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About a Problem of Optimal Investment of the Stock Market

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Abstract. The article analyzes the investment portfolio containing 3 assets from the Russian stock market. This analysis provides an opportunity to determine how to make an effective portfolio with less risk. The article takes into account the real data on financial instruments. The calculations are presented in Microsoft Excel program.

Key Words and Phrases: correlation, covariance, risk, expected return, portfolio variance, standard deviation, effective portfolio, effective frontier, Markowitz model.

2010 Mathematics Subject Classifications: 74P05; 91G10

In this article, the problem of constructing an optimal portfolio of stocks for a given level of return is solved. To solve the problem, two methods were used: the Solver command from the Excel application program and the Lagrange multiplier method. What is more, the first method is considered for both cases, when short sales of assets are allowed and prohibited. The application of the second method is described only for the case of the possibility of short sales, in that the condition that the Lagrange method is not negative is not considered. Comparison of the effectiveness of these products in all cases can be performed by comparing the final results of portfolio management. One of the classic problems of optimization of the portfolio of these securities is the task of determining the relative densities of each of these securities.

Suppose an investor has a portfolio of N kinds of assets. It is necessary to decide what specific weight each of these securities should have, so that the investor can obtain the desired return, and the portfolio risk at the same time would be reduced to a minimum. As it is known [1], [2], to solve this problem the expected return of each of the securities and their matrix of pairwise covariance will be needed. The whole task is led to minimizing the portfolio variance.

$$\sigma_\rho^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \theta_i \theta_j cov_{(r_i r_j)}. \tag{1}$$

Under two limiting conditions:

1) the expected return on the portfolio (r_p) is equals to:

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$$\overline{r_p} = \sum_{i=1}^{n} \theta_i \overline{r}_i \tag{2}$$

where - r_i - the profitability of the i-th security, and θ_i - its weight in the portfolio.
2) the sum of the weights of all assets is equal to one:

$$\sum_{i=1}^{n} \theta_i = 1. \tag{3}$$

In this paper, we examine some of the important tasks for the numerical computer calculation of optimal portfolio shares. As we observe the three companies as Aeroflot, Megafon, and Rosneft for 2016. The goal of the task is to find the optimal proportion of these shares using the method of Harry Markowitz.

In order to determine the expected return, we need the real yields of assets that we would like to include in the portfolio. For example, take the data in a percentage of Aeroflot, Megafon, and Rosneft for 2016. The investment period equals to one month in 2016.

Using formulas (1), (2) and (3), we will compile a model for this portfolio of three stocks. Find $\theta = (\theta_1, \theta_2, \theta_3)$:

$$\sigma_\rho^2 = \theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2 + \theta_3^2\sigma_3^2 + 2\theta_1\theta_2 cov_{1,2} + 2\theta_1\theta_3 cov_{1,3} + 2\theta_2\theta_3 cov_{2,3} \ \rightarrow min \tag{4}$$

$$\begin{cases} \theta_1\overline{r_1} + \theta_2\overline{r_2} + \theta_3\overline{r_3} = \overline{r_p} \\ \sum_i^3 \theta_i = 1 \\ \theta_i \geq 0.1 \quad i = \overline{1,3} \end{cases}$$

	A	B	C	D
1	**month**	**Aeroflot**	**Megafon**	**Rosneft**
2	January	-9.43	1.41	7.10
3	February	12.18	-0.46	5.64
4	March	30.21	-13.79	6.42
5	April	4.35	-1.07	15.94
6	May	6.04	1.22	-8.84
7	June	3.52	-10.06	4.38
8	July	0.69	-0.07	-1.17
9	August	15.04	-0.37	5.57
10	September	18.23	-8.83	-0.57
11	October	11.13	-0.82	0.77
12	November	3.60	-6.27	-2.31
13	December	12.41	1.60	17.38

Table 1. Monthly growth for 2016

For the case of the absence of dividend payments, the yield coefficient can be determined from the formula:

$$r = (P_1 - P_0)/P_0, \tag{5}$$

where P_0 - is the value of the security in the initial period; P_1 - the value of the security at the end of the period. [6]
Knowing the monthly returns, we will try to determine the expected return on each of the assets. To find the expected (average) yield of Aeroflot, Megafon, and Rosneft, we need the following table:

AEROFLOT				MEGAFON				ROSNEFT			
return	frequency	probability		return	frequency	probability		return	frequency	probability	
-9.43	1.00	0.08	-0.79	1.41	1	0.08	0.12	7.10	1	0.08	0.59
12.18	1.00	0.08	1.02	-0.46	1	0.08	-0.04	5.64	1	0.08	0.47
30.21	1.00	0.08	2.52	-13.79	1	0.08	-1.15	6.42	1	0.08	0.53
4.35	1.00	0.08	0.36	-1.07	1	0.08	-0.09	15.94	1	0.08	1.33
6.04	1.00	0.08	0.50	1.22	1	0.08	0.10	-8.84	1	0.08	-0.74
3.52	1.00	0.08	0.29	-10.06	1	0.08	-0.84	4.38	1	0.08	0.37
0.69	1.00	0.08	0.06	-0.07	1	0.08	-0.01	-1.17	1	0.08	-0.10
15.04	1.00	0.08	1.25	-0.37	1	0.08	-0.03	5.57	1	0.08	0.46
18.23	1.00	0.08	1.52	-8.83	1	0.08	-0.74	-0.57	1	0.08	-0.05
11.13	1.00	0.08	0.93	-0.82	1	0.08	-0.07	0.77	1	0.08	0.06
3.60	1.00	0.08	0.30	-6.27	1	0.08	-0.52	-2.31	1	0.08	-0.19
12.41	1.00	0.08	1.03	1.60	1	0.08	0.13	17.38	1	0.08	1.45
Sum	**12.00**	**1.00**	**9.00**	**Sum**	**12**	**1**	**-3.13**	**Sum**	**12**	**1**	**4.19**

Table 2. Expected returns

After determining the expected return values of assets, we can proceed to the calculation of risks, which are determined by dispersion and standard deviations, as a measure of the spread of possible outcomes relative to the expected value. Consequently, the higher the dispersion, the greater the spreading, and hence the risk. To calculate the dispersion, the following formula is chosen:

$$\sigma^2 = \frac{\sum_{i=1}^{n} r_i - r}{(n-1)} \tag{6}$$

r_i – return of the asset, r_{aver}-expected (average) return on an asset, n-number of observations.
To calculate the "standard quadratic deviation" (standard deviation), which is the square root of the dispersion, is used:

$$\sigma_p = \sqrt{\sigma^2} \tag{7}$$

Here is an example of calculation of dispersion and standard deviation using Microsoft Excel based on the available data of Aeroflot, Megafon, and Rosneft, using the built-in functions of **VAR** and **STDEV**.
As a result, we get:

AEROFLOT				MEGAFON				ROSNEFT			
return	frequency	probability		return	frequency	probability		return	frequency	probability	
-9.43	1.00	0.08	-0.79	1.41	1	0.08	0.12	7.10	1	0.08	0.59
12.18	1.00	0.08	1.02	-0.46	1	0.08	-0.04	5.64	1	0.08	0.47
30.21	1.00	0.08	2.52	-13.79	1	0.08	-1.15	6.42	1	0.08	0.53
4.35	1.00	0.08	0.36	-1.07	1	0.08	-0.09	15.94	1	0.08	1.33
6.04	1.00	0.08	0.50	1.22	1	0.08	0.10	-8.84	1	0.08	-0.74
3.52	1.00	0.08	0.29	-10.06	1	0.08	-0.84	4.38	1	0.08	0.37
0.69	1.00	0.08	0.06	-0.07	1	0.08	-0.01	-1.17	1	0.08	-0.10
15.04	1.00	0.08	1.25	-0.37	1	0.08	-0.03	5.57	1	0.08	0.46
18.23	1.00	0.08	1.52	-8.83	1	0.08	-0.74	-0.57	1	0.08	-0.05
11.13	1.00	0.08	0.93	-0.82	1	0.08	-0.07	0.77	1	0.08	0.06
3.60	1.00	0.08	0.30	-6.27	1	0.08	-0.52	-2.31	1	0.08	-0.19
12.41	1.00	0.08	1.03	1.60	1	0.08	0.13	17.38	1	0.08	1.45
Sum	**12.00**	**1.00**	**9.00**	**Sum**	**12**	**1**	**-3.13**	**Sum**	**12**	**1**	**4.19**
Dispersion		**99.66**		**27.23**				**54.97**			
Standard deviation		**9.98**		**5.22**				**7.41**			

Table 3.Dispersions and standard deviations

Final calculations allow us to state that the least risky paper is Megafon. The expected monthly return is -3.13% at a risk of 5.22%. And the riskiest paper is Aeroflot. It should be noted that not always the asset having the highest standard deviation is the riskiest. Therefore, before using the standard deviation as a measure of relative risk, you need to calculate the risk per unit of return using the coefficient of variation. This indicator will not be analyzed, as applied to the formation of our portfolio, it is not necessary.

Knowing the expected returns and risk indicators (standard deviation), it is necessary to make a number of calculations to determine the coefficients of covariance and correlation. After calculating these coefficients, it becomes possible to form portfolios that meet our requirements for risk and return. Note that covariance is a measure that takes into account the dispersion of individual values of the return of a paper and the strength of the links between changes in the returns of this paper and others.

The formula for calculating the covariance is the following:

$$cov\left(\theta_i\theta_j\right) = \frac{\sum_{i=1}^{n}\left(r_{\theta_i} - r_{\theta_i}\right) * \left(r_{\theta_j} - r_{\theta_j}\right)}{n-1} \tag{8}$$

where r_X and r_Y – returns of assets X and Y, rx_{aver} and rY_{aver}– expected (average) returns of the assets X and Y, n – number of observations.

The positive value of covariance indicates that the values of the returns of these assets change in one direction, the negative value of covariance indicates multidirectional movements between returns. Covariance is low if the fluctuations in the returns of the two assets in any direction are random. To interpret covariance, as well as dispersion, is quite difficult due to large numerical values, so almost always a correlation coefficient is used to measure the strength of the relationship between the two assets.

The correlation coefficient lies in the interval from -1 to +1. A correlation value of +1 indicates a strong relationship, as assets go the same way. The value -1, on the contrary, indicates a different direction, to wit, the growth of one of the assets is accompanied by the fall of the other. A value of 0 indicates no correlation.
To calculate the correlation in the EXCEL environment, we will use the following formula:

$$\rho = \frac{cov(\theta_i, \theta_j)}{\sigma_{\theta_i} * \sigma_{\theta_j}} \tag{9}$$

where $cov(X, Y)$ – covariance between to assets X and Y in the denominator are standard deviations of the assets X and Y.
Here are the calculations of covariance and correlation using Excel between Aeroflot, Megafon, and Rosneft, using the built-in **COVAR** and **CORREL** functions.

month	Aeroflot	Megafon	Rosneft
January	-9.43	1.41	7.10
February	12.18	-0.46	5.64
March	30.21	-13.79	6.42
April	4.35	-1.07	15.94
May	6.04	1.22	-8.84
June	3.52	-10.06	4.38
July	0.69	-0.07	-1.17
August	15.04	-0.37	5.57
September	18.23	-8.83	-0.57
October	11.13	-0.82	0.77
November	3.60	-6.27	-2.31
December	12.41	1.60	17.38

Covariance	Aeroflot	Megafon	Rosneft
Aeroflot	91.36	-25.76	5.26
Megafon	-25.76	24.96	4.08
Rosneft	5.26	4.08	50.39

Correlation	Aeroflot	Megafon	Rosneft
Aeroflot	1.00	-0.54	0.08
Megafon	-0.54	1.00	0.11
Rosneft	0.08	0.11	1.00

Table 4. Covariance and Correlations

As can be seen from the correlation table, the monthly returns of our assets in the segment of 2015 are not quite positively correlated, which, of course, is not very good, but even the inclusion of positively correlated assets in the portfolio can significantly reduce the risk of the entire portfolio. Having all the data that you can now transfer to the formation of the portfolio and the problems associated with it.

An investment portfolio - is a purposefully formed set of objects of real and financial investment for the implementation of the investment policy of the enterprise in the forthcoming period. The main objective in the formation of the investment portfolio is to ensure the implementation of the main areas of the investment activity of the enterprise by selecting the most profitable and safe investment objects. Taking into consideration the main goal, a system of specific local goals is being constructed, the main of which are:

1. high level of income in the current period
2. minimization of investment risks
3. sufficient liquidity of the investment portfolio

These specific objectives of forming an investment portfolio are largely alternative. Thus, ensuring a high growth rate of capital, in the long run, is achieved to a certain extent by reducing the current return level of the investment portfolio, and vice versa. The level of the current return on the investment portfolio directly depends on the level of investment risks. Ensuring sufficient liquidity can prevent the inclusion of investment projects in the portfolio aimed at obtaining high capital growth in the long-term. Given the alternatives of the objectives of forming an investment portfolio, each investor himself sets their priorities.

The alternative nature of the objectives of forming an investment portfolio determines the differences in the company's financial investment policy, which in turn provides a specifically formed type of investment portfolio. The income portfolio is formed by the criterion of maximizing the level of investment profit in the current period, regardless of the growth rate of the invested capital in the long term. In other words, this portfolio is focused on high current return on investment costs, despite the fact that in the future period these costs could provide a higher rate of investment return on invested capital. [7]

Now, let's begin to search for the optimal portfolio.

According to Markowitz, any investor should base his choice exclusively on the expected return and the standard deviation when choosing a portfolio. Thus, having assessed various combinations of portfolios, he should choose the "best", based on the ratio of expected returns and standard deviation of these portfolios. At the same time, the ratio of return on portfolio risk remains the same: the higher the return, the higher the risk.

Also, before starting to form a portfolio, it is necessary to define the term "effective portfolio". An effective portfolio is a portfolio that provides: the maximum expected return for a certain level of risk, or the minimum level of risk for some expected return.

In the future, we will find efficient portfolios in the Excel environment in accordance with the second principle - with a minimum level of risk for any expected return. In order to find the optimal portfolio, it is necessary to determine the acceptable set of risk-return ratios for the investor, which is achieved by constructing a minimal dispersion portfolio frontier, that is, the border on which portfolios with a minimum risk for a given return lies.

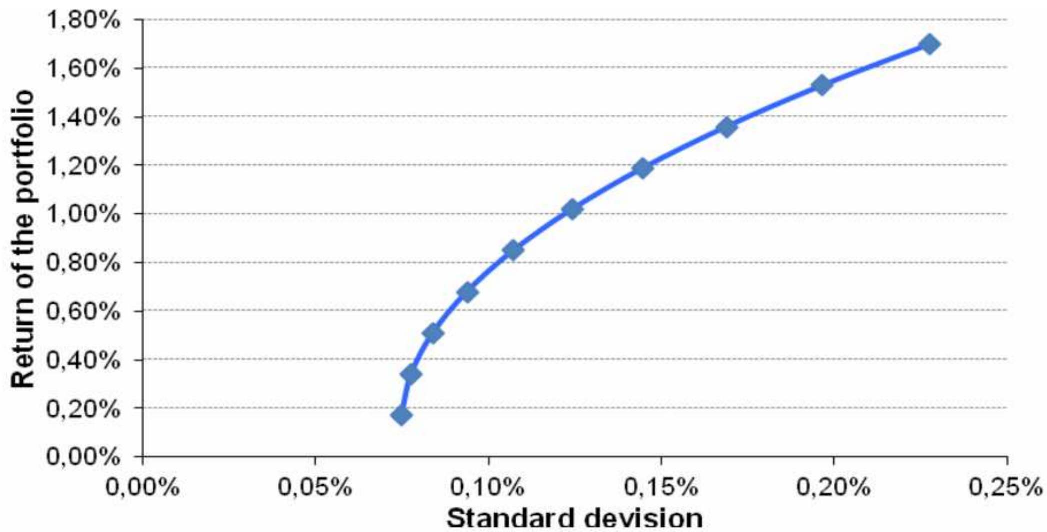

Table 5.Effective frontier

In the table above, the bold line shows the "effective frontier", and the big points indicate possible combinations of portfolios.

An effective frontier - is a boundary that defines an effective set of portfolios. Portfolios lying to the left of the effective boundary can not be applied, that is, they do not belong to an admissible set. Portfolios on the right (internal portfolios) and below the effective border are ineffective because there are portfolios that at a given level of risk provide higher returns or lower risk for a given level of profitability.
Let's start by calculating the expected return on the portfolio using the formula:

$$E\left(r_{\rho}\right) = \sum_{i=1}^{n} \theta_i E\left(r_i\right), \tag{10}$$

where X_i – the share of the i-th paper in the portfolio:
$E(r_i)$ – the expected return on the i-th paper:
And then determine the dispersion of the portfolio, in the formula which uses double summation:

$$\sigma_{\rho}^{2} = \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_i \theta_j cov_{(r_i r_j)}, \tag{11}$$

where $\sigma\ \rho 2$ – dispersion of portfolio:
$X_i X_j$ – the proportion of the i-th and j-th paper in the portfolio:
$cov(ri, rj)$ – covariance of returns of papers i and j.
As a consequence, we find the standard deviation of the portfolio which is the square root of the dispersion of the formula:

$$\sigma = \sqrt{\sigma^2} \tag{12}$$

Applying the above formula, we set the share of each asset in our initial portfolio in proportion to their number. Therefore, the share of each asset in the portfolio will be 1/3, to wit, 33%. The total share should be equal to 1, both for portfolios in which "short" positions are allowed and for those that are prohibited. If "short" positions are allowed, the share of the asset will be displayed as -0.33 and the proceeds from its sale must be invested in another asset, so the share of assets in the portfolio will, in any case, be 1. [8]

We calculate the expected return, dispersion and standard deviation of the average weighted portfolio. As can be seen from the table, in order to determine the portfolio dispersion, you just need to sum the data in cells B28-D28, and the square root of the cell value C31 will give us the standard deviation of the portfolio in cell C32. The multiplication of shares of papers by their expected return will give us the expected return on our portfolio, which is reflected in cell C33. The final result of the average weighted portfolio is presented below.

	A	B	C	D
16		Аэрофлот	Мегафон	Роснефть
17	Аэрофлот	91.36	-25.76	5.26
18	Мегафон	-25.76	24.96	4.08
19	Роснефть	5.26	4.08	50.39
20				
21				
22				
23		Аэрофлот	Мегафон	Роснефть
24	доля	0.33	0.33	0.33
25	0.33	10.15	-2.86	0.58
26	0.33	-2.86	2.77	0.45
27	0.33	0.58	0.45	5.60
28	**1.00**	**7.87**	**0.36**	**6.64**
29				
30				
31	Дисперсия портфеля		**14.87**	
32	Стандарт отклон. Порт		**3.86**	
33	Ожидаемая доход. Порт.		**3.35**	
34				

	G	H	I	J
16		Аэрофлот	Мегафон	Роснефть
17	Аэрофлот	1.00	-0.54	0.08
18	Мегафон	-0.54	1.00	0.11
19	Роснефть	0.08	0.11	1.00

	G	Ожидаемая доходность	Стандарт отклонение
24	Аэрофлот	9.00	9.98
25	Мегафон	-3.13	5.22
26	Роснефть	4.19	7.41

Table 6.

The average (expected) monthly return of the average weighted portfolio is 3.35% at a risk of 3.86%. Now we can apply the very second principle, which was written above, that is, to ensure a minimum risk at a given level of profitability. To do this, use the "Find Solutions" function from the "Tools" menu. [9]

We launch the "Find solutions", in the item "Set the specified cell" we specify cell C32, which we will minimize by changing the shares of assets in the portfolio, that is, by varying the values in cells A25-A27. Then we need to add two conditions, namely:
1) the sum of shares must equal 1; cell A28 = 1,

2) To set the profitability that interests us, for example, the return of 3.35% (cell 33), which turned out when calculating the average weighted portfolio.
As we prohibit the presence of "short" positions on securities in the "Options" menu, it is necessary to put a tick in the box "Non-negative values". This should look like this:

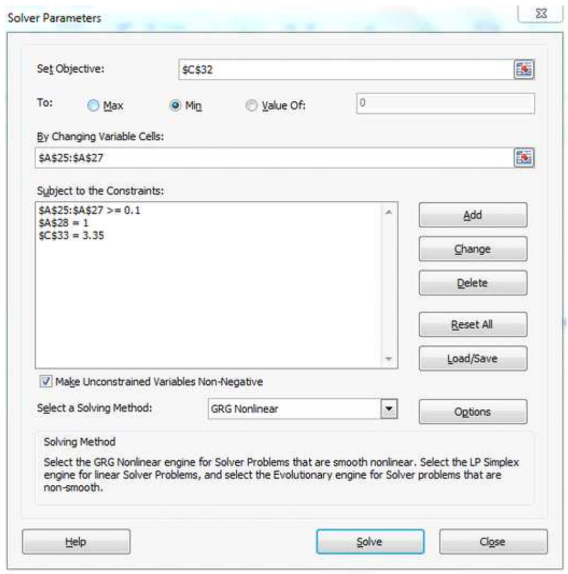

Table 7. "Solver"

Table 8. Solver options

As a result, we get:

Covariance

	Aeroflot	Megafon	Rosneft
Aeroflot	91.36	-25.76	5.26
Megafon	-25.76	24.96	4.08
Rosneft	5.26	4.08	50.39

Correlation

	Aeroflot	Megafon	Rosneft
Aeroflot	1.00	-0.54	0.08
Megafon	-0.54	1.00	0.11
Rosneft	0.08	0.11	1.00

	Aeroflot	Megafon	Rosneft
shares	0.33	0.33	0.33
0.47	14.43	-4.07	0.83
0.43	-3.66	3.55	0.58
0.10	0.18	0.14	1.68
1.00	**10.95**	**-0.39**	**3.09**

Dispersion of portfolio	**13.65**
Standard deviation of port.	**3.69**
Expected return of port.	**3.35**

	Expected returns	Standard deviation
Aeroflot	9.00	9.98
Megafon	-3.13	5.22
Rosneft	4.19	7.41

Table 9. Result of "Finding Solutions"

So, having set "Finding solutions" to find the minimum standard deviation for a given expected return of 0.33%, we got the optimal portfolio consisting of 47% of Aeroflot, 43% of Megafon and 10% of Rosneft. Despite the fact that the level of profitability is the same as with the average weighted portfolio, the risk has decreased.

Let's observe a number of drawbacks inherent in the model H. Markowitz.

This model was developed for efficient capital markets, where there is a constant growth in the value of assets and there are no sharp fluctuations in rates, which was more typical for the economy of the developed countries of the 50-80s. The correlation between shares is not constant and changes with time, as a result in the future this does not reduce the systematic risk of the investment portfolio.

The future profitability of financial instruments (assets) is determined as the arithmetic mean. This forecast is based only on the historical significance of the stock's returns and does not include the impact of macroeconomic (GDP, inflation, unemployment, sectoral indices of prices for commodities and materials, etc.) and microeconomic factors (liquidity, profitability, financial stability, business activity of the company).

The risk of a financial instrument is estimated using a measure of return variability related to the arithmetic mean, but a change in return above is not a risk but represents a stock's super-returns.

For the case where short sales of assets are allowed, an analytically effective boundary and an effective portfolio can be found using the Lagrange multiplier method. The problem is led to minimizing the dispersion of the portfolio [5] and [10]:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\theta_i\theta_j cov_{ij} \tag{13}$$

Under two limiting conditions:

1) the expected return on the portfolio (r_p) is:

$$\overline{r_p} = \sum_{i=1}^{n} \theta_i \overline{r}_i \tag{14}$$

2) the sum of the weights of all assets is equal to one:

$$\sum_{i=1}^{n} \theta_i = 1 \tag{15}$$

Artificially created and minimized the Lagrange function in the form:

$$L = G + \lambda_1 C_1 + \lambda_2 C_2 \tag{16}$$

Where L is the Lagrange function;

G - objective function;

$\lambda 1$ and $\lambda 2$ are the Lagrange multipliers for the first and second constraints;

C_1, C_2 - first and second constraints.

The objective function represented by the function (14), first constraints - the equation (15), the second - (16). In the Lagrange function, we include the first, second and third constraints in the following form:

$$\begin{cases} \sum_{i=1}^{3} \theta_i \overline{r}_i - \overline{r_p} = 0 \\ \sum_{i=1}^{3} \theta_i - 1 = 0 \\ \theta_i \geq 0.1 \quad i = 1,3 \end{cases} \tag{17}$$

The standard deviation of the return of the first asset (in decimal values) is 9.98, the second - 5.22, the third - 7.41. The covariance of the return of the first and second stocks is - (-25.7624), the first and third - 5.2605, the second and the third - 4.0757. The return of the first paper (in decimal values) is 9.00, the second - (-3.13), the third - 4.19. Determine the specific weight of assets in the portfolio with a return of 3.35.

As an objective function to be optimized in this task performs a function:

$$F(\theta) = \theta_1^2 9,98^2 + \theta_2^2 5,22^2 + \theta_3^2 7,41^2 + 2\theta_1\theta_2(-25.76) + 2\theta_1\theta_3 5.26 + 2\theta_2\theta_3 4.08 \tag{18}$$

We rewrite the restriction of the problem in an implicit form:

$$\begin{cases} \varphi_1(\theta) = 3.35 - (9\theta_1 - 3.13\theta_2 + 4.19\theta_3) = 0 \\ \varphi_2(\theta) = 1 - (\theta_1 + \theta_2 + \theta_3) = 0 \\ \varphi_3(\theta) = 0.1 - (\theta_3) = 0 \end{cases} \tag{19}$$

Let us compose the auxiliary Lagrange function. Since the objective function is not linear in the formula. In this case, the Kuhn-Tucker method [12] should be applied in order to solve the problem posed:

$$\mathrm{L}(\theta,\ \lambda,\ \mu) = 9,98^2\theta_1^2 + 5.22^2\theta_2^2 + 7.41^2\theta_3^2 + 2\theta_1\theta_2(-25.76) +$$

$$+2\theta_1\theta_3\,(5.26)+2\theta_2\theta_3(4.08)\ +\ \lambda_1(3.35-(9\theta_1-3.13\theta_2+4.19\theta_3))+$$
$$+\lambda_2(1-(\theta_1+\theta_2+\theta_3))\ +\ \mu_3\,(0.1-(\theta_3)) \tag{20}$$

A necessary condition for the extremum of the Lagrange function is the vanishing of its partial derivatives with respect to the variables and θ_i, i = 1,2,3 undetermined factors.

Set up a system:

$$\begin{cases} \frac{\partial L}{\partial \theta_1} = 199,2\theta_1 - 51,52\theta_2 + 10,52\theta_3 - 9\lambda_1 - \lambda_2 = 0 \\ \frac{\partial L}{\partial \theta_2} = -51,52\theta_1 + 54,5\theta_2 + 8,16\theta_3 + 3,13\lambda_1 - \lambda_2 = 0 \\ \frac{\partial L}{\partial \theta_3} = 10,52\theta_1 + 8,16\theta_2 + 109,8\theta_3 - 4,19\lambda_1 - \lambda_2 - \mu_3 = 0 \\ \frac{\partial L}{\partial \lambda_1} = 3,35 - (9\theta_1 - 3,13\theta_2 + 4,19\theta_3) = 0 \\ \frac{\partial L}{\partial \lambda_2} = 1 - (\theta_1 + \theta_2 + \theta_3) = 0 \\ \mu_3\,(0,1 - (\theta_3)) = 0, \qquad \mu_3 \geq 0 \end{cases} \tag{21}$$

Let us solve the following system of equations:

$$\begin{cases} 199,2\theta_1 - 51,52\theta_2 + 10,52\theta_3 - 9\lambda_1 - \lambda_2 = 0 \\ -51,52\theta_1 + 54,5\theta_2 + 8,16\theta_3 + 3,13\lambda_1 - \lambda_2 = 0 \\ 10,52\theta_1 + 8,16\theta_2 + 109,8\theta_3 - 4,19\lambda_1 - \lambda_2 - \mu_3 = 03,35 - (9\theta_1 - 3,13\theta_2 + 4,19\theta_3) = 0 \\ 1 - (\theta_1 + \theta_2 + \theta_3) = 0 \\ \mu_3\,(0,1 - (\theta_3)) = 0, \qquad \mu_3 \geq 0 \end{cases} \tag{22}$$

Solving the system of equations (22) by the inverse matrix method we finally obtain:
θ_1= 0,4739 or 47.39 %
θ_2= 0,4261 or 42.61 %
θ_3= 0,10 or 10 %

Similarly, we set restrictions with a weight of 10% for the second and first shares in the portfolio, and find a solution using the Kuhn-Tucker method:

$$\mathrm{L}(\theta,\ \lambda,\ \mu)\ = 9,98^2\theta_1^2{+}5.22^2\theta_2^2{+}7.41^2\theta_3^2{+}2\theta_1\theta_2(-25.76) + 2\theta_1\theta_3\,(5.26) +$$
$$+2\theta_2\theta_3(4.08)\ +\ \lambda_1(3.35-(9\theta_1-3.13\theta_2+4.19\theta_3))\ +\ \lambda_2(1-(\theta_1+\theta_2+\theta_3))\ +\ \mu_3\,(0.1-(\theta_2)) \tag{23}$$

Let's make the system:

$$\begin{cases} \frac{\partial L}{\partial \theta_1} = 199,2\theta_1 - 51,52\theta_2 + 10,52\theta_3 - 9\lambda_1 - \lambda_2 = 0 \\ \frac{\partial L}{\partial \theta_2} = -51,52\theta_1 + 54,5\theta_2 + 8,16\theta_3 + 3,13\lambda_1 - \lambda_2 = 0 \\ \frac{\partial L}{\partial \theta_3} = 10,52\theta_1 + 8,16\theta_2 + 109,8\theta_3 - 4,19\lambda_1 - \lambda_2 - \mu_3 = 0 \\ \frac{\partial L}{\partial \lambda_1} = 3,35 - (9\theta_1 - 3,13\theta_2 + 4,19\theta_3) = 0 \\ \frac{\partial L}{\partial \lambda_2} = 1 - (\theta_1 + \theta_2 + \theta_3) = 0 \\ \mu_3\,(0,1 - (\theta_2)) = 0, \qquad \mu_3 \geq 0 \end{cases} \tag{24}$$

Let us solve the following system of equations:

$$\left\{\begin{array}{c} 199,2\theta_1 - 51,52\theta_2 + 10,52\theta_3 - 9\lambda_1 - \lambda_2 = 0 \\ -51,52\theta_1 + 54,5\theta_2 + 8,16\theta_3 + 3,13\lambda_1 - \lambda_2 = 0 \\ 10,52\theta_1 + 8,16\theta_2 + 109,8\theta_3 - 4,19\lambda_1 - \lambda_2 - \mu_3 = 0 \\ 3,35 - (9\theta_1 - 3,13\theta_2 + 4,19\theta_3) = 0 \\ 1 - (\theta_1 + \theta_2 + \theta_3) = 0 \\ \mu_3\,(0,1 - (\theta_2)) = 0, \qquad \mu_3 \geq 0 \end{array}\right. \tag{25}$$

Solving the system of equations (25) by the inverse matrix method we finally obtain:
θ_1= - 0, 0225 or -2.25 %
θ_2= 0,1 or 10 %
θ_3= 0,9225 or 92,25 %

$$\mathrm{L}(\theta,\ \lambda,\ \mu)\ = 9,98^2\theta_1^2{+}5.22^2\theta_2^2{+}7.41^2\theta_3^2{+}2\theta_1\theta_2(-25.76) + 2\theta_1\theta_3\,(5.26) +$$

$$+2\theta_2\theta_3(4.08)\ +\ \lambda_1(3.35 - (9\theta_1{-}3.13\theta_2{+}4.19\theta_3))\ +\ \lambda_2(1 - (\theta_1{+}\theta_2{+}\theta_3))\ +\ \mu_3\,(0.1{-}\,(\theta_1)) \tag{26}$$

Let's make the system:

$$\left\{\begin{array}{c} \frac{\partial L}{\partial \theta_1} = 199,2\theta_1 - 51,52\theta_2 + 10,52\theta_3 - 9\lambda_1 - \lambda_2 = 0 \\ \frac{\partial L}{\partial \theta_2} = -51,52\theta_1 + 54,5\theta_2 + 8,16\theta_3 + 3,13\lambda_1 - \lambda_2 = 0 \\ \frac{\partial L}{\partial \theta_3} = 10,52\theta_1 + 8,16\theta_2 + 109,8\theta_3 - 4,19\lambda_1 - \lambda_2 - \mu_3 = 0 \\ \frac{\partial L}{\partial \lambda_1} = 3,35 - (9\theta_1 - 3,13\theta_2 + 4,19\theta_3) = 0 \\ \frac{\partial L}{\partial \lambda_2} = 1 - (\theta_1 + \theta_2 + \theta_3) = 0 \\ \mu_3\,(0,1 - (\theta_1)) = 0, \qquad \mu_3 \geq 0 \end{array}\right. \tag{27}$$

Let us solve the following system of equations:

$$\left\{\begin{array}{c} 199,2\theta_1 - 51,52\theta_2 + 10,52\theta_3 - 9\lambda_1 - \lambda_2 = 0 \\ -51,52\theta_1 + 54,5\theta_2 + 8,16\theta_3 + 3,13\lambda_1 - \lambda_2 = 0 \\ 10,52\theta_1 + 8,16\theta_2 + 109,8\theta_3 - 4,19\lambda_1 - \lambda_2 - \mu_3 = 0 \\ 3,35 - (9\theta_1 - 3,13\theta_2 + 4,19\theta_3) = 0 \\ 1 - (\theta_1 + \theta_2 + \theta_3) = 0 \\ \mu_3\,(0,1 - (\theta_1)) = 0, \qquad \mu_3 \geq 0 \end{array}\right. \tag{28}$$

Solving the system of equations (28) by the inverse matrix method we finally obtain:
θ_1= 0,1 or 10 %
θ_2= 0,1805 or 18,05 %
θ_3= 0,7195 or 71,95 %

Comparing the first, second and third decisions, we see that for an investor with a rational strategy it is important to make such a decision that with a minimum risk to obtain an allowable amount of profit. However, for an investor with a conservative plan of action, it is most likely that it is essential to choose such a combination of risk and income ratios that the maximum profitability will be the priority.

