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\,\bar{\varphi}'(\psi(\varphi))}{A^-(\psi(\varphi))\,\omega^-(\psi(\varphi))\,\varphi'(\psi(\varphi))}.$$

Thus, if the system (1) is not complete in $L_{p,\rho}(a,b)$, then the homogeneous conjugation problem (3) is non-trivially solvable in the classes $E_{p,\rho^\pm}(D)$.

Now suppose to the contrary that the problem (3) is non-trivially solvable in the classes $E_{p,\rho^\pm}(D)$. From the definition of the classes $E_{p,\rho^\pm}(D)$ and from $F_i(z)\in E_1(D)\,,\ i=\overline{1,2}$, it follows that

$$\int_\Gamma F_i^+(\varphi)\,\varphi^n d\varphi = 0, n\geq 0.$$

Taking into account the expression for the function $G(\tau)$, we have

$$\frac{F_1^+(\varphi)\,\varphi'(\psi(\varphi))}{A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\,\rho(\psi(\varphi))} = \frac{F_2^+(\varphi)\,\varphi'(\psi(\varphi))}{A^-(\psi(\varphi))\,\omega^-(\psi(\varphi))\,\rho(\psi(\varphi))}.$$

Denoting the last expression by $\overline{f(\varphi)}$, we obtain

$$\int_\Gamma A^+(\psi(\varphi))\,\omega^+(\psi(\varphi))\,\frac{\overline{f(\varphi)}}{\varphi'(\psi(\varphi))}\varphi^n\rho(\psi(\varphi))\,d\varphi =$$

$$= \int_a^b A^+(t)\,\omega^+(t)\,\overline{f(\varphi(t))}\varphi^n(t)\,\rho(t)\,dt = 0\,,\ n\geq 0.$$

Similarly we have

$$\int_a^b A^-(t)\,\omega^-(t)\,\overline{f(\varphi(t))}\bar{\varphi}^n(t)\,\rho(t)\,dt = 0\,,\ n\geq 0.$$

From the conditions of the theorem, by the definition of the classes $E_{p,\rho^{\pm}}(D)$ it follows that the function $f(\varphi(t))$ belongs to the space $L_{q,\rho}(a,b)$. It is absolutely clear that this function is different from zero. Then, the previous relations imply that the system (1) is not complete in $L_{p,\rho}(a,b)$. ◄

Remark 1. *One of the results obtained by Smirnov implies that if the domain D belongs to the Smirnov class and $\rho^{\pm} \equiv 1$, $p \geq 1$, then the definition of the classes $E_{p,\rho^{\pm}}$ is equivalent to the classical definition for the classes E_p.*

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Azerbaijan in the Context of the Consumer Confidence Index

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Abstract. An article analyses the Consumer Confidence Index that is observed in the context of Azerbaijan economy. It has been studied the connection between the changes in oil prices (Azeri Light) and the Index value.

Key Words and Phrases: the consumer confidence index, Azerbaijan economy, regression equation, time-series analysis, trend, forecast, Azeri light, oil prices.

The Consumer Confidence Index as an economic indicator shows the degree of optimism for the current period which can be computed using a certain methodology of measurement the consumer's activities on saving and spending [1]. There are three well-known methodologies – of Michigan University, ABC News/Money Magazine and The Conference Board. We will figure out the most common features of these indices.

The Consumer Confidence Index provides information about stages of economic cycles, the inflection points which show the change to the positive or negative trend. It's a very sensitive index, which can predict the upcoming recession on a very early stage. In theory, if the country is on stage of economic growth, the level of consumption increases, as the consumers spend more money and even buy elastic goods such as luxury goods etc. This feature found realization in index: because the consumers are optimistic, they spend more, and have positive expectations about economy, what has the direct connection to the index. And vice-versa, if there is a certain tension in economy, when, for example, the exchange rate of national currency dropped, or the world economic crisis occurred, or the prices for the goods started growing – this all influences the consumers' spirits and they start saving more than consume, and if they face the choice, whether to buy or not to buy a certain thing which may seem important, they would rather prefer to postpone this type of purchase, and keep these money as savings. This influences the index, which changes the dynamics from positive to the negative.

It's interesting to find a relation of this index not only to the economic growth, but also to other indicators. It can be a GDP value, unemployment level, the inflation rate.

In our study we took a model of connection between Azeri Light oil prices and the Index itself for Azerbaijan. We have the following data by the end of the month for

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Azeri Light oil prices (USD/barrel): September 2013 –112,22 [2]; March, 2014 -109,64$ [3]; September 2014 – 97,48$ [4]; December 2014 – 61,41$ [5]; 57,27$ in March, 2015 [9], 50,06$ in September, 2015 [10], 38,23$ in December 2015 [11], 41,41$ in March 2016 [12].

Let's put the data we have in table form:

Periods	**Month, year**	**Azeri Light Price, $/barrel**
1	September 2013	112,22
2	March 2014	109,64
3	September 2014	97,48
4	December 2014	61,41
5	March 2015	57,27
6	September 2015	50,06
7	December 2015	38,23
8	March 2016	41,41

We have the following values of Index for Azerbaijan: for September 2013 and March 2014 - 25,85 units respectively, and 26,35 [6, p.17]; for September 2014 - 25 [7, p.16]. For December 2014 - 23.8 [8, p.10]. The data in table form will be represented as:

t, period	**Month, year**	**Consumer Confidence Index**
1	September 2013	25,85
2	March 2014	26,35
3	September 2014	25
4	December 2014	23,8

Let's try to build a time-series regression model (which is close to the simple linear regression model):

$$y_t = \beta_0, \beta_1 z_t, u_t, t = \overline{1, n},$$

where y_t denotes Consumer Confidence Index, z_t-the price for Azeri Light crude oil; u_t is a disturbance (error term); β_0, β_1 are model parameters.

Let's build a table using the data we have for 4 periods ($t = \overline{1, 4}$):

t	z_t	y_t
1	112,22	25,85
2	109,64	26,35
3	97,48	25
4	61,41	23,8

Now we will use the data we have to obtain OLS estimators. So, we need the following auxiliary table:

t	z_t	y_t	z_t^2	y_t^2	$y_t * z_t$
1	112,22	25,85	12593,33	668,2225	2900,887
2	109,64	26,35	12020,93	694,3225	2889,014
3	97,48	25	9502,35	625	2437
4	61,41	23,8	3771,188	566,44	1461,558
Sum	380,75	101	37887,8	2553,985	9688,459

First, let's calculate the correlation coefficient and check, whether the data we have, can be approximated by the simple linear regression. The formula looks like this:

$$r_{xy} = \frac{n\sum xy - \sum x \sum y}{\sqrt{[(n\sum x^2 - (\sum x)^2) * (n\sum y^2 - (\sum y)^2)]}}.$$

For the model we use z_t as x, and yt as y. Formula will change the look and the applied calculations will be (using the previous table data):

$$r_{z_t y_t} = \frac{n\sum z_t y_t - \sum z_t \sum y_t}{\sqrt{\left[(n\sum z_t^2 - (\sum z_t)^2) * (n\sum y_t^2 - (\sum y_t)^2)\right]}} =$$

$$= \frac{4 * 9688,459 - 380,75 * 101}{\sqrt{[4 * 37887,8 - (380,75)^2] * [4 * 2553,985 - 101^2]}} = 0,95067.$$

The correlation coefficient value that we obtained (0,95) shows that there is a strong direct (positive) connection between two variables y_t and z_t.

The coefficient of determination R^2 can be calculated from the coefficient of correlation. It is equal to:

$$R^2 = (r_{z_t y_t})^2 = 0,95067^2 = 0,9038.$$

This coefficient shows that the variation in dependent variable (Consumer Confidence Index) can be explained by the variation in the explanatory variable (Azeri Light oil price) for 90%. It is highly fitting the simple linear regression model.

Let's return to OLS estimations of b_0 and b_1. The results of the table we will put in our system of equations:

$$\begin{cases} n\beta_0 + \beta_1 \sum z_t = \sum y_t, \\ \\ \beta_0 \sum z_t + \beta_1 \sum z_t^2 = \sum y_t * z_t. \end{cases}$$

Let's put the data we have in our model:

$$\begin{cases} 4\beta_0 + 380,75\beta_1 = 101, \\ \\ 380,75\beta_0 + 37887,8\beta_1 = 9688,459. \end{cases}$$

We can use Cramer's rule. We have the following matrices:

$$Z = \begin{bmatrix} 4 & 380,75 \\ 380,75 & 37887,8 \end{bmatrix};\ Y = \begin{bmatrix} 101 \\ 9688,459 \end{bmatrix}.$$

Mathematically, $det(Z) = 6580,637$; $det(\beta_0) = 137787$; $det(\beta_1) = 298,086$. So, $\beta_0 = \frac{det(\beta_0)}{det(Z)} = 209382504$; β_1 using the same analogue equals to $0,045297435$.

So, the model that we obtain looks like this:

$$y_t = 20,9382504 + 0,045297435 * z_t + u_t; t = \overline{1,n}.$$

Using the above model, we can make a certain forecast. Let's fill the table with the results we know:

t, period	z_t	y_t
1	112,22	25,85
2	109,64	26,35
3	97,48	25
4	61,41	23,8
5	57,27	-
6	50,06	-
7	38,23	-
8	41,41	-

We will put the price of Azeri Light crude oil for a period $t = 5$ to obtain the Consumer Confidence Index for this period:

$$y_{t=5} = 20,9382504 + 0,045297435 * 57,27 \approx 23,53243085.$$

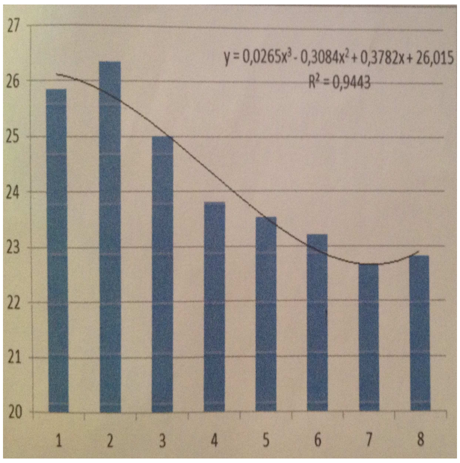
For the periods $t = \overline{1,8}$ the results will be the following (analogical calculations):

$$y_{t=6} \approx 23,20583565;\ y_{t=7} \approx 22,66996585;\ y_{t=8} \approx 22,814012.$$

Finally, we obtain the table and a corresponding graph:

Periods, t	Azeri Light price, $/barrel	Consumer Confidence Index
1	112,22	25,85
2	109,64	26,35
3	97,48	25
4	61,41	23,8
5	57,27	23,53
6	50,06	23,21
7	38,23	22,67
8	41,41	22,81

As we can see, the Index started to increase in the current period. It shows that the recession in our economy has finished and the economic growth is about to start again.



In conclusion, we would like to point out that the economy of our country still has the connection with the dynamics of the World oil prices. There's a need to say that this connection is getting weaker from year to year, so, in future we say that our country will overcome this barrier and will choose non-oil sector as the base of development.

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General Minkowski type and Related Inequalities for Semiconormed Fuzzy Integrals

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Abstract. A general Minkowski type inequalities for the semiconormed fuzzy integrals on abstract spaces are studied. Some related inequalities to this type one and Chebyshev type one for the semiconormed fuzzy integral are also discussed. Also a related inequalities for the Minkoski type inequality are obtained.

Key Words and Phrases: fuzzy measure, Sugeno integral, Minkowski's inequality, comonotone function, seminormed fuzzy integrals, semiconormed fuzzy integrals.

2010 Mathematics Subject Classifications: 03E72, 26E50, 28E10

1. Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [15]. The properties and applications of the Sugeno integral have been studied by many authors, including Ralescu and Adams [9] in the study of several equivalent definitions of fuzzy integrals, Román-Flores et al. [13] and Wang and Klir [16], among others. Many authors generalized the Sugeno integral by using some other operators to replace the special operators $\wedge$ and/or $\vee$ [17, 3, 4]. In [14] Suárez and Gil presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals.

The study of inequalities for Sugeno integral was initiated by Román-flores et al. [10, 11, 12] and then followed by the authors [1, 2, 8]. Recently Ouyang et al. [1] proved a general Minkowski type inequality for comonotone functions and arbitrary fuzzy measure-based Sugeno integrals and then they provided the inverse of this inequality for the same conditions [8]. In [7], Chebyshev type inequality for seminormed fuzzy integrals and a related inequality for semiconormed fuzzy integral were proposed in a rather general form by Ouyang and Mesiar.

Theorem 1. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0, 1]$ two comonotone measurable functions. Let $\star : [0, 1]^2 \to [0, 1]$ be continuous and nondecreasing in both arguments. If the seminorm T satisfies*

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$T(a \star b, c) \geq (T(a,c) \star b) \vee (a \star T(b,c))$,
then

$$\int_{T,A} f \star g d\mu \geq \int_{T,A} f d\mu \star \int_{T,A} g d\mu$$

holds for any $A \in \mathcal{F}$.

Theorem 2. *Let$(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the semiconorm S satisfies*
$S(a \star b, c) \leq (S(a,c) \star b) \wedge (a \star S(b,c))$,
then

$$\int_{S,A} f \star g d\mu \leq \int_{S,A} f d\mu \star \int_{S,A} g d\mu$$

holds for any $A \in \mathcal{F}$.
This paper is organized as follows: In Section 2 some preliminaries and summarization of some previous known results are given. Section 3 proposes general Minkowski type inequalities for semiconormed fuzzy integrals. Section 4 includes a revers inequality for this type of integrals. Section 5 contains a short conclusion.

2. Preliminaries

In this section, we recall some basic definitions and previous results that will be used in the next sections. Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X. Throughout this paper, all considered subsets are supposed to belong to $\mathcal{F}$.

Definition 1 (Sugeno [15])**.** *A set function $\mu : \mathcal{F} \to [0,1]$ is called a fuzzy measure if the following properties are satisfied:*
(FM1) $\mu(\emptyset) = 0$ and $\mu(X) = 1$
(FM2) $A \subset B$ implies $\mu(A) \leq \mu(B)$
(FM3) $A_n \to A$ implies $\mu(A_n) \to \mu(A)$.

When μ is a fuzzy measure, the triple (X,$\mathcal{F}, \mu$) is called a fuzzy measure space.

Let (X,$\mathcal{F}, \mu$) be a fuzzy measure space, and $\mathcal{F}_+(X) = \{f | f : X \to [0,\ 1]$ is measurable with respect to $\mathcal{F}\}$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$. For any $\alpha \in [0,1]$, we will denote the set $\{x \in X | f(x) \geq \alpha\}$ by F_α and $\{x \in X | f(x) > \alpha\}$ by $F_{\bar{\alpha}}$. Clearly, both F_α and $F_{\bar{\alpha}}$ are non-increasing with respect to α, i.e., $\alpha \leq \beta$ implies $F_\alpha \supseteqq F_\beta$ and $F_{\bar{\alpha}} \supseteqq F_{\bar{\beta}}$.

Definition 2 (Sugeno [15])**.** *Let (X,$\mathcal{F}, \mu$) be a fuzzy measure space and $A \in \mathcal{F}$. The Sugeno integral of f over A with respect to the fuzzy measure μ, is defined by*

$$\fint_A f d\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When A=X, then

$$\fint_X f d\mu = \fint f d\mu = \bigvee_{\alpha\in[0,1]} (\alpha \wedge \mu(F_\alpha)).$$

Notice that Ralescu and Adams (see [9]) extended the range of fuzzy measures and the Sugeno integrals from [0, 1] to [0, ∞]. But we only deal with the original fuzzy measure and the Sugeno integrals which was introduced by Sugeno in 1974.

Note that in the above definition, $\wedge$ is just the prototypical t-norm minimum and $\vee$ the prototypical t-conorm maximum. A t-conorm [6] is a function $S : [0,1]\times[0,1] \to [0,1]$ satisfying the following condition:

(S1) $S(x,\ 0) \ = \ S(0,\ x) \ = \ x \quad \forall x \in [0,\ 1]$.

(S2) $\forall x_1, x_2, y_1, y_2$ in [0, 1], if $x_1 \le x_2,\ y_1 \le y_2$ then $S(x_1,\ y_1) \le S(x_2,\ y_2)$.

(S3) $S(x,\ y) \ = \ S(y,\ x)$.

(S4) $S(S(x,\ y),\ z) \ = \ S(x,\ S(y,\ z))$.

A t-norm [6] is a function $T : [0,\ 1]\times[0,1] \to [0,\ 1]$ satisfying the following condition:

(T1) $T(x,\ 1) \ = \ T(1,\ x) \ = \ x \quad \forall x \in [0,\ 1]$.

(T2) $\forall x_1, x_2, y_1, y_2$ in [0, 1], if $x_1 \le x_2,\ y_1 \le y_2$ then $T(x_1,\ y_1) \le T(x_2,\ y_2)$.

(T3) $T(x,\ y) \ = \ T(y,\ x)$.

(T4) $T(T(x,\ y), z) \ = \ T(x,\ T(y,\ z))$.

A binary operator S (T) on [0, 1] is called a t-semiconorm (t-seminorm) [14] if it satisfies the above conditions (S1) and(S2) ((T1) and (T2)). Using the concepts of t-seminorm and t-semiconorm, Suárez and Gil proposed two families of fuzzy integrals:

Definition 3. *Let S be a t-semiconorm, then the semiconormed fuzzy integral of f over A with respect to S and the fuzzy measure μ is defined by*

$$\int_{S,A} f d\mu = \bigwedge_{\alpha\in[0,\ 1]} S(\alpha, \mu(A\cap F_{\bar{\alpha}})).$$

Definition 4. *Let T be a t-seminorm, then the seminormed fuzzy integral of f over A with respect to T and the fuzzy measure μ is defined by*

$$\int_{T,A} f d\mu = \bigvee_{\alpha\in[0,1]} T(\alpha, \mu(A\cap F_\alpha)).$$

It is easy to see that the Sugeno integral is a special seminormed fuzzy integral. Moreover, Kandel and Byatt (see [5]) showed another expression of the Sugeno integral as follows:

$$\fint_A f d\mu = \bigwedge_{\alpha\in[0,\ 1]} (\alpha \vee \mu(A\cap F_{\bar{\alpha}})).$$

So the semiconormed fuzzy integrals also generalized the concept of the Sugeno integral. Note that if $\int_{S,A} f d\mu = a$, then $S(\alpha, \mu(A\cap F_{\bar{\alpha}})) \ge a$ for all $\alpha \in [0,\ 1]$ and, for $\varepsilon > 0$ there exists α_ε such that $S(\alpha_\varepsilon, \mu(A\cap F_{\bar{\alpha}_\varepsilon})) \le a+\varepsilon$.