Summary

The studied stock packages in the article are owned by leading Russian stock market companies in real time. The optimization of the investment portfolio here has been resolved both with the Microsoft Excel software package technologically and with the Kunn-Takker method by adding Lagrangian multipliers. Optimization problems by changing restrictive conditions have been investigated, the dynamic structure of the optimal investment portfolio for aggressive investors has been identified.

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The Boundedness Weighted Hardy Operator in the Orlicz-Morrey Spaces

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Abstract. We prove the boundedness of the weighted Hardy operator in the locally Orlicz-Morrey spaces $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$.

Key Words and Phrases: Orlicz-Morrey space, Hardy operator.
2010 Mathematics Subject Classifications: 42B20; 42B25; 42B35

1. Introduction

Inequalities involving classical operators of harmonic analysis, such as maximal functions, fractional integrals and singular integrals of convolution type have been extensively investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue spaces are classical and can be found for example in [1, 16, 17]. Generalizations of these results to Zygmund spaces are presented in [1]. As far as Orlicz spaces are concerned, we refer to the books [8, 9] and note that a characterization of Young functions A for which the Hardy-Littlewood maximal operator or the Hilbert and Riesz transforms are of weak or strong type in Orlicz space L_A is known (see for example [2, 8]). In paper [4] give necessary and sufficient conditions for the strong and weak boundedness of the Riesz potential operator I_α on Orlicz spaces. Also in paper [2] found necessary and sufficient conditions on general Young functions Φ and Ψ ensuring that this operator is of weak or strong type from L^Φ into L^Ψ. Our characterizations for the boundedness of the abovementioned operator are different from the ones in [2]. In paper [4] as an application of these results, we consider the boundedness of the commutators of Riesz potential operator $[b, I_\alpha]$ on Orlicz spaces when b belongs to the BMO and Lipschitz spaces, respectively.

The classical Morrey spaces were originally introduced by Morrey in [10] to study the local behavior of solutions to second order elliptic partial differential equations. We recall its definition as

$$M_{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L_p^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\}, \tag{1.1}$$

*Corresponding author.

where $0 \leq \lambda \leq n$, $1 \leq p < \infty$. Here and everywhere in the sequel $B(x,r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$. $M_{p,\lambda}(\mathbb{R}^n)$ was an expansion of $L_p(\mathbb{R}^n)$ in the sense that $M_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space. Morrey found that many properties of solutions to PDE can be attributed to the boundedness of some operators on Morrey spaces. Hardy operators, maximal functions and singular integrals play a key role in harmonic analysis since maximal functions could control crucial quantitative information concerning the given functions, despite their larger size, while singular integrals, Hilbert transform as its prototype, nowadays intimately connected with PDE, operator theory and other fields.

2. Some preliminaries on Orlicz and Orlicz-Morrey spaces

Definition 2.1. A function $\Phi : [0,+\infty] \to [0,+\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to+\infty} \Phi(r) = \Phi(+\infty) = +\infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

We say that $\Phi \in \Delta_2$, if for any $a > 1$, there exists a constant $C_a > 0$ such that $\Phi(at) \leq C_a \Phi(t)$ for all $t > 0$.

Recall that a function Φ is said to be quasiconvex if there exist a convex function ω and a constant $c > 0$ such that

$$\omega(t) \leq \Phi(t) \leq c\omega(ct),\ t \in [0,+\infty).$$

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \qquad \text{for} \qquad 0 < r < +\infty \tag{2.1}$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,+\infty)$ and bijective from $[0,+\infty)$ to itself.

Definition 2.2. (Orlicz Space). For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < +\infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. The space $L_\Phi^{\rm loc}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_{_B} \in L_\Phi(\mathbb{R}^n))$ for all balls $B \subset \mathbb{R}^n$.

Note that, $L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1\right\},$$

see, for example, [14], Section 3, Theorem 10, so that

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_\Phi}}\Big) dx \le 1.$$

Definition 2.3. The weak Orlicz space

$$WL_\Phi(\mathbb{R}^n) = \{f \in L^1_{1\mathrm{oc}}(\mathbb{R}^n) : \|f\|_{WL_\Phi} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf\left\{\lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \ \le 1\right\},$$

where $m(f,\ t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|$.

For Young functions Φ and Ψ, we write $\Phi \sim \Psi$ if there exists a constant $C \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \qquad \text{for all } r \ge 0$$

If $\Phi \approx \Psi$, then $L_\Phi(\mathbb{R}^n) = L_\Psi(\mathbb{R}^n)$ with equivalent norms.

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{ for } 0 \le r < +\infty.$$

A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & , \quad r \in [0,\infty) \\ +\infty & , \quad r = +\infty. \end{cases} \tag{2.2}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$.

Remark 2.4. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. Also, if Φ is a Young function, then $\Phi \in \nabla_2$ if and only if Φ^γ be quasiconvex for some $\gamma \in (0,1)$ (see, for example, [8], p. 15).

It is known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \qquad \text{for } r \geq 0. \tag{2.3}$$

Definition 2.5. (Orlicz-Morrey Space). For a Young function Φ and $0 \leq \lambda \leq n$, we denote by $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ the locally Orlicz-Morrey space, defined as the space of all functions $L^{loc}_{\Phi}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)} = \sup_{r>0} \Phi^{-1}(r^{-\lambda}) \|f\chi_{B(0,r)}\|_{L_\Phi}.$$

Also by $WL^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ we denote the weak Orlicz-Morrey space of all functions $f \in WL^{\mathrm{loc}}_{\Phi}(\mathbb{R}^n)$ for which

$$\|f\|_{WL^{0,loc}_{\Phi,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}(r^{-\lambda})\, \|f\|_{WL_\Phi(B(x,r))} < \infty.$$

If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n) = L^{0,loc}_{p,\lambda}(\mathbb{R}^n)$. If $\lambda = 0$, then $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n) = L_\Phi(\mathbb{R}^n)$.

3. The weighted Hardy operator in the spaces $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$

We consider the following weighted Hardy operator

$$H_\beta f(x) = |x|^{\beta-n} \int_{|y|<|x|} \frac{f(y)}{|y|^\beta} dy.$$

Theorem 3.1. *Let Φ any Young function and $0 \leq \lambda < n$. Suppose also that the function $\frac{r^{n-\beta}\Phi^{-1}(r^{-n})}{\Phi^{-1}(r^{-\lambda})}$ is almost increasing and*

$$\int_0^r \frac{t^{n-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt \leq C \frac{r^n \Phi^{-1}(r^{-n})}{\Phi^{-1}(r^{-\lambda})}, \tag{3.1}$$

$$\int_0^\varepsilon \frac{s^{n-\beta-1}\Phi^{-1}(s^{-n})}{\Phi^{-1}(s^{-\lambda})} ds < \infty, \tag{3.2}$$

where $\varepsilon > 0$ and $r > 0$.

1) Then the operator H_β is bounded from $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ to $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ if and only if $\frac{\Phi^{-1}(|\cdot|^{-n})}{\Phi^{-1}(|\cdot|^{-\lambda})} \in L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$.

2) Then the operator H_β is bounded from $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ to $WL^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ if and only if $\frac{\Phi^{-1}(|\cdot|^{-n})}{\Phi^{-1}(|\cdot|^{-\lambda})} \in WL^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$.

Proof. 1) Let $f \in L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$. We have

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz = \sum_{k=0}^{\infty} \int\limits_{B_k(y)} \frac{|f(z)|}{|z|^\beta} dz, \tag{3.3}$$

where $B_k(y) = \left\{z : 2^{-k-1}r < |z| < 2^{-k}r\right\}$. Applying this in (3.3) and making use of the following Hölder's inequality

$$\|f\|_{L_1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L_\Phi(B)}$$

we obtain

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz \le \sum_{k=0}^{\infty} \frac{2|B(0,2^{-k}r)|\Phi^{-1}\left(|B(0,2^{-k}r)|^{-1}\right)||f||_{L_\Phi(B(0,2^{-k}r))}}{(2^{-k}r)^\beta}$$

so that

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz \le C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \sum_{k=0}^{\infty} \frac{(2^{-k}r)^{n-\beta}\Phi^{-1}\left((2^{-k}r)^{-n}\right)}{\Phi^{-1}\left((2^{-k}r)^{-\lambda}\right)}.$$

We have $\mathcal{A}(r) = \sum\limits_{k=0}^{\infty} \int\limits_{2^{-k-1}r}^{2^{-k}r} \frac{t^{n-\beta-1}\Phi^{-1}\left(t^{-n}\right)}{\Phi^{-1}\left(t^{-\lambda}\right)} dt$. Since the function $\frac{t^{n-\beta}\Phi^{-1}\left(t^{-n}\right)}{\Phi^{-1}\left(t^{-\lambda}\right)}$ is almost decreasing, we obtain

$$\mathcal{A}(r) \ge C \sum_{k=0}^{\infty} \frac{(2^{-k}r)^{n-\beta}\Phi^{-1}\left((2^{-k}r)^{-n}\right)}{\Phi^{-1}\left((2^{-k}r)^{-\lambda}\right)} \ge C,$$

Therefore

$$\sum_{k=0}^{\infty} \frac{(2^{-k}r)^{n-\beta}\Phi^{-1}\left((2^{-k}r)^{-n}\right)}{\Phi^{-1}\left((2^{-k}r)^{-\lambda}\right)} \le C\mathcal{A}(r)$$

or

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz \le C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \int\limits_0^r \frac{t^{n-\beta-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt.$$

We proved that

$$|H_\beta f(x)| \le C\|f\|_{L^{0,loc}_{\Phi,\lambda}} |x|^{\beta-n} \int\limits_0^{|x|} \frac{t^{n-\beta-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt = C\|f\|_{L^{0,loc}_{\Phi,\lambda}} B(|x|). \tag{3.4}$$

Then by the (3.4), we have

$$\|H_\beta f\|_{L_\Phi(B(0,r))} \le C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| |\cdot|^{\beta-n} \int\limits_0^{|\cdot|} \frac{t^{n-\beta-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt \right\|_{L_\Phi(B(0,r))}$$

$$\leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| |\cdot|^{\beta-n} \frac{|\cdot|^{n-\beta}\Phi^{-1}(|\cdot|^{-n})\chi_{B(0,r)}}{\Phi^{-1}(|\cdot|^{-\lambda})} \right\|_{L_\Phi}$$

$$\leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| \frac{\Phi^{-1}(|\cdot|^{-n})\chi_{B(0,r)}}{\Phi^{-1}(|\cdot|^{-\lambda})} \right\|_{L_\Phi} \leq \frac{C}{\Phi^{-1}(r^{-\lambda})}\|f\|_{L^{0,loc}_{\Phi,\lambda}}.$$

2) Let $f \in L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$. The preceding division will prove to be the proper analogy. We have

$$\|H_\beta f\|_{L_{W\Phi}(B(0,r))} \leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| |\cdot|^{\beta-n} \int\limits_0^{|\cdot|} \frac{t^{n-\beta-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt \right\|_{WL_\Phi(B(0,r))}$$

$$\leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| |\cdot|^{\beta-n} \frac{|\cdot|^{n-\beta}\Phi^{-1}(|\cdot|^{-n})\chi_{B(0,r)}}{\Phi^{-1}(|\cdot|^{-\lambda})} \right\|_{WL_\Phi}$$

$$\leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| \frac{\Phi^{-1}(|\cdot|^{-n})\chi_{B(0,r)}}{\Phi^{-1}(|\cdot|^{-\lambda})} \right\|_{WL_\Phi} \leq \frac{C}{\Phi^{-1}(r^{-\lambda})}\|f\|_{L^{0,loc}_{\Phi,\lambda}}.$$

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Oscillatory Integral Operators in Morrey Spaces with Variable Exponent

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Abstract. In case of unbounded sets $\Omega \subset \mathbb{R}^n$ we prove the boundedness of the conditions in terms of Calderón-Zygmund-type integral inequalities for oscillatory integral operators in the Morrey spaces with variable exponent.

Key Words and Phrases: Calderón-Zygmund-type integral inequalities for oscillatory integral operators, Morrey space with variable exponent.

2010 Mathematics Subject Classifications: 42B20; 42B25; 42B35

1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [18] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [8, 9, 10, 18].

As it is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [7, 14, 20], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein. For mapping properties of maximal functions and singular integrals on Lebesgue spaces with variable exponent we refer to [3, 5, 6].

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [2] in the Euclidean setting in case of bounded sets. The boundedness of the maximal operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$, $\lambda(\cdot)$ was proved in [2]. P. Hästö in [12] used his new "local-to-global" approach to extend the result of [2] on the maximal operator to the case of the whole space $\mathbb{R}^n$. The boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [13].

In the case of constant p and λ , the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [19], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [4].

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A distribution kernel $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n} \text{ for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \ \ \sigma > 0, \ \text{ if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \ \ \sigma > 0, \ \text{ if } |x-y| > 2|x-\xi|$$

Calderón-Zygmund type singular operator and the oscillatory integral operator are defined by

$$Tf(x) = \int_{\Omega} K(x,y)f(y)dy, \tag{1}$$

$$Sf(x) = \int_{\Omega} e^{P(x,y)}K(x,y)f(y)dy, \tag{2}$$

where $P(x,y)$ is a real valued polynomial defined on $\Omega \times \Omega$. Lu and Zhang [17] used L^2 -boundedness of T to get L^p - boundedness of S with $1 < p < \infty$.

Let

$$T^*f(x) = \sup_{\varepsilon > 0} |T_{\varepsilon}f(x)|$$

be the maximal singular operator, where $T_{\varepsilon}f(x)$ is the usual truncation

$$T_{\varepsilon}f(x) = \int_{\{y \in \Omega : |x-y| \geq \varepsilon\}} K(x,y)f(y)dy.$$

We use the following notation: $\mathbb{R}^n$ is the n -dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, by c, C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line.

2. Preliminaries on variable exponent weighted Lebesgue and Morrey spaces

We refer to the book [5] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $(1,\infty)$. An open set Ω which may be unbounded throughout the whole paper. We mainly suppose that

$$1 < p_- \leq p(x) \leq p_+ < \infty, \tag{3}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x)$, $p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \eta > 0 : \; I_{p(\cdot)} \left(\frac{f}{\eta} \right) \leq 1 \right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega,$ we denote the conjugate exponent.

The space $L^{p(\cdot)}(\Omega)$ coincides with the space

$$\left\{ f(x) : \left| \int_\Omega f(y) g(y) dy \right| < \infty \quad for\ all \quad g \in L^{p'(\cdot)}(\Omega) \right\} \tag{4}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}(\Omega)} \approx \sup_{\|g\|_{L^{p'(\cdot)}} \leq 1} \left| \int_\Omega f(y) g(y) dy \right| \tag{5}$$

see [15, Theorem 2.3], or [21, Theorem 3.5].

For the basics on variable exponent Lebesgue spaces we refer to [22], [15].
$\mathcal{P}(\Omega)$ is the set of bounded measurable functions $p : \Omega \to [1, \infty)$;
$\mathcal{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln |x-y|}, \quad |x-y| \leq \frac{1}{2} \;\; x, y \in \Omega, \tag{6}$$

where $A = A(p) > 0$ does not depend on x, y ;
$\mathcal{A}^{log}(\Omega)$ is the set of bounded exponents $p : \Omega \to \mathbb{R}^n$ satisfying the condition (6);
$\mathbb{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{log}(\Omega)$ with $1 < p_- \leq p_+ < \infty$;
for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}^{log}_\infty(\Omega)$, $\mathbb{P}^{log}_\infty(\Omega)$, $\mathcal{A}^{log}_\infty(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \leq \frac{A_\infty}{\ln(2 + |x|)}, \quad x \in \mathbb{R}^n. \tag{7}$$

where $p_\infty = \lim_{x\to\infty} p(x) > 1$.

We will also make use of the estimate provided by the following lemma (see [5], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leq C r^{\theta_p(x,r)}, \qquad x \in \Omega, \; p \in \mathbb{P}^{log}_\infty(\Omega), \tag{8}$$

where $\theta_p(x,r)=\begin{cases}\frac{n}{p(x)}, & r\le 1,\\ \frac{n}{p(\infty)}, & r\ge 1\end{cases}$.

A locally integrable function $\omega:\Omega\to(0,\infty)$ is called a weight. We say that $\omega\in A_p(\Omega)$, $1<p<\infty$, if there is a constant $C>0$ such that

$$\left(\frac{1}{|\widetilde{B}(x,t)|}\int_{\widetilde{B}(x,t)}\omega(x)dx\right)\left(\frac{1}{|\widetilde{B}(x,t)|}\int_{\widetilde{B}(x,t)}\omega^{1-p'}(x)dx\right)^{p-1}\le C,$$

where $1/p+1/p'=1$. We say that $\omega\in A_1(\Omega)$ if there is a constant $C>0$ such that $M\omega(x)\le C\omega(x)$ almost everywhere.

The extrapolation theorems (Lemma 1 and Lemma 2 below) are originally due to Cruz-Uribe, Fiorenza, Martell and Pérez [3]. Here we use the form in [5], see Theorem 7.2.1 and Theorem 7.2.3 in [5].

Lemma 1. *([5]). Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0<p_0<\infty$, every $(f,g)\in\mathcal{F}$ and every $\omega\in A_1$,*

$$\int_\Omega|f(x)|^{p_0}\omega(x)dx\le C_0\int\Omega|g(x)|^{p_0}\omega(x)dx.$$

Let $p(\cdot)\in P(\Omega)$ with $p_0\le p_-$. If maximal operator is bounded on $L^{\left(\frac{p(\cdot)}{p_0}\right)'}(\Omega)$, then there exists a constant $C>0$ such that for all $(f,g)\in\mathcal{F}$,

$$\|f\|_{L^{p(\cdot)}(\Omega)}\le C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Lemma 2. *([5]). Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0<p_0<q_0<\infty$, every $(f,g)\in\mathcal{F}$ and every $\omega\in A_1$,*

$$\left(\int_\Omega|f(x)|^{q_0}\omega(x)dx\right)^{\frac{1}{q_0}}\le C_0\left(\int_\Omega|g(x)|^{p_0}\omega^{\frac{p_0}{q_0}}(x)dx\right)^{\frac{1}{p_0}}.$$

Let $p(\cdot)\in P(\Omega)$ with $p_0\le p_-$ and $\frac{1}{p_0}-\frac{1}{q_0}<\frac{1}{p_+}$, and define $q(x)$ by

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{1}{p_0}-\frac{1}{q_0}.$$

If maximal operator is bounded on $L^{\left(\frac{q(\cdot)}{q_0}\right)'}(\Omega)$, then there exists a constant $C>0$ such that for all $(f,g)\in\mathcal{F}$,

$$\|f\|_{L^{q(\cdot)}(\Omega)}\le C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Singular operators within the framework of the spaces with variable exponents were studied in [6]. From Theorem 4.8 and Remark 4.6 of [6] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω , but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

Theorem 1. *([6, Theorem 4.8]) Let* $\Omega \subset \mathbb{R}^n$ *be a unbounded open set and* $p \in \mathbb{P}^{log}(\Omega)$ *. Then the singular integral operator* T *is bounded in* $L^{p(\cdot)}(\Omega)$ *.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{\frac{\lambda(x)}{p(x)} - \theta_p(x,t)} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

respectively.

We will use the following results on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [11].

Theorem 2. *Let* v_1 *,* v_2 *and* w *be weights on* $(0, \infty)$ *and* $v_1(t)$ *be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H^*_w g(t) \leq C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0, \infty)$ *if and only if*

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3. Oscillatory integral operators in $\mathcal{L}^{p(\cdot),\lambda}(\Omega)$

Lemma 3. *(see [16]). If* K *is a standard Calderón-Zygmund kernel and the Calderón-Zygmund singular integral operator* T *is of type* $(L^2(\Omega), L^2(\Omega))$ *, then for any real polynomial* $P(x, y)$ *and* $\omega \in A_p$ $(1 < p < \infty)$ *, there exists constants* $C > 0$ *independent of the coefficients of* P *such that*

$$\|Sf\|_{L^p_\omega(\Omega)} \leq C\|f\|_{L^p_\omega(\Omega)}.$$

Theorem 1. *Let* $\Omega \subset \mathbb{R}^n$ *be an open unbounded set,* $p \in \mathbb{P}^{log}_\infty(\Omega)$ *. Then the operator* S *is bounded in the space* $L^{p(\cdot)}(\Omega)$ *.*

Proof. By the Lemma 1 and Lemma 3, we derive the operator S is bounded in the space $L^{p(\cdot)}(\Omega)$.

The following local estimates are valid.

Theorem 2. *Let* $\Omega \subset \mathbb{R}^n$ *be an open unbounded set,* $p \in \mathbb{P}^{log}_\infty(\Omega)$ *and* $f \in L^{p(\cdot)}(\Omega)$ *. Then*

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}, \tag{1}$$

where C *does not depend on* f *,* $x \in \Omega$ *and* t *.*

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\Omega\setminus\widetilde{B}(x,2t)}(y), \quad t > 0, \tag{2}$$

and have

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} + \|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By the Theorem 1 we obtain

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)},$$

so that

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}.$$

Taking into account the inequality

$$\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

we get

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{3}$$

To estimate $\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|Sf_2(z)| \le C \int\limits_{\Omega\setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$|Sf_2(z)| \le C \int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n}|f(y)|dy,$$

To estimate Sf_2 , we first prove the following auxiliary inequality

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \\ \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{4}$$

To this end, we choose $\delta > 0$ and proceed as follows

$$\int_{\Omega\backslash \widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \le \delta \int_{\Omega\backslash \widetilde{B}(x,t)} |x-y|^{-n+\delta}|f(y)|dy \int_{|x-y|}^{\infty} s^{-\delta-1}ds$$

$$\le C\int_t^{\infty} s^{-n}\frac{ds}{s}\int_{\{y\in\Omega: 2t\le|x-y|\le s\}}|f(y)|dy \le C\int_t^{\infty} s^{-n}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\|\chi_{\widetilde{B}(x,s)}\|_{L^{p'(\cdot)}(\Omega)}\frac{ds}{s}$$

$$\le C\int_t^{\infty} s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \tag{5}$$

Hence by inequality (5), we get

$$\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}\int_t^{\infty} s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}$$

$$= Ct^{\theta_p(x,t)}\int_t^{\infty} s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \tag{6}$$

From (3) and (6) we arrive at (1).

Theorem 3. *Let $\Omega\subset\mathbb{R}^n$ be an open unbounded set, $p\in\mathbb{P}_{\infty}^{log}(\Omega)$ and $0\le\lambda(x)<n$. Then the singular integral operator S is bounded from the space $\mathcal{L}^{p(\cdot),\lambda}(\Omega)$ to the space $\mathcal{L}^{p(\cdot),\lambda}(\Omega)$.*

Proof. Let $f\in\mathcal{L}^{p(\cdot),\lambda}(\Omega)$. As usual, when estimating the norm

$$\|Sf\|_{\mathcal{L}^{p(\cdot),\lambda}(\Omega)} = \sup_{x\in\Omega,\ t>0} t^{\frac{\lambda(x)}{p(x)}-\theta_p(x,t)}\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}. \tag{7}$$

We estimate $\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}$ in (7) by means of Theorems 2, 2 and obtain

$$\|Sf\|_{\mathcal{L}^{p(\cdot),\lambda}(\Omega)}$$
$$\le C\sup_{x\in\Omega,\ t>0} t^{\frac{\lambda(x)}{p(x)}-\theta_p(x,t)}t^{\theta_p(x,t)}\int_t^{\infty} s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}$$
$$\le C\sup_{x\in\Omega,\ t>0} t^{\frac{\lambda(x)}{p(x)}-\theta_p(x,t)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} = C\|f\|_{\mathcal{L}^{p(\cdot),\lambda}(\Omega)}.$$

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Factors Affecting the Dynamics of Exchange Rates and Their Econometric Analysis

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Abstract. The present work is devoted to the study of the interrelations of the exchange rate of manat with the main indicators of the monetary system in Azerbaijan. A mathematical-statistical model of the exchange rate has been constructed, allowing the study by econometric methods of its interrelations with the basic monetary aggregates and the interest rate. The model is analyzed for adequacy.

Key Words and Phrases: exchange rate, devaluation of manat, statistical-mathematical model, monetary aggregates, interest rate.

2010 Mathematics Subject Classifications: C1, C3, E4, E5, G1

Currency rates are considered as an important component of the international monetary system, which can affect the macroeconomic circumstance of each country, acts as a tool between the global and national value indicators in the system of world economic relations.

Nowadays, exchange rates change every day, which creates the necessity of studying their interpretation laws.

One of the key factors affecting the dynamics of exchange rates is the condition of the balance of payments. The balance of payments (In the statistical sources of Azerbaijan are presented quarterly and yearly) indicates the amount of payments inflows to and outflows from the country within a given time period and, in other words, the movement dynamics of the currency. The increase in demand for the national currency by creating a positive balance, in turn, enhances the offer and reinforces it. Naturally, the negative balance weakens the national currency, as the demand for foreign currency is increasing.

In general, the formation of a currency course [1, p. 639-685] can be characterized as a multinational process. In addition to the balance of payments, the rate of exchange rate inflation, trade balance deficit, unemployment rate, GDP, investment environment, interest rates, state debt, volume of state orders, economic productivity, industrial production index, oil price in world markets, government economic policy, changes in the management system, and so on. are exposed to indicators, and these factors are not interrupted separately but at the same time interfere the exchange rate formation process.

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Insurance, investment, and social security funds use more stable currencies to safe financial resources and the exchange rate of foreign currency used by them increases due to the large money consuming, while the national currency weakens.

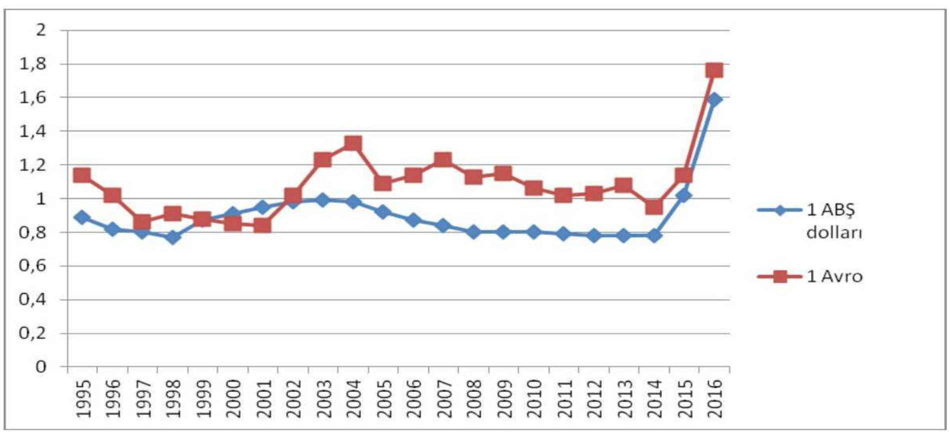
The investigation of the influence of the macroeconomic policy leading by the government to the occurring and increasing systematic risks process was taken an important place for the positive and negative sides [2, 3] of anticrisis financial events. Within managing explorers, especially, the financial tools such as the government currency resources, the main aggregates of money volume and interest rate have been probed in the role of eliminating financial-economic crisis and supporting the microeconomic stability processes with the help of world experience, were given essential advises.

Experience shows that the exchange rate significantly depends on the external factors. In particular, the dynamics of the national currency rate is explained by the volume of exports in many aspects in countries such as Azerbaijan.