In [1] Agahi et al. proved the following inequality for the Sugeno integral (with respect to a fuzzy measure in the sense of Ralescu and Adams [9]):

Theorem 3. *Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $⨍_A f \star g d\mu$ is finite. Let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f, g are comonotone, then the inequality*

$$\Big(⨍_A (f \star g)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(⨍_A f^s d\mu\Big)^{\frac{1}{s}} \star \Big(⨍_A g^s d\mu\Big)^{\frac{1}{s}}. \tag{1}$$

holds for all $0 < s < \infty$.

Theorem 4 (Ouyang et al. [8])**.** *Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $⨍_A f d\mu$ and $⨍_A g d\mu$ are finite. Let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$\Big(⨍_A (f \star g)^s d\mu\Big)^{\frac{1}{s}} \geq \Big(⨍_A f^s d\mu\Big)^{\frac{1}{s}} \star \Big(⨍_A g^s d\mu\Big)^{\frac{1}{s}}. \tag{2}$$

holds for all $0 < s < \infty$.
It should be pointed out that Inequalities (1) *and* (2) *also hold for the original Sugeno integral.*

3. Minkowski type inequality

In this section, we prove the Minkowski type inequality for the semiconormed fuzzy integrals.

Theorem 5. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and non-decreasing in both arguments. If the semiconorm S satisfies*

$$S(a \star b, c) \leq (S(a,c) \star b) \wedge (a \star S(b,c)), \tag{3}$$

then the inequality

$$\Big(\int_{S,A} (f \star g)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(\int_{S,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu\Big)^{\frac{1}{s}} \tag{4}$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

Proof. Let $\int_{S,A} f^s d\mu = a$ and $\int_{S,A} g^s d\mu = b$, then for any $\varepsilon > 0$, there exist a_ε and b_ε such that $\mu(A \cap F_{\overline{(a_\varepsilon)^{\frac{1}{s}}}}) = a_1$ and $\mu(A \cap G_{\overline{(b_\varepsilon)^{\frac{1}{s}}}}) = b_1$, where $S(a_\varepsilon, a_1) \leq a + \varepsilon$ and $S(b_\varepsilon, b_1) \leq b + \varepsilon$. The fact of $H_{\overline{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}} \subset F_{\overline{(a_\varepsilon)^{\frac{1}{s}}}} \cup G_{\overline{(b_\varepsilon)^{\frac{1}{s}}}}$ and the comonotonicity of f, g imply that $\mu(A \cap H_{\overline{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}}) \leq a_1 \vee b_1$, where $F_{\overline{(a_\varepsilon)^{\frac{1}{s}}}} = \{x \mid f(x) > a_\varepsilon^{\frac{1}{s}}\} = \{x \mid$

$f^s(x) > a_\varepsilon\}$, $G_{\overline{(b_\varepsilon)^{\frac{1}{s}}}} = \{x \mid g(x) > b_\varepsilon^{\frac{1}{s}}\} = \{x \mid g^s(x) > b_\varepsilon\}$ and $H_{\overline{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}} = \{x \mid f \star g > a_\varepsilon^{\frac{1}{s}} \star b_\varepsilon^{\frac{1}{s}}\} = \{x \mid (f \star g)^s(x) > a_\varepsilon \star b_\varepsilon\}$. Hence

$$\begin{aligned}
\int_{S,A} (f \star g)^s d\mu &= \inf_{\alpha \in [0,1]} S(\alpha, \mu(A \cap H_{\overline{\alpha^{\frac{1}{s}}}})) \\
&\leq S(a_\varepsilon \star b_\varepsilon, a_1 \vee b_1) \\
&= S(a_\varepsilon \star b_\varepsilon, a_1) \vee S(a_\varepsilon \star b_\varepsilon, b_1) \\
&\leq [S(a_\varepsilon, a_1) \star b_\varepsilon] \vee [a_\varepsilon \star S(b_\varepsilon, b_1)] \\
&\leq [(a+\varepsilon) \star b_\varepsilon] \vee [a_\varepsilon \star (b+\varepsilon)] \\
&\leq (a+\varepsilon) \star (b+\varepsilon),
\end{aligned}$$

whence$\int_{S,A} (f \star g)^s d\mu \leq (a^{\frac{1}{s}} \star b^{\frac{1}{s}})^s$ follows from the continuity of $\star$ and the arbitrariness of ε. It follows that

$$\Big(\int_{S,A} (f \star g)^s d\mu\Big)^{\frac{1}{s}} \leq a^{\frac{1}{s}} \star b^{\frac{1}{s}} = \Big(\int_{S,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu\Big)^{\frac{1}{s}}.$$

Example 1. *Let $X = [0,1]$ and the fuzzy measure μ be the Lebesgue measure. Let $\star$ and S be defined as $\star(x,y) = x + y - xy$ and $S(x,y) = x + y - xy$. Let $f(x) = x$, $g(x) = \frac{1}{2}$ and $s = 2$. A straightforward calculus shows that*

$$\begin{aligned}
\int_{S,X} f^2 d\mu = \int_{S,X} x^2 d\mu &= \inf_{\alpha \in [0,1]} S(\alpha, \mu(\{x \in [0,1] \mid x^2 > \alpha\})) \\
&= \inf_{\alpha \in [0,1]} S(\alpha, (1 - \sqrt{\alpha})) \\
&= 0.6151.
\end{aligned}$$

Also we have

$$\begin{aligned}
\int_{S,X} g^2 d\mu = \int_{S,X} \frac{1}{4} d\mu &= \inf_{\alpha \in [0,1]} S(\alpha, \mu([0, \frac{1}{4}))) \\
&= \inf_{\alpha \in [0,1]} S(\alpha, \frac{1}{4}) \\
&= 0.25
\end{aligned}$$

$$\begin{aligned}
\int_{S,X} (f \star g)^2 d\mu &= \int_{S,X} \frac{1}{4}(x+1)^2 d\mu \\
&= \inf_{\alpha \in [\frac{1}{4},1]} S(\alpha, \mu(\{x \in [0,1] \mid \frac{1}{4}(x+1)^2 > \alpha\})) \\
&= \inf_{\alpha \in [\frac{1}{4},1]} S(\alpha, \mu(\{x \in [0,1] \mid x > 2\sqrt{\alpha} - 1\}))
\end{aligned}$$

$$= \inf_{\alpha \in [\frac{1}{4}, 1]} S(\alpha, 2 - 2\sqrt{\alpha})$$
$$= 0.780145.$$

Therefore,

$$0.883259 = \Big(\int_{S,X} (f \star g)^2 d\mu\Big)^{\frac{1}{2}} \leq \Big(\int_{S,X} f^2 d\mu\Big)^{\frac{1}{2}} \star \Big(\int_{S,X} g^2 d\mu\Big)^{\frac{1}{2}}$$
$$= 0.784283 \star 0.5 = 0.8921415$$

Notice that if the semiconormed S is maximum (i.e. for the sugeno integral) and $\star$ is bounded from below by maximum. Then S dominated by $\star$. Thus the following result holds.

Corollary 1. *Let $f, g : X \to [0,1]$ be two comonotone measurable functions. And let $\star : [0,1]^2 \to [0,1]$ be continuous and non-decreasing in both arguments and bounded from below by maximum. Then the inequality*

$$\Big(⨍_A (f \star g)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(⨍_A f^s d\mu\Big)^{\frac{1}{s}} \star \Big(⨍_A g^s d\mu\Big)^{\frac{1}{s}}.$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

Corollary 2 (Ouyang and Mesiar [7])**.** *Let$(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and non-decreasing in both arguments. If the semiconorm S satisfies*

$$S(a \star b, c) \leq (S(a,c) \star b) \wedge (a \star S(b,c))$$

then

$$\int_{S,A} f \star g d\mu \leq \Big(\int_{S,A} f d\mu\Big) \star \Big(\int_{S,A} g d\mu\Big)$$

for any $A \in \mathcal{F}$.

Theorem 6. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measures space and $f, g : X \to [0,1]$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and non-decreasing in both arguments and $\varphi : [0,1] \to [0,1]$ be a continuous and strictly increasing function such that φ commutes whit $\star$. If the semiconorm S satisfies*

$$S(a \star b, c) \leq (S(a,c) \star b) \wedge (a \star S(b,c)),$$

then

$$\varphi^{-1}\big(\int_{S,A} \varphi(f \star g) d\mu\big) \leq \varphi^{-1}\big(\int_{S,A} \varphi(f) d\mu\big) \star \ \varphi^{-1}\big(\int_{S,A} \varphi(g) d\mu\big) \tag{5}$$

holds for any $A \in \mathcal{F}$.

Proof. Since φ commutes with $\star$, so we have

$$\int_{S,A} \varphi(f \star g) d\mu = \int_{S,A} (\varphi(f) \star \varphi(g)) d\mu. \tag{6}$$

If f, g are comonotone functions and φ is continuous and strictly increasing function, then $\varphi(f)$ and $\varphi(g)$ are also comonotone. From (6) and using the Corollary 2, we have

$$\begin{aligned}\int_{S,A} (\varphi(f) \star \varphi(g)) d\mu &\le (\int_{S,A} \varphi(f) d\mu) \star (\int_{S,A} \varphi(f) d\mu)\\ &= \varphi[\varphi^{-1}(\int_{S,A} \varphi(f) d\mu) \star \varphi^{-1}(\int_{S,A} \varphi(g) d\mu)]\end{aligned}$$

where φ commutes with $\star$. Hence (5) is valid.