The investigations which have been occurred was explored the influence mechanism of the balance of payment by the factors such as the export-import operations of manat course, the managing of total and foreign investment, the quality tendencies of model were investigated by creating an econometric model [4].

Figure 1 the dynamics of the exchange rate of manat is illustrated compared to the leading foreign currencies (US dollars, euro) from 1995 to 2016.

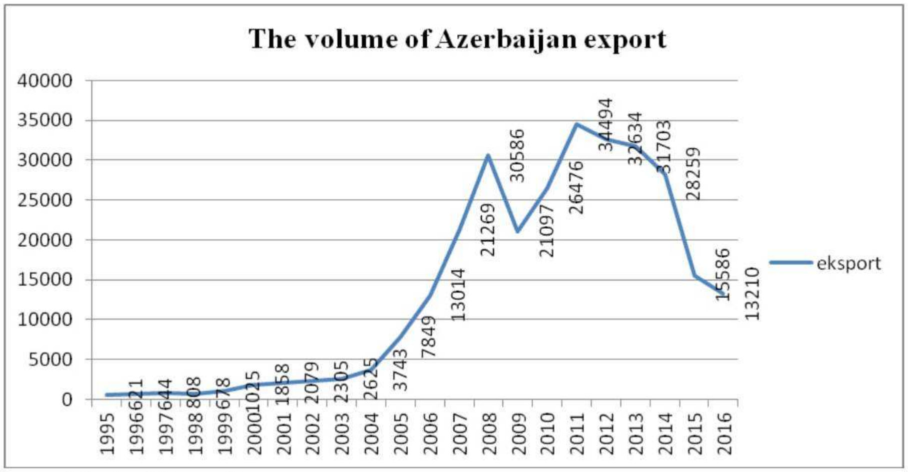
Figure 1. Dynamics of exchange rate of Azerbaijani manat according to the foreign currencies within 1995-2016



Source: Work of the authors

As can be seen from Figure 1, the exchange rate of manat from 1995 to 2014 can be relatively stable assumed compared with the leading currencies. The negative consequences of the global economic crisis since 2015 began to be felt in the Azerbaijani economy. The Azerbaijani manat was unable to maintain its firmness after the known processes and weakened by being times devaluated in 2015. This, in its turn, has been reflected in almost all macroeconomic indicators of Azerbaijan. With the sharp decline in oil prices in the world markets, the export of Azerbaijan also started to weaken.

Figure 2. Dynamics of Azerbaijani Exports for 1995-2016 (in mln USD)



Source: Work of the authors

As we have noted, the volume of exports is one of the major factors affecting the national currency. The drop in exports starting since 2012 and being felt severely in 2014 hit the Azerbaijani manat overwhelmingly. The deviation from the main trends in the dynamics of the manat wasn't made wait itself and once again reaffirmed the sensitivity of the national currency in front of external factors.

We present the estimated value of manat depend on export volumes for 1996-2016 on the basis of correlation coefficients for the US dollar and the euro exchange rate in Table 1.

Table 1. Correlation coefficients between the exchange rate and export volumes

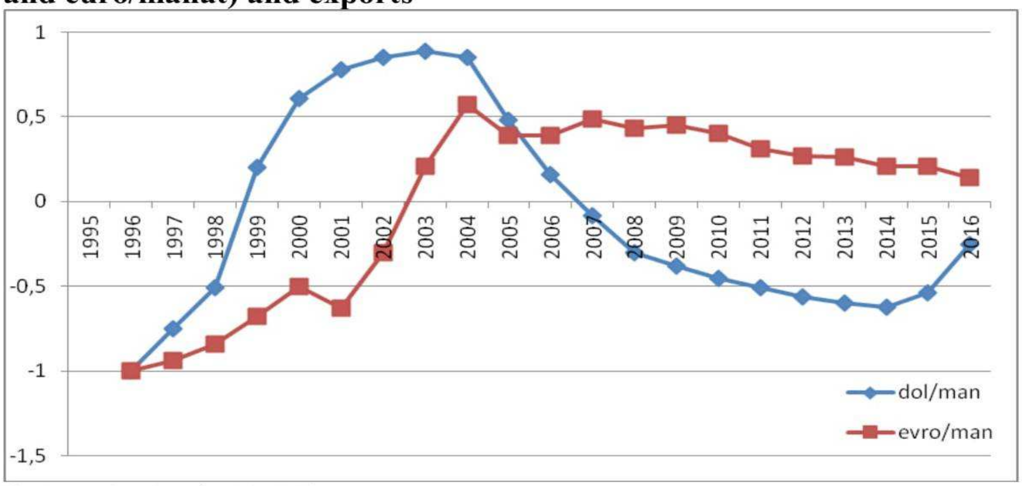
illər	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
dol/man	-1	-0,75	-0,51	0,2	0,61	0,78	0,85	0,89	0,85	0,48
evro/man	-1	-0,94	-0,84	-0,68	-0,5	-0,63	-0,3	0,21	0,57	0,39

illər	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
dol/man	0,16	-0,08	-0,3	-0,38	-0,45	-0,51	-0,56	-0,6	-0,62	-0,54	-0,25
evro/man	0,39	0,49	0,43	0,45	0,4	0,31	0,27	0,26	0,21	0,21	0,14

Source: Authors' calculations

When analyzing the results, we see the periods in which currency is strongly dependent on the volume of exports. Thus, the exchange rate of manat in relation to the exports is strong enough by 2005 are expressed by the volume of ratios. The rise in this indicator to 0,89 (US dollar / manat dependency on imports and exports) in 2003 indicates strong dependence and density. Looking at statistical indicators since 2005, "invisible", but after computations gradual decrease of correlation of correlation coefficients is observed.Even since 2007, correlation indicators have a negative mark, which means their passing in a converse state. Thus, the processes taking place in the world markets had had an impact on the manat since that period. Currently examining figure, the dependence between the rate of the Azerbaijani manat (US dollar/manat) and the export is relatively higher than the peak edge of global economic crisis in 2007 (-0,08) and is -0,25. The same weakening trend of the rate of manat in front of the euro and the volume of exports, however with a slower pace, has continued since 2007. The correlation dependence between the studied parameters is graphically shown in Figure 3.

Figure 3. The correlation dependence between manat's course (US $/manats and euro/manat) and exports

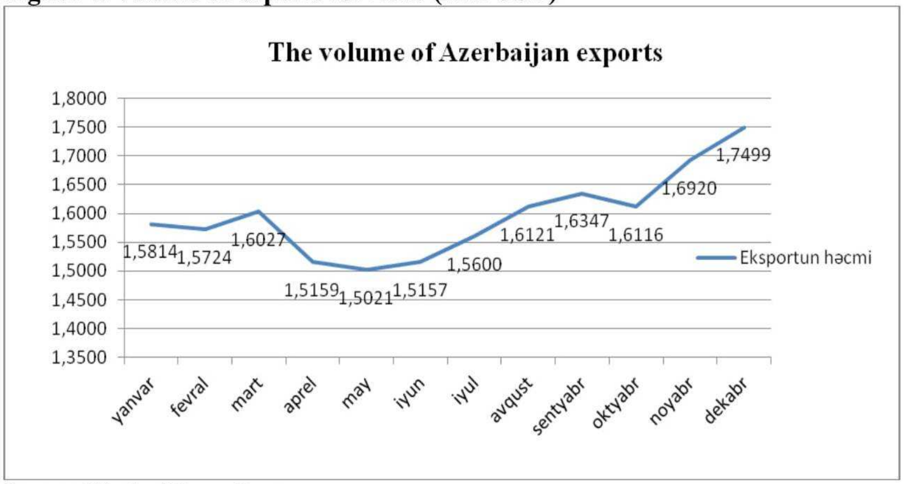


Source: Authors' calculations

Figure 3 illustrates a similar attitude to exports by both indicators, even with different frequency and intensity. The fluctuation amplitude and velocity of the floating exchange rate currencies are usually considered to be normally accepted for development tendencies. This is a multidimensional, complex, challenging and hard-predictable process. This is explained by the nature, mobility and dependence of many internal and external factors forming the exchange rate. In this regard, as it is always important to analyze the world economic crisis of recent decades, to examine countries with developed economies, evaluate the dependence on factors by applying mathematical-statistical methods in this field, expand the scope of scientific research and is even more important in the period when the exchange rate of manat weakened.

Analyzing the situation and evaluating the situation properly, the country's economic and social development strategy has focused on increasing export volume regardless of oil, creating new production areas, keeping the agrarian sector in focus, developing tourism, and building multidimensional industries. The results of the new strategy of the state in exports have started to be observed since 2016, even with small pace. An increase in the volume of exports for months is seen obviously in Figure 4.

Figure 4. Volume of exports for 2016 (mln USD)



Source: Work of the authors

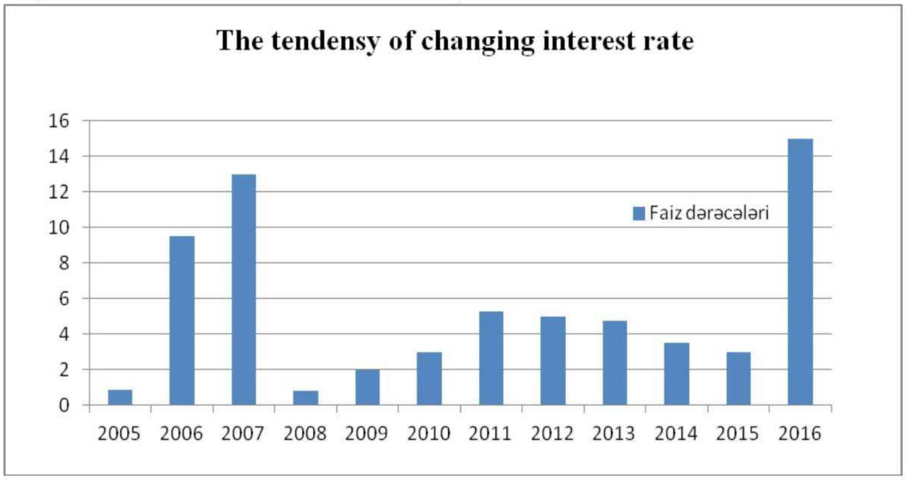
The principal institution that affecting directly the exchange rate and has its management tools is the central bank that forms the monetary policy of the state. The central bank can directly intervene in these processes by increasing or decreasing the volume of money circulation. Along with regulation of the money supply, it also manages important mechanisms in the formation of the national currency environment in the country, even by changing interest rates.

The Central Bank of Azerbaijan (CBA) carries out the role of important regulatory aggregate in the implementation of key economic strategies such as maintenance of stability of economic development in the Republic of Azerbaijan, secure and long-term investment climate, and sustainable macroeconomic stability. The main goals of the CBA are currently stabilization of the national currency by means of instruments such as monetary and national currency regulation policies, low inflation rates or a certain optimisation, encourage the banking sector which having recession after the depreciation of manat in 2015 and restore them as a major and active financial sector in the economic, financial environment, monetary and credit relations of republic, establishment of interbank money market in Azerbaijan, strengthening the trust among credit institutions, and developing non-cash payments.

Strengthening of the Azerbaijani manat, development of the banking sector and provision of a balanced financial environment are the leading directions of the CBA's macroprudential policy. The monetary policy of the Central Bank is aimed at maintaining the stability of prices by influencing the inflation processes, adjusting the exchange rate, money supply, and interest rates. As a direct tool, control over interest rates, loan corridor, loans directly or indirectly, as a roundabout instrument the open market operations, the money supply are applied to the issue [5]. The CBA chairman E.Rustamov mentioned [6] that during the floating course the exchange rate $ 1 US dollar due to manat fell to 1,92 manat. However, stabilized and 11% below than the limit. The real effective exchange rate of the manat has decreased by 50%, which is also reasonable. It is possible that strong manat can create new problems by increasing imports, while the increase in exports from the non-oil sector requires the manat to be more favorable for market participants.

It should be noted that interest rates in Azerbaijan are 15% since September 14, 2016. CB's head [7] noted that due to the decrease of inflation rates the interest rates has declined, updates the issue of transition to the softer regime. Figure 5 presents the dynamics of interest rates in Azerbaijan.

Fiqure 5. Interest rate in 2005-2016 years



Source: Work of the authors

As it is clear from the graph, interest rates at 3% in 2015 are raised to 15% by the end of 2016 within the framework of regulatory arrangements of the CBA.

In this case, we look at quantifying the dependence between the national currency rate and interest rates in Azerbaijan and the regression analysis of the statistical indices in order to determine how the exchange rate changes depending on the interest rate.

According to the results of the regression analysis of the exchange rate and interest rates, the Student criterion was t=*2,54*; the significance level was p=*0,03;* the standard deviation was S=*0,01.* The p-significance level is sufficiently low and does not exceed *0,05*, the results are considered satisfactorily. The double regression model is formed as follows, given that the free boundary a_0=*0,69* and the regression coefficient a_1 =*0,03*:

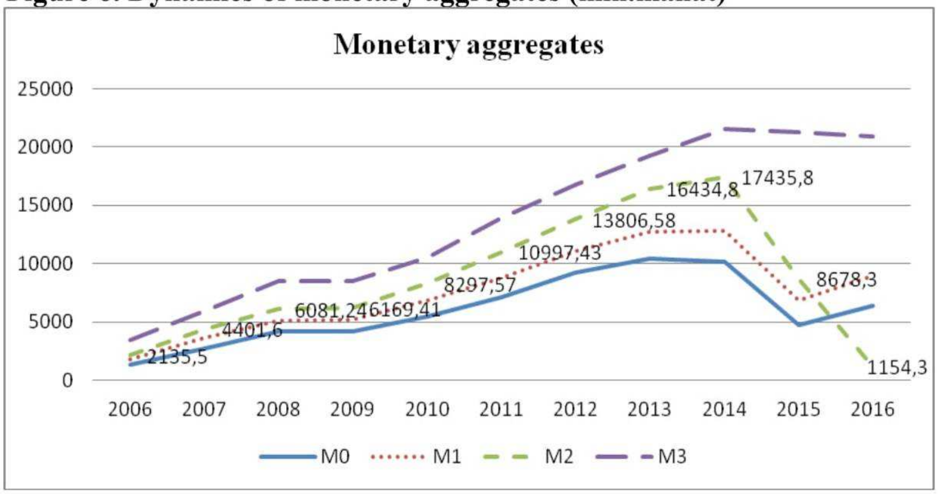
$$MR = 0,69 + 0,03IR$$

where *MR*-manat rate; *IR*-rates interest rates. According to the model, the correlation coefficient r = *0,69*; determinate coefficient R^2 = *0,41.*

41% of determinate coefficients indicate that 41% of exchange rate fluctuations occurred on the basis of interest rates, and 59% were due to factors not included in the model. This created a necessity to continue research.

Now let's look at the money aggregates. The sharp decline of key monetary aggregates in Azerbaijan after the devaluation of manat is graphically illustrated in Figure 6. Particularly, the M2 aggregate, which amounts to 17435,8 million manats in 2014, having fallen to 8678.3 million manats in 2015 and 1154,3 million in 2016 is observed.

Figure 6. Dynamics of monetary aggregates (mln.manat)



Source: Work of the authors

here

M0 - the most liquid part of the money supply, banknotes and coins in circulation;

*M1-M0+*demand deposits and deposits denominated;

*M2- M1+*denominated term deposits and savings:

M3-M2+ is expressed in a freely convertible currency savings deposits [5]

We conducted a multidimensional regression analysis based on the fact that there was a significant dependence on the currency exchange rate between the major monetary aggregates in Azerbaijan. The main results are presented in Table 2.

Table 2. Results of regression analysis for major monetary aggregates and exchange rate

	Coeff.s	*Stand.error*	*t-stat.*	*P-value*
Y	0,766017	0,022012	34,80011	3,75E-08
M0	-0,00015	5,79E-05	-2,65195	0,03793
M1	0,000185	5,33E-05	3,464232	0,013398
M2	-5,8E-05	3,21E-06	-18,0756	1,85E-06
M3	1,01E-05	5,01E-06	2,027254	0,089002

Source: Authors' calculations

Taking the results, we have the following multidimensional regression model:

$$MK = 0,76 - 0,00015M0 + 0,0002M1 - 5,8M2 + 1,01M3$$

According to the model, the correlation coefficient *r=0,99*; the determinant coefficient R^2*=0,99.*

Considering the fact that the interest rate factor is essential in the dynamics of the exchange rate (in the previous model R^2*=0,41*), we added the interest rate dependent variable to the latter model.

The results of the multidimensional regression model describing the impact of the major monetary aggregates in Azerbaijan (*M0, M1, M2, M3*) and interest rates (*IR*) on the Azerbaijani manat rate are presented in Table 3.

Thus, based on the results we have obtained, we have set up the next multi-dimensional regression model:

$$MR = 0,74 - 9,8M0 + 0,0001M1 - 5,4M2 + 1,53M3 + 0,0048IR$$

According to the model we received, *r=0,99;* R^2*=0,99; S=0,01* indicates that the model is sufficiently adequate. Thus, the fact that the correlation coefficient obtained by almost the maximum price dependent variables is very strong, dense, and the determination factor R^2*=0,99* The *M1, M2, M3, M4, IR* factors change the maximum rate of manat, explaining that 99% of these factors are the result. *S=0,01* indicates that the standard error in the model is low. The *F=1,28* price of the Fischer criteria for the model is compared to the table price to evaluate the overall quality of the model. If the evaluation of the model is more than an estimate, the model is considered more qualitative.

Table 3. The results of the regression analysis performed among the M0, M1, M2, M3, IR factors and MR-result marks

RESULTS									
Reqres. stat.									
R	0,998881								
R^2	0,997762								
Norm. R^2	0,995525								
Stand. error	0,016112								
Observation	11								
Variance analysis									
	Df	*SS*	*MS*	*F*	*F*				
Reqression	5	0,57877	0,115755	445,891	1,284E-06				
Remainder	5	0,0013	0,00026						
Total	10	0,58007							
	Coeff.	*Stand.error*	*t-stat.*	*P-value*	*lower 95%*	*Upper 95%*	*Lowre95,0%*	*Upper95,0%*
Y-	0,743649	0,01705	43,61322	1,2E-07	0,6998184	0,78748	0,699818	0,78748
M0	-9,8E-05	4,4E-05	-2,2217	0,07696	-0,000212	1,54E-05	-0,00021	1,54E-05
M1	0,000128	4,2E-05	3,052305	0,02835	2,015E-05	0,000235	2,01E-05	0,000235
M2	-5,4E-05	2,5E-06	-21,5252	4E-06	-6,099E-05	-4,8E-05	-6,1E-05	-4,8E-05
M3	1,53E-05	3,9E-06	3,93207	0,01105	5,287E-06	2,53E-05	5,29E-06	2,53E-05
FD	0,004803	0,00172	2,796489	0,03815	0,000388	0,009218	0,000388	0,009218

The regression ratios of the *M1, M2, M3, M4, IR* factors, corresponding to the course of the manat, are either flat or reverse, ie the elasticity values are as follows:

$$a_0 = 0.74,\ a_1 = -9.8,\ a_2 = 0.00013\,,\ a_3 = -5.4,\ a_4 = 1.53,\ a_5 = 0.005$$

Note that positive signage regression coefficients show the direct relationship between the variables and the negative regression coefficients show the inverse.
The default errors for *M1, M2, M3, M4, IR* are the following prices.

$$S_1 = 4.4,\ S_2 = 4.2,\ S_3 = 2.5,\ S_4 = 3.9,\ S_5 = 0.001$$

estimated for Student criterion

$$t_1 = -2.22,\ t_2 = 3.05,\ t_3 = -21.5,\ t_4 = 3.93,\ t_5 = 2.79$$

prices are compared with table prices. Factors that are bigger than the table price criteria are considered to be more important for the model. The relatively high price of the estimated values indicates the acceptance of the hypothesis that the factors are important. The fairly low prices of the obtained value for p significance levels are also a positive factor for the model. This means that the results have been gained with higher probability.

Thus, the multi-dimensional linear regression model we have established can be used to predict the course of the manat, depending on the interest rate and key aggregates of the money supply. We believe that our research can be regarded as important analytical calculations in the investigation and management of the financial and economic system and its key assets, planning and forecasting development dynamics, and regulation of the manat's course.

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The Existence of Global Solutions of a Semi Linear Parabolic Equation with a Singular Potential

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Abstract. In the domain $Q_R^{'} = \{x; |x| > R\} \times (0; +\infty)$ we consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} = -\Delta^2 u + \frac{C_0}{|x|^4} u + |x|^\sigma |u|^q \\ u|_{t=0} = u_0(x) \geq 0 \\ \int\limits_0^\infty \int\limits_{\partial B_R} u dx dt \geq 0, \quad \int\limits_0^\infty \int\limits_{\partial B_R} \Delta u dx dt \leq 0. \end{cases}$$

Nonexistence of global solutions is analyzed.

Key Words and Phrases: Semilinear parabolic equation, biharmonic operator, global solution, singular potential, critical exponent, method of test functions.

2010 Mathematics Subject Classifications: 35B33; 35K58; 35K91

1. Introduction

Let us introduce the following denotations:$x = (x_1, ..., x_n) \in R^n$, $n > 4$, $r = |x| = \sqrt{x_1^2 + ... + x_n^2}$, $B_R = \{x; |x| < R\}$, $B_R^{'} = \{x; |x| > R\}$, $B_{R_1,R_2} = \{x; R_1 < |x| < R_2\}$, $Q_R = B_R \times (0; +\infty)$, $Q_R^{'} = B_R^{'} \times (0; +\infty)$, $\partial B_R = \{x; |x| = R\}$, $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$, $C_{x,t}^{4,1}\left(Q_R^{'}\right)$ is the set of functions four times continuously differentiable with respect to x and continuously differentiable with respect to t.

In the domain $Q_R^{'}$ consider the following problem:

$$\frac{\partial u}{\partial t} = -\Delta^2 u + \frac{C_0}{|x|^4} u + |x|^\sigma |u|^q \tag{1.1}$$

$$u|_{t=0} = u_0(x) \geq 0 \tag{1.2}$$

$$\int\limits_0^\infty \int\limits_{\partial B_R} u dx dt \geq 0, \qquad \int\limits_0^\infty \int\limits_{\partial B_R} \Delta u dx dt \leq 0, \tag{1.3}$$

*Corresponding author.

where $q > 1$, $0 \le C_0 \le \left(\frac{n(n-4)}{4}\right)^2$, $\sigma > -4$, $u_0(x) \in C\left(B'_R\right)$, $\Delta^2 u = \Delta(\Delta u)$, $\Delta u = \sum\limits_{i=1}^{n} \frac{\partial^2 u}{\partial x_t^2}$.

We will study the existence of non-negative global solutions of problem (1.1)-(1.3). We will understand the solution of the problem in the classic sense. The function $u(x,t) \in C^{4,1}_{x,t}(Q'_R) \cap C(B'_R \times [0,+\infty))$ will be said the solution of problem (1.1)-(1.3) if $u(x,t)$ satisfies equation (1.1) at each point of Q'_R, condition (1.2) for $t = 0$ and condition (1.3) for $|x| = R$.

The problems of non-existence of global solutions for different classes of differential equations and inequalities play a key role in theory and applications. Therefor, they are at constant attention of mathematicians and a great number of works were devoted to them. Survey of such results are in the monograph [1]. In the classical paper [2] Fujita considered the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u^p, (x,t) \in R^n \times (0,+\infty), \\ \\ \qquad u|_{t=0} = u_0(x), x \in R^n. \end{cases} \tag{1.4}$$

And it is proved that positive global solutions of problem (1.4) do not exist for $1 < p < p^* = 1 + \frac{2}{n}$, and for $p > p*$ for small $u_0(x)$ there are positive global solutions.The case $p = p*$ was investigated in [3], [4] and it is proved that in this case there also do not exist positive global solutions.The results of Fujita's work [2] aroused great interest in the problem of the absence of global solutions, and they were expanded in several directions.For example, instead of R^n, various bounded and unbounded domains are considered, or more general operators were considered than the Laplace operator and nonlinearities of a different type.A survey of such papers is available in [5], in the monograph [1] and in the book [6]. Weakly nonlinear equations with a biharmonic operator were considered by many authors. In the paper [7] for $c = 0$ problem (1.1)-(1.3) is considered in the domain Q_R and it is proved that if $\sigma \le -4$, $q > 1$, then the solution is absent. In this paper we consider problem (1.1)-(1.3) for $0 \le c \le \frac{(n-2)^2}{4}$, $\sigma > -4$ and also in the papers [7], using the technique of test function, worked out Mitidieri and Pohozaev in the papers [1],[8], find an exact exponent of absence of a global solutions.

2. Auxiliary facts

Let us consider in $R^n \setminus \{0\}$ the linear equation

$$\Delta^2 u - \frac{C_0}{|x|^4} u = 0. \tag{2.1}$$

If $u(x) = u(r)$ is a radial solution of equation (2.1), then

$$\Delta^2 u - \frac{C_0}{|x|^4} u = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r}\right) - \frac{C_0}{|x|^4} u =$$

$$= \frac{\partial^4 u}{\partial r^4} + \frac{2\,(n-1)}{r}\frac{\partial^3 u}{\partial r^3} + \frac{(n-1)\,(n-3)}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{(n-1)\,(n-3)}{r^3}\frac{\partial u}{\partial r} - \frac{C_0}{r^4}u = 0. \qquad (2.2)$$

This is the Euler equation. Its characteristically equation has the following form:

$$\lambda\,(\lambda-1)\,(\lambda-2)\,(\lambda-3) + 2\,(n-1)\,\lambda\,(\lambda-1)\,(\lambda-2) +$$
$$+ (n-1)\,(n-3)\left(\lambda^2 - 2\lambda\right) - C_0 = 0. \qquad (2.3)$$

Make the substitution $\lambda - 1 = t$. Then we get

$$t\,(t-2)\left(t^2-1\right) + 2\,(n-1)\left(t^2-1\right)t + (n-1)\,(n-3)\left(t^2-1\right) - C_0 = 0. \qquad (2.4)$$

Hence

$$t^4 + 2\,(n-2)\,t^3 + \left((n-2)^2 - 2\right)t^2 - 2\,(n-2)\,t - (n-2)^2 + 1 - C_0 = 0,$$

$$\left(t^2 + (n-2)\,t - 1\right)^2 - \left((n-2)^2 + C_0\right) = 0,$$

$$\left(t^2 + (n-2)\,t - 1 - \sqrt{(n-2)^2 + C_0}\right)\left(t^2 + (n-2)\,t - 1 + \sqrt{(n-2)^2 + C_0}\right) = 0.$$

So,

$$t = -\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 + \sqrt{(n-2)^2 + C_0}},$$

$$t = -\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 - \sqrt{(n-2)^2 + C_0}}.$$

are the all roots of equation (2.4).

Hence,

$$\lambda = -\frac{n-4}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 + \sqrt{(n-2)^2 + C_0}},$$

$$\lambda = \frac{n-4}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 - \sqrt{(n-2)^2 + C_0}}.$$

all the roots of equation (2.3).