4. A related inequality

In this section, we prove a related inequality for the Minkowski's inequality in semiconormed fuzzy integrals.

Theorem 7. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,\ 1]$ be two comonotone measurable functions. Let $\star : [0,\ 1]^2 \to [0,\ 1]$ be continuous and nondecreasing in both arguments. If the semiconorm S satisfies*

$$S(a \star b, c) \ge (S(a, c) \star b) \vee (a \star S(b, c)), \tag{7}$$

then the inequality

$$\Big(\int_{S,A} (f \star g)^s d\mu\Big)^{\frac{1}{s}} \ge \Big(\int_{S,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu\Big)^{\frac{1}{s}} \tag{8}$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

Proof. Let $\int_{S,A} f^s d\mu = a,\ \int_{S,A} g^s d\mu = b$ and $\int_{S,A} (f \star g)^s d\mu = c$. Then for all $\alpha \in [0,\ 1]$ we have $S(\alpha, \mu(A \cap F_{\alpha^{\frac{1}{s}}})) \ge a,\ S(\alpha, \mu(A \cap G_{\alpha^{\frac{1}{s}}})) \ge b$ and $S(\alpha, \mu(A \cap H_{\alpha^{\frac{1}{s}}})) \ge c$, where $H_{\alpha^{\frac{1}{s}}} = \{x \mid (f \star g)(x) > \alpha^{\frac{1}{s}}\}$. Hence for any $\varepsilon > 0$, there exist $a_\varepsilon,\ b_\varepsilon$ and $c_\varepsilon = a_\varepsilon \star b_\varepsilon$ such that $\mu(A \cap F_{(a_\varepsilon)^{\frac{1}{s}}}) = a_1,\ \mu(A \cap G_{(b_\varepsilon)^{\frac{1}{s}}}) = b_1$ and $\mu(A \cap H_{(c_\varepsilon)^{\frac{1}{s}}}) = c_1$, where $S(c_\varepsilon, c_1) \le c + \varepsilon$. (Thus $a_\varepsilon \ge a,\ b_\varepsilon \ge b,\ S(a_\varepsilon, a_1) \ge a$ and $S(b_\varepsilon, b_1) \ge b$). The fact of $H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}} \supset F_{(a_\varepsilon)^{\frac{1}{s}}} \cap G_{(b_\varepsilon)^{\frac{1}{s}}}$ and the comonotonicity of f, g imply that $\mu(A \cap H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}) \ge a_1 \wedge b_1$. Hence

$$\begin{aligned}c + \varepsilon &\ge S(c_\varepsilon,\ c_1)\\ &= S(a_\varepsilon \star b_\varepsilon, \mu(A \cap H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}))\end{aligned}$$

$$
\begin{aligned}
&\geq S(a_\varepsilon \star b_\varepsilon, a_1 \wedge b_1\\
&= S(a_\varepsilon \star b_\varepsilon, a_1) \wedge S(a_\varepsilon \star b_\varepsilon, b_1)\\
&\geq (S(a_\varepsilon, a_1) \star b_\varepsilon) \wedge (a_\varepsilon \star S(b_\varepsilon, b_1))\\
&\geq (a \star b_\varepsilon) \wedge (a_\varepsilon \star b)\\
&\geq (a \star b) \wedge (a \star b)\\
&= a \star b.
\end{aligned}
$$

Hence $c \geq a \star b$ follows from the arbitrariness of ε. Consequently from the continuity of $\star$ we have $c \geq (a^{\frac{1}{s}} \star b^{\frac{1}{s}})^s$. This implies

$$\Big(\int_{S,A} (f \star g)^s d\mu\Big)^{\frac{1}{s}} \geq \Big(\int_{S,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu\Big)^{\frac{1}{s}}.$$

Example 2. *Let $A = [0,1]$ and μ be the Lebesgue measure. Let $\star$ be the usual product and S be the maximum. Let $f(x) = x^2$, $g(x) = \frac{1}{4}$ and $s = \frac{1}{2}$. A straightforward calculus shows that*

$$
\begin{aligned}
\int_{S,A} f^{\frac{1}{2}} d\mu = \int_{S,A} x d\mu &= \inf_{\alpha\in[0,1]} (\alpha \vee \mu(A \cap \{x \mid x > \alpha\}))\\
&= \inf_{\alpha\in[0,1]} (\alpha \vee (1-\alpha))\\
&= 0.5,
\end{aligned}
$$

$$
\begin{aligned}
\int_{S,A} g^{\frac{1}{2}} d\mu = \int_{S,A} \frac{1}{2} d\mu &= \inf_{\alpha\in[0,1]} (\alpha \vee \mu(A \cap [0, \frac{1}{2})))\\
&= 0.5
\end{aligned}
$$

and we have

$$
\begin{aligned}
\int_{S,A} (f \star g)^{\frac{1}{2}} d\mu &= \int_{S,A} \frac{1}{2} x d\mu\\
&= \inf_{\alpha\in[0,1]} (\alpha \vee \mu(A \cap \{x \in [0,1] \mid \frac{1}{2}x > \alpha\}))\\
&= \inf_{\alpha\in[0,1]} (\alpha \vee (1-2\alpha))\\
&= 0.33333.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0.11111 = \Big(\int_{S,X} (f \star g)^{\frac{1}{2}} d\mu\Big)^2 &\geq \Big(\int_{S,X} f^{\frac{1}{2}} d\mu\Big)^2 \star \Big(\int_{S,X} g^{\frac{1}{2}} d\mu\Big)^2\\
&= 0.25 \ .\ 0.25 = 0.0625.
\end{aligned}
$$

If in the Theorem 7 we assume $s = 1$, then we get the Chebyshev type inequality for the semiconormed fuzzy integrals.

Corollary 3. *Let* $(X, \mathcal{F}, \mu)$ *be a fuzzy measure space and* $f, g : X \to [0,\ 1]$ *be two comonotone measurable functions. Let* $\star : [0,\ 1]^2 \to [0,\ 1]$ *be continuous and non-decreasing in both arguments. If the semiconorm* S *satisfies*

$$S(a \star b, c) \geq (S(a, c) \star b) \vee (a \star S(b, c)),$$

then the inequality

$$\int_{S,A} (f \star g) d\mu \geq \int_{S,A} f d\mu \star \int_{S,A} g d\mu$$

holds for any $A \in \mathcal{F}$ *and for all* $0 < s < \infty$.

If $\star$ bounded from above by maximum, then $\star$ is dominated by maximum. Thus Corollary 4.4 gives us a general Minkowski type inequality for the Sugeno integral which appears in [8].

Corollary 4. *Let* $f, g : X \to [0, 1]$ *be two comonotone measurable functions. And let* $\star : [0, 1]^2 \to [0, 1]$ *be continuous and non-decreasing in both arguments and bounded from above by minimum. Then the inequality*

$$\Big(\fint_A (f \star g)^s d\mu \Big)^{\frac{1}{s}} \geq \Big(\fint_A f^s d\mu \Big)^{\frac{1}{s}} \star \Big(\fint_A g^s d\mu \Big)^{\frac{1}{s}}$$

holds for any $A \in \mathcal{F}$ *and for all* $0 < s < \infty$.

5. Conclusion

We have proved general Minkowski type inequalities for semiconormed fuzzy integrals on an abstract fuzzy measure space $(X, \mathcal{F}, \mu)$ based on a product like operator $\star$ and an inequality relevant for it. The semiconormed fuzzy integrals generalize the Sugeno integral, so it remains that when the following inequality

$$\Big(\int_{S,A} (f \star g)^s d\mu \Big)^{\frac{1}{s}} = \Big(\int_{S,A} f^s d\mu \Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu \Big)^{\frac{1}{s}}$$

holds.

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Parabolic Fractional Integral Operators with Rough Kernels in Parabolic Local Generalized Morrey Spaces

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Abstract. Let P be a real $n \times n$ matrix, whose all the eigenvalues have positive real part, $A_t = t^P$, $t > 0$, $\gamma = \mathrm{tr} P$ is the homogeneous dimension on $\mathbb{R}^n$ and Ω is an A_t-homogeneous of degree zero function, integrable to a power $s > 1$ on the unit sphere generated by the corresponding parabolic metric. We study the parabolic fractional integral operator $I^P_{\Omega,\alpha}$, $0 < \alpha < \gamma$ with rough kernels in the parabolic local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)$. We find conditions on the pair (φ_1, φ_2) for the boundedness of the operator $I^P_{\Omega,\alpha}$ from the space $LM^{\{x_0\}}_{p,\varphi_1,P}(\mathbb{R}^n)$ to another one $LM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $LM^{\{x_0\}}_{1,\varphi_1,P}(\mathbb{R}^n)$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 \le q < \infty$, $1 - 1/q = \alpha/\gamma$.

Key Words and Phrases: parabolic fractional integral, parabolic local generalized Morrey space.

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1. Introduction

The boundedness of classical operators of the real analysis, such as the fractional integral operators, from one weighted Lebesgue space to another one is well studied by now, and there are well known various applications of such results in partial differential equations. Besides Lebesgue spaces, Morrey spaces, both the classical ones (the idea od their definition having appeared in [13]) and generalized ones, also play an important role in the theory of partial differential equations.

In this paper, we find conditions for the boundedness of the parabolic fractional integral operators with rough kernel from a parabolic local generalized Morrey space to another one, including also the case of weak boundedness.

Note that we deal not exactly with the parabolic metric, but with a general anisotropic metric ρ of generalized homogeneity, the parabolic metric being its particular case, but we keep the term "parabolic in the title and text of the paper, following the existing tradition, see for instance [4].

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For $x \in \mathbb{R}^n$ and $r > 0$, we denote the open ball centered at x of radius r by $B(x,r)$, its complement by ${}^{\complement}B(x,r)$ and $|B(x,r)|$ will stand for the Lebesgue measure of $B(x,r)$.

Let P be a real $n \times n$ matrix, whose all the eigenvalues have positive real part. Let $A_t = t^P \ \ (t > 0)$, and set

$$\gamma = \mathrm{tr} P.$$

Then, there exists a quasi-distance ρ associated with P such that (see [5])

$$\begin{aligned}
&(a) \ \ \rho(A_t x) = t\rho(x), \quad t > 0, \quad \text{for every} \quad x \in \mathbb{R}^n;\\
&(b) \ \ \rho(0) = 0, \quad \rho(x) = \rho(-x) \geq 0\\
&\qquad \text{and} \quad \rho(x-y) \leq k(\rho(x-z) + \rho(y-z));\\
&(c) \ \ dx = \rho^{\gamma-1} d\sigma(w) d\rho, \quad \text{where} \quad \rho = \rho(x), w = A_{\rho^{-1}} x\\
&\qquad \text{and } d\sigma(w) \text{ is a measure on the unit ellipsoid } \ S_\rho = \{w : \rho(w) = 1\}.
\end{aligned}$$

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss (see [5]) and a homogeneous group in the sense of Folland-Stein (see [7]).

In the standard parabolic case $P_0 = \mathrm{diag}(1, \ldots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

The balls $\mathcal{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$ with respect to the quasidistance ρ are ellipsoids. For its Lebesgue measure one has

$$|\mathcal{E}(x,r)| = v_\rho r^\gamma,$$

where v_ρ is the volume of the unit ellipsoid. By ${}^{\complement}\mathcal{E}(x,r) = \mathbb{R}^n \setminus \mathcal{E}(x,r)$ we denote the complement of $\mathcal{E}(x,r)$.

Everywhere in the sequel $A \lesssim B$ means that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

1.1. Parabolic local generalized Morrey spaces

In the doctoral thesis [8], 1994 by Guliyev (see, also [9], [1]-[3]) introduced the local Morrey-type space $LM_{p\theta,w}$ given by

$$\|f\|_{LM_{p\theta,w}} = \big\| w(r) \, \|f\|_{L_p(B(0,r))} \big\|_{L_\theta(0,\infty)},$$

where w is a positive measurable function defined on $(0,\infty)$. If $\theta = \infty$, it denotes $LM_{p,w} \equiv LM_{p\infty,w}$. In [8] Guliyev intensively studied the classical operators in the local Morrey-type space $LM_{p\theta,w}$ (see also the book [9] (1999)), where these results were presented for the case when the underlying space is the Heisenberg group or a homogeneous group, respectively. Note that, the generalized local (central) Morrey spaces $LM_{p,\varphi}(\mathbb{R}^n) = LM_{p,\varphi}^{\{0\}}(\mathbb{R}^n)$ introduced by Guliyev in [8] (see also, [9], [1]-[3]).

We define the parabolic local Morrey space $LM_{p,\lambda,P}(\mathbb{R}^n)$ via the norm

$$\|f\|_{LM_{p,\lambda,P}} = \sup_{t>0} \left(t^{-\lambda} \int_{\mathcal{E}(0,t)} |f(y)|^p dy \right)^{1/p} < \infty,$$

where $1 \le p \le \infty$ and $0 \le \lambda \le \gamma$.

If $\lambda = 0$, then $LM_{p,0,P}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$; if $\lambda = \gamma$, then $LM_{p,\gamma,P}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$; if $\lambda < 0$ or $\lambda > \gamma$, then $LM_{p,\lambda,P} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WLM_{p,\lambda,P}(\mathbb{R}^n)$ the weak parabolic Morrey space of functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\lambda,P}} = \sup_{t>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(0,r))} < \infty,$$

where $WL_p(\mathcal{E}(0,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(\mathcal{E}(0,r))} = \sup_{t>0} t \left|\{y \in \mathcal{E}(0,r) : |f(y)| > t\}\right|^{1/p}.$$

Note that $WL_p(\mathbb{R}^n) = WLM_{p,0,P}(\mathbb{R}^n)$,

$$LM_{p,\lambda,P}(\mathbb{R}^n) \subset WLM_{p,\lambda,P}(\mathbb{R}^n) \ \text{ and } \ \|f\|_{WLM_{p,\lambda,P}} \le \|f\|_{LM_{p,\lambda,P}}.$$

If $P = I$, then $LM_{p,\lambda}(\mathbb{R}^n) \equiv LM_{p,\lambda,I}(\mathbb{R}^n)$ is the local Morrey space.

We introduce the parabolic local generalized Morrey spaces following the known ideas of defining local generalized Morrey spaces ([8, 11, 12] etc).

Definition 1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. The space $LM_{p,\varphi,P} \equiv LM_{p,\varphi,P}(\mathbb{R}^n)$, called the parabolic local generalized Morrey space, is defined by the norm*

$$\|f\|_{LM_{p,\varphi,P}} = \sup_{t>0} \varphi(0,t)^{-1} \left|\mathcal{E}(0,t)\right|^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}(0,t))}.$$

Definition 2. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. The space $WLM_{p,\varphi,P} \equiv WLM_{p,\varphi,P}(\mathbb{R}^n)$, called the weak parabolic local generalized Morrey space, is defined by the norm*

$$\|f\|_{WLM_{p,\varphi,P}} = \sup_{t>0} \varphi(0,t)^{-1} \left|\mathcal{E}(0,t)\right|^{-\frac{1}{p}} \|f\|_{WL_p(\mathcal{E}(0,t))}.$$

If $P = I$, then $LM_{p,\varphi}(\mathbb{R}^n) \equiv LM_{p,\varphi,I}(\mathbb{R}^n)$ and $WLM_{p,\varphi}(\mathbb{R}^n) \equiv WLM_{p,\varphi,I}(\mathbb{R}^n)$ are the generalized local Morrey space and the weak generalized local Morrey space, respectively.

Definition 3. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM^{\{x_0\}}_{p,\varphi,P} \equiv LM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)$ the parabolic generalized local Morrey space, the space of all functions $f \in L^{\rm loc}_p(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM^{\{x_0\}}_{p,\varphi,P}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi,P}}.$$

Also by $WLM^{\{x_0\}}_{p,\varphi,P} \equiv WLM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^{\rm loc}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WLM^{\{x_0\}}_{p,\varphi,P}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi,P}} < \infty.$$

According to this definition, we recover the space $LM^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n)$ under the choice $\varphi(0,r) = r^{\frac{\lambda-\gamma}{p}}$:

$$LM^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n) = LM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-\gamma}{p}}}.$$

Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ $(n \ge 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$ and Ω be A_t-homogeneous of degree zero, i.e. $\Omega(A_t x) \equiv \Omega(x)$, $x \in \mathbb{R}^n$, $t > 0$. The parabolic fractional integral $I^P_{\Omega,\alpha} f$ by with rough kernels, $0 < \alpha < \gamma$, of a function $f \in L^{\rm loc}_1(\mathbb{R}^n)$ is defined by

$$I^P_\alpha f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)\ f(y)}{\rho(x-y)^{\gamma-\alpha}} dy.$$

We prove the boundedness of the parabolic integral operator $I^P_{\Omega,\alpha}$ with rough kernel from one parabolic local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi_1,P}(\mathbb{R}^n)$ to another one $LM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $LM^{\{x_0\}}_{1,\varphi_1,P}(\mathbb{R}^n)$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 \le q < \infty$, $1 - 1/q = \alpha/\gamma$.

2. Preliminaries

Let v be a weight on $(0,\infty)$. We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)|g(t)|$$

and write $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions. By $\mathfrak{M}^+(0,\infty;\uparrow)$ we denote the cone of all functions in $\mathfrak{M}^+(0,\infty)$ non-decreasing on $(0,\infty)$ and introduce also the set

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a non-negative continuous function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [2].

Theorem 1. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{1}$$

We are going to use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty g(s) w(s) ds, \ \ 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.

The following theorem in the case $w = 1$ was proved in [3].

Theorem 2. *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H^*_w g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{2}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{3}$$

Moreover, if C^* is the minimal value of C in (2), then $C^* = B$.

Remark 1. *In (2) and (3) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.*

3. Parabolic fractional integral operator with rough kernels in the spaces $LM^{\{x_0\}}_{p,\varphi,P}$

In [10] was proved the (p,p)-boundedness of the operator M^P_Ω and the (p,q)-boundedness of the operator $M^P_{\Omega,\alpha}$.

Theorem 1. *[10] Let $\Omega \in L_s(S_\rho)$, $1 < s \le \infty$, be A_t-homogeneous of degree zero. Then the operator M^P_Ω is bounded in the space $L_p(\mathbb{R}^n)$, $p > s'$.*

Theorem 2. *[10] Suppose that $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$, is A_t-homogeneous of degree zero. Let $1 \le p < \frac{\gamma}{\alpha}$ and $1/p - 1/q = \alpha/\gamma$. Then the fractional integration operator I^P_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$ and from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$.*

The following lemma is valid.