For brevity of notation we denote:

$$(n-2)^2 + C_0 = D, \quad \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 \pm \sqrt{(n-2)^2 + C_0}} = \alpha_\pm.$$

We consider the function

$$\xi\,(|x|) = \frac{1}{2}\left(1 + \frac{\sqrt{D} - \alpha_+}{\alpha_-}\right)|x|^{-\frac{n-4}{2}+\alpha_-} +$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)|x|^{-\frac{n-4}{2}-\alpha_{-}}-|x|^{-\frac{n-4}{2}-\alpha_{+}}.$$

Obviously, $\xi(|x|)$ is a radial solution of equation (2.1) in $R^n\backslash\{0\}$.
Show that $\xi(x)$ satisfies the following conditions:

$$\xi|_{|x|=1}=0,\quad \left.\frac{\partial\xi}{\partial r}\right|_{|x|=1}\geq 0,\quad \Delta\xi|_{|x|=1}=0,\quad \left.\frac{\partial(\Delta\xi)}{\partial r}\right|_{|x|=1}\leq 0. \tag{2.5}$$

$$\xi(x)|_{|x|=1}=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)-1=0,$$

$$\left.\frac{\partial\xi}{\partial r}\right|_{|x|=1}=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)\left(-\frac{n-4}{2}+\alpha_{-}\right)+$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)\left(-\frac{n-4}{2}-\alpha_{-}\right)-\left(-\frac{n-4}{2}-\alpha_{+}\right)=$$

$$=\frac{1}{2}\left(\alpha_{-}+\sqrt{D}-\alpha_{+}\right)-\frac{1}{2}\left(\alpha_{-}-\sqrt{D}+\alpha_{+}\right)+\alpha_{+}=\sqrt{D}\geq 0.$$

$$\Delta\xi|_{|x|=1}=\left.\left(\frac{\partial^2\xi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\xi}{\partial r}\right)\right|_{|x|=1}=$$

$$=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)\left(-\frac{n-4}{2}+\alpha_{-}\right)\left(\frac{n}{2}+\alpha_{-}\right)+$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)\left(-\frac{n-4}{2}-\alpha_{-}\right)\left(\frac{n}{2}-\alpha_{-}\right)-\left(-\frac{n-4}{2}-\alpha_{+}\right)\left(\frac{n}{2}-\alpha_{+}\right)=$$

$$=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)\left(-\frac{n(n-4)}{4}+\alpha_{-}^2+2\alpha_{-}\right)+$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)\left(-\frac{n(n-4)}{4}+\alpha_{-}^2-2\alpha_{-}\right)-\left(-\frac{n(n-4)}{4}+\alpha_{+}^2-2\alpha_{+}\right)=$$

$$=-\frac{n(n-4)}{4}\left(\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_{+}}{\alpha_{-}}\right)-1\right)+$$

$$+\alpha_{-}^2-\alpha_{+}^2+\alpha_{-}+\sqrt{D}-\alpha_{+}-\alpha_{-}+\sqrt{D}-\alpha_{+}+2\alpha_{+}=$$

$$=-\sqrt{D}-\sqrt{D}+\sqrt{D}+\sqrt{D}=0.$$

$$\frac{\partial}{\partial r}\left(\Delta\xi\right)\bigg|_{|x|=1} = \left(\frac{\partial^3\xi}{\partial r^3}+\frac{n-1}{r}\frac{\partial^2\xi}{\partial r^2}-\frac{n-1}{r^2}\frac{\partial\xi}{\partial r}\right)\bigg|_{|x|=1} =$$

$$=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left[\left(-\frac{n-4}{2}+\alpha_-\right)\left(-\frac{n-4}{2}+\alpha_--1\right)\left(-\frac{n}{2}+\alpha_-\right)+\right.$$

$$\left.+(n-1)\left(-\frac{n-4}{2}+\alpha_+\right)\left(-\frac{n}{2}+\alpha_-\right)\right]+$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left[\left(-\frac{n-4}{2}-\alpha_-\right)\left(-\frac{n-4}{2}-\alpha_--1\right)\left(-\frac{n}{2}-\alpha_-\right)+\right.$$

$$\left.+(n-1)\left(-\frac{n-4}{2}-\alpha_-\right)\left(-\frac{n}{2}-\alpha_-\right)\right]-$$

$$-\left[\left(-\frac{n-4}{2}-\alpha_+\right)\left(-\frac{n-4}{2}-\alpha_+-1\right)\left(-\frac{n}{2}-\alpha_+\right)+\right.$$

$$\left.+(n-1)\left(-\frac{n-4}{2}-\alpha_+\right)\left(-\frac{n}{2}-\alpha_+\right)\right]=$$

$$=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}+\alpha_-\right)\left(-\frac{n}{2}+\alpha_-\right)\left(\frac{n}{2}+\alpha_-\right)+$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}-\alpha_-\right)\left(-\frac{n}{2}-\alpha_-\right)\left(\frac{n}{2}-\alpha_+\right)-$$

$$-\left(-\frac{n-4}{2}-\alpha_+\right)\left(-\frac{n}{2}-\alpha_+\right)\left(\frac{n}{2}-\alpha_+\right)=$$

$$=\frac{1}{2}\left(1+\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}+\alpha_-\right)\left(\alpha_-^2-\frac{n^2}{4}\right)+$$

$$+\frac{1}{2}\left(1-\frac{\sqrt{D}-\alpha_+}{\alpha_-}\right)\left(-\frac{n-4}{2}-\alpha_-\right)\left(\alpha_-^2-\frac{n^2}{4}\right)-\left(-\frac{n-4}{2}-\alpha_+\right)\left(\alpha_+^2-\frac{n^2}{4}\right)=$$

$$=\left(2-n-\sqrt{D}\right)\left(-\frac{n-4}{2}+\frac{1}{2}\left(\alpha_-+\sqrt{D}-\alpha_+-\alpha_-+\sqrt{D}-\alpha_+\right)\right)+$$

$$+\left(2-n+\sqrt{D}\right)\left(\frac{n-4}{2}+\alpha_+\right)=\left(2-n-\sqrt{D}\right)\left(-\frac{n-4}{2}+\sqrt{D}-\alpha_+\right)+$$

$$+\left(2-n+\sqrt{D}\right)\left(\frac{n-4}{2}+\alpha_+\right)=-\left(2-n-\sqrt{D}\right)\left(\frac{n-4}{2}+\alpha_+\right)+(2-n)\sqrt{D}-D+$$

$$+\left(2-n+\sqrt{D}\right)\left(\frac{n-4}{2}+\alpha_+\right)=\sqrt{D}\frac{n-4}{2}-\alpha_+\left(2-n\right)+\sqrt{D}\alpha_+ +\left(2-n\right)\sqrt{D}-D+$$

$$+\sqrt{D}\frac{n-4}{2}+\left(2-n\right)\alpha_+ +\sqrt{D}\alpha_+ = 2\sqrt{D}\alpha_+ - 2\sqrt{D}-D=$$

$$=\sqrt{D}\left(2\alpha_+ - \sqrt{D}-2\right)\leq 0.$$

Indeed, as $C_0 \geq 0$, then

$$(n-2)^2 \leq (n-2)^2 + C_0 = D.$$

Then

$$4+4\sqrt{D}+(n-2)^2 \leq 4+4\sqrt{D}+D.$$

Hence

$$4\left(\left(\frac{n-2}{2}\right)^2+1+\sqrt{D}\right)\leq\left(2+\sqrt{D}\right)^2$$

$$4\alpha_+^2 \leq \left(2+\sqrt{D}\right)^2,$$

$$2\alpha_+ \leq 2+\sqrt{D}.$$

So,

$$2\alpha_+ - \sqrt{D}-2\leq 0.$$

3. Formulation of the basic result and proof.

The following theorem is the basic result of this paper.

Theorem. Let $n > 4$, $\sigma > -4$, $0 \leq C_0 \leq (\frac{n(n-4)}{4})^2$ and $1 < q \leq 1 + \frac{\sigma+4}{\frac{n+4}{2}+\alpha_-}$. If $u(x,t)$ is the solution of problem (1.1)-(1.3), then $u(x,t) \equiv 0$.

Proof.

For simplicity of notation we take $R = 1$. Assume that $u(x) \geq 0$ is the solution of problem (1.1)-(1.3) in Q_R' .

Let us consider the following functions:

$$\varphi\left(x\right)=\left\{\begin{array}{c} 1,\ \text{for } 1\leq|x|\leq\rho \\ \left(2-\frac{|x|}{\rho}\right)^{\beta},\ \text{for } \rho\leq|x|\leq 2\rho \\ 0,\ \text{for } |x|\geq 2\rho \end{array}\right.$$

$$T_\rho\left(t\right)=\left\{\begin{array}{c} 1,\ \text{for } 0\leq t\leq\rho^{\chi} \\ \left(2-\rho^{-\chi}t\right)^{\mu},\ \text{for } \rho^{\chi}\leq t\leq 2\rho^{\chi} \\ 0,\ \text{for } t\geq 2\rho^{\chi} \end{array}\right.$$

where β,μ are larger positive numbers, moreover β is such that for $|x| = 2\rho$

$$\psi = \frac{\partial\psi}{\partial r} = \frac{\partial^2\psi}{\partial r^2} = \frac{\partial^3\psi}{\partial r^3} = 0, \tag{3.1}$$

and χ will be defined later.

Multiply, equation (1.1) by the function

$$\psi\left(x,t\right) = T_\rho\left(t\right)\xi\left(x\right)\varphi\left(x\right)$$

and integrate in domain $Q_1^{'}$.

After integrating by parts, we get

$$\begin{gathered}\int\limits_{Q_1^{'}} u^q\left|x\right|^\sigma T_\rho\xi\varphi dxdt = -\int\limits_{Q_1^{'}} u\xi\varphi\frac{dT\rho}{dt}dxdt+ \\ +\int\limits_{Q_1^{'}} uT_\rho\Delta^2\left(\xi\varphi\right)dxdt - \int\limits_{Q_1^{'}}\frac{C_0}{|x|^4}uT_\rho\xi\varphi dxdt- \\ -\int\limits_{B_1^{'}} u_0\left(x\right)\xi\left(x\right)\varphi\left(x\right)dx + \int\limits_0^\infty T_\rho\left(t\right)dt\times \\ \times\left[\int\limits_{\partial B_{1,2\rho}}\frac{\partial\left(\Delta u\right)}{\partial\nu}\xi\varphi ds - \int\limits_{\partial B_{1,2\rho}}\Delta u\frac{\partial\left(\xi\varphi\right)}{\partial\nu}ds+\right. \\ \left.+\int\limits_{\partial B_{1,2\rho}}\frac{\partial u}{\partial\nu}\Delta\left(\xi\varphi\right)ds - \int\limits_{\partial B_{1,2\rho}} u\frac{\partial}{\partial\nu}\left(\Delta\left(\xi\varphi\right)\right)ds\right]\end{gathered} \tag{3.2}$$

Estimate the integrals in the square bracket, taking into account (2.5), (3.1) and condition (1.3), we get:

$$\int\limits_{\partial B_{1,2\rho}}\frac{\partial\left(\Delta u\right)}{\partial\nu}\xi\varphi ds = 0,$$

$$\begin{gathered}-\int\limits_{\partial B_{1,2\rho}}\Delta u\frac{\partial\left(\xi\varphi\right)}{\partial\nu}ds = -\int\limits_{|x|=1}\Delta u\frac{\partial\left(\xi\varphi\right)}{\partial\nu}ds - \int\limits_{|x|=2\rho}\Delta u\frac{\partial\left(\xi\varphi\right)}{\partial\nu}ds = \\ = \int\limits_{|x|=1}\Delta u\left(\frac{\partial\xi}{\partial r}\varphi + \xi\frac{\partial\varphi}{\partial r}\right)ds - \int\limits_{|x|=2\rho}\Delta u\left(\frac{\partial\xi}{\partial r}\varphi + \xi\frac{\partial\varphi}{\partial r}\right)ds = \\ = \int\limits_{|x|=1}\Delta u\frac{\partial\xi}{\partial r}ds = \sqrt{D}\int\limits_{|x|=1}\Delta u ds \leq 0,\end{gathered}$$

$$\int\limits_{\partial B_{1,2\rho}} \frac{\partial u}{\partial \nu} \Delta\left(\xi\varphi\right) ds = \int\limits_{\partial B_{1,2\rho}} \frac{\partial u}{\partial \nu}\left(\Delta\xi\varphi + 2\left(\nabla\xi, \nabla\varphi\right) + \xi\Delta\varphi\right) ds =$$

$$= -\int\limits_{|x|=1} \frac{\partial u}{\partial r} \Delta\xi ds = 0,$$

$$-\int\limits_{\partial B_{1,2\rho}} u\frac{\partial}{\partial \nu}\left(\Delta\left(\xi\varphi\right)\right) ds = -\int\limits_{|x|=1} u\frac{\partial}{\partial \nu}\left(\Delta\xi\varphi + 2\left(\nabla\xi, \nabla\varphi\right) + \xi\Delta\varphi\right) ds =$$

$$= \int\limits_{|x|=1} u\frac{\partial\left(\Delta\xi\right)}{\partial r} ds = \sqrt{D}\left(2\alpha_{+} - \sqrt{D} - 2\right) \int\limits_{|x|=1} u ds \leq 0.$$

As $\int\limits_{B_1'} u_0\left(x\right)\zeta\left(x\right)\varphi\left(x\right)dx \geq 0$, $\int\limits_0^{\infty} T_{\rho}\left(t\right)dt > 0$, then from (3.2)

$$\int\limits_{Q_1'} u^q\left|x\right|^{\sigma} T_{\rho}\xi\varphi dx dt \leq -\int\limits_{Q_1'} u\xi\varphi\frac{dT_{\rho}}{dt} dx dt +$$

$$+\int\limits_{Q_1'} uT_{\rho}\Delta^2\left(\xi\varphi\right) dx dt - \int\limits_{Q_1'} \frac{C_0}{\left|x\right|^4} uT_{\rho}\xi\varphi dx dt =$$

$$= -\int\limits_{Q_1'} u\xi\varphi\frac{dT_{\rho}}{dt} dx dt + \int\limits_{Q_1'} uT_{\rho}\varphi\left(\Delta^2\xi - \frac{C_0}{\left|x\right|^4}\xi\right) dx dt +$$

$$+\int\limits_{Q_1'} uT_{\rho}\left[4\left(\nabla\left(\Delta\xi\right), \nabla\varphi\right) + 4\left(\nabla\xi, \nabla\left(\Delta\varphi\right)\right) + 2\Delta\xi\Delta\varphi + \right.$$

$$\left. +4\sum_{i,j=1}^{n} \frac{\partial^2\xi}{\partial x_i\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right] dx dt \leq$$

$$\leq -\int\limits_{\rho^{\varkappa}}^{2\rho^{\varkappa}}\int\limits_{B_1'} u\xi\varphi\frac{dT_{\rho}}{dt} dx dt + \int\limits_0^{2\rho^{\varkappa}}\int\limits_{B_{\rho,2\rho}} uT_{\rho}J\left(\xi,\varphi\right) dx dt, \tag{3.3}$$

where $J(\xi,\varphi)$ denotes the expression in the square bracket, i.e.

$$J\left(\xi,\varphi\right) \equiv 4\left(\nabla\left(\Delta\xi\right), \nabla\varphi\right) + 4\left(\nabla\xi, \nabla\left(\Delta\varphi\right)\right) +$$

$$+2\Delta\xi\Delta\varphi + 4\sum_{i,j=1}^{n} \frac{\partial^2\xi}{\partial x_i\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j}$$

Using the Holder inequality, from (3.3) we get

$$\int\limits_{Q_1'} u^q |x|^\sigma T_\rho \xi\varphi dxdt \le \left(\int\limits_{\rho^\chi}^{2\rho^\chi} \int\limits_{B_1'} u^q |x|^\sigma T_\rho \xi\varphi dxdt \right)^{\frac{1}{q}} \times$$

$$\times \left(\int\limits_{\rho^\chi}^{2\rho^\chi} \int\limits_{B_1'} \frac{\left|\frac{dT_\rho}{dt}\right|^{q'} \xi\varphi}{T_\rho^{q'-1} |x|^{\sigma(q'-1)}} dxdt \right)^{\frac{1}{q'}} +$$

$$+ \left(\int\limits_0^{2\rho^\chi} \int\limits_{B_{\rho,2\rho}} u^q |x|^\sigma T_\rho \xi\varphi dxdt \right)^{\frac{1}{q}} \left(\int\limits_0^{2\rho^\chi} \int\limits_{B_{\rho,2\rho}} \frac{|J(\xi,\varphi)|^{q'} T_\rho}{\xi^{q'-1}\varphi^{q'-1} |x|^{\sigma(q'-1)}} dxdt \right)^{\frac{1}{q'}}, \tag{3.4}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Hence we get

$$\int\limits_{Q_1'} u^q |x|^\sigma T_\rho \xi\varphi dxdt \le C_1 \int\limits_{\rho^\chi}^{2\rho^\chi} \int\limits_{B_1'} \frac{\left|\frac{dT_\rho}{dt}\right|^{q'} \xi\varphi}{T_\rho^{q'-1} |x|^{\sigma(q'-1)}} dxdt +$$

$$+C_2 \int\limits_0^{2\rho^\chi} \int\limits_{B_{\rho,2\rho}} \frac{|J(\xi,\varphi)|^{q'}}{\xi^{q'-1}\varphi^{q'-1} |x|^{\sigma(q'-1)}} dxdt. \tag{3.5}$$

Making the substitution $t = \rho^\chi \tau$, $r = \rho\theta$, $\widetilde{T}(\tau) = T_\rho(\rho^\chi \tau)$, $\widetilde{\xi}(\theta) = \xi(\rho\theta)$, $\widetilde{\varphi}(\theta) = \varphi(\rho\theta)$, we estimate the integrals in the right hand side of (3.5).

$$I_1 \equiv \int\limits_{\rho^\chi}^{2\rho^\chi} \int\limits_{B_1'} \frac{\left|\frac{dT_\rho}{dt}\right|^{q'} \xi\varphi}{T_\rho^{q'-1} |x|^{\sigma(q'-1)}} dxdt \le$$

$$\le \int\limits_{\rho^\chi}^{2\rho^\chi} \frac{\left|\frac{dT_\rho}{dt}\right|^{q'}}{T_\rho^{q'-1}} dt \int\limits_{B_{1,2\rho}} |x|^{-\sigma(q'-1)} \xi\varphi dx \le$$

$$\le C_3 \rho^{-\chi q' + \chi} \int\limits_1^2 \frac{\left|\frac{d\widetilde{T}}{d\tau}\right|^{q'}}{\widetilde{T}^{(q'-1)}} d\tau \int\limits_1^{2\rho} r^{-\frac{n-4}{2}+\alpha_-} r^{-\sigma(q'-1)} r^{n-1} dr \le$$

$$\le C_3 \rho^{\chi(1-q') - \frac{n-4}{2} + \alpha_- - \sigma(q'-1) + n} A_1\left(\widetilde{T}\right) =$$

$$= C_3 \rho^{\chi(1-q') + \frac{n+4}{2} + \alpha_- - \sigma(q'-1)} A_1\left(\widetilde{T}\right), \tag{3.6}$$

where

$$A_1\left(\widetilde{T}\right)=\int\limits_1^2\frac{\left|\frac{d\widetilde{T}}{d\tau}\right|^{q'}}{\widetilde{T}^{(q'-1)}}d\tau.$$

$$I_2\equiv\int\limits_0^{2\rho^{\chi}}\int\limits_{B_{\rho,2\rho}}\frac{|J(\xi,\varphi)|^{q'}T_\rho}{\xi^{q'-1}\varphi^{q'-1}|x|^{\sigma(q'-1)}}dxdt=$$

$$=\int\limits_0^{2\rho^{\chi}}T_\rho(t)\,dt\int\limits_{B_{\rho,2\rho}}\frac{|J(\xi,\varphi)|^{q'}}{\xi^{q'-1}\varphi^{q'-1}|x|^{\sigma(q'-1)}}dx \qquad (3.7)$$

Estimate each addend of $J(\xi,\varphi)$ separately

$$|(\nabla(\Delta\xi),\nabla\varphi)|=\left|\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{\partial^2\xi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\xi}{\partial r}\right)\frac{\partial\varphi}{\partial x_i}\right|=$$

$$=\left|\left(\frac{\partial^3\xi}{\partial r^3}+\frac{n-1}{r}\frac{\partial^2\xi}{\partial r^2}-\frac{n-1}{r^2}\frac{\partial\xi}{\partial r}\right)\frac{\partial\varphi}{\partial r}\right|\le$$

$$\le C_4r^{-\frac{n-4}{2}+\alpha_--3}\left|\frac{\partial\varphi}{\partial r}\right|$$

$$|\Delta\xi\Delta\varphi|=\left|\left(\frac{\partial^2\xi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\xi}{\partial r}\right)\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\varphi}{\partial r}\right)\right|\le$$

$$\le C_5r^{-\frac{n-4}{2}+\alpha_--2}\left|\frac{\partial^2\varphi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\varphi}{\partial r}\right|,$$

$$|(\nabla\xi,\nabla(\Delta\varphi))|\le$$

$$\le C_6r^{-\frac{n-4}{2}+\alpha_--1}\left(\left|\frac{\partial^3\varphi}{\partial r^3}\right|+\frac{n-1}{r}\left|\frac{\partial^2\varphi}{\partial r^2}\right|+\frac{n-1}{r^2}\left|\frac{\partial\varphi}{\partial r}\right|\right),$$

$$\left|\sum_{i,j=1}^n\frac{\partial^2\xi}{\partial x_i\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|=$$

$$=\left|\sum_{i,j=1}^n\frac{\partial}{\partial x_j}\left(\frac{\partial\xi}{\partial r}\frac{x_i}{r}\right)\frac{\partial}{\partial x_j}\left(\frac{\partial\varphi}{\partial r}\frac{x_i}{r}\right)\right|=$$

$$=\left|\sum_{i,j=1}^n\left(\frac{\partial^2\xi}{\partial r^2}\frac{x_ix_j}{r^2}+\frac{\partial\xi}{\partial r}\left(\frac{\delta_{ij}}{r}-\frac{x_ix_j}{r^3}\right)\right)\times\right.$$

$$\left.\times\left(\frac{\partial^2\varphi}{\partial r^2}\frac{x_ix_j}{r^2}+\frac{\partial\varphi}{\partial r}\left(\frac{\delta_{ij}}{r}-\frac{x_ix_j}{r^3}\right)\right)\right|\le$$

$$\leq C_7 \left(\left| \frac{\partial^2 \xi}{\partial r^2} \right| + \frac{1}{r} \left| \frac{\partial \xi}{\partial r} \right| \right) \left(\left| \frac{\partial^2 \varphi}{\partial r^2} \right| + \frac{1}{r} \left| \frac{\partial \varphi}{\partial r} \right| \right) \leq$$

$$\leq C_8 r^{-\frac{n-4}{2}+\alpha_- -2} \left(\left| \frac{\partial^2 \varphi}{\partial r^2} \right| + \frac{1}{r} \left| \frac{\partial \varphi}{\partial r} \right| \right).$$

Using all of these ones, from (3.7) we get

$$I_2 \leq C_9 \rho^{\chi} \int_{\rho}^{2\rho} \frac{r^{\left(-\frac{n-4}{2}+\alpha_- -4\right)q'} \left(r \left| \frac{\partial \varphi}{\partial r} \right| + r^2 \left| \frac{\partial^2 \varphi}{\partial r^2} \right| + r^3 \left| \frac{\partial^3 \varphi}{\partial r^3} \right| \right)^{q'} r^{n-1}}{r^{\left(-\frac{n-4}{2}+\alpha_-\right)(q'-1)+\sigma(q'-1)} \varphi^{q'-1}} dr.$$

Hence

$$I_2 \leq C_{10} \rho^{\chi - \frac{n-4}{2}+\alpha_- -4q' - \sigma(q'-1)+n} \int_1^2 \frac{\left(\theta \left| \frac{\partial \widetilde{\varphi}}{\partial \theta} \right| + \theta^2 \left| \frac{\partial^2 \widetilde{\varphi}}{\partial \theta^2} \right| + \theta^3 \left| \frac{\partial^3 \widetilde{\varphi}}{\partial \theta^3} \right| \right)^{q'}}{\theta^{\frac{n-4}{2}-\alpha_- +4q'+\sigma(q'-1)-n+1} \widetilde{\varphi}^{(q'-1)}} d\theta \leq$$

$$\leq C_{10} \rho^{\chi + \frac{n+4}{2}+\alpha_- -4q' - \sigma(q'-1)} A_2\left(\widetilde{\varphi}\right), \tag{3.8}$$

where $A_2\left(\widetilde{\varphi}\right)$ denotes the last integral.

Obviously, for large μ and β, $A_1\left(\widetilde{T}\right) < \infty$, $A_2\left(\widetilde{\varphi}\right) < \infty$.

We take χ so that

$$\chi - 4q' - \sigma\left(q'-1\right) = \chi - \left(1-q'\right) - \sigma\left(q'-1\right).$$

Hence $\chi = 4$.

Using (3.6), (3.8) and (3.5) we get

$$\int_{Q_1'} |u|^q |x|^{\sigma} T_\rho \xi \varphi dx dt \leq \left(C_{11} A_1\left(\widetilde{T}\right) + C_{12} A_2\left(\widetilde{\varphi}\right) \right) \rho^{\frac{n+4}{2}+\alpha_- -(\sigma+4)(q'-1)}. \tag{3.9}$$

Let now $(\sigma+4)\left(q'-1\right) - \frac{n+4}{2} - \alpha_- > 0$.

Then

$$(\sigma+4)\frac{1}{q-1} > \frac{n+4}{2} + \alpha_-$$

and

$$q < 1 + \frac{\sigma+4}{\frac{n+4}{2}+\alpha_-}.$$

In this case, tending ρ to $+\infty$ from (3.9) we get, that

$$\int_{Q_1'} |u|^q |x|^{\sigma} \xi dx dt \leq 0.$$

This means that $u \equiv 0$.

Let now $(\sigma+4)(q'-1) - \frac{n+4}{2} - \alpha_- = 0$.

Then from (3.6), (3.8) we get $I_1 < C$, $I_2 < C$ and therefore

$$\int\limits_{Q_1'} u^q |x|^\sigma \xi dxdt < C.$$

From the property of the integral we get

$$\int\limits_0^\infty \int\limits_{B_{\rho,2\rho}} u^q |x|^\sigma \xi dxdt \to 0, \tag{3.10}$$

and

$$\int\limits_{\rho^4}^{2\rho^4} \int\limits_{B_1'} u^q |x|^\sigma \xi dxdt \to 0. \tag{3.11}$$

Then from (3.4)

$$\int\limits_{Q_1'} u^q |x|^\sigma T_\rho \xi \varphi dxdt \le \left(\int\limits_{\rho^4}^{2\rho^4} \int\limits_{B_1'} u^q |x|^\sigma \xi T_\rho \varphi dxdt \right)^{\frac{1}{q}} I_1^{\frac{1}{q'}} +$$

$$+ \left(\int\limits_0^{2\rho^4} \int\limits_{B_{\rho,2\rho}} u^q |x|^\sigma T_\rho \xi \varphi dxdt \right)^{\frac{1}{q}} I_2^{\frac{1}{q'}} \le$$

$$\le \left(\int\limits_{\rho^4}^{2\rho^4} \int\limits_{B_1'} u^q |x|^\sigma \xi dxdt \right)^{\frac{1}{q}} I_1^{\frac{1}{q'}} + \left(\int\limits_0^{\infty} \int\limits_{B_{\rho,2\rho}} u^q |x|^\sigma T_\rho \xi \varphi dxdt \right)^{\frac{1}{q}} I_2^{\frac{1}{q'}} \to 0$$

as $\rho \to +\infty$ by (3.9), (3.10).

Hence it follows that $u \equiv 0$.

This completely proves the theorem.

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On Completeness of Double Exponential System in Generalized Weighted Lebesgue Spaces

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Abstract. A double exponential system with complex-valued complex coefficients is considered in generalized weighted Lebesgue spaces. Completeness of this system in $L_{p(\cdot);\rho}$ spaces is studied.

Key Words and Phrases: exponential system, basicity, variable exponent, generalized Lebesgue space.

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1. Introduction

In the context of applications to some problems of mechanics and mathematical physics, since recently there arose great interest in the study of different problems in generalized Lebesgue spaces $L_{p(\cdot)}$ of variable summability rate $p(\cdot)$. Some fundamental results of classical harmonic analysis have been extended to the case of $L_{p(\cdot)}$ (for more details see [9, 10, 11, 12]). Note that the use of Fourier method in solving some problems for partial differential equations in generalized Sobolev classes requires the study of approximative properties of perturbed exponential systems in generalized Lebesgue spaces. Some approximation problems in these spaces have been studied by I.I. Sharapudinov (see, e.g., [11]).

In this work, we consider the completeness of a double exponential system with complex-valued complex coefficients in the spaces $L_{p(\cdot);\rho}$. The completeness is reduced to trivial solvability of the corresponding homogeneous Riemann problem in the classes $H^{+}_{q(\cdot);\rho} \times_{-1} H^{-}_{q(\cdot);\rho}$, where $q\,(t)$ is a conjugate function of $p\,(t)$. Note that when considering the basicity of such systems in $L_{p(\cdot);\rho}$, unlike in the case of completeness, the solvability of corresponding Riemann problem is studied in the classes $H^{+}_{p(\cdot);\rho} \times_{-1} H^{-}_{p(\cdot);\rho}$. That's why we treat the completeness separately. The scheme we use is not new. We just follow the works [2;4].

2. Needful Information

Let $\omega \equiv \{z : |z| < 1\}$ be a unit ball in the complex plane and $\Gamma = \partial\omega$ be a unit circumference. Let $p : [-\pi, \pi] \to [1, +\infty)$ be some Lebesgue measurable function. The class of all Lebesgue measurable functions on $[-\pi, \pi]$ is denoted by $\mathscr{L}_0$. Denote

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Let

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

For $p^+ = \sup\underset{[-\pi,\pi]}{vrai}\, p(t) < +\infty$, $\mathscr{L}$ becomes a linear space with the usual linear operations of addition of functions and multiplication by a number. Equipped with the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

$\mathscr{L}$ becomes a Banach space which we denote by $L_{p(\cdot)}$. Let

$$WL \stackrel{def}{\equiv} \left\{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow \right.$$
$$\left. \Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\right\}.$$

Throughout this work, $q(\cdot)$ denotes a conjugate function of $p(\cdot)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Denote$p^- = \inf\underset{[-\pi,\pi]}{vrai}\, p(t)$.

The following generalized Hölder inequality is true:

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where

$$c\left(p^-; p^+\right) = 1 + \frac{1}{p^-} - \frac{1}{p^+}.$$

We will significantly use the following easy-to-prove statement:

Statement 1. *Suppose*

$$p \in WL,\ p(t) > 0, \forall t \in [-\pi, \pi]; \{\alpha_i\}_0^m \subset R.$$

The weight function

$$\rho(t) = |t|^{\alpha_0} \prod_{i=1}^{m} |t - \tau_i|^{\alpha_i} \tag{1}$$

belongs to the space $L_{p(\cdot)}$ *if the following inequalities are true:*

$$\alpha_i > -\frac{1}{p(\tau_i)}, \forall i = \overline{0, m};$$

where $-\pi = \tau_1 < \tau_2 < ... < \tau_m = \pi$, $\tau_0 = 0$, $t_i \neq 0, \forall i = \overline{1, m}.$

To obtain our main results, we will also use the following important fact:

Property 1. *If* $p(t) : 1 < p^- \leq p^+ < +\infty$, *then the class* $C_0^\infty(-\pi, \pi)$ *(class of finite, infinitely differentiable functions on* $(-\pi, \pi)$*) is everywhere dense in* $L_{p(\cdot)}$.

Define the weighted class $h_{p(\cdot),\rho}$ of functions which are harmonic inside the unit circle ω with the variable summability rate $p(\cdot)$, where the weight function $\rho(\cdot)$ is defined by (1).

Denote

$$h_{p(\cdot),\rho} \equiv \left\{ u : \Delta u = 0 \ in \ \omega \ and \ \|u\|_{p(\cdot),\rho} = \sup_{0<r<1} \left\| u\left(re^{it}\right)\right\|_{p(\cdot),\rho} < +\infty \right\}.$$

We will need the following

Lemma 1. *Let* $p \in WL, p^- \geq 1$, *and the weight* $\rho(\cdot)$ *satisfy the condition*

$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{q(t_k)}, \ k = \overline{0, m}. \tag{2}$$

If $f \in L_{p(\cdot),\rho}$, *then* $\exists p_0 \geq 1 : f \in L_{p_0}$.

The following lemma is also true:

Lemma 2. *Let* $p \in WL, p^- \geq 1$, *and the weight* $\rho(\cdot)$ *satisfy the condition* (2). *If* $u \in h_{p(\cdot),\rho}$, *then* $\exists p_0 \in [1, +\infty]$:$u \in h_{p_0}$.

Using these lemmas, one can prove the following theorem:

Theorem 1. *Let* $p \in WL, p^- > 1$, *and the inequalities* (2) *be fulfilled. If* $u \in h_{p(\cdot),\rho}$, *then* $\exists f \in L_{p(\cdot),\rho}$:

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(t) dt, \tag{3}$$

where

$P_r(\alpha) = \frac{1-r^2}{1+r^2-2r\cos\alpha}$ *is a Poisson kernel.*

On the contrary, if $f \in L_{p(\cdot),\rho}$, *then the function* u *defined by* (3) *belongs to the class* $h_{p(\cdot),\rho}$.

Similarly we define the weighted Hardy classes $H^{\pm}_{p(\cdot),\rho}$. By $H^{+}_{p_0}$ we denote the usual Hardy class, where $p_0 \in [1, +\infty)$ is some number. Let

$$H^{\pm}_{p(\cdot),\rho} \equiv \left\{ f \in H^{+}_{1} : f^{+} \in L_{p(\cdot),\rho}(\partial\omega) \right\},$$

where f^{+} are nontangential boundary values of $f(\cdot)$ on $\partial\omega$.