Lemma 1. *Suppose that $x_0 \in \mathbb{R}^n$, $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$, is A_t-homogeneous of degree zero. Let $1 \le p < \frac{\gamma}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. Then for all $f \in L^{\rm loc}_p$ there hold the inequalities*

$$\|I^P_{\Omega,\alpha} f\|_{L_q(\mathcal{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^\infty t^{-\frac{\gamma}{q}-1} \|f\|_{L_p(\mathcal{E}(x_0,t))} dt, \quad p > 1$$

and

$$\|I^P_{\Omega,\alpha} f\|_{WL_q(\mathcal{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^\infty t^{-\frac{\gamma}{q}-1} \|f\|_{L_1(\mathcal{E}(x_0,t))} dt, \quad p = 1. \tag{1}$$

Proof. For a given ball $\mathcal{E} = \mathcal{E}(x_0, r)$ f, we represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2k\mathcal{E}}(y), \quad f_2(y) = f(y)\chi_{\complement(2k\mathcal{E})}(y), \quad r > 0,$$

and have

$$\|I^P_{\Omega,\alpha} f\|_{L_q(\mathcal{E})} \le \|I^P_{\Omega,\alpha} f_1\|_{L_q(\mathcal{E})} + \|I^P_{\Omega,\alpha} f_2\|_{L_q(\mathcal{E})}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, by the boundedness of $I^P_{\Omega,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that

$$\|I^P_{\Omega,\alpha} f_1\|_{L_q(\mathcal{E})} \le \|I^P_{\Omega,\alpha} f_1\|_{L_q(\mathbb{R}^n)} \le C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2k\mathcal{E})}.$$

Observe that the conditions $x \in \mathcal{E}$, $y \in \complement(2k\mathcal{E})$ imply

$$\frac{1}{2k}\rho(x_0 - y) \le \rho(x - y) \le \frac{3k}{2}\rho(x_0 - y).$$

We then get

$$|I^P_{\Omega,\alpha} f_2(x)| \le 2^{\gamma-\alpha} c_1 \int_{\complement(2k\mathcal{E})} \frac{|f(y)||\Omega(x-y)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy.$$

By Fubini's theorem we have

$$\begin{aligned}
\int_{\complement(2k\mathcal{E})} \frac{|f(y)||\Omega(x-y)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy &\approx \int_{\complement(2k\mathcal{E})} |f(y)||\Omega(x-y)| \int_{\rho(x_0-y)}^\infty \frac{dt}{t^{\gamma+1-\alpha}} dy \\
&\approx \int_{2kr}^\infty \int_{2kr \le \rho(x_0-y) < t} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{\gamma+1-\alpha}} \\
&\lesssim \int_{2kr}^\infty \int_{\mathcal{E}(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{\gamma+1-\alpha}}.
\end{aligned}$$

Applying Hölder's inequality with $1/p + (\gamma-\alpha)/\gamma + (\alpha p - \gamma)/\gamma p = 1$ taken into account, we get

$$\begin{aligned}
&\int_{\complement_{(2k\mathcal{E})}} \frac{|f(y)||\Omega(x-y)|}{\rho(x_0-y)^{\gamma-\alpha}} dy \\
&\lesssim \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \, \|\Omega(\cdot - y)\|_{L_{\frac{\gamma}{\gamma-\alpha}}(\mathcal{E}(x_0,t))} \, |\mathcal{E}(x_0,t)|^{\frac{\alpha}{\gamma}-\frac{1}{p}} \, \frac{dt}{t^{\gamma+1-\alpha}} \\
&\lesssim \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \, \frac{dt}{t^{\frac{\gamma}{q}+1}}.
\end{aligned}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|I^P_{\Omega,\alpha} f_2\|_{L_q(\mathcal{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{2}$$

is valid. Thus

$$\|I^P_{\Omega,\alpha} f\|_{L_q(\mathcal{E})} \lesssim \|f\|_{L_p(2k\mathcal{E})} + r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.$$

On the other hand,

$$\begin{aligned}
\|f\|_{L_p(2k\mathcal{E})} &\approx \; r^{\frac{\gamma}{q}} \|f\|_{L_p(2k\mathcal{E})} \int_{2kr}^{\infty} \frac{dt}{t^{\frac{\gamma}{q}+1}} \\
&\leq \; r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.
\end{aligned} \tag{3}$$

Thus

$$\|I^P_{\Omega,\alpha} f\|_{L_q(\mathcal{E})} \lesssim \; r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.$$

By Fubini's theorem and the Minkowski inequality, we get

$$\begin{aligned}
\|I^P_{\Omega,\alpha} f_2\|_{L_q(\mathcal{E})} &\leq \Big(\int_{\mathcal{E}} \Big| \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{\gamma+1-\alpha}} \Big|^q \Big)^{\frac{1}{q}} \\
&\leq \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_q(\mathcal{E})} dy \frac{dt}{t^{\gamma+1-\alpha}} \\
&\lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_1(\mathcal{E}(x_0,t))} \frac{dt}{t^{\gamma+1-\alpha}} \\
&\lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}.
\end{aligned}$$

Finally, in the case $p = 1$ by the weak $(1,q)$-boundedness of $I^P_{\Omega,\alpha}$ and the inequality (3) it follows that

$$\|I^P_{\Omega,\alpha} f_1\|_{WL_q(\mathcal{E})} \leq \|I^P_{\Omega,\alpha} f_1\|_{WL_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)}$$

$$= \|f\|_{L_1(2k\mathcal{E})} \lesssim r^{\frac{\gamma}{q}} \int_{2kr}^{\infty} \|f\|_{L_1(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}}. \tag{4}$$

Then from (2) and (4) we get the inequality (1).

Theorem 3. *Suppose that $x_0 \in \mathbb{R}^n$, $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero. Let $1 \le p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{\gamma}{q}+1}} dt \le C\, \varphi_2(x_0,r), \tag{5}$$

where C does not depend on x_0 and r. Then the operator $I^P_{\Omega,\alpha}$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1,P}$ to $LM^{\{x_0\}}_{q,\varphi_2,P}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1,P}$ to $WLM^{\{x_0\}}_{q,\varphi_2,P}$ for $p = 1$.

Proof. By Lemma 1 and Theorem 2 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1(r) = \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p}}$ and $w(r) = r^{-\frac{\gamma}{q}}$ we have for $p > 1$

$$\begin{aligned}\|I^P_{\Omega,\alpha} f\|_{M_{q,\varphi_2,P}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}} \\ &\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\frac{\gamma}{p}}\, \|f\|_{L_p(\mathcal{E}(x_0,r))} = \|f\|_{M_{p,\varphi_1,P}}\end{aligned}$$

and for $p = 1$

$$\begin{aligned}\|I^P_{\Omega,\alpha} f\|_{WM_{q,\varphi_2,P}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_1(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{\gamma}{q}+1}} \\ &\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\gamma}\, \|f\|_{L_1(\mathcal{E}(x_0,r))} = \|f\|_{M_{1,\varphi_1,P}}.\end{aligned}$$

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Atomic Decomposition in a Direct Sum of Banach Spaces and Their Application to Discontinuous Differential Operators

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Abstract. An atomic decomposition is considered in Banach space. A method for constructing an atomic decomposition of Banach space, proceeding from atomic decomposition of subspaces is presented. Some relations between them are established. The proposed method is used in the study of the frame properties of systems of eigenfunctions and associated functions of discontinuous differential operators.

Key Words and Phrases: p-frames, $\tilde{X}$-frames, conjugate systems to $\tilde{X}$.
2010 Mathematics Subject Classifications: 34L10, 41A58, 46A35

1. Introduction

One of the commonly used methods for solving differential equations is the method of separation of variables (Fourier method). This method yields the appropriate spectral problem (usually with respect to the space variables). To justify the method is very important the question of the expansion of functions of a certain class on eigenfunctions of the spectral problem. That is why many mathematicians have been paying so much attention lately to the study of frame properties (such as completeness, minimality, basicity, atomic decomposition) of the systems of special functions, mostly eigenfunctions and associated functions of differential operators. Various methods have been developed for establishing these properties. For more information we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9]. In case of discontinuous differential operator, there arise the systems of eigenfunctions that cannot be treated for frameness by the earlier methods. To shed some light on this situation, we consider the following model spectral problem for second order discontinuous differential operator

$$-y''(x) = \lambda y(x), \, x \in (-1, 0) \cup (0, 1), \tag{1}$$

with the boundary conditions

$$y(-1) = y(1) = 0; y(-0) = y(+0), \tag{2}$$

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$$y'(-0) - y(+0) = \lambda m y(0).$$

This spectral problem has two sets of eigenfunctions [9]:

$$u_n^{(1)}(x) = \sin \pi n x, x \in [-1, 1], n \in N,$$

and

$$\tilde{u}_n^{(2)}(x) = \begin{cases} \sin \pi n x + 0\left(\frac{1}{n}\right), & x \in [-1, 0], \\ -\sin \pi n x + 0\left(\frac{1}{n}\right), & x \in [0, 1], \quad n \in N. \end{cases}$$

Such spectral problems arise when solving the problem of a loaded string fixed at both ends with a load placed in the middle of the string by the Fourier method [10, 11]. The use of this method requires the study of basis properties of the double system $\left\{u_n^{(1)}; \tilde{u}_n^{(2)}\right\}_{n \in N}$ in corresponding spaces of functions (usually in the Lebesgue or Sobolev spaces). Of course, it should be started with the basis properties of the system $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n \in N}$, which is the principal part of the asymptotics of the system $\left\{u_n^{(1)}; \tilde{u}_n^{(2)}\right\}_{n \in N}$, where

$$u_n^{(2)} = \begin{cases} \sin \pi n x, & x \in [-1, 0), \\ -\sin \pi n x, & x \in [0, 1]. \end{cases}$$

This is usually followed by the application of various perturbation methods. This approach is well studied (see, e.g., the articles [5, 6, 7, 8, 9, 13, 14, 15, 16] and the monographs [12, 17, 18, 19, 20, 21, 22, 23]). On the other hand, it is not difficult to see that the principal part $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n \in N}$ is not a standard (in other words, classical) system. It turns out that the form of the system $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n \in N}$ is not special, i.e. it can be derived from the general case. The general approach to these systems allows introducing a new approach for constructing bases with a lot of applications in the spectral theory of differential operators. It should be noted that some problems of an atomic decomposition and frames with respect to the specific systems have been previously studied in [27, 28, 29, 30, 31].