It is absolutely clear that $f(\cdot)$ belongs to the space $H^{+}_{p(\cdot),\rho}$ only when Ref and Imf belong to the space $h_{p(\cdot),\rho}$. Therefore, many properties of the functions from $h_{p(\cdot),\rho}$ are transferred to the functions from $H^{+}_{p(\cdot),\rho}$. Taking into account the relationship between the Poisson kernel $P_r(\alpha)$ and the Cauchy kernel $K_z(t) = \frac{e^{it}}{e^{it}-z}$, it is easy to derive from Theorem 1 the validity of the following one:

Theorem 2. *Let $p \in WL, p^{-} > 1$, and the inequalities* (2) *be fulfilled. If $F \in H^{+}_{p(\cdot),\rho}$, then $F^{+} \in L_{p(\cdot),\rho}$:*

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F^{+}(t)dt}{1 - ze^{-it}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_z(t) F^{+}(t)dt. \tag{4}$$

On the contrary, if $F^{+} \in L_{p(\cdot),\rho}$, then the function F defined by (4) *belongs to the class $H^{+}_{p(\cdot),\rho}$, where $F^{+}(\cdot)$ are nontangential boundary values of $F(\cdot)$ on $\partial\omega$.*

Following the classics, we define the weighted Hardy class ${}_{m}H^{-}_{p(\cdot),\rho}$ of analytic functions on $C\backslash\bar{\omega}$ of order $k \leq m$ at infinity. Let $f(z)$ be an analytic function on $C\backslash\bar{\omega}$ of finite order $k \leq m$ at infinity, i.e.

$$f(z) = f_1(z) + f_2(z),$$

where $f_1(z)$ is a polynomial of degree $k \leq m$, $f_2(z)$ is the principal part of Laurent decomposition of the function $f(z)$ at infinity. If the function $\varphi(z) \equiv \overline{f_2\left(\frac{1}{\bar{z}}\right)}$ belongs to the class $H^{+}_{p(\cdot),\rho}$, then we will say that the function $f(z)$ belongs to the class ${}_{m}H^{-}_{p(\cdot),\rho}$.

Absolutely similar to the classical case, one can prove the following theorem:

Theorem 3. *Let $p \in WL, p^{-} > 1$, and the inequalities* (2) *be fulfilled. If $f \in H^{+}_{p(\cdot),\rho}$, then*

$$\left\| f(re^{it}) - f^{+}(e^{it}) \right\|_{p(\cdot),\rho} \to 0,\ r \to 1-0,$$

$$\left\| f\left(re^{it}\right) \right\|_{p(\cdot),\rho} \to \left\| f^{+}\left(e^{it}\right) \right\|_{p(\cdot),\rho},\ r \to 1-0,$$

where f^{+} are nontangential boundary values of f on $\partial\omega$.

The similar fact is true also in ${}_{m}H^{-}_{p(\cdot),\rho}$ classes.

Theorem 4. *Let $p \in WL, p^{-} > 1$, and the inequalities* (2) *be fulfilled. If $f \in {}_{m}H^{-}_{p(\cdot),\rho}$, then*

$$\left\|f(re^{it})-f^{-}(e^{it})\right\|_{p(\cdot),\rho}\to 0,\ r\to 1+0,$$

$$\left\|f\left(re^{it}\right)\right\|_{p(\cdot),\rho}\to\left\|f^{-}\left(e^{it}\right)\right\|_{p(\cdot),\rho},\ r\to 1+0,$$

where f^{-} are nontangential boundary values of $\theta(t)\equiv\arg G\left(e^{it}\right)$ on $\partial\omega$ from outside ω.

The following analog of the classical Smirnov theorem is valid:

Theorem 5. *Let $p\in WL, p^{-}>1$, and the inequalities (2) be fulfilled. If $u\in H_1^{+}$ and $L_{p(\cdot),\rho}$, then $u\in H^{+}_{p(\cdot),\rho}$. Denote the restrictions of the classes $H^{+}_{p(\cdot),\rho}, {}_mH^{-}_{p(\cdot),\rho}$ to $\partial\omega$ by $L^{+}_{p(\cdot),\rho}$ and ${}_mL^{-}_{p(\cdot),\rho}$, respectively, i.e.*

$$L^{+}_{p(\cdot),\rho}=H^{+}_{p(\cdot),\rho}/_{\partial\omega}\,;\ {}_mL^{-}_{p(\cdot),\rho}={}_mH^{-}_{p(\cdot),\rho}/_{\partial\omega}\,.$$

We will need the following result:

Theorem 6. *Let $k=\overline{1,r}$, and the inequalities (2) be fulfilled. Then the system $E_{+}^{(0)}=\left\{e^{int}\right\}_{n\ge 0}$($E_{-}^{(m)}=\left\{e^{-int}\right\}_{n\ge m}$) forms a basis for $L^{+}_{p(\cdot),\rho}({}_mL^{-}_{p(\cdot),\rho})$,$1<p<+\infty$.*

We will also need the following easy-to-prove lemma, which is derived immediately from the definition of weighted space $L_{p(\cdot),\rho}$.

Lemma 3. *Let $p\in C\left[-\pi,\pi\right]$ and $p(t)>0$, $\forall t\in\left[-\pi,\pi\right]$. Then the function $\xi(t)=|t-c|^{\alpha}$ belongs to $L_{p(\cdot),\rho}$, if $\alpha>-\frac{1}{p(c)}$, for $c\ne\tau_k$, $\forall\,k=\overline{1,m}$, and $\alpha+\alpha_{k_0}>-\frac{1}{p(c)}$, for $c=\tau_{k_0}$.*

3. Main Assumptions and Riemann Problem Statement

Let's state the Riemann problem in the classes $H^{\pm}_{p(\cdot);\rho}$. Let the complex-valued function $G(t)$ on $[-\pi,\pi]$ satisfy the following conditions:
i) Function $|G(t)|$ belongs to the space $L_{r(\cdot)}$ for some $r:\ 0<r^{-}\le r^{+}<+\infty$, and $|G(t)|^{-1}\in L_{\omega(\cdot)}$ for $\omega:\ 0<\omega^{-}\le\omega^{+}<+\infty$.
ii) Argument $\theta(t)\equiv\arg G(t)$ has a following decomposition:

$$\theta(t)=\theta_0(t)+\theta_1(t)\,,$$

where $\theta_0(t)$ is a continuous function on $[-\pi,\pi]$ and $\theta_1(t)$ is a function of bounded variation on $[-\pi,\pi]$.

It is required to find a piecewise analytic function $F^{\pm}(z)$ on the complex plane with a cut $\partial\omega$ which satisfies the following conditions:

a) $F^{+}(z)\in H^{+}_{p(\cdot)}:0<p^{-}\le p^{+}<+\infty$;

b) $F^{-}(z)\in{}_mH^{-}_{\nu(\cdot)}$; $0<\nu^{-}\le\nu^{+}<+\infty$;

c) nontangential boundary values on the unit circumference $\partial\omega$ satisfy the relation $F^{+}\left(e^{it}\right)+G(t)\,F^{-}\left(e^{it}\right)=g(t)$, for a.e. $t\in(-\pi,\pi)$, *where* $g \in L_{\rho(\cdot)} : 0 < \rho^- \le \rho^+ < +\infty$ *is some given function.*

Note that in the case of constant summability rate, the theory of such problems has been well studied (see [3].

Consider the following homogeneous Riemann problem in the classes $H^+_{p(\cdot),\rho} \times {}_m H^-_{p(\cdot),\rho}$:

$$F^+(\tau) - G(\tau)F^-(\tau) = 0, \ \tau \in \partial\omega. \tag{5}$$

By the solution of the problem (5) we mean a pair of analytic functions

$$(F^+(z); F^-(z)) \in H^+_{p(\cdot),\rho} \times {}_m H^-_{p(\cdot),\rho},$$

whose boundary values satisfy a.e. the equation (5). Introduce the following functions $X_i(z)$ analytic inside (with the sign +) and outside (with the sign -) the unit circle:

$$X_1(z) \equiv \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G\left(e^{it}\right)\right| \frac{e^{it}+z}{e^{it}-z} dt\right\},$$

$$X_2(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta(t) \frac{e^{it}+z}{e^{it}-z} dt\right\},$$

where $\theta(t) \equiv \arg G\left(e^{it}\right)$. Define

$$Z_i(z) \equiv \left\{ \begin{array}{l} X_i(z), \ |z| < 1, \\ \left[X_i(z)\right]^{-1}, \ |z| > 1, \quad i = 1, 2. \end{array} \right.$$

Let $\{s_k\}_1^r : -\pi < s_1 < ...s_r < \pi$ be points of discontinuity of the function $\theta(t)$ and

$$\{h_k\}_1^r : \ h_k = \theta(s_k + 0) - \theta(s_k - 0), k = \overline{1, r},$$

be the corresponding jumps of this function at these points. Denote

$$h_0 = \theta(-\pi) - \theta(\pi); h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi).$$

Let

$$u_0(t) \equiv \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi} \theta_0(\tau)\, ctg\frac{t-\tau}{2} d\tau\right\}$$

and

$u(t) = \prod_{k=0}^r \left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{\frac{h_k}{2\pi}}$, where $s_0 = \pi$.

As is known, (see [3]), the boundary values $\left|Z_2^-(\tau)\right|$ are defined by the formula

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\left[u(t)\right]^{-1}\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}},$$

i.e.

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\prod_{k=0}^r \left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}.$$

It follows directly from Sokhotskii-Plemelj formula that

$$\sup_{(-\pi,\pi)} vrai \left\{ \left| Z_1^- \left(e^{it} \right) \right|^{\pm 1} \right\} < +\infty.$$

Thus, for $\left| Z^- \left(e^{it} \right) \right|^{-1}$ we have the representation

$$\left| Z^- \left(e^{it} \right) \right|^{-1} = \left| Z_1^- \left(e^{it} \right) \right|^{-1} \left| u_0 \left(t \right) \right|^{-1} \prod_{k=0}^{r} \left| \sin \frac{t - s_k}{2} \right|^{\frac{h_k}{2\pi}} . \tag{6}$$

Represent the product $\left| Z^- \rho \right|^{-1}$ as follows:

$$\left| Z^- \rho \right|^{-1} = \left| Z_1^- \right|^{-1} \left| u_0 \right|^{-1} \prod_{k=0}^{l} \left| t - t_k \right|^{\beta_k} ,$$

where

$$\left\{ t_k \right\}_{k=0}^{l} \equiv \left\{ \tau_k \right\}_{k=1}^{m} \bigcup \left\{ s_k \right\}_{k=0}^{r} ,$$

and β_k's are defined by

$$\beta_k = - \sum_{i=1}^{m} \alpha_i \chi_{\{t_k\}} \left(\tau_i \right) + \frac{1}{2\pi} \sum_{i=0}^{r} h_i \chi_{\{t_k\}} \left(s_i \right) , \;\; k = \overline{0, l} . \tag{7}$$

By virtue of Lemma 3, we obtain that if the inequalities

$$\beta_k > - \frac{1}{q(t_k)} \; , \;\; k = \overline{0, r}, \tag{8}$$

are true, then the product $\left| Z^- \rho \right|^{-1}$ belongs to the space $L_{q(\cdot)}$, i.e. $\left| Z^- \right|^{-1} \in L_{q(\cdot),\rho^{-1}}$. So, if the inequalities (8) are true, then the function $\Phi(z)$ belongs to the classes $H_1^{\pm}$. Then, according to [3], $\Phi(z)$ is a polynomial $P_{m_0}(z)$ of degree $m_0 \leq m$. Thus,

$$F^-(z) = P_{m_0}(z) Z^-(z).$$

Let's find out under which conditions the function $F^-(z)$ belongs to the space $H^-_{p(\cdot),\rho}$. We have

$$\left| Z^- \rho \right| = \left| Z_1 \right| \left| u_0 \right| \prod_{k=0}^{l} \left| t - t_k \right|^{-\beta_k} .$$

Consequently, if the inequalities

$$\beta_k < \frac{1}{p(t_k)} \; , \; k = \overline{0, r},$$

are true, then it is clear that $F^-(\tau) \in L_{p(\cdot),\rho}$, and hence $F^- \in {}_mH^-_{p(\cdot),\rho}$. So, if the inequalities

$$-\frac{1}{q(t_k)} < \beta_k < \frac{1}{p\,(t_k)},\ k = \overline{0, r}, \tag{9}$$

are true, then the general solution of homogeneous problem

$$F_0^+ (\tau) = G_1 (\tau) F_0^- (\tau) , \quad \tau \in \partial\omega,$$

in the classes $H^+_{p(\cdot),\,\rho} \times_m H^-_{p(\cdot),\,\rho}$ can be represented as follows:

$$F_0 (z) = P_{m_0} (z) Z (z) ,$$

where $P_{m_0} (z)$ is an arbitrary polynomial of degree $m_0 \le m$. So the following theorem is valid:

Theorem 7. *Let the* $\{\beta_k\}_1^r$*'s be defined by* (7) *and the inequalities* (9) *be true. If*

$$-\frac{1}{p\,(\tau_k)} < \alpha_k < \frac{1}{q\,(\tau_k)}, \quad k = \overline{1, m},$$

then the general solution of the homogeneous Riemann problem (5) *in the classes* $H^+_{p(\cdot),\,\rho} \times_m H^-_{p(\cdot),\,\rho}$ *can be represented as*

$$F (z) = P_{m_0} (z) Z (z) ,$$

where $Z (\cdot)$ *is a canonical solution of homogeneous problem, and* $P_{m_0} (\cdot)$ *is a polynomial of degree* $m_0 \le m$.

This theorem has the following direct corollary:

Corollary 1. *Let all the conditions of Theorem 7 be satisfied. Then the homogeneous Riemann problem* (5) *is trivially solvable in the Hardy classes* $H^+_{p(\cdot),\rho} \times {}_-H^-_{p(\cdot),\rho}$.

4. Reducing The Completeness of Exponential System with Complex Coefficients to Boundary Value Problems

Consider the following exponential system:

$$\left\{A (t) e^{int};\, B (t) e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{10}$$

where $A (t) \equiv |A (t)| e^{i\alpha(t)}$; $B (t) \equiv |B (t)| e^{i\beta(t)}$ are complex-valued functions on $[-\pi, \pi]$. We will consider the completeness of the system (10) in the space $L_{p(\cdot);\rho}$. It is known [6] that the conjugate space of $L_{p(\cdot);\rho}$ is isometrically isomorphic to the space $L_{q(\cdot);\rho}$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Therefore, the completeness of the system (10) in $L_{p(\cdot);\rho}$ is equivalent to the equality to zero of any function $f (t)$ from the space $L_{q(\cdot);\rho}$ which satisfies the relations

$$\int_{-\pi}^{\pi} A (t) e^{int}\overline{f (t)}dt = 0; \quad \int_{-\pi}^{\pi} B (t) e^{-i(n+1)t}\overline{f (t)}dt = 0, \forall n \in Z_+. \tag{11}$$

Assume that the following main condition is satisfied:

$$\underset{[-\pi,\pi]}{ess\sup}\left\{\,|A\,(t)|^{\pm 1};\,|B\,(t)|^{\pm 1}\right\}<+\infty. \tag{12}$$

From the first of equalities (11) we have

$$\int_{-\pi}^{\pi}A\,(t)\,e^{int}\overline{f}\,(t)\,dt=\frac{1}{i}\int_{\partial\omega}f^{+}\,(\tau)\,\tau^{n}d\tau=0,\forall n\in Z_{+}, \tag{13}$$

where $f^{+}\,(\tau)\equiv A\,(\arg\tau)\;\overline{f}\,(\arg\tau)\;\bar{\tau}\;,\;\;\tau\in\partial\omega$.

It is absolutely clear that $f^{+}\,(\tau)\in L_1\,(\partial\omega)$. Then it is well known (see [5, p.205]) that the conditions (13) are equivalent to the existence of a function $F^{+}\,(z)$ from H_1^{+} whose nontangential boundary values on $\partial\omega$ coincide with $f^{+}\,(\tau):F^{+}\,(\tau)=f^{+}\,(\tau)$ a.e. on $\partial\omega$.

Similarly, from the second of equalities (11) we have

$$\int_{-\pi}^{\pi}\overline{B\,(t)}e^{i(n+1)t}f\,(t)\,dt=\frac{1}{i}\int f^{-}\,(\tau)\,\tau^{n}d\tau=0,\forall n\in Z_{+}, \tag{14}$$

where $f^{-}\,(\tau)\;=\;\overline{B\,(\arg\tau)}\,f\,(\arg\tau)\;,\;\;\tau\;\in\;\partial\omega$. For the reason stated above, the equalities (14) are equivalent to the existence of a function $\Phi^{+}\,(z)\in H_1^{+}$ whose nontangential boundary values $\Phi^{+}\,(\tau)$ on $\partial\omega$ coincide with $f^{-}\,(\tau):\Phi^{+}\,(\tau)=f^{-}\,(\tau)$ a.e. on$\partial\omega$.

It is absolutely clear that $F^{+}\,(\tau)\,;\,\Phi^{+}\,(\tau)\;\in\;L_{q(\cdot);\rho}\,(\partial\omega)$. Consequently, if we additionally require that $p\,(t)\in WL$, then from theorem in [7] we obtain the inclusion $F^{+}\,(z)\,;\,\Phi^{+}\,(z)\in H^{+}_{q(\cdot);\rho}$. Representing $f\,(t)$ in terms of $F^{+}\,(\tau)$ and $\Phi^{+}\,(\tau)$, we obtain the following conjugation problem:

$$F^{+}\,(\tau)-\frac{A\,(\arg\tau)}{B\,(\arg\tau)}\overline{\tau\,\Phi^{+}\,(\tau)}=0,\tau\in\partial\omega.$$

Define the function $F^{-}\,(z)$ analytic outside the unit circle:

$$F^{-}\,(z)=\frac{1}{z}\,\overline{\Phi^{+}\left(\frac{1}{\bar{z}}\right)},|z|>1.$$

It is absolutely clear that $F^{-}\,(\infty)=0$. Moreover, $F^{-}\,(\tau)=\bar{\tau}\,\overline{\Phi^{+}\,(\tau)}$, $\tau\in\partial\omega$. Then we arrive at the following Riemann problem:

$$\left\{\begin{array}{l}F^{+}\,(\tau)-G\,(\tau)\,F^{-}\,(\tau)=0\;,\;\;\tau\in\partial\omega,\\F^{-}\,(\infty)=0\,,\end{array}\right. \tag{15}$$

where

$$G\,(\tau)\equiv\frac{A\,(\arg\tau)}{B\,(\arg\tau)},\tau\in\partial\omega.$$

By definition, we have $F^{-}\,(z)\;\in_{-1}\;H^{-}_{q(\cdot);\rho}$. Consequently, if the system (10) is incomplete in $L_{p(\cdot);\rho}$, then the Riemann problem (15) is non-trivially solvable in the classes $\left(H^{+}_{q(\cdot);\rho};\;{}_{-1}H^{-}_{q(\cdot);\rho}\right)$.

Now let's assume that the problem (15) is non-trivially solvable in the classes $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$, i.e. $F^+(z) \in H^+_{q(\cdot);\rho}$, $F^-(z) \in_{-1} H^-_{q(\cdot);\rho}$. Define

$$\Phi_1^+(z) \equiv \overline{F^-\left(\frac{1}{\bar{z}}\right)} \quad \text{where} \quad |z| < 1.$$

We have $F^-(\tau) = \overline{\Phi_1^+(\tau)}$, $\tau \in \partial\omega$ and $\Phi^+(0) = 0$. Then it is clear that the function $\Phi^+(z) = z^{-1}\Phi_1^+(z)$ will be analytic when $|z| < 1$, and moreover, $\Phi^+(z) \in H^+_{q(\cdot);\rho}$. Thus,

$$F^+(\tau) - G(\tau)\,\overline{\tau\,\Phi^+(\tau)} = 0\ ,\ \ \tau \in \partial\omega,$$

or

$$\frac{F^+(\tau)}{A(\arg\tau)\,\bar{\tau}} = \frac{\overline{\Phi^+(\tau)}}{B(\arg\tau)}, \tau \in \omega.$$

Denote

$$f(t) = \frac{\overline{F^+(e^{it})}}{\overline{A(t)}e^{it}} = \frac{\Phi^+\left(e^{it}\right)}{B(t)}.$$

It is absolutely clear that $f(t) \in L_{q(\cdot);\rho}$. From $F^+(z),\ \Phi^+(z) \in H_1^+$ we obtain the equalities

$$\int_{\partial\omega} F^+(\tau)\,\tau^n d\tau = 0; \int_{\partial\omega} \Phi^+(\tau)\,\tau^n d\tau = 0, \forall n \in Z_+.$$

Expressing $F^+(\tau)$ and $\Phi^+(\tau)$ in terms of $f(\arg\tau)$ as $\tau \in \partial\omega$, we have

$$\int_{\partial\omega} A(t)\,e^{-it}\overline{f(t)}e^{int}de^{it} = i\int_{-\pi}^{\pi} A(t)\,e^{int}\overline{f}(t)\,dt = 0, \forall n \in Z_+;$$

$$\int_{\partial\omega} \overline{B(t)}f(t)\,e^{int}de^{it} = i\int_{-\pi}^{\pi} \overline{B(t)}e^{i(n+1)t}f(t)\,dt = 0, \forall n \in Z_+.$$

Obviously, $f(t) \neq 0$ on $[-\pi, \pi]$. Then these relations imply that the system (10) is incomplete in $L_{p(\cdot);\rho}$. So we have the following theorem:

Theorem 8. *Let $p:\ 1 < p^- \leq p^+ < +\infty$, $p(t) \in WL$, and complex-valued coefficients $A(t)$; $B(t)$ satisfy the condition* (12). *Then the exponential system* (10) *is complete in $L_{p(\cdot);\rho}$ only if the Riemann problem* (15) *is only trivially solvable in the classes $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$.*

5. Completeness of Exponential System with Complex Coefficients in $L_{p(\cdot);\rho}$

In this section, we apply the results of previous sections to obtain the sufficient conditions for the completeness of exponential system with complex coefficients in $L_{p(\cdot);\rho}$. So let's consider the system

$$\left\lfloor A(t)\, e^{int};\; B(t)\, e^{-i(n+1)t} \right\rfloor_{n\in Z_+}, \tag{16}$$

where $A(t) \equiv |A(t)|\, e^{i\alpha(t)};\; B(t) \equiv |B(t)|\, e^{i\beta(t)}$ are complex-valued functions on $[-\pi,\pi]$. Assume that the following conditions are satisfied:

$$1) \sup_{[-\pi,\pi]} vrai \left\{ \left(A^{\pm 1};\; B^{\pm 1}\right) \right\} < +\infty,$$

2) The function $\theta(t) \equiv \alpha(t) - \beta(t)$ is piecewise continuous on $[-\pi,\pi]$ with points of discontinuity $\{s_i\}_1^r : \; -\pi < s_1 < ... < s_r < \pi$. Let $\{h_k\}_1^r = \theta(s_k+0) - \theta(s_k-0)$, $k = \overline{1,r}$, be the jumps of the function $\theta(t)$ at these points and $h_0 = \theta(-\pi) - \theta(\pi)$.

3) $\frac{h_k}{2\pi} + \frac{1}{p(s_k)} \notin Z$ (Z is a set of all integers), where $h_k$is a jump of the function $\theta(t) \equiv \alpha(t) - \beta(t)$ at the discontinuity point s_k, $k = \overline{0,r}$; $s_0 = \pi$.

Define the integers n_i, $i = \overline{1,r}$, from the following inequalities:

$$\begin{cases} -\frac{1}{p(s_k)} < \frac{h_k}{2\pi} + n_k - n_{k-1} < \frac{1}{q(s_k)}\,, \; k = \overline{1,r}\,, \\ \\ n_0 = 0\,. \end{cases} \tag{17}$$

Let

$$\Delta_r = \frac{1}{2\pi}\left[\alpha(-\pi) - \alpha(\pi) + \beta(\pi) - \beta(-\pi)\right] + n_r.$$

The following theorem is true:

Theorem 9. *Let the coefficients $A(t)$ and $B(t)$ of the system* (16) *satisfy the conditions 1)-3), where $G\left(e^{it}\right) \equiv \frac{A(t)}{B(t)}$, the integer n_r is defined by* (17), *$p(t) \in WL$, $1 < p^- \le p^+ < +\infty$. Then, if $\Delta_r \notin Z$ and $\Delta_r > -\frac{1}{p(\pi)}$, then the system* (10) *is complete in the space $L_{p(\cdot);\rho}$.*

Let's apply the obtained theorem to the special case

$$\left\{ e^{i[n+\alpha\, sign\, n]t} \right\}_{n\in Z}, \tag{18}$$

where $\alpha \in C$ is a complex parameter. Basis properties of the system (18) in the spaces $L_{p(\cdot);\rho}$ have been well studied. From Theorem 9 we have

Corollary 2. *Let $p(t) \in WL_\pi$, $1 < p^- \le p^+ < +\infty$ and $Re\alpha \in Z$. If $Re\alpha < \frac{1}{2p(\pi)}$, then the system* (18) *is complete in $L_{p(\cdot);\rho}$.*

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On Completeness of Double Exponential System in Generalized Weighted Lebesgue Spaces

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Abstract. A double exponential system with complex-valued complex coefficients is considered in generalized weighted Lebesgue spaces. Completeness of this system in $L_{p(\cdot);\rho}$ spaces is studied.

Key Words and Phrases: exponential system, basicity, variable exponent, generalized Lebesgue space.

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1. Introduction

In the context of applications to some problems of mechanics and mathematical physics, since recently there arose great interest in the study of different problems in generalized Lebesgue spaces $L_{p(\cdot)}$ of variable summability rate $p(\cdot)$. Some fundamental results of classical harmonic analysis have been extended to the case of $L_{p(\cdot)}$ (for more details see [9, 10, 11, 12]). Note that the use of Fourier method in solving some problems for partial differential equations in generalized Sobolev classes requires the study of approximative properties of perturbed exponential systems in generalized Lebesgue spaces. Some approximation problems in these spaces have been studied by I.I. Sharapudinov (see, e.g., [11]).

In this work, we consider the completeness of a double exponential system with complex-valued complex coefficients in the spaces $L_{p(\cdot);\rho}$. The completeness is reduced to trivial solvability of the corresponding homogeneous Riemann problem in the classes $H^{+}_{q(\cdot);\rho} \times_{-1} H^{-}_{q(\cdot);\rho}$, where $q\left(t\right)$ is a conjugate function of $p\left(t\right)$. Note that when considering the basicity of such systems in $L_{p(\cdot);\rho}$, unlike in the case of completeness, the solvability of corresponding Riemann problem is studied in the classes $H^{+}_{p(\cdot);\rho} \times_{-1} H^{-}_{p(\cdot);\rho}$. That's why we treat the completeness separately. The scheme we use is not new. We just follow the works [2;4].

2. Needful Information

Let $\omega \equiv \{z : |z| < 1\}$ be a unit ball in the complex plane and $\Gamma = \partial\omega$ be a unit circumference. Let $p : [-\pi, \pi] \to [1, +\infty)$ be some Lebesgue measurable function. The class of all Lebesgue measurable functions on $[-\pi, \pi]$ is denoted by $\mathscr{L}_0$. Denote

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Let

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

For $p^+ = \sup\underset{[-\pi,\pi]}{vrai}\, p(t) < +\infty$, $\mathscr{L}$ becomes a linear space with the usual linear operations of addition of functions and multiplication by a number. Equipped with the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

$\mathscr{L}$ becomes a Banach space which we denote by $L_{p(\cdot)}$. Let

$$\begin{aligned} WL \stackrel{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow \\ \Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\Big\}. \end{aligned}$$

Throughout this work, $q(\cdot)$ denotes a conjugate function of $p(\cdot)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Denote$p^- = \inf\underset{[-\pi,\pi]}{vrai}\, p(t)$.

The following generalized Hölder inequality is true:

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where

$$c\left(p^-; p^+\right) = 1 + \frac{1}{p^-} - \frac{1}{p^+}.$$

We will significantly use the following easy-to-prove statement:

Statement 1. *Suppose*

$$p \in WL,\ p(t) > 0, \forall t \in [-\pi, \pi]; \{\alpha_i\}_0^m \subset R.$$

The weight function

$$\rho(t) = |t|^{\alpha_0} \prod_{i=1}^{m} |t - \tau_i|^{\alpha_i} \tag{1}$$

belongs to the space $L_{p(\cdot)}$ *if the following inequalities are true:*

$$\alpha_i > -\frac{1}{p(\tau_i)}, \forall i = \overline{0,m};$$

where $-\pi = \tau_1 < \tau_2 < ... < \tau_m = \pi$, $\tau_0 = 0$, $t_i \neq 0, \forall i = \overline{1,m}.$

To obtain our main results, we will also use the following important fact:

Property 1. *If* $p(t) : 1 < p^- \leq p^+ < +\infty$, *then the class* $C_0^\infty(-\pi,\pi)$ *(class of finite, infinitely differentiable functions on* $(-\pi,\pi)$*) is everywhere dense in* $L_{p(\cdot)}$.

Define the weighted class $h_{p(\cdot),\rho}$ of functions which are harmonic inside the unit circle ω with the variable summability rate $p(\cdot)$, where the weight function $\rho(\cdot)$ is defined by (1).

Denote

$$h_{p(\cdot),\rho} \equiv \left\{ u : \Delta u = 0 \ in \ \omega \ and \ \|u\|_{p(\cdot),\rho} = \sup_{0<r<1} \left\| u\left(re^{it}\right)\right\|_{p(\cdot),\rho} < +\infty \right\}.$$

We will need the following

Lemma 1. *Let* $p \in WL, p^- \geq 1$, *and the weight* $\rho(\cdot)$ *satisfy the condition*

$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{q(t_k)}, \ k = \overline{0,m}. \tag{2}$$

If $f \in L_{p(\cdot),\rho}$, *then* $\exists p_0 \geq 1 : f \in L_{p_0}$.

The following lemma is also true:

Lemma 2. *Let* $p \in WL, p^- \geq 1$, *and the weight* $\rho(\cdot)$ *satisfy the condition* (2). *If* $u \in h_{p(\cdot),\rho}$, *then* $\exists p_0 \in [1,+\infty]$:$u \in h_{p_0}$.

Using these lemmas, one can prove the following theorem:

Theorem 1. *Let* $p \in WL, p^- > 1$, *and the inequalities* (2) *be fulfilled. If* $u \in h_{p(\cdot),\rho}$, *then* $\exists f \in L_{p(\cdot),\rho}$:

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(t)dt, \tag{3}$$

where

$P_r(\alpha) = \frac{1-r^2}{1+r^2-2r\cos\alpha}$ *is a Poisson kernel.*

On the contrary, if $f \in L_{p(\cdot),\rho}$, *then the function* u *defined by* (3) *belongs to the class* $h_{p(\cdot),\rho}$.

Similarly we define the weighted Hardy classes $H^{\pm}_{p(\cdot),\rho}$. By $H^{+}_{p_0}$ we denote the usual Hardy class, where $p_0 \in [1, +\infty)$ is some number. Let

$$H^{\pm}_{p(\cdot),\rho} \equiv \left\{ f \in H_1^+ : f^+ \in L_{p(\cdot),\rho}(\partial\omega) \right\},$$

where f^+ are nontangential boundary values of $f(\cdot)$ on $\partial\omega$.

It is absolutely clear that $f(\cdot)$ belongs to the space $H^{+}_{p(\cdot),\rho}$ only when Ref and Imf belong to the space $h_{p(\cdot),\rho}$. Therefore, many properties of the functions from $h_{p(\cdot),\rho}$ are transferred to the functions from $H^{+}_{p(\cdot),\rho}$. Taking into account the relationship between the Poisson kernel $P_r(\alpha)$ and the Cauchy kernel $K_z(t) = \frac{e^{it}}{e^{it}-z}$, it is easy to derive from Theorem 1 the validity of the following one:

Theorem 2. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $F \in H^{+}_{p(\cdot),\rho}$, then $F^+ \in L_{p(\cdot),\rho}$:*

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F^+(t)dt}{1 - ze^{-it}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_z(t) F^+(t) dt. \tag{4}$$

On the contrary, if $F^+ \in L_{p(\cdot),\rho}$, then the function F defined by (4) *belongs to the class $H^{+}_{p(\cdot),\rho}$, where $F^+(\cdot)$ are nontangential boundary values of $F(\cdot)$ on $\partial\omega$.*

Following the classics, we define the weighted Hardy class ${}_mH^{-}_{p(\cdot),\rho}$ of analytic functions on $C\backslash\bar{\omega}$ of order $k \leq m$ at infinity. Let $f(z)$ be an analytic function on $C\backslash\bar{\omega}$ of finite order $k \leq m$ at infinity, i.e.

$$f(z) = f_1(z) + f_2(z),$$

where $f_1(z)$ is a polynomial of degree $k \leq m$, $f_2(z)$ is the principal part of Laurent decomposition of the function $f(z)$ at infinity. If the function $\varphi(z) \equiv \overline{f_2\left(\frac{1}{\bar{z}}\right)}$ belongs to the class $H^{+}_{p(\cdot),\rho}$, then we will say that the function $f(z)$ belongs to the class ${}_mH^{-}_{p(\cdot),\rho}$.

Absolutely similar to the classical case, one can prove the following theorem:

Theorem 3. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $f \in H^{+}_{p(\cdot),\rho}$, then*

$$\left\| f(re^{it}) - f^+(e^{it}) \right\|_{p(\cdot),\rho} \to 0, \; r \to 1-0,$$

$$\left\| f\left(re^{it}\right) \right\|_{p(\cdot),\rho} \to \left\| f^+\left(e^{it}\right) \right\|_{p(\cdot),\rho}, \; r \to 1-0,$$

where f^+ are nontangential boundary values of f on $\partial\omega$.

The similar fact is true also in ${}_mH^{-}_{p(\cdot),\rho}$ classes.

Theorem 4. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $f \in {}_mH^{-}_{p(\cdot),\rho}$, then*

$$\left\|f(re^{it})-f^{-}(e^{it})\right\|_{p(\cdot),\rho}\rightarrow 0,\ r\rightarrow 1+0,$$

$$\left\|f\left(re^{it}\right)\right\|_{p(\cdot),\rho}\rightarrow\left\|f^{-}\left(e^{it}\right)\right\|_{p(\cdot),\rho},\ r\rightarrow 1+0,$$

where f^{-} *are nontangential boundary values of* $\theta\left(t\right)\equiv\arg G\left(e^{it}\right)$ *on* $\partial\omega$ *from outside* ω.

The following analog of the classical Smirnov theorem is valid:

Theorem 5. *Let* $p\in WL, p^{-}>1$, *and the inequalities* (2) *be fulfilled. If* $u\in H_{1}^{+}$ *and* $L_{p(\cdot),\rho}$ *, then* $u\in H_{p(\cdot),\rho}^{+}$. *Denote the restrictions of the classes* $H_{p(\cdot),\rho}^{+}$,${}_{m}H_{p(\cdot),\rho}^{-}$ *to* $\partial\omega$ *by* $L_{p(\cdot),\rho}^{+}$ *and* ${}_{m}L_{p(\cdot),\rho}^{-}$, *respectively, i.e.*

$$L_{p(\cdot),\rho}^{+}=H_{p(\cdot),\rho}^{+}/_{\partial\omega}\,;\ {}_{m}L_{p(\cdot),\rho}^{-}={}_{m}H_{p(\cdot),\rho}^{-}/_{\partial\omega}\,.$$

We will need the following result:

Theorem 6. *Let* $k=\overline{1,r}$, *and the inequalities* (2) *be fulfilled. Then the system* $E_{+}^{(0)}=\left\{e^{int}\right\}_{n\geq0}$*(* $E_{-}^{(m)}=\left\{e^{-int}\right\}_{n\geq m}$*) forms a basis for* $L_{p(\cdot),\rho}^{+}$*(*${}_{m}L_{p(\cdot),\rho}^{-}$*)*,$1<p<+\infty$.

We will also need the following easy-to-prove lemma, which is derived immediately from the definition of weighted space $L_{p(\cdot),\rho}$.

Lemma 3. *Let* $p\in C\left[-\pi,\pi\right]$ *and* $p(t)>0$, $\forall t\in\left[-\pi,\pi\right]$. *Then the function* $\xi(t)=\left|t-c\right|^{\alpha}$ *belongs to* $L_{p(\cdot),\rho}$, *if* $\alpha>-\frac{1}{p(c)}$, *for* $c\neq\tau_{k}$, $\forall\,k=\overline{1,m}$, *and* $\alpha+\alpha_{k_0}>-\frac{1}{p(c)}$, *for* $c=\tau_{k_0}$.

3. Main Assumptions and Riemann Problem Statement

Let's state the Riemann problem in the classes $H_{p(\cdot);\rho}^{\pm}$. Let the complex-valued function $G\left(t\right)$ on $\left[-\pi,\pi\right]$ satisfy the following conditions:
i) Function $\left|G\left(t\right)\right|$ belongs to the space $L_{r(\cdot)}$ for some $r\,:\ 0<r^{-}\leq r^{+}<+\infty$, and $\left|G\left(t\right)\right|^{-1}\in L_{\omega(\cdot)}$ for $\omega:\ 0<\omega^{-}\leq\omega^{+}<+\infty$.
ii) Argument $\theta\left(t\right)\equiv\arg G\left(t\right)$ has a following decomposition:

$$\theta\left(t\right)=\theta_{0}\left(t\right)+\theta_{1}\left(t\right),$$

where $\theta_{0}\left(t\right)$ is a continuous function on $\left[-\pi,\pi\right]$ and $\theta_{1}\left(t\right)$ is a function of bounded variation on $\left[-\pi,\pi\right]$.

It is required to find a piecewise analytic function $F^{\pm}\left(z\right)$ on the complex plane with a cut $\partial\omega$ which satisfies the following conditions:

a) $F^{+}\left(z\right)\in H_{p(\cdot)}^{+}:0<p^{-}\leq p^{+}<+\infty$;

b) $F^{-}\left(z\right)\in{}_{m}H_{\nu(\cdot)}^{-}$; $0<\nu^{-}\leq\nu^{+}<+\infty$;

c) nontangential boundary values on the unit circumference $\partial\omega$ *satisfy the relation* $F^{+}\left(e^{it}\right)+G\left(t\right)F^{-}\left(e^{it}\right)=g\left(t\right)$, *for a.e.* $t\in\left(-\pi,\pi\right)$, *where* $g \in L_{\rho(\cdot)} : 0 < \rho^- \leq \rho^+ < +\infty$ *is some given function.*

Note that in the case of constant summability rate, the theory of such problems has been well studied (see [3].

Consider the following homogeneous Riemann problem in the classes $H^+_{p(\cdot),\rho} \times {}_mH^-_{p(\cdot),\rho}$:

$$F^+(\tau) - G(\tau)F^-(\tau) = 0,\ \tau \in \partial\omega. \tag{5}$$

By the solution of the problem (5) we mean a pair of analytic functions

$$(F^+(z); F^-(z)) \in H^+_{p(\cdot),\rho} \times {}_mH^-_{p(\cdot),\rho},$$

whose boundary values satisfy a.e. the equation (5). Introduce the following functions $X_i(z)$ analytic inside (with the sign +) and outside (with the sign -) the unit circle:

$$X_1(z) \equiv \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G\left(e^{it}\right)\right| \frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta(t) \frac{e^{it}+z}{e^{it}-z}dt\right\},$$

where $\theta(t) \equiv \arg G\left(e^{it}\right)$. Define

$$Z_i(z) \equiv \begin{cases} X_i(z), & |z| < 1, \\ \left[X_i(z)\right]^{-1}, & |z| > 1, \quad i = 1, 2. \end{cases}$$

Let $\{s_k\}_1^r : -\pi < s_1 < ...s_r < \pi$ be points of discontinuity of the function $\theta(t)$ and

$$\{h_k\}_1^r : h_k = \theta(s_k + 0) - \theta(s_k - 0), k = \overline{1, r},$$

be the corresponding jumps of this function at these points. Denote

$$h_0 = \theta(-\pi) - \theta(\pi); h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi).$$

Let

$$u_0(t) \equiv \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi} \theta_0(\tau)\, ctg\frac{t-\tau}{2}d\tau\right\}$$

and

$u(t) = \prod_{k=0}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{\frac{h_k}{2\pi}}$, where $s_0 = \pi$.

As is known, (see [3]), the boundary values $\left|Z_2^-(\tau)\right|$ are defined by the formula

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\left[u(t)\right]^{-1}\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}},$$

i.e.

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}.$$

It follows directly from Sokhotskii-Plemelj formula that

$$\sup_{(-\pi,\pi)} vrai \left\{ \left| Z_1^- \left(e^{it}\right) \right|^{\pm 1} \right\} < +\infty.$$

Thus, for $\left| Z^- \left(e^{it}\right) \right|^{-1}$ we have the representation

$$\left| Z^- \left(e^{it}\right) \right|^{-1} = \left| Z_1^- \left(e^{it}\right) \right|^{-1} \left| u_0 \left(t\right) \right|^{-1} \prod_{k=0}^{r} \left| \sin \frac{t - s_k}{2} \right|^{\frac{h_k}{2\pi}} . \tag{6}$$

Represent the product $\left| Z^- \rho \right|^{-1}$ as follows:

$$\left| Z^- \rho \right|^{-1} = \left| Z_1^- \right|^{-1} \left| u_0 \right|^{-1} \prod_{k=0}^{l} \left| t - t_k \right|^{\beta_k} ,$$

where

$$\{t_k\}_{k=0}^{l} \equiv \{\, \tau_k\}_{k=1}^{m} \bigcup \{s_k\}_{k=0}^{r} ,$$

and β_k's are defined by

$$\beta_k = -\sum_{i=1}^{m} \alpha_i \chi_{\{t_k\}} \left(\tau_i\right) + \frac{1}{2\pi} \sum_{i=0}^{r} h_i \chi_{\{t_k\}} \left(s_i\right), \quad k = \overline{0, l} \, . \tag{7}$$

By virtue of Lemma 3, we obtain that if the inequalities

$$\beta_k > -\frac{1}{q(t_k)} \ , \quad k = \overline{0, r}, \tag{8}$$

are true, then the product $\left| Z^- \rho \right|^{-1}$ belongs to the space $L_{q(\cdot)}$, i.e. $\left| Z^- \right|^{-1} \in L_{q(\cdot),\rho^{-1}}$. So, if the inequalities (8) are true, then the function $\Phi(z)$ belongs to the classes $H_1^{\pm}$. Then, according to [3], $\Phi(z)$ is a polynomial $P_{m_0}(z)$ of degree $m_0 \leq m$. Thus,

$$F^-(z) = P_{m_0}(z) Z^-(z).$$

Let's find out under which conditions the function $F^-(z)$ belongs to the space $H^-_{p(\cdot),\rho}$. We have

$$\left| Z^- \rho \right| = \left| Z_1 \right| \left| u_0 \right| \prod_{k=0}^{l} \left| t - t_k \right|^{-\beta_k} .$$

Consequently, if the inequalities

$$\beta_k < \frac{1}{p(t_k)} \ , \ k = \overline{0, r},$$

are true, then it is clear that $F^-(\tau) \in L_{p(\cdot),\rho}$, and hence $F^- \in {}_m H^-_{p(\cdot),\rho}$. So, if the inequalities

$$-\frac{1}{q(t_k)} < \beta_k < \frac{1}{p\,(t_k)},\ k = \overline{0,r}, \tag{9}$$

are true, then the general solution of homogeneous problem

$$F_0^+(\tau) = G_1(\tau)\, F_0^-(\tau)\,, \quad \tau \in \partial\omega,$$

in the classes $H^+_{p(\cdot),\,\rho} \times_m H^-_{p(\cdot),\,\rho}$ can be represented as follows:

$$F_0(z) = P_{m_0}(z)\, Z(z)\,,$$

where $P_{m_0}(z)$ is an arbitrary polynomial of degree $m_0 \le m$. So the following theorem is valid:

Theorem 7. *Let the* $\{\beta_k\}_1^r$*'s be defined by* (7) *and the inequalities* (9) *be true. If*

$$-\frac{1}{p\,(\tau_k)} < \alpha_k < \frac{1}{q\,(\tau_k)}, \quad k = \overline{1,m},$$

then the general solution of the homogeneous Riemann problem (5) *in the classes* $H^+_{p(\cdot),\,\rho}\times_m H^-_{p(\cdot),\,\rho}$ *can be represented as*

$$F(z) = P_{m_0}(z)\, Z(z)\,,$$

where $Z(\cdot)$ *is a canonical solution of homogeneous problem, and* $P_{m_0}(\cdot)$ *is a polynomial of degree* $m_0 \le m$.

This theorem has the following direct corollary:

Corollary 1. *Let all the conditions of Theorem 7 be satisfied. Then the homogeneous Riemann problem* (5) *is trivially solvable in the Hardy classes* $H^+_{p(\cdot),\rho} \times {}_{-}H^-_{p(\cdot),\rho}$.

4. Reducing The Completeness of Exponential System with Complex Coefficients to Boundary Value Problems

Consider the following exponential system:

$$\left\{A(t)\, e^{int};\ B(t)\, e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{10}$$

where $A(t) \equiv |A(t)|\, e^{i\alpha(t)}$; $B(t) \equiv |B(t)|\, e^{i\beta(t)}$ are complex-valued functions on $[-\pi,\pi]$. We will consider the completeness of the system (10) in the space $L_{p(\cdot);\rho}$. It is known [6] that the conjugate space of $L_{p(\cdot);\rho}$ is isometrically isomorphic to the space $L_{q(\cdot);\rho}$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Therefore, the completeness of the system (10) in $L_{p(\cdot);\rho}$ is equivalent to the equality to zero of any function $f(t)$ from the space $L_{q(\cdot);\rho}$ which satisfies the relations

$$\int_{-\pi}^{\pi} A(t)\, e^{int}\overline{f(t)}dt = 0; \quad \int_{-\pi}^{\pi} B(t)\, e^{-i(n+1)t}\overline{f(t)}dt = 0,\ \forall n \in Z_+. \tag{11}$$

Assume that the following main condition is satisfied:

$$\underset{[-\pi,\pi]}{ess\sup}\left\{ |A(t)|^{\pm 1};\ |B(t)|^{\pm 1}\right\} < +\infty. \tag{12}$$

From the first of equalities (11) we have

$$\int_{-\pi}^{\pi} A(t)\, e^{int}\overline{f}(t)\, dt = \frac{1}{i}\int_{\partial\omega} f^{+}(\tau)\, \tau^{n} d\tau = 0, \forall n \in Z_{+}, \tag{13}$$

where $f^{+}(\tau) \equiv A(\arg\tau)\ \overline{f}(\arg\tau)\ \bar{\tau}\ ,\ \ \tau \in \partial\omega.$

It is absolutely clear that $f^{+}(\tau) \in L_1(\partial\omega)$. Then it is well known (see [5, p.205]) that the conditions (13) are equivalent to the existence of a function $F^{+}(z)$ from H_1^{+} whose nontangential boundary values on $\partial\omega$ coincide with $f^{+}(\tau) : F^{+}(\tau) = f^{+}(\tau)$ a.e. on $\partial\omega$.

Similarly, from the second of equalities (11) we have

$$\int_{-\pi}^{\pi} \overline{B(t)} e^{i(n+1)t} f(t)\, dt = \frac{1}{i}\int f^{-}(\tau)\, \tau^{n} d\tau = 0, \forall n \in Z_{+}, \tag{14}$$

where $f^{-}(\tau) = \overline{B(\arg\tau)}\, f(\arg\tau)\ ,\ \ \tau \in \partial\omega$. For the reason stated above, the equalities (14) are equivalent to the existence of a function $\Phi^{+}(z) \in H_1^{+}$ whose nontangential boundary values $\Phi^{+}(\tau)$ on $\partial\omega$ coincide with $f^{-}(\tau) : \Phi^{+}(\tau) = f^{-}(\tau)$ a.e. on$\partial\omega$.

It is absolutely clear that $F^{+}(\tau);\ \Phi^{+}(\tau) \in L_{q(\cdot);\rho}(\partial\omega)$. Consequently, if we additionally require that $p(t) \in WL$, then from theorem in [7] we obtain the inclusion $F^{+}(z);\ \Phi^{+}(z) \in H^{+}_{q(\cdot);\rho}$. Representing $f(t)$ in terms of $F^{+}(\tau)$ and $\Phi^{+}(\tau)$, we obtain the following conjugation problem:

$$F^{+}(\tau) - \frac{A(\arg\tau)}{B(\arg\tau)}\overline{\tau\, \Phi^{+}(\tau)} = 0, \tau \in \partial\omega.$$

Define the function $F^{-}(z)$ analytic outside the unit circle:

$$F^{-}(z) = \frac{1}{z}\,\overline{\Phi^{+}\left(\frac{1}{\bar{z}}\right)}, |z| > 1.$$

It is absolutely clear that $F^{-}(\infty) = 0$. Moreover, $F^{-}(\tau) = \bar{\tau}\,\overline{\Phi^{+}(\tau)}$, $\tau \in \partial\omega$. Then we arrive at the following Riemann problem:

$$\left\{\begin{array}{l} F^{+}(\tau) - G(\tau)\, F^{-}(\tau) = 0\ ,\ \ \tau \in \partial\omega, \\ F^{-}(\infty) = 0\,, \end{array}\right. \tag{15}$$

where

$$G(\tau) \equiv \frac{A(\arg\tau)}{B(\arg\tau)}, \tau \in \partial\omega.$$

By definition, we have $F^{-}(z) \in_{-1} H^{-}_{q(\cdot);\rho}$. Consequently, if the system (10) is incomplete in $L_{p(\cdot);\rho}$, then the Riemann problem (15) is non-trivially solvable in the classes $\left(H^{+}_{q(\cdot);\rho};\ {}_{-1}H^{-}_{q(\cdot);\rho}\right)$.

Now let's assume that the problem (15) is non-trivially solvable in the classes $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$, i.e. $F^+(z) \in H^+_{q(\cdot);\rho}$, $F^-(z) \in_{-1} H^-_{q(\cdot);\rho}$. Define

$$\Phi_1^+(z) \equiv \overline{F^-\left(\frac{1}{\bar{z}}\right)} \quad \text{where} \quad |z| < 1.$$

We have $F^-(\tau) = \overline{\Phi_1^+(\tau)}$, $\tau \in \partial\omega$ and $\Phi^+(0) = 0$. Then it is clear that the function $\Phi^+(z) = z^{-1}\Phi_1^+(z)$ will be analytic when $|z| < 1$, and moreover, $\Phi^+(z) \in H^+_{q(\cdot);\rho}$. Thus,

$$F^+(\tau) - G(\tau)\, \overline{\tau\, \Phi^+(\tau)} = 0\ , \ \ \tau \in \partial\omega,$$

or

$$\frac{F^+(\tau)}{A(\arg\tau)\,\bar{\tau}} = \frac{\overline{\Phi^+(\tau)}}{B(\arg\tau)}, \tau \in \omega.$$

Denote

$$f(t) = \frac{\overline{F^+(e^{it})}}{\overline{A(t)}e^{it}} = \frac{\Phi^+\left(e^{it}\right)}{B(t)}.$$

It is absolutely clear that $f(t) \in L_{q(\cdot);\rho}$. From $F^+(z)$, $\Phi^+(z) \in H_1^+$ we obtain the equalities

$$\int_{\partial\omega} F^+(\tau)\,\tau^n d\tau = 0; \int_{\partial\omega} \Phi^+(\tau)\,\tau^n d\tau = 0, \forall n \in Z_+.$$

Expressing $F^+(\tau)$ and $\Phi^+(\tau)$ in terms of $f(\arg\tau)$ as $\tau \in \partial\omega$, we have

$$\int_{\partial\omega} A(t)\, e^{-it}\overline{f(t)}e^{int} de^{it} = i\int_{-\pi}^{\pi} A(t)\, e^{int}\overline{f}(t)\, dt = 0, \forall n \in Z_+;$$

$$\int_{\partial\omega} \overline{B(t)} f(t)\, e^{int} de^{it} = i\int_{-\pi}^{\pi} \overline{B(t)} e^{i(n+1)t} f(t)\, dt = 0, \forall n \in Z_+.$$

Obviously, $f(t) \neq 0$ on $[-\pi, \pi]$. Then these relations imply that the system (10) is incomplete in $L_{p(\cdot);\rho}$. So we have the following theorem:

Theorem 8. *Let $p:\ 1 < p^- \leq p^+ < +\infty$, $p(t) \in WL$, and complex-valued coefficients $A(t)$; $B(t)$ satisfy the condition* (12). *Then the exponential system* (10) *is complete in $L_{p(\cdot);\rho}$ only if the Riemann problem* (15) *is only trivially solvable in the classes $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$.*

5. Completeness of Exponential System with Complex Coefficients in $L_{p(\cdot);\rho}$

In this section, we apply the results of previous sections to obtain the sufficient conditions for the completeness of exponential system with complex coefficients in $L_{p(\cdot);\rho}$. So let's consider the system

$$\left\lfloor A(t)\, e^{int};\, B(t)\, e^{-i(n+1)t} \right\rfloor_{n\in Z_+}, \tag{16}$$

where $A(t) \equiv |A(t)|\, e^{i\alpha(t)}$; $B(t) \equiv |B(t)|\, e^{i\beta(t)}$ are complex-valued functions on $[-\pi,\pi]$. Assume that the following conditions are satisfied:

$$1) \sup_{[-\pi,\pi]} vrai \left\{ \left(A^{\pm 1};\, B^{\pm 1} \right) \right\} < +\infty,$$

2) The function $\theta(t) \equiv \alpha(t) - \beta(t)$ is piecewise continuous on $[-\pi,\pi]$ with points of discontinuity $\{s_i\}_1^r$: $-\pi < s_1 < ... < s_r < \pi$. Let $\{h_k\}_1^r = \theta(s_k+0) - \theta(s_k-0)$, $k=\overline{1,r}$, be the jumps of the function $\theta(t)$ at these points and $h_0 = \theta(-\pi) - \theta(\pi)$.

3) $\frac{h_k}{2\pi} + \frac{1}{p(s_k)} \notin Z$ (Z is a set of all integers), where $h_k$is a jump of the function $\theta(t) \equiv \alpha(t) - \beta(t)$ at the discontinuity point s_k, $k=\overline{0,r}$; $s_0 = \pi$.

Define the integers n_i, $i=\overline{1,r}$, from the following inequalities:

$$\left\{ \begin{array}{l} -\frac{1}{p(s_k)} < \frac{h_k}{2\pi} + n_k - n_{k-1} < \frac{1}{q(s_k)}\,,\ k=\overline{1,r}\,, \\ \\ n_0 = 0\,. \end{array} \right. \tag{17}$$

Let

$$\Delta_r = \frac{1}{2\pi}\left[\alpha(-\pi) - \alpha(\pi) + \beta(\pi) - \beta(-\pi)\right] + n_r.$$

The following theorem is true:

Theorem 9. *Let the coefficients $A(t)$ and $B(t)$ of the system* (16) *satisfy the conditions 1)-3), where $G\left(e^{it}\right) \equiv \frac{A(t)}{B(t)}$, the integer n_r is defined by* (17), *$p(t) \in WL$, $1 < p^- \le p^+ < +\infty$. Then, if $\Delta_r \notin Z$ and $\Delta_r > -\frac{1}{p(\pi)}$, then the system* (10) *is complete in the space $L_{p(\cdot);\rho}$.*

Let's apply the obtained theorem to the special case

$$\left\{ e^{i[n+\alpha\, sign\, n]t} \right\}_{n\in Z}, \tag{18}$$

where $\alpha \in C$ is a complex parameter. Basis properties of the system (18) in the spaces $L_{p(\cdot);\rho}$ have been well studied. From Theorem 9 we have

Corollary 2. *Let $p(t) \in WL_\pi$, $1 < p^- \le p^+ < +\infty$ and $Re\alpha \in Z$. If $Re\alpha < \frac{1}{2p(\pi)}$, then the system* (18) *is complete in $L_{p(\cdot);\rho}$.*

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On Some Properties of Harmonic Functions from Hardy-Morrey type Classes

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Abstract. In this paper Morrey-Poisson class of harmonic functions in the unit circle is introduced, the Dirichlet problem with the boundary value from the Morrey Lebesgue space is considered.

Key Words and Phrases: Dirichlet problem, Morrey-Poisson class, maximal function, Morrey-Lebesgue space.

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1. Introduction

Let $\omega = \{z \in C : |z| < 1\}$ be the unit disk on the complex plane C and $\gamma = \partial\omega$ be its circumference.

Consider the following Dirichlet problem for the Laplace equation

$$\left.\begin{array}{l} \Delta u = 0, \ \ in \ \ \omega \,, \\ u\Big/_{\gamma} = f \,, \end{array}\right\} \tag{1}$$

where $f : \gamma \to R$ some real function. Assume $u_r(t) = u\left(re^{it}\right)$ and let

$$h_p = \left\{ u : \Delta u = 0 \quad in \quad \omega \,, \quad and \quad \|u\|_{h_p} < +\infty \right\},$$

where

$$\|u\|_{h_p} = \sup_{0<r<1} \|u_r\|_p \,,$$

$$\|g\|_p = \left(\int_{-\pi}^{\pi} |g(t)|^p \, dt \right)^{\frac{1}{p}}, 1 \le p < +\infty.$$

By $P_z(\varphi)$ denote a Poisson kernel for the unit circle

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$$P_z(\varphi) = Re\frac{e^{i\varphi}+re^{it}}{e^{i\varphi}-re^{it}} = \frac{1-r^2}{1-2r\cos(t-\varphi)+r^2}, z = re^{it} \in \omega.$$

If $f \in L_p(\gamma) =: L_p$, then the problem (1) is solvable in class h_p, and its solution can be represented as a Poisson-Lebesgue integral

$$u\left(re^{it}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_z(\varphi) f(\varphi)\, d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(t-\varphi)+r^2} f(\varphi)\, d\varphi,$$

wherein a boundary value $u\big/_{\gamma} = f$ in (1) is understood in the sense that nontangential values on γ:

$$u\left(e^{it}\right) = \lim_{z\to e^{it}} u(z),$$

exist and a.e. on γ coincides with $f\left(e^{it}\right)$, i.e.

$$u\left(e^{it}\right) = f\left(e^{it}\right), \text{ a.e. } t \in (-\pi,\pi), \tag{2}$$

and moreover

$$\lim_{r\to 1-0} \|u_r(\cdot) - f(\cdot)\|_p = 0. \tag{3}$$

These results are well known and illuminated, e.g., in the monograph I.I.Danilyuk [27] .

It should be noted that the concept of Morrey space was introduced by C. Morrey [1] in 1938 in the study of qualitative properties of the solutions of elliptic type equations with BMO (Bounded Mean Oscillations) coefficients (see also [2, 3]). This space provides a large class of weak solutions to the Navier-Stokes system [4]. In the context of fluid dynamics, Morrey-type spaces have been used to model the fluid flow in case where the vorticity is a singular measure supported on some sets in R^n [5]. There appeared lately a large number of research works which considered many problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc. in Morrey-type spaces (for more details see [2-26]). It should be noted that the matter of approximation in Morrey-type spaces has only started to be studied recently (see, e.g., [11, 12, 16, 17]), and many problems in this field are still unsolved.

In the present paper non-tangential maximal function is considered and it is estimated from above a maximum operator, and the proof is carried out for the Poisson-Stieltjes integral, when the density belongs to the corresponding Morrey-Lebesgue space.

It should be noted that similar problems with respect to the analytical functions from Hardy classes were considered in [16, 17, 31].

2. Needful Information

We will need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$. All the constants throughout this paper (can be different in different places) will be denoted by c.

By Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0<\alpha\le 1$, $p\ge 1$, we mean the normed space of all measurable functions $f(\cdot)$ on Γ with the finite norm

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_B \left(\left| B \bigcap \Gamma \right|_\Gamma^{\alpha-1} \int_{B\bigcap\Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty,$$

where sup is taken all over the balls B with the centre on Γ. $L^{p,\alpha}(\Gamma)$ is a Banach space with $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$. Similarly we define the weighted Morrey-Lebesgue space $L_\mu^{p,\alpha}(\Gamma)$ with the weight function $\mu(\cdot)$ on Γ equipped with the norm

$$\|f\|_{L_\mu^{p,\alpha}(\Gamma)} = \|f\mu\|_{L^{p,\alpha}(\Gamma)}, f \in L_\mu^{p,\alpha}(\Gamma).$$

The inclusion $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0<\alpha_1\le\alpha_2\le 1$. Thus, $L^{p,\alpha}(\Gamma)\subset L_1(\Gamma)$, $\forall \alpha \in (0,1]$, $\forall p \ge 1$. For $\Gamma=\gamma$ we will use the notation $L^{p,\alpha}(\gamma) = L^{p,\alpha}$ and the spaces $L^{p,\alpha}(\gamma)$ and $L^{p,\alpha}(-\pi,\pi)$ we will identify by usual method.

More details on Morrey-type spaces can be found in [2-26].

We will use the following concepts. Let $\Gamma \subset C$ be some bounded rectifiable curve, $t=t(\sigma)$, $0\le\sigma\le 1$, be its parametric representation with respect to the arc length σ, and l be the length of Γ. Let $d\mu(t) = d\sigma$, i.e. let $\mu(\cdot)$ be a linear measure on Γ. Let

$$\Gamma_t(r) = \{\tau \in \Gamma : |\tau - t| < r\}, \Gamma_{t(s)}(r) = \{\tau(\sigma) \in \Gamma : |\sigma - s| < r\}.$$

It is absolutely clear that $\Gamma_{t(s)}(r) \subset \Gamma_t(r)$.