In this work we consider an abstract approach to the above problem. We consider a direct expansion of a Banach space with respect to subspaces. We offer a method for constructing an atomic decomposition for a space proceeding from atomic decomposition for subspaces.

2. Notation and needful information

We will use the standard notation. N will be a set of all positive integers; $L[M]$ will denote the linear span of the set M and $\overline{M}$ will stand for the closure of M; X^* will denote a space conjugate to X; $L(X_1, X_2)$ will be a space of linear bounded operators from X_1 to X_2 with $L(X, X) = L(X)$; D_T will denote a domain of the operator T and R_T will be the range of T; $KerT$ will stand for the kernel of the operator T; $< x, f >= f(x)$ will denote the value of the functional f at the point x; Banach space will be referred to as

B-space; $\|\cdot\|_X$ will denote a norm in X; $\Leftrightarrow$ will mean "if and only if"; $1:n \equiv \{1;...;n\}$; δ_{ij} will be the Kronecker symbol.

We will also use the concept of the space of coefficients. We define it as follows. Let $\vec{x} \equiv \{x_n\}_{n\in N} \subset X$ be a non-degenerate system in a B-space X, i.e. $x_n \neq 0$, $\forall n \in N$. Define

$$\mathscr{K}_{\vec{x}} \equiv \left\{ \{\lambda_n\}_{n\in N} : \textit{ the series } \sum_{n=1}^{\infty} \lambda_n x_n \textit{ is convergent in } X \right\}.$$

Introduce the norm in $\mathscr{K}_{\vec{x}}$:

$$\left\|\vec{\lambda}\right\|_{\mathscr{K}_{\vec{x}}} = \sup_m \left\| \sum_{n=1}^{m} \lambda_n x_n \right\|, \quad \text{where } \vec{\lambda} = \{\lambda_n\}_{n\in N}.$$

With respect to the usual operations of addition and multiplication by a complex number, $\mathscr{K}_{\vec{x}}$ is a B- space. Take $\forall \vec{\lambda} \in \mathscr{K}_{\vec{x}}$ and consider the operator $T : \mathscr{K}_{\vec{x}} \to X$:

$$T\vec{\lambda} = \sum_{n=1}^{\infty} \lambda_n x_n, \; \vec{\lambda} = \{\lambda_n\}_{n\in N}.$$

Denote by $\{e_n\}_{n\in N} \subset \mathscr{K}_{\vec{x}}$ a canonical system in $\mathscr{K}_{\vec{x}}$, where $e_n = \{\delta_{nk}\}_{k\in N}$. It is absolutely clear that $Te_n = x_n$, $\forall n \in N$. The following statement is true.

Statement 1. *Space of coefficients $\mathscr{K}_{\vec{x}}$ is a B-space with the canonical basis $\{e_n\}_{n\in N}$. Moreover, the system $\vec{x}$ forms a basis for $X \Leftrightarrow T$ performs an isomorphism between $\mathscr{K}_{\vec{x}}$ and X.*

Let's recall some concepts and facts from the frames theory . First, let us give a definition of atomic decomposition in Banach spaces.

Definition 1. *Let X be a B-space and $\mathscr{K}$ be a B- space of the sequences of scalars. Let $\{f_k\}_{k\in N} \subset X$, $\{g_k\}_{k\in N} \subset X^*$. Then $\left(\{g_k\}_{k\in N} \; ; \; \{f_k\}_{k\in N}\right)$ is an atomic decomposition of X with respect to $\mathscr{K}$, if :*

(i) $\{g_k(f)\}_{k\in N} \in \mathscr{K}$, $\forall f \in X$;

(ii) $\exists A, B > 0 : A\|f\|_X \leq \left\| \{g_k(f)\}_{k\in N} \right\|_{\mathscr{K}} \leq B\|f\|_X$, $\quad \forall f \in X$;

(iii) $f = \sum_{k=1}^{\infty} g_k(f) f_k$, $\forall f \in X$.

The concept of the frame is a generalization of the concept of an atomic decomposition.

Definition 2. *Let X be a B-space and $\mathscr{K}$ be a B-space of the sequences of scalars. Let $\{g_k\}_{k\in N} \subset X^*$ and be a bounded operator. Then $\left(\{g_k\}_{k\in N} \; ; S\right)$ forms a Banach frame for X with respect to $\mathscr{K}$, if:*

(i) $\{g_k(f)\}_{k\in N} \in \mathscr{K}$, $\forall f \in X$;

(ii) $\exists A, B > 0 : A\|f\|_X \leq \left\| \{g_k(f)\}_{k\in N} \right\|_{\mathscr{K}} \leq B\|f\|_X$, $\quad \forall f \in X$;

(iii) $S\ \left[\{g_k(f)\}_{k\in N}\right] = f$, $\forall f \in X$.

It is true the following

Proposition 1. *Let X be a B-space and $\mathscr{K}$ a B-space of the sequences of scalars with canonical basis $\{\delta_n\}_{n\in N}$. Let $\{g_k\}_{k\in N} \subset X^*$ and $S \in L(\mathscr{K};;X)$. Then the following statements are equivalent:*

(i) $\left(\{g_k\}_{k\in N}\ ;S\right)$ is a Banach frame for X with respect to $\mathscr{K}$;

(ii)$\left(\{g_k\}_{k\in N}\ ;\{S\ (\delta_k)\}_{k\in N}\right)$ is an atomic decomposition of X with respect to $\mathscr{K}$.

More information about the above facts can be found in [17, 18, 19, 20, 21, 22, 23, 24].

In the sequel, we will use the following construction and some obvious facts. Let the following direct sum hold

$$X = X_1 \oplus \ldots \oplus X_m,$$

where X_i, $i=\overline{1,\,m}$, are some B-spaces. For convenience, we will represent the elements of the space X in the form of a vector

$$x \in X \ \ \Leftrightarrow \ \ x = (x_1, x_2, ..., x_m)\,,$$

where $x_k \in X_k$, $k=\overline{1,m}$. The norm in X will be defined by the formula

$$\|x\|_X = \sqrt{\sum_{i=1}^{m} \|x_i\|_{X_i}^2}.$$

Then we have $X^* = X_1^* \oplus \ldots \oplus X_m^*$ (see [13]), and for $\vartheta \in X^*$ and $x \in X$ it holds

$$< x,\, \vartheta >= \sum_{i=1}^{m} < x_i,\, \vartheta_i >,$$

where $\vartheta = (\vartheta_1, ..., \vartheta_m)$ and

$$\|\vartheta\|_{X^*} = \sqrt{\sum_{i=1}^{m} \|\vartheta_i\|_{X_i^*}^2}.$$

Let some system $\left\{u_n^{(i)}\right\}_{n\in N} \subset X_i$ be given for every $i \in 1:m$. Consider the following system in the space X :

$$u_{in}^0 = \left(\underbrace{0, ..., 0, u_n^{(i)}}_{i}, 0, ..., 0\right)\ ,\ \ i=\overline{1,\,m}\,;\ n \in N.$$

Let the pair $\left(\left\{u_{in}^0\right\}_{i=\overline{1,\,m};n\in N}; \{\vartheta_{in}\}_{i=\overline{1,\,m};n\in N}\right)$ be an atomic decomposition of X with respect to the space of coefficients $\mathscr{K}$, i.e. $\forall x \in X$ has a decomposition of the form

$$x = \sum_{i=1}^{m}\sum_{n=1}^{\infty} \vartheta_{in}\,(x)\, u_{in}^0, \tag{3}$$

moreover, the following inequality holds

$$A\,\|\{\vartheta_{in}\,(x)\}\|_{\mathscr{K}} \le \|x\|_X \le B\,\|\{\vartheta_{in}\,(x)\}\|_{\mathscr{K}}\,. \tag{4}$$