Definition 1. Curve Γ is said to be Carleson if $\exists c>0$:

$$\sup_{t\in\Gamma} \mu(\Gamma_t(r)) \le cr, \forall r > 0.$$

Curve Γ is said to satisfy the chord-arc condition at the point $t_0 = t(s_0) \in \Gamma$ if there exists a constant $m>0$ independent of t such that $|s-s_0| \le m\,|t(s)-t(s_0)|$, $\forall t(s) \in \Gamma$. Γ satisfies a chord-arc condition uniformly on Γ if $\exists m>0 : |s-\sigma| \le m\,|t(s)-t(\sigma)|$, $\forall t(s), t(\sigma) \in \Gamma$.

Let's recall some facts about the homogeneous Morrey-type spaces from the work [10]. Let $(X; d; \nu)$ be a homogeneous space equipped with the quasi-distance $d(\cdot;\cdot)$ and the measure $\nu(\cdot)$. Recall that the quasi-distance $d : X^2 \to R_+$ is a function which satisfies the following conditions:

i) $d(x;y)\ge 0$ & $d(x;y)=0 \Leftrightarrow x=y$; $\forall x, y \in X$;

ii) $d(x;y) \le c(d(x;z)+d(z;y))$, $\forall x, y \in X$.

Let $B_r(x)$ be an open ball

$$B_r(x) = \{y \in X : d(x;y) < r\}.$$

Set

$$\nu\left(B_r\left(x\right)\right)=\int_{B_r(x)} 1\,d\nu.$$

Assume that X has a constant homogeneous dimension æ >0, i.e. $\exists c_1;\, c_2>0$:

$$c_1 r^{æ} \le \nu\left(B_r\left(x\right)\right) \le c_2 r^{æ}, \forall x\in X, \forall r>0. \qquad (æ)$$

In this case, the Morrey space $L^{p,\,\lambda}\left(X\right)$ is defined by means of the norm

$$\|f\|_{L^{p,\,\lambda}(X)}=\sup_{x\in X,\, r>0}\left\{\frac{1}{r^{\lambda}}\int_{B_r(x)}\left|f\left(y\right)\right|^p d\nu\left(y\right)\right\}^{1/p}.$$

Theorem 1 ([10])**.** *Let $(X;\, d;\, \nu)$ be a homogeneous space equipped with the quasi-metrics d and the measure ν with $\nu\left(X\right)=+\infty$, and the condition (æ) be true. Then the maximal operator $\left(\left|B_r\left(x\right)\right|_{\nu} =: \nu\left(B_r\left(x\right)\right)\right)$:*

$$M_{\nu} f\left(x\right)=\sup_{r>0}\frac{1}{\left|B_r\left(x\right)\right|_{\nu}}\int_{B_r(x)}\left|f\left(y\right)\right| d\nu\left(y\right),$$

is bounded in $L^{p,\,\lambda}\left(X\right)$ for $1<p<+\infty$, $0\le\lambda<$ æ.

3. Weighted Morrey-type space $h_{\rho}^{p,\,\alpha}$ and Hardy-Littlewood operator

Let $\rho : [-\pi,\, \pi]\to R_+=(0,\, +\infty)$, be some weight function. Consider the weighted Morrey-type space $h_{\rho}^{p,\,\alpha}$ of harmonic functions in ω furnished with the norm

$$\|u\|_{h_{\rho}^{p,\,\alpha}}=\sup_{0<r<1}\left\|u_r\left(\cdot\right)\rho\left(\cdot\right)\right\|_{p,\,\alpha},$$

where

$$u_r\left(t\right)=u\left(re^{it}\right)=u\left(r\cos t;\, r\sin t\right).$$

Assume that the weight $\rho\left(\cdot\right)$ satisfies the following condition

$$\rho^{-1}\in L_q, \frac{1}{p}+\frac{1}{q}=1. \qquad (4)$$

Applying Hölder inequality we obtain

$$\int_{-\pi}^{\pi}\left|u_r\left(\cdot\right)\right| dt \le \left(\int_{-\pi}^{\pi}\left|u_r\left(\cdot\right)\rho\left(\cdot\right)\right|^p dt\right)^{1/p}\left(\int_{-\pi}^{\pi}\rho^{-q}\left(t\right) dt\right)^{1/q}\le$$

$$\le\left(2\pi\right)^{\frac{1-\alpha}{p}}\sup_{I\in[-\pi,\pi]}\left(\frac{1}{\left|I\right|^{1-\alpha}}\int_I\left|u_r\rho\right|^p dt\right)^{1/p}\left\|\rho^{-1}\right\|_{L_q}=$$

$$= (2\pi)^{\frac{1-\alpha}{p}} \left\|\rho^{-1}\right\|_{L_q} \|u_r\|_{h_\rho^{p,\alpha}} .$$

It follows immediately that if the condition (4) is true, then $u \in h_1$. Consequently, every function $u \in h_\rho^{p,\alpha}$ has nontangential boundary values $u^+\left(e^{it}\right)$ on γ. Then, by Fatou's lemma (see e.g. [28, 29, 30]) we have $u_r\left(e^{it}\right) \to u^+\left(e^{it}\right)$ as $r \to 1-0$ a.e. in $[-\pi, \pi]$. Applying Fatou theorem on passage to the limit, we obtain

$$\int_I \left|u^+\left(e^{it}\right)\rho(t)\right|^p dt \le \varliminf_{r\to 1-0} \int_I \left|u_r\left(e^{it}\right)\rho(t)\right|^p dt \le$$

$$\le \|u\|^p_{h_\rho^{p,\alpha}} |I|^{1-\alpha},$$

because

$$\left|u_r\left(e^{it}\right)\rho(t)\right| \to \left|u^+\left(e^{it}\right)\rho(t)\right|, r \to 1-0, \quad \text{for a.e.}\, t \in [-\pi, \pi] .$$

It follows immediately that $u^+ \in L_\rho^{p,\alpha}$ and

$$\left\|u^+\right\|_{p,\alpha;\rho} \le \|u\|_{h_\rho^{p,\alpha}} .$$

If the relation

$$\rho^{-1} \in L_{q+0}(-\pi,\pi), \ \text{ i.e.} \, \exists\varepsilon > 0 : \rho^{-1} \in L_{q+\varepsilon}(-\pi,\pi), \tag{5}$$

true, then we have

$$\int_\pi^\pi |u_r(\cdot)|^{1+\delta} dt \le \left(\int_{-\pi}^\pi |u_r(\cdot)\rho(\cdot)|^p dt\right)^{\frac{1+\delta}{p}} \left(\int_{-\pi}^\pi |\rho(\cdot)|^{-\frac{pq}{p-q\delta}} dt\right)^{\frac{1}{q}-\frac{\delta}{p}} \le c_\delta \|u\|^{1+\delta}_{h_\rho^{p,\alpha}},$$

where $\delta > 0$ is a sufficiently small number, and c_δ is a constant depending only on δ. Then, in view of the classical results, the representation

$$u\left(re^{it}\right) = \frac{1}{2\pi}\int_{-\pi}^\pi u^+(s)\,P(r;s-t)\,ds, \tag{6}$$

is true, where $u^+(s) =: u^+\left(e^{is}\right)$, $s \in [-\pi,\pi]$, and $P_z(\varphi) =: P(r;\theta-\varphi)$ is a Poisson kernel for the unit disk

$$P_z(\varphi) = P_r(\theta-\varphi) = P(r;\theta-\varphi) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}, \ z = re^{i\theta}.$$

Thus, if $u \in h_\rho^{p,\alpha}$ and $\rho(\cdot)$ satisfies the condition (5), then $u^+ \in L_\rho^{p,\alpha}$ and the relation (6) holds.

Now let's prove the converse. In other words, let's prove that if $u^+ \in L_\rho^{p,\alpha}$ and the representation (6) holds, then $u \in h_\rho^{p,\alpha}$. To do so, we need some auxiliary facts.

Consider the arbitrary nontangential internal angle θ_0 with a vertex at the point $z = e^{it} \in \gamma$, $t \in [-\pi, \pi]$. Denote by $M_\mu f(t)$ the Hardy-Littlewood type maximal function (or Hardy-Littlewood operator) of the function $f(\cdot)$:

$$M_\mu f(x) = \sup_{I \ni x} \frac{1}{\mu(I)} \int_I |f(t)|\, d\mu(t),$$

where sup is taken over all intervals $I \subset [-\pi, \pi]$ which contain x, and $\mu(\cdot)$ is a Borel measure on $[-\pi, \pi]$, which satisfies the condition

$$\mu(I) > 0, \text{ for } \forall I : |I| > 0.$$

It is shown that there exists a positive constant C_{θ_0}, depending only on θ_0 such that

$$\sup_{z \in \theta_0} |u_\mu(z)| \le C_{\theta_0} M_\mu f(t), \forall t \in [-\pi, \pi],$$

where

$$u_\mu(z) = u\left(re^{it}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r;\, s - t)\, u^+(s)\, d\mu(s).$$

For a usual maximal operator, this fact was established in [29, p.237] and [30, p.30].

Consider the Poisson kernel $P_z(t)$ in the upper half-plane

$$P_z(t) =: P_y(x - t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}, z = x + iy, y > 0.$$

Let $f \in L_1\left(\frac{d\mu(t)}{1+t^2}\right)$ and consider the Poisson integral

$$u_\mu(x;\, y) = \int_R P_y(x - s)\, f(s)\, d\mu(s).$$

The following main lemma is proved.

Lemma 1. *Let $\mu(\cdot)$ be a Borel measure on R with*

$$\mu(I) > 0, \forall I : |I| > 0; \sup_{y>0;\, x \in R} \int_R P_y(s - |x|)\, d\mu < +\infty.$$

Then, for $f \in L_1\left(\frac{d\mu(t)}{1+t^2}\right)$, the function

$$u_\mu(x;\, y) = \int_R P_y(x - s)\, f(s)\, d\mu(s),$$

which is harmonic on the upper half-plane, satisfies the relation

$$\sup_{z \in \Gamma_{\mu;\, \alpha_0}(t)} |u_\mu(z)| \le A_{\alpha_0} M_\mu f(t), t \in R,$$

where M_μ is the Hardy-Littlewood type maximal function

$$M_\mu f(x) = \sup_{I \ni x} \frac{1}{\mu(I)} \int_I |f(t)|\, d\mu(t),$$

$$\Gamma_{\mu;\alpha_0}(t) = \{(x;\, y) \in C : \mu((-|x-t|,\, |x-t|)) < \alpha_0 y;\, y > 0\},\, \alpha_0 > 0,$$

and A_{α_0} is a constant depending only on α_0.

By M we denote the usual Hardy-Littlewood operator, i.e.

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)|\, dt,$$

where $|I|$ is a Lebesgue measure of the interval $I \subset [-\pi, \pi]$.

It is not difficult to see that the Lebesgue measure on R satisfies all the conditions of Lemma 1.

Let's go back to Theorem 1 [10] . Let the condition (æ) be fulfilled. Note that Theorem 1 [10] is true in case $\mu(X) < +\infty$, too. Because its proof is based on the Fefferman-Stein inequality which is true also in case $\mu(X) < +\infty$. Let's apply this theorem to our case. In our case we have $X = R$, $d(x;\, y) = |x-y|$ and æ $= 1$. So, if the measure $\mu(\cdot)$ satisfies the conditions of Theorem 1 [10] in our case, then we have

$$\int_I |M_\mu f|^p\, d\mu \le c\, |I|^{1-\alpha},$$

where $|I|$ is a Lebesgue measure of the set $I \subset R$. Then from (**??**) it directly follows that $u_\mu \in h^{p,\,\alpha}(d\mu)$, where $h^{p,\,\alpha}(d\mu)$ is a class of harmonic functions on the upper half-plane equipped with the norm

$$\|u_\mu\|_{h^{p,\,\alpha}(d\mu)} = \sup_{y>0} \sup_{I \subset R} \left(\frac{1}{|I|^{1-\alpha}} \int_I |u_\mu(x;\, y)|^p\, d\mu(x) \right)^{1/p}.$$

So we get the validity of the following theorem.

Theorem 2. *Assume that the measure $\mu(\cdot)$ satisfies the conditions ($I$is an interval)*

$$\mu(I) \sim |I|,\, \forall I \subset R;\ \sup_{y>0;\, x \in R} \int_R P_y(s - |x|)\, d\mu(s) < +\infty.$$

Let

$$u_\mu(z) = u_\mu(x;\, y) = \int_R P_y(x-t)\, f(t)\, d\mu(t),\, f \in L^{p,\,\alpha}(d\mu),\, 0 \le 1-\alpha < 1,$$

where $L^{p,\,\alpha}(d\mu)$ is a Morrey space equipped with the norm

$$\|f\|_{p,\,\alpha;\, d\mu} = \sup_{I \subset R} \left\{ \frac{1}{|I|^{1-\alpha}} \int_I |f(y)|^p\, d\mu(y) \right\}^{1/p}.$$

Then for $\forall\alpha_0 > 0$, $\exists A_{\alpha_0} > 0$*:*

$$\sup_{(x;\,y)\in\Gamma_{\alpha_0}(t)} |u_\mu(x;y)| \le A_{\alpha_0} M_\mu f(t)\,, \forall t \in R, \tag{7}$$

and $u_\mu^* \in h^{p,\,\alpha}(d\mu)$*:*

$$\left\| u_\mu^* \right\|_{h^{p,\,\alpha}(d\mu)} \le A_{\alpha_0} \left\| f \right\|_{p,\,\alpha;\,d\mu}\,, \tag{8}$$

where $u_\mu^*(\cdot)$ *is a nontangential maximal function for* u*:*

$$u_\mu^*(t) = \sup_{z\in\Gamma_{\alpha_0}(t)} |u_\mu(z)|\,, t \in R.$$

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Generating Function of the Number of Jumps at which Complex Process of Semi-Markov Walk Achieves First the Level "a"$(a > 0)$

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Abstract. Using the sequence of independent random variables, we construct difference process of semi-markov walk. The generating function of the number of jumps under which complex process of semi-markov walk achieves first the level "a" $(a > 0)$, is found.

Key Words and Phrases: random variable, process of semi-markov walk, generating function.

2010 Mathematics Subject Classifications: 60A10; 60J25; 60D10

1. Introduction

There are a few papers devoted to studying generating function of the number of jumps under which complex process of semi-markov walk achieves first the level "a" $(a > 0)$.

In the paper [1,p. 61-63], asymptotic behavior of random walks in a random medium with delaying barrier was studied. Random walk in a band was studied in [2, p. 160-165]. In [3, p. 26-51], asymptotic expansion of distributions determined on Markov chains, was found. Different semi-makrov chains with delaying barrier and functionals of these processes were studied in the paper [4, p. 61-63]. In [5,p. 77-84], Laplace transform of distribution of the lower boundary functional of the process of semi-markov walk with delaying barrier in zero, was found. The Laplace transform of ergodic distribution of the process of semi-markov walk with negative drift, nonnegative jumps and delaying barrier in zero was found in [6,p. 49-60].

In the present paper we find a generating function of the number of jumps under which the complex process of semi-markov walk achieves first the level "a" $(a > 0)$.

As far as we know, a generating function of the number of jumps at which it achieves first the level "a" $(a > 0)$ was not found for complex process of semi-markov walk.

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2. Mathematical statement of the problem.

On probability space $(\Omega, F, P(\cdot))$ we are given the sequence of independent identically distributed positive random variables $\xi_k^+, \eta_k^+, \xi_k^-, \eta_k^-$, $k = \overline{1, \infty}$.

Introduce the following denotation $\nu\pm(t) = \min\left\{k : \sum\limits_{i=1}^{k+1} \xi_i^\pm > t\right\}$ is the number of positive jumps of the process $\mathrm{X}^\pm(t)$ for time t

$$\mathrm{X}^\pm(t) = \sum_{i=1}^{\nu^\pm(t)} \eta_i^\pm$$

$$\mathrm{X}(t) = \mathrm{X}^+(t) - \mathrm{X}^-(t)$$

The process $\mathrm{X}(t) = \mathrm{X}^+(t) - \mathrm{X}^-(t)$ is called a complex process of semi-markov walk.

The goal of the paper is to find explicit form of the generating function of the number of jumps under which the process $X(t)$ achieves first the level "a".

We denote it by ν_1^a.

Let $X(0) = z > 0$.

3. Setting-up integral equation for generating function of the number of jumps of the process $X(t)$ under which it achieves first the level $''a''(a > 0)$.

Denote by ν_1^a the number of jumps of the process $X(t)$ under which it achieves first the level "a"$(a > 0)$.

Denote

$$\Psi(u|z) = \sum_{k=1}^{\infty} u^k P\left\{\nu_1^a = k | X(0) = z\right\}, |u| \leq 1$$

Theorem 1. $\Psi(u|z)$ *satisifies the following integral equation*

$$\Psi(u|z) = uP\left\{\eta_1^+ > a - z\right\} + u\int\limits_{y=z}^{a} \Psi(u|y)d_yP\left\{\eta_1^+ < y - z\right\}P\left\{\xi_1^+ < \xi_1^-\right\} +$$

$$+u\int_{y=z}^{a} \Psi(u|y)\int\limits_{x=y}^{\infty} dy\sum_{m=1}^{\infty} P\left\{\eta_1^- + ... + \eta_m^- < x - y\right\} \times$$

$$\times \int\limits_{t=0}^{\infty} P\left\{\nu^-(t) = m\right\} d_tP\left\{\xi_1^+ - \xi_1^- < t\right\} d_yP\left\{\eta_1^+ < y\right\} +$$

$$+u\int\limits_{y=-\infty}^{z}\Psi(u|y)\int\limits_{x=z}^{\infty}dy\sum_{m=1}^{\infty}P\left\{\eta_1^-+...+\eta_m^-<x-y\right\}\times$$

$$\times\int\limits_{t=0}^{\infty}P\left\{\nu^-(t)=m\right\}d_tP\left\{\xi_1^+-\xi_1^-<t\right\}d_yP\left\{\eta_1^+<y\right\}. \qquad (1)$$

Proof. Let $k\geq 2$. Then by the total probability formula we have

$$\begin{array}{l}P\left\{\nu_1^a=k|X(0)=z\right\}=P\left\{\nu_1^a=k;(\xi_1^+<\xi_1^-)\bigcup(\xi_1^-<\xi_1^+)|X(0)=z\right\}=\\=P\left\{\nu_1^a=k;(\xi_1^+<\xi_1^-)|X(0)=z\right\}+P\left\{\nu_1^a=k;(\xi_1^-<\xi_1^+)|X(0)=z\right\}=\\=\int\limits_{y=z}^{a}P\left\{\xi_1^+<\xi_1^-;z+\eta_1^+<a;z+\eta_1^-\in dy\right\}P\left\{\nu_1^a=k-1|X(0)=y\right\}+\\+\int\limits_{y=z}^{a}P\left\{\xi_1^-<\xi_1^+;z-\eta_1^--\eta_2^--...-\eta_{\nu^-(\xi_1^+-\xi_1^-)}+\eta_1^+\in dy\right\}P\left\{\nu_1^a=k-1|X(0)=y\right\}\end{array}.$$

So, by $x-y>0$ and $x-z>0$ we get

$$P\left\{\nu_1^a=k|X(0)=z\right\}=\int\limits_{y=z}^{a}P\left\{\nu_1^a=k-1|X(0)=y\right\}d_yP\left\{\eta_1^+<y-z\right\}P\left\{\xi_1^+<\xi_1^-\right\}-$$

$$-\int\limits_{y=z}^{a}P\left\{\nu_1^a=k-1|X(0)=y\right\}\int\limits_{x=\max(y,z)}^{\infty}d_y\sum_{m=1}^{\infty}P\left\{\eta_1^-+...+\eta_m^-<x-y\right\}$$

$$\int\limits_{t=0}^{\infty}P\left\{\nu^-(t)=m\right\}d_tP\left\{\xi_1^+-\xi_1^-<t\right\}d_xP\left\{\eta_1^+<x-z\right\}.$$

Then we have

$$P\left\{\nu_1^a=k|X(0)=z\right\}=\int\limits_{y=z}^{a}P\left\{\nu_1^a=k-1|X(0)=y\right\}d_yP\left\{\eta_1^+<y-z\right\}P\left\{\xi_1^--\xi_1^+<t\right\}-$$

$$-\int\limits_{y=z}^{a}P\left\{\nu_1^a=k-1|X(0)=y\right\}\int\limits_{x=y}^{\infty}d_yP\sum_{m=1}^{\infty}P\left\{\eta_1^-+...+\eta_m^-<x-y\right\}$$

$$\int\limits_{t=0}^{\infty}P\left\{\nu^-(t)=m\right\}d_tP\left\{\xi_1^+-\xi_1^-<t\right\}d_xP\left\{\eta_1^+<x-z\right\}+$$

$$-\int\limits_{y=-\infty}^{z} P\left\{\nu_1^a = k-1|X(0)=y\right\}\int\limits_{x=z}^{\infty} d_y P\sum_{m=1}^{\infty} P\left\{\eta_1^- + ... + \eta_m^- < x-y\right\}$$

$$\int\limits_{t=0}^{\infty} P\left\{\nu^-(t)=m\right\} d_t P\left\{\xi_1^+ - \xi_1^- < t\right\} d_y P\left\{\eta_1^+ < y-z\right\}$$

We multiply the both hand sides of (1) by u^k and sum over $k\geq 2$.

$$\sum_{k=2}^{\infty} u^k P\left\{\nu_1^a = k|X(0)=z\right\} =$$

$$= \int\limits_{y=z}^{a}\sum_{k=2}^{\infty} u^k P\left\{\nu_1^a = k-1|X(0)=y\right\} d_y P\left\{\eta_1^+ < y-z\right\} P\left\{\xi_1^+ < \xi_1^-\right\} +$$

$$-\int\limits_{y=z}^{a}\sum_{k=2}^{\infty} u^k P\left\{\nu_1^a = k-1|X(0)=y\right\}\times$$

$$\times\int\limits_{x=y}^{\infty} d_y \sum_{m=1}^{\infty} P\left\{\eta_1^- + ... + \eta_m^- < x-y\right\}$$

$$\int\limits_{t=0}^{\infty} P\left\{\nu^-(t)=m\right\} d_t P\left\{\xi_1^+ - \xi_1^- < t\right\} d_x P\left\{\eta_1^+ < x-z\right\} -$$

$$-\int\limits_{y=-\infty}^{z}\sum_{k=2}^{\infty} u^k P\left\{\nu_1^a = k-1|X(0)=y\right\}\int\limits_{x=z}^{\infty} d_y \sum_{m=1}^{\infty} P\left\{\eta_1^- + ... + \eta_m^- < x-y\right\}\times$$

$$\times\int\limits_{t=0}^{\infty} P\left\{\nu^-(t)=m\right\} d_t P\left\{\xi_1^+ - \xi_1^- < t\right\} d_y P\left\{\eta_1^+ < y-z\right\}. \qquad (2)$$

Obviously,

$$P\{z+\eta_1^+ > a\} = P\{\nu_1^a = 1|X(0)=z\}. \qquad (3)$$

Adding (3) to both hand sides of (2), we complete the proof of the theorem.

We will solve the equation with respect to $\Psi(u|z)$ if the random variables ξ_k^+ , η_k^+,ξ_k^-,η_k^- have exponential distribution with the parameters $\lambda_+ > 0$, $\lambda_- > 0$, $\mu_+ > 0$, $\mu_- > 0$, respectively.

$$P\{\xi_1^{\pm} < t\} = \begin{cases} 0, t < 0 \\ 1 - e^{-\lambda_{\pm} t} \end{cases}$$

$$t > 0, \lambda_+ > 0, \lambda_- > 0 P\{\eta_1^{\pm} < x\} = \begin{cases} 0, x < 0 \\ 1 - e^{-\mu_{\pm} t} \end{cases} \quad x > 0, \mu_+ > 0, \mu_- > 0. \tag{4}$$

It is easy to find that under supposition (4),

$$P\left\{\xi_1^+ < \xi_1^-\right\} = \frac{\lambda_+}{\lambda_+ + \lambda_-},$$

$$P\left\{\xi_1^- < \xi_1^+\right\} = \frac{\lambda_-}{\lambda_+ + \lambda_-},$$

$$d_t P\left\{\xi_1^+ - \xi_1^- < t\right\} = \frac{\lambda_+ \lambda_-}{\lambda_+ + \lambda_-} e^{-\lambda_+ t} dt.$$

From references it is known that

$$P\left\{\nu^{\pm}(t) = m\right\} = \frac{(\lambda_{\pm} t)^m}{m!} e^{-\lambda_{\pm} t}$$

$$d_y P\left\{\eta_1^- + \eta_2^- + ... + \eta_m^- < y\right\} = \mu_- \frac{(\mu_- y)^{m-1}}{(m-1)!} e^{-\mu_- y} dy.$$

We substitute these formuls in equation (1)

$$\Psi(u|z) = u e^{-\mu_+ a} e^{\mu_+ z} + \frac{\lambda_+ \mu_+}{\lambda_+ + \lambda_-} u e^{\mu_+ z} \int\limits_{y=z}^{a} e^{-\mu_+ y} \Psi(u|y) dy +$$

$$+ \frac{\lambda_+ \lambda_-^2 \mu_+ \mu_-}{(\lambda_+ + \lambda_-)^3} u e^{\mu_+ z} \int\limits_{y=-\infty}^{z} e^{\mu_- y} \Psi(u|y) \int\limits_{x=z}^{\infty} e^{-(\mu_+ + \mu_-) x} e^{\frac{\lambda_- \mu_- (x-y)}{\lambda_+ + \lambda_-}} dx dy +$$

$$+ \frac{\lambda_+ \lambda_-^2 \mu_+ \mu_-}{(\lambda_+ + \lambda_-)^3} u e^{\mu_+ z} \int_{y=z}^{\infty} e^{\mu_- y} \Psi(u|y) \int\limits_{x=y}^{\infty} e^{-(\mu_+ + \mu_-) x} e^{\frac{\lambda_- \mu_- (x-y)}{\lambda_+ + \lambda_-}} dx dy \tag{5}$$

Having multiplied the both hand sides by $e^{-\mu, z}$ and differentiated with respect to z, we get

$$\Psi'(u|z) - \mu_+ \Psi(u|z) = -\frac{\lambda_+ \mu_+}{\lambda_+ + \lambda_-} u \Psi(u|z) -$$

$$- \frac{\lambda_+ \lambda_-^2 \mu_+ \mu_-}{(\lambda_+ + \lambda_-)^3} u e^{(\frac{\lambda_- \mu_-}{\lambda_+ + \lambda_-} - \mu_-) z} \int\limits_{y=-\infty}^{z} e^{\mu_- y} \Psi(u|y) e^{-\frac{\lambda_- \mu_-}{\lambda_+ + \lambda_-} y} dy. \tag{6}$$

We multiply the both hand sides by $e^{(\mu_- - \frac{\lambda_-\mu_-}{\lambda_++\lambda_-})z}$

$$\left[\Psi'(u|z) - \mu_+\Psi(u|z)\right] e^{(\mu_- - \frac{\lambda_-\mu_-}{\lambda_++\lambda_-})z} = -\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_-}u\Psi(u|z)e^{(\mu_- - \frac{\lambda_-\mu_-}{\lambda_++\lambda_-})z} -$$

$$-\frac{\lambda_+\lambda_-^2\mu_+\mu_-}{(\lambda_+ + \lambda_-)^3}u \int\limits_{y=-\infty}^{z} e^{\mu_- y}\Psi(u|y)e^{-\frac{\lambda_-\mu_-}{\lambda_++\lambda_-}y}dy.$$

Differentiate both hand sides with respect to z.

$$\left[(\Psi'(u|z) - \mu_+\Psi(u|z))(\mu_- - \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_-}) + \Psi''(u|z) - \mu_+\Psi'(u|z)\right] e^{(\mu_- - \frac{\lambda_-\mu_-}{\lambda_++\lambda_-})z} =$$

$$-\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_-}ue^{(\mu_- - \frac{\lambda_-\mu_-}{\lambda_++\lambda_-})z}[\Psi'(u|z) + (\mu_- - \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_-})\Psi(u|z)] -$$

$$-\frac{\lambda_+\lambda_-^2\mu_+\mu_-}{(\lambda_+ + \lambda_-)^3}ue^{\mu_- z}\Psi(u|z)e^{-\frac{\lambda_-\mu_-}{\lambda_++\lambda_-}z}.$$

Multiply both hand sides by $e^{-(\mu_- - \frac{\lambda_-\mu_-}{\lambda_++\lambda_-})z}$

$$(\mu_- - \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_-})[\Psi'(u|z) - \mu_+\Psi(u|z)] + \Psi''(u|z) - \mu_+\Psi'(u|z)$$

$$= -\frac{\lambda_+\mu_+}{\lambda_+ + \lambda_-}u[\Psi'(u|z) + (\mu_- - \frac{\lambda_-\mu_-}{\lambda_+ + \lambda_-})\Psi(u|z)] - \frac{\lambda_+\lambda_-^2\mu_+\mu_-}{(\lambda_+ + \lambda_-)^3}u\Psi(u|z).$$

We get a second order homogeneous differential equation

$$\Psi''(u|z) + [-\mu_+ + \frac{\lambda_+\mu_-}{\lambda_+ + \lambda_-} + \frac{\lambda_+\mu_+}{\lambda_+ + \lambda_-}u]\Psi'(u|z) + [-\frac{\lambda_+\mu_+\mu_-}{\lambda_+ + \lambda_-} + \frac{\lambda_+^2\mu_+\mu_-}{(\lambda_+ + \lambda_-)^2}u +$$

$$+\frac{\lambda_+\lambda_-^2\mu_+\mu_-}{(\lambda_+ + \lambda_-)^3}u]\Psi(u|z) = 0. \tag{7}$$

Let us solve this equation.
Characteristic equation and the roots

$$K^2(u) + [-\mu_+ + \frac{\lambda_+\mu_-}{\lambda_+ + \lambda_-} + \frac{\lambda_+\mu_+}{\lambda_+ + \lambda_-}u]\times$$

$$\times K(u) + [-\frac{\lambda_+\mu_+\mu_-}{\lambda_+ + \lambda_-} + \frac{\lambda_+^2\mu_+\mu_-}{(\lambda_+ + \lambda_-)^2}u + \frac{\lambda_+\lambda_-^2\mu_+\mu_-}{(\lambda_+ + \lambda_-)^3}u] = 0 \tag{8}$$

$$K_{1,2}(1) = \frac{\frac{\lambda_+\mu_- + \lambda_-\mu_+}{\lambda_+ + \lambda_-} \pm \frac{1}{\lambda_+ + \lambda_-}\sqrt{\lambda_+^2\mu_-^2 + \lambda_-^2\mu_+^2 + \frac{2\lambda_+\lambda_-\mu_+\mu_-(\lambda_+ - \lambda_-)}{\lambda_+ + \lambda_-}}}{2}$$

if $u = 1$ we get,

$$K_{1,2}(1) = \frac{\lambda_+\mu_- + \lambda_-\mu_+ \pm \sqrt{\lambda_+^2\mu_-^2 + \lambda_-^2\mu_+^2 + \frac{2\lambda_+\lambda_-\mu_+\mu_-(\lambda_+ - \lambda_-)}{\lambda_+ + \lambda_-}}}{2(\lambda_+ + \lambda_-)}$$