Suppose

$$\vartheta_{in} = \left(\vartheta_{in}^{(1)}; ...; \vartheta_{in}^{(m)}\right) \in X^*,$$

where $\vartheta_{in}^{(k)} \in X_k^*, \forall k \in 1:m$. We have $(x = (x_1, ..., x_m))$: $\vartheta_{in}(x) = \sum_{k=1}^{m} \vartheta_{in}^{(k)}(x_k)$, $\quad i = \overline{1, m}$; $n \in N$. Take $\forall k \in 1:m$, and let

$$x_k^0 = \left(\underbrace{0; ...; 0; x_k}_{k}; 0; ...; 0\right).$$

We have

$$\vartheta_{in}\left(x_k^0\right) = \vartheta_{in}^{(k)}(x_k).$$

Then from (4) we obtain

$$A\left\|\left\{\vartheta_{in}^{(k)}(x_k)\right\}\right\|_{\mathscr{K}} \leq \|x_k\|_{X_k} \leq B\left\|\left\{\vartheta_{in}^{(k)}(x_k)\right\}\right\|_{\mathscr{K}}.$$

Paying attention to the decomposition (3), we obtain

$$x_k^0 = (0, ..., 0, x_k, 0, ..., 0) = \sum_{i=1}^{m}\left(\underbrace{0, ..., 0, \sum_{n=1}^{\infty} \vartheta_{in}^{(k)}(x_k)\, u_n^{(i)}}_{i}, 0, ..., 0\right) =$$

$$= \left(\sum_{n=1}^{\infty} \vartheta_{1n}^{(k)}(x_k)\, u_n^{(1)}, ..., \sum_{n=1}^{\infty} \vartheta_{kn}^{(k)}(x_k)\, u_n^{(k)}, ..., \sum_{n=1}^{\infty} \vartheta_{mn}^{(m)}(x_k)\, u_n^{(m)}\right) \Rightarrow$$

$$\sum_{n=1}^{\infty} \vartheta_{in}^{(k)}(x_k)\, u_n^{(i)} = \begin{cases} x_k, & i = k, \\ 0, & i \neq k. \end{cases} \tag{5}$$

As a result, we obtain that

$$\left(\left\{u_n^{(k)}\right\}_{n\in N}; \left\{\vartheta_{kn}^{(k)}\right\}_{n\in N}\right), \tag{6}$$

is an atomic decomposition of X_k with respect to the space of coefficients $\mathscr{K}$, for every $k \in 1:m$.

Accept the following

Definition 3. *The pair* $\left(\{u_n\}; \left\{\vec{\vartheta}_n\right\}\right)$ $\left(u_n \in X \wedge \vec{\vartheta}_n \in X^*\right)$ *is called an atomic decomposition of* X *with respect to* $\mathscr{K}^m$, *if the following conditions are fulfilled:*

i) $\left\{\vec{\vartheta}_n(x)\right\} \in \mathscr{K}^m, \quad \forall x \in X$;

ii) $\exists\, A; B > 0: \quad A\left\|\left\{\vec{\vartheta}_n(x)\right\}\right\|_{\mathscr{K}^m} \leq \|x\|_X \leq B\left\|\left\{\vec{\vartheta}_n(x)\right\}\right\|_{\mathscr{K}^m}$;

iii) $x = \sum_{n=1}^{\infty} \vec{\vartheta}_n(x)\, u_n, \quad \forall x \in X.$

The following theorem is true.

Theorem 1. *i) Let the pair* $\left(\left\{u_{in}^{0}\right\}_{i=\overline{1,m};\ n\in N};\{\vartheta_{in}\}_{i=\overline{1,m};\ n\in N}\right)$ *be an atomic decomposition of* X *with respect to* $\mathscr{K}$, *where* $\vartheta_{in}=\left(\vartheta_{in}^{(1)};...;\vartheta_{in}^{(m)}\right)\in X^{*}$, $i\in 1:m$; $n\in N$. *Then the relation* (5) *holds and system* (6) *is an atomic decomposition of* X_k *with respect to* $\mathscr{K}$.
ii) Let the pair (6) *be an atomic decomposition of* X_k *for every* $k\in 1:m$ *with respect to* $\mathscr{K}$ *and the relation* (5) *holds. Then* $\left(\left\{\vec{\vartheta}_n\right\};\{u_n\}\right)$ *is an atomic decomposition of* X *with respect to* $\mathscr{K}^m$ *in the sense of Definition 3.*

3. Main results

Let the following direct sum hold

$$X=X_1\dot{+}...\dot{+}X_m,$$

where X_k, $k=\overline{1,\,m}$—are some B-spaces. Consider the system $\{u_{in}\}_{n\in N}\subset X_i$, $i=\overline{1,\,m}$; and form

$$\vec{u}_{in}=(a_{i1}u_{1n};a_{i2}u_{2n};...;a_{im}u_{mn}),\ \ i=\overline{1,\,m};\ n\in N.$$

Let

$$A=(a_{ij})\ \ i,j=\overline{1,\,m};\quad \Delta=\det A.$$

We will need the following easy-to-prove lemma.

Lemma 1. *Let* $(\{u_n\};\{\vartheta_n\})$ *be an atomic decomposition of* X *with respect to* $\mathscr{K}$ *and* $T\in L(X)$ *be some automorphism. Then* $\left(\{Tu_n\};\left\{(T^{*})^{-1}\vartheta_n\right\}\right)$ *is also an atomic decomposition of* X *with respect to* $\mathscr{K}$.

Let $T_{ij}:X_i\to X_j$ be some operators. Consider the system

$$\sum_{i=1}^{m}a_{ij}T_{ij}x_i=y_j,\quad j=\overline{1,\,m}, \tag{7}$$

where $y_j\in X_j$, $j=\overline{1,\,m}$ are the given, and $x_i\in X_i$, $i=\overline{1,\,m}$ are the sought elements. Assume that the spaces X_k, $k=\overline{1,\,m}$, are pairwise isomorphic and T_{ij} performs a corresponding isomorphism. Besides, assume that the following conditions are satisfied:

α) $T_{ii}=I_i$, $T_{ij}=T_{ji}^{-1}$, $T_{jk}T_{ij}=T_{ik}$, $\forall i,j=\overline{1,m}$, *where* I_i *is the identity operator in* X_i.

Applying the operator $T_{j1}=T_{1j}^{-1}$ to the j-th equation in the system (7), we obtain the following system

$$\sum_{i=1}^{m}a_{ij}T_{i1}x_i=T_{j1}y_j,\quad j=\overline{1,\,m}.$$

Let $\tilde{x}_i = T_{i1}x_i$, $\tilde{y}_j = T_{j1}y_j$. It is clear that $\tilde{x}_i, \tilde{y}_j \in X_1$. As a result, we obtain the following system of linear equations in the space X_1:

$$\sum_{i=1}^{m} a_{ij}\tilde{x}_i = \tilde{y}_j \, , \; j = \overline{1, m}.$$

If the determinant of this system $\Delta = \det(a_{ij}) \neq 0$, then it is clear that this system is uniquely solvable with respect to the unknowns $\tilde{x}_i$. Then the system (7) is also uniquely solvable.

The following lemma is true.

Lemma 2. *Let the operators $T_{ij} : X_i \to X_j$ perform an isomorphism between X_i and X_j, the conditions α) be satisfied and $\Delta \neq 0$. Then the system* (7) *is uniquely solvable for $\forall y \in X$, $y = (y_1, \ldots, y_m)$ and, moreover, $\exists M > 0$:*

$$\|x\|_X \leq M \, \|y\|_X \, , \tag{8}$$

where $x = (x_1, \ldots, x_m)$.

The validity of the estimate (8) follows immediately from the following representation for the solution of the system (7):

$$x_i = \sum_{j=1}^{m} b_{ij} T_{ji} y_j, \;\; i = \overline{1, \, m},$$

where b_{ij} are the elements of the inverse matrix A^{-1}.

Consider the operator $T : X \to X$ defined by the matrix $(a_{ij}T_{ij})_{i,\,j=1}^{m}$. Let all the conditions of Lemma 2 be satisfied. It follows from this lemma that $KerT = \{0\}$, $R_T = X$, and the estimate (8) means $T \in L(X)$. Then it follows from Banach's theorem on the inverse operator that T is an automorphism in X. So the following theorem is true.

Theorem 2. *Let $T_{ij} \in L(X_i, X_j)$ be an isomorphism, the conditions α) be satisfied and $\Delta \neq 0$. Then the operator $T : X \to X$ defined by the matrix $(a_{ij}T_{ij})_{i,\,j=1}^{m}$ is an automorphism in $X = X_1 \oplus \ldots \oplus X_m$.*

The following theorem is true.

Theorem 3. *Let the direct sum $X = X_1 \dot{+} ... \dot{+} X_m$ hold, the pair $\left(\{u_{in}\}_{n\in N}; \{\vartheta_{in}\}_{n\in N}\right)$ be an atomic decomposition of X_i, $i = \overline{1, m}$; with respect to $\mathscr{K}$, $T_{ij} \in L(X_i; X_j)$ be an isomorphism and $T_{ij}u_{in} = u_{jn}$, $\forall n \in N$, for $i \neq j$. Let $\det(a_{ij})_{i,j=\overline{1,m}} \neq 0$, and operators $T_{ij}; i, j = \overline{1, m}$; satisfy the condition α) and the operator $T \in L(X)$ defined by the matrix $(a_{ij}T_{ij})_{i,j=\overline{1,m}}$. Then the pair $\left(\left\{\{Tu_{in}^0\}_{n\in N}\right\}_{i=\overline{1,m}}; \left\{\left\{(T^*)^{-1}\vartheta_{in}^0\right\}_{n\in N}\right\}_{i=\overline{1,m}}\right)$, is also an atomic decomposition of X with respect to $\mathscr{K}^m$.*

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