We get the solution of the differential equation

$$\Psi(u|z) = C_1(u)e^{k_1(u)z} + C_2(u)e^{k_2(u)z} \tag{9}$$

If in equations (5) and (6) we substitute $z = 0$, we get the system of equations

$$\begin{cases} C_1(u)\Big[1 - \frac{\lambda_+\mu_+u}{(\lambda_+ + \lambda_-)(k_1(u) - \mu_+)}\left(e^{(k_1-\mu_+)a} - 1\right) - \\ -\frac{\lambda_-^2\mu_-u}{(\lambda_+ + \lambda_-)(k_1(u)(\lambda_+ + \lambda_-) + \lambda_+\mu_-)} + \frac{\lambda_+\lambda_-^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^2(\mu_+(\lambda_+ + \lambda_-) + \lambda_+\mu_-(k_1(u) - \mu_+))} + \Big] \\ +C_2(u)\Big[1 - \frac{\lambda_+\mu_+u}{(\lambda_+ + \lambda_-)(k_2(u) - \mu_+)}\left(e^{(k_2-\mu_+)a} - 1\right) - \\ -\frac{\lambda_-^2\mu_-u}{(\lambda_+ + \lambda_-)(k_2(u)(\lambda_+ + \lambda_-) + \lambda_+\mu_-)} + \frac{\lambda_+\lambda_-^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^2(\mu_+(\lambda_+ + \lambda_-) + \lambda_+\mu_-(k_2(u) - \mu_+))}\Big] = \\ = ue^{-\mu_+a} \\ C_1(u)\left[k_1(u) - \mu_+ + \frac{\lambda_+\mu_+u}{\lambda_+ + \lambda_-} - \frac{\lambda_+\lambda_-^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^2(k_1(u)(\lambda_+ + \lambda_-) + \lambda_+\mu_-)}\right] + \\ +C_2(u)\left[k_2(u) - \mu_+ + \frac{\lambda_+\mu_+u}{\lambda_+ + \lambda_-} - \frac{\lambda_+\lambda_-^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^2(k_2(u)(\lambda_+ + \lambda_-) + \lambda_+\mu_-)}\right] = 0 \end{cases}$$

Simplify the second equation of system (10). For that we use the characteristic equation and get

$$k_1(u) - \mu_+ + \frac{\lambda_+\mu_+u}{\lambda_+ + \lambda_-} - \frac{\lambda_+\lambda_-^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^2[k_1(u)(\lambda_+ + \lambda_-) + \lambda_+\mu_-]} = 0$$

$$k_2(u) - \mu_+ + \frac{\lambda_+\mu_+u}{\lambda_+ + \lambda_-} - \frac{\lambda_+\lambda_-^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^2[k_2(u)(\lambda_+ + \lambda_-) + \lambda_+\mu_-]} = 0$$

$$C_1(u)\Bigg[k_1^2(u) + k_1(u)[-\mu_+ + \frac{\lambda_+\mu_-}{\lambda_+ + \lambda_-} + \frac{\lambda_+\mu_+u}{\lambda_+ + \lambda_-}] -$$

$$-\frac{\lambda_+\mu_+\mu_-}{\lambda_+ + \lambda_-} + \frac{\lambda_+^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^2} + \frac{\lambda_+\lambda_-^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^3}\Bigg] +$$

$$+C_2(u)\Bigg[k_2^2(u) + k_2(u)[-\mu_+ + \frac{\lambda_+\mu_-}{\lambda_+ + \lambda_-} + \frac{\lambda_+\mu_+u}{\lambda_+ + \lambda_-}] -$$

$$\frac{\lambda_+\mu_+\mu_-}{\lambda_+ + \lambda_-} + \frac{\lambda_+^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^2} + \frac{\lambda_+\lambda_-^2\mu_+\mu_-u}{(\lambda_+ + \lambda_-)^3}\Bigg] = 0.$$

We get

$$C_1(u) \cdot 0 + C_2(u) \cdot 0 = 0$$

Then we substitute $C_2(u) = 0$ in the first equation and get an expression for $C_1(u)$

$$C_1(u) = e^{-\mu_+ a} / \left[1 - \frac{\lambda_+ \mu_+}{(\lambda_+ + \lambda_-)(k_1(u) - \mu_+)}(e^{(k_1 - \mu_+)a} - 1) - \right.$$

$$\left. - \frac{\lambda_-^2 \mu_-}{(\lambda_+ + \lambda_-)[k_1(u)(\lambda_+ + \lambda_-) + \lambda_+ \mu_-]} + \frac{\lambda_+ \lambda_-^2 \mu_+ \mu_-}{(\lambda_+ + \lambda_-)^2(\mu_+(\lambda_+ + \lambda_-) + \lambda_+ \mu_-(k_1(u) - \mu_+))}\right].$$

We simplify it using the roots of the characteristic equation and get

$$C_1(u) =$$

$$= \frac{e^{-\mu_+ a}}{\frac{\lambda_+ \mu_+}{(\lambda_+ + \lambda_-)(k_1(u) - \mu_+)} e^{(k_1 - \mu_+)a} - \frac{\lambda_-^2 \mu_-(\lambda_- \mu_+ + \lambda_+ \mu_-)(k_1(u) + \mu_+)}{(\lambda_+ + \lambda_-)[k_1(u)(\lambda_+ + \lambda_-) + \lambda_+ \mu_-](\mu_+(\lambda_+ + \lambda_-) + \lambda_+ \mu_-)(k_1(u) - \mu_+)}}$$

If we substitute the values of $C_1(u)$ and $C_2(u)$ in equation (9), we get

$$\Psi(u|z) =$$

$$= \frac{e^{k_1(u)z}}{\frac{\lambda_+ \mu_+}{(\lambda_+ + \lambda_-)(k_1(u) - \mu_+)} e^{k_1(u)a} - \frac{\lambda_-^2 \mu_-(\lambda_- \mu_+ + \lambda_+ \mu_-)(k_1(u) + \mu_+)}{(\lambda_+ + \lambda_-)[k_1(u)(\lambda_+ + \lambda_-) + \lambda_+ \mu_-](\mu_+(\lambda_+ + \lambda_-) + \lambda_+ \mu_-)(k_1(u) - \mu_+)} e^{\mu_+ a}}$$

or

$$\Psi(u|z) = \frac{(\lambda_+ + \lambda_-)(k_1(u) - \mu_+)e^{k_1(u)z}}{\lambda_+ \mu_+ e^{k_1(u)a} - \frac{\lambda_-^2 \mu_-(\lambda_- \mu_+ + \lambda_+ \mu_-)(k_1(u) + \mu_+)}{[k_1(u)(\lambda_+ + \lambda_-) + \lambda_+ \mu_-](\mu_+(\lambda_+ + \lambda_-) + \lambda_+ \mu_-)} e^{\mu_+ a}}$$

4. Conclusion

Using the sequence of independent random variables, we constructed difference process of semi-markov process. We found generating function of the number of jumps under which complex process of semi-markov walk achieves first the level $''a''(a > 0)$.

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Constructive Method for Solving the External Dirichlet Boundary – Value Problem for the Helmholtz Equation

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Abstract. This work presents the justification of collocation method for the boundary integral equation of the external Dirichlet boundary – value problem for the Helmholtz equation. Besides, the sequence of approximate solutions is built which converges to the exact solution of the original problem and the estimate for the rate of convergence is obtained.

Key Words and Phrases: Helmholtz equation, external Dirichlet boundary – value problem, surface integral, cubature formula, collocation method.
2010 Mathematics Subject Classifications: 45E05; 31B10

1. Introduction and Problem Statement

It is known that one of the methods for solving the external Dirichlet boundary – value problem for the Helmholtz equation is its reduction to the boundary integral equation (BIE). Integral equation methods play a central role in the study of boundary – value problems associated with the scattering of acoustic or electromagnetic waves by bounded obstacles. This is primarily due to the fact that the mathematical formulation of such problems leads to equations defined over unbounded domains, and hence their reformulation in terms of boundary integral equations not only reduces the dimensionality of the problem, but also allows one to replace a problem over an unbounded domain by one over a bounded domain. Since BIE is solved only in very rare cases, it is therefore of paramount importance to develop approximate methods for solving BIE with an appropriate theoretical justification.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a twice continuously differentiable boundary S. Consider the external Dirichlet boundary – value problem for the Helmholtz equation: us to find a function u which is twice continuously differentiable in $\mathbb{R}^3 \backslash \bar{D}$ and continuous on S, satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^3 \backslash \bar{D}$, the Sommerfeld radiation condition

$$\left(\frac{x}{|x|}\, , \, grad\, u\,(x) \right) - i\, k\, u\,(x) = o\, \left(\frac{1}{|x|} \right), \quad |x| \to \infty,$$

and the boundary condition

$$u\,(x) = f\ (x)\ \text{ on } S,$$

where k is a wave number with $Im\, k \geq 0$, and f is a given continuous function on S.

It is proved in [1] that the potential of double layer

$$u\ (x) = \int\limits_S \frac{\partial \Phi_k\,(x, y)}{\partial \vec{n}\ (y)}\, \varphi\,(y)\ dS_y, \quad x \in \mathbb{R}^3 \backslash \bar{D},$$

is a solution of the external Dirichlet boundary – value problem for the Helmholtz equation if the density φ is a solution of BIE

$$\varphi + K\,\varphi = 2f, \tag{1}$$

where

$$(K\varphi)\,(x) = \ 2 \int\limits_S \frac{\partial \Phi_k\,(x, y)}{\partial \vec{n}\ (y)}\, \varphi\,(y)\ dS_y, \quad x \in S,$$

and $\Phi_k\,(x, y)$ is fundamental solution the Helmholtz equation, i.e.

$$\Phi_k\,(x, y) = \ e^{i\,k\,|x-y|} / \ (4\pi\ |x-y|)\ ,\ \ x, y \in \mathbb{R}^3,\ x \neq y.$$

Let us note that the integral equations of boundary – value problems for the Helmholtz equation in the two – dimensional case were first considered by Kupradse [2-4]. In the present paper, we study an approximate solution of the external Dirichlet boundary – value problem for the Helmholtz equation by the integral equations method (1).

2. Main Results

Divide S into elementary domains $S = \bigcup\limits_{l=1}^{N} S_l^N$ in such a way that:

(1) for every $l = \overline{1, N}$ the domain S_l^N is closed and the set of its internal points $\overset{0}{S}{}_l^N$ with respect to S is nonempty, with $mes\, \overset{0}{S}{}_l^N = mes S_l^N$ and $\overset{0}{S}{}_l^N \cap \overset{0}{S}{}_j^N = \varnothing$ for $j \in \{1,\, 2,\, ...N\}\,,\ j \neq l$;

(2) for every $l = \overline{1, N}$ the domain S_l^N is a connected piece of the surface S with a continuous boundary;

(3) for every $l = \overline{1, N}$ there exists a so-called control point $x_l \in S_l^N$ such that:

(3.1) $r_l(N) \sim R_l(N)$ ($r_l(N) \sim R_l(N) \ \Leftrightarrow\ C_1 \leq r_l\,(N)\,/R_l\,(N) \leq C_2$, C_1 and C_2 are positive constants independent of N), where $r_l(N) = \min\limits_{x \in \partial S_l^N} |x - x_l|$ and $R_l(N) = \max\limits_{x \in \partial S_l^N} |x - x_l|$;

(3.2) $R_l(N) \leq d/2$, where d is the radius of a standard sphere (see [5]);

(3.3) for every $j = \overline{1, N}$, $r_j(N) \sim r_l(N)$.

It is clear that $r(N) \sim R(N)$ and $\lim\limits_{N\to\infty} r(N) = \lim\limits_{N\to\infty} R(N) = 0$, where $R(N) = \max\limits_{l=\overline{1,\,N}} R_l(N)$, $\quad r(N) = \min\limits_{l=\overline{1,\,N}} r_l(N)$.

Such a partition, as well as the partition of the unit sphere into elementary parts, has been carried out earlier in [6].

Let $S_d(x)$ and $\Gamma_d(x)$ be the parts of the surface S and the tangential plane $\Gamma(x)$, respectively, at the point $x \in S$, contained inside the sphere $B_d(x)$ of radius d centered at the point x. Besides, let $\tilde{y} \in \Gamma(x)$ be the projection of the point $y \in S$. Then

$$|x - \tilde{y}| \le |x - y| \le C_1(S)\ |x - \tilde{y}| \text{ and } mes S_d(x) \le C_2(S)\, mes \Gamma_d(x), \tag{1}$$

where $C_1(S)$ and $C_2(S)$ are positive constants that depend only on S (if S is a sphere, then $C_1(S) = \sqrt{2}$ and $C_2(S) = 2$).

Lemma 2.1 ([6]). *There exist the constants $C_0' > 0$ and $C_1' > 0$, independent of N, such that for $\forall\ l, j \in \{1, 2, ..., N\}$, $j \ne l$, and $\forall\ y \in S_j^N$ the inequality $C_0'\ |y - x_l| \le |x_j - x_l| \le C_1'\ |y - x_l|$ holds.*

For a continuous function $\varphi\,(x)$ on S, we introduce the modulus of continuity, which has the following form:

$$\omega(\varphi,\, \delta) = \delta \sup_{\tau \ge \delta} \frac{\overline{\omega}(\varphi,\, \tau)}{\tau}, \quad \delta > 0,$$

where $\bar{\omega}(\varphi,\, \tau) = \max\limits_{\substack{|x-y|\le\tau \\ x,\, y \in S}} |\varphi\,(x) - \varphi\,(y)|$.

Let

$$k_{l\,j} = 2\ |sgn\,(l - j)\,|\ \frac{\partial \Phi_k\,(x_l, x_j)}{\partial \vec{n}\ (x_j)}\, mes\, S_j^N \text{ for } l\,, j = \overline{1\,,\,N}.$$

It is proved in [7] that the expression

$$\left(K^N \varphi\right)\,(x_l) = \sum_{j=1}^{N} k_{l\,j}\, \varphi\,(x_j)$$

are cubature formula at the points x_l, $\ l = \overline{1,\,N}$, for the integral $(K\varphi)\,(x)$, with

$$\max_{l=\overline{1,\,N}} \left|\, (K\varphi)\,(x_l) - \left(K^N\varphi\right)\,(x_l)\,\right| \le M^*\,(\,\|\varphi\|_\infty\, R\,(N)\ |\ln R\,(N)\,| + \omega\,(\varphi, R\,(N))) \tag{2}$$

Let $\mathbb{C}^N -$ be a space of vectors $z^N = \left(z_1^N, z_2^N, \ldots, z_N^N\right)^{\mathrm{T}}$, $z_l^N \in \mathbb{C}$, $l = \overline{1, N}$, equipped with the norm $\left\|\, z^N \right\| = \max\limits_{l=\overline{1,N}} \left| z_l^N \right|$, and

$$K_l^N\, z^N = \sum_{j=1}^{N} k_{l\,j}\, z_j^N\,,\ \ l = \overline{1, N}\,, \qquad K^N\, z^N = \left(K_1^N\, z^N,\ K_2^N\, z^N\,, \ldots, K_N^N\, z^N\right)\,.$$

---

*Here and after, M denotes positive constants which can be different in different inequalities.

Then the BIE (1) by the system of algebraic equations with respect to z_l^N, approximate values of $\varphi\left(x_l\right)$, $l=\overline{1,N}$, stated as follows:

$$z^N+K^N z^N=2p^N f, \tag{3}$$

where $p^N f=\left(f\left(x_1\right),\, f\left(x_2\right),\, \ldots,\, f\left(x_N\right)\right)$.

To justify the collocation method, we will use Vainikko's convergence theorem for linear operator equations (see [8]). To formulate that theorem, we need some definitions and a theorem from [8].

Definition 2.1 ([8]). *A system $Q=\left\{q^N\right\}$ of operators $q^N: C\left(S\right)\to\mathbb{C}^N$ is called a connecting system for $C\left(S\right)$ and $\mathbb{C}^N$ if*

$$\left\|q^N\varphi\right\|\to\left\|\varphi\right\|_\infty \text{ as } N\to\infty\,,\quad \forall\varphi\in C\left(S\right);$$

$$\left\|q^N\left(a\varphi+a'\varphi'\right)-\left(a\,q^N\varphi+a'q^N\varphi'\right)\right\|\to 0 \text{ as } N\to\infty\,,\quad \forall\varphi\,,\,\varphi'\in C\left(S\right)\,,\ \ a\,,a'\in\mathbb{C}.$$

Definition 2.2 ([8]). *A sequence $\left\{\varphi_N\right\}$ of elements $\varphi_N\in\mathbb{C}^N$ is called $Q-$convergent to $\varphi\in C\left(S\right)$ if $\left\|\varphi_N-q^N\varphi\right\|\to 0$ as $N\to\infty$. We denote this fact by $\varphi_N\overset{Q}{\to}\varphi$.*

Definition 2.3 ([8]). *A sequence $\left\{\varphi_N\right\}$ of elements $\varphi_N\in\mathbb{C}^N$ is called $Q-$compact if every subsequence of it $\left\{\varphi_{N_m}\right\}$ contains a $Q-$convergent subsequence $\left\{\varphi_{N_{m_k}}\right\}$.*

Proposition 2.1 ([8]). *Let $q^N: C\left(S\right)\to\mathbb{C}^N$ be linear and bounded. Then the following conditions are equivalent:*

(1) *the sequence $\left\{\varphi_N\right\}$ is $Q-$compact and the set of its $Q-$limit points is compact in $C\left(S\right)$;*

(2) *there exists a relatively compact sequence $\left\{\varphi^{(N)}\right\}\subset C\left(S\right)$ such that*

$$\left\|\varphi_N-q^N\varphi^{(N)}\right\|\to 0 \text{ as } N\to\infty.$$

Definition 2.4 ([8]). *A sequence of operators $\mathrm{E}^N:\mathbb{C}^N\to\mathbb{C}^N$ is called $QQ-$convergent to the operator $\mathrm{E}: C\left(S\right)\to C\left(S\right)$ if for every $Q-$convergent sequence $\left\{\varphi_N\right\}$ the relation $\varphi_N\overset{Q}{\to}\varphi\ \Rightarrow\ \ \mathrm{E}^N\varphi_N\overset{Q}{\to}\mathrm{E}\varphi$ holds. We denote this fact by $\mathrm{E}^N\overset{QQ}{\to}\mathrm{E}$.*

Definition 2.5 ([8]). *We say that a sequence of linear bounded operators $\mathrm{E}^N:\mathbb{C}^N\to\mathbb{C}^N$ converges compactly to the linear bounded operator $\mathrm{E}: C\left(S\right)\to C\left(S\right)$ if $\mathrm{E}^N\overset{QQ}{\to}\mathrm{E}$ and the following compactness condition holds:*

$$\varphi_N\in\mathbb{C}^N,\quad\left\|\varphi_N\right\|\le M\quad\Rightarrow\left\{\mathrm{E}^N\varphi_N\right\}\ \text{is}\ Q-\text{compact}.$$

Theorem 2.3 ([8]). *Let the following conditions hold:*

1) $Ker\ \left(I+\mathrm{E}\right)=\left\{0\right\}$;

2) $I^N+\mathrm{E}^{N'}s\ \ \left(N\ge N_0\right)$ *are Fredholm operators of index zero;*

3) $\vartheta_N\overset{Q}{\to}\vartheta,\ \ \vartheta_N\in\mathbb{C}^N,\ \ \vartheta\in C\left(S\right)$;

4) $\mathrm{E}^N\to\mathrm{E}$ *compactly.*

Then the equation $(I+\mathrm{E})\,\varphi = \vartheta$ *has a unique solution* $\tilde{\varphi}\in C(S)$, *the equation* $\left(I^N+\mathrm{E}^N\right)\,\varphi_N=\vartheta_N \quad (N\ge N_0)$ *has a unique solution* $\tilde{\varphi}_N\in\mathbb{C}^N$, *and* $\tilde{\varphi}_N \overset{Q}{\to} \tilde{\varphi}$ *with*

$$c_1\,\left\|\left(I^N+\mathrm{E}^N\right)\,q^N\tilde{\varphi}-\vartheta_N\right\|\le\left\|\tilde{\varphi}_N-q^N\tilde{\varphi}\right\|\le c_2\,\left\|\left(I^N+\mathrm{E}^N\right)\,q^N\tilde{\varphi}-\vartheta_N\right\|,$$

where

$$c_1=1/\sup_{N\ge N_0}\left\|I^N+\mathrm{E}^N\right\|>0\,,\quad c_2=\sup_{N\ge N_0}\left\|\left(I^N+\mathrm{E}^N\right)^{-1}\right\|<+\infty.$$

Theorem 2.2. *Let* $Im\,k>0$, *then the equations (1) and (3) have unique solutions* $\varphi_*\in C(S)$ *and* $z_*^N\in\mathbb{C}^N \quad (N\ge N_0)$, *respectively, and* $\left\|z_*^N-p^N\varphi_*\right\|\to 0$ *as* $N\to\infty$ *with the following estimate for the rate of convergence:*

$$\left\|z_*^N-p^N\varphi_*\right\|\le M\;\left[\,\|f\|_\infty\,R(N)\;\;|\ln R(N)\,|+\omega\,\left(f,R(N)\right)\right].$$

Proof. Let's verify that the conditions of Theorem 2.1 are satisfied. It is proved in [1] that if $Im\,k>0$, then $Ker\ (I+K)=\{0\}$. Obviously, the operators I^N+B^N are Fredholm operators of index zero and the system operators $P=\left\{p^N\right\}$ is a connecting system for the spaces $C(S)$ and $\mathbb{C}^N$. Then $I^N+K^N\overset{PP}{\to} I+K$. By Definition 2.5, it remains only to verify the compactness condition, which in view of Proposition 2.1 is equivalent to the following one: $\forall\left\{z^N\right\}$, $z^N\in\mathbb{C}^N$, $\left\|z^N\right\|\le M$, there exists a relatively compact sequence$\left\{K_N\,z^N\right\}\subset C(S)$ such that

$$\left\|K^N\,z^N-p^N\left(\;K_N\,z^N\right)\;\right\|\to 0\,\text{as}\ N\to\infty.$$

As $\left\{K_N\,z^N\right\}$, we choose the sequence

$$\left(K_N\,z^N\right)\;(x)=2\sum_{j=1}^{N}z_j^N\int\limits_{S_j^N}\frac{\partial\Phi_k\left(x,y\right)}{\partial\vec{n}\left(y\right)}dS_y.$$

Take arbitrary points $x',\ x''\in S$ such that $|x'-x''|=\delta<d/2$. Then

$$\left|\left(K_N\,z^N\right)\;(x')-\left(K_N\,z^N\right)\;(x'')\,\right|\le\;M\;\left\|z^N\right\|\;\int\limits_S\left|\frac{\partial\Phi_k\left(x',y\right)}{\partial\vec{n}\left(y\right)}-\frac{\partial\Phi_k\left(x'',y\right)}{\partial\vec{n}\left(y\right)}\right|dS_y\le$$

$$M\;\left\|z^N\right\|\int\limits_{S_{\delta/2}(x')}\left|\frac{\partial\Phi_k\left(x',y\right)}{\partial\vec{n}\left(y\right)}\right|dS_y+M\;\left\|z^N\right\|\int\limits_{S_{\delta/2}(x'')}\left|\frac{\partial\Phi_k\left(x'',y\right)}{\partial\vec{n}\left(y\right)}\right|dS_y+$$

$$M\;\left\|z^N\right\|\int\limits_{S_{\delta/2}(x')}\left|\frac{\partial\Phi_k\left(x'',y\right)}{\partial\vec{n}\left(y\right)}\right|dS_y+M\;\left\|z^N\right\|\int\limits_{S_{\delta/2}(x'')}\left|\frac{\partial\Phi_k\left(x',y\right)}{\partial\vec{n}\left(y\right)}\right|dS_y+$$

$$M\;\left\|z^N\right\|\int\limits_{S\backslash\left(S_{\delta/2}(x')\cup S_{\delta/2}(x'')\right)}\left|\frac{\partial\Phi_k\left(x',y\right)}{\partial\vec{n}\left(y\right)}-\frac{\partial\Phi_k\left(x'',y\right)}{\partial\vec{n}\left(y\right)}\right|dS_y.$$

Using the inequality

$$\left| \frac{\partial \Phi_k\,(x,y)}{\partial \vec{n}\,(y)} \right| \le \frac{M}{|x-y|}, \quad \forall\, x\,, y \in S,\ x \ne y,$$

and the formula for reducing surface integral to a double integral, we obtain:

$$\int\limits_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k\,(x',y)}{\partial \vec{n}\,(y)} \right| dS_y \le M \int\limits_{S_{\delta/2}(x')} \frac{1}{|x'-y|}\, dS_y \le\ M\,\delta,$$

$$\int\limits_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k\,(x'',y)}{\partial \vec{n}\,(y)} \right| dS_y \le\ M\,\delta.$$

Besides, taking into account the inequalities $|x''-y| \ \ge\ \delta/2\,,\ \ \forall y \in S_{\delta/2}\,(x')$ and $|x'-y| \ge\ \delta/2\,,\ \ \forall y \in S_{\delta/2}\,(x'')$, we have:

$$\int\limits_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k\,(x'',y)}{\partial \vec{n}\,(y)} \right| dS_y \le M \int\limits_{S_{\delta/2}(x')} \frac{1}{|x''-y|}\, dS_y \le\ \frac{2M}{\delta}\, mes\left(S_{\delta/2}\,(x')\right) \le M\,\delta,$$

$$\int\limits_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k\,(x',y)}{\partial \vec{n}\,(y)} \right| dS_y \le M\,\delta.$$

It is easy to show that

$$\left| \frac{\partial \Phi_k\,(x',y)}{\partial \vec{n}\,(y)} - \frac{\partial \Phi_k\,(x'',y)}{\partial \vec{n}\,(y)} \right| \le \frac{M\,\delta}{|x'-y|^2}\,, \quad \forall y \in S\backslash\left(S_{\delta/2}\left(x'\right) \cup S_{\delta/2}\left(x''\right)\right).$$

Hence we find

$$\int\limits_{S\backslash\left(S_{\delta/2}(x')\cup S_{\delta/2}(x'')\right)} \left| \frac{\partial \Phi_k\,(x',y)}{\partial \vec{n}\,(y)} - \frac{\partial \Phi_k\,(x'',y)}{\partial \vec{n}\,(y)} \right| dS_y \le M\,\delta\ |\ln\delta|\,.$$

Then

$$\left|\ \left(K_N\, z^N\right)\ \left(x'\right) - \left(K_N\, z^N\right)\ \left(x''\right)\ \right| \le M\ \left\|z^N\right\|\ \delta\ |\ln\delta|\,, \tag{4}$$

and, consequently, $\left\{K_N\, z^N\right\} \subset C\,(S)$.

The relative compactness of the sequence $\left\{K_N\, z^N\right\}$ follows from the Arzela theorem. In fact, the uniform boundedness follows directly from the condition $\left\|z^N\right\| \le M$, and the equicontinuity follows from the estimate (4). Then, applying Theorem 2.1 we obtain that the equations (1) and (3) have unique solutions $\varphi_* \in C\,(S)$ and $z_*^N \in \mathbb{C}^N$ $(N \ge N_0)$, respectively, with

$$c_1\,\delta_N \le \left\|\, z_*^N - p^N \varphi_*\right\| \le\ c_2\,\delta_N,$$

where

$$c_1 = 1/\sup_{N\ge N_0} \left\| I^N + K^N \right\| > 0\,, \quad c_2 = \sup_{N\ge N_0} \left\| \left(I^N + K^N\right)^{-1} \right\| < +\infty,$$

$$\delta_N = \max_{l=\overline{1,N}} \left|(K\varphi_*)\ (x_l) - \left(K^N\varphi_*\right)(x_l)\right|.$$

Using the inequality (2), we obtain:

$$\delta_N \le M\ \left[\|\varphi_*\|_\infty R(N)\ |\ln R(N)| + \omega\ (\varphi_*, R(N)) + \right.$$

$$\left. \|f\|_\infty R(N)\ |\ln R(N)| + \omega\ (f, R(N))\right].$$

As $\varphi_* = 2\ (I+K)^{-1} f$, we have

$$\|\varphi_*\|_\infty \le 2 \left\| (I+K)^{-1} \right\|\ \ \|f\|_\infty .$$

Besides, taking into account the estimate

$$\omega\ (K\varphi_*, R\,(N)) \le M\ \|\varphi_*\|_\infty\ R\,(N)\ |\ln R\,(N)\,|\,,$$

we obtain:

$$\omega\ (\varphi_*\,,\, R\,(N)) = \omega\ (2f - K\varphi_*, R\,(N)) \le 2\omega\ (f, R\,(N)) + \omega\ (K\varphi_*, R\,(N)) \le$$

$$M\ \ \|\,f\,\|_\infty\ R\,(N)\ |\ln R\,(N)\,|\,,$$

consequently

$$\delta_N \le M\ [\ \|\,f\,\|_\infty\ R\,(N)\ |\ln R\,(N)\,| + \omega\ (f, R\,(N))\,]\,.$$

Theorem is proved.

Let's state the main result of this work.

Theorem 2.3. *Let* $Im\,k > 0, x_0 \in \mathbb{R}^3 \backslash \bar{D}$ *and* $z_*^N = (z_1^*, z_2^*, \ldots, z_N^*)^{\mathrm{T}}$ *be a solution of the system of algebraic equations (3). Then the sequence*

$$u_N(x_0) = \sum_{j=1}^{N} \frac{\partial \Phi_k\,(x_0, x_j)}{\partial \vec{n}\,(x_j)}\ z_j^*\, mes S_j^N$$

converges to the value of the solution $u\,(x)$ *of the external Dirichlet boundary - value problem for the Helmholtz equation at the point* x_0*, with*

$$|u_N\,(x_0) - u\,(x_0)| \le M\ [\ \|\,f\,\|_\infty\ R\,(N)\ |\ln R\,(N)\,| + \omega\ (f, R\,(N))\ ]\,.$$

Proof. Let the function $\varphi_* \in C\,(S)$be a solution of the equation (1). Then, as is known, the function

$$u\,(x) = \int_S \frac{\partial \Phi_k\,(x, y)}{\partial \vec{n}\,(y)} \varphi_*\,(y)\ dS_y\,, \quad x \in \mathbb{R}^3 \backslash \bar{D},$$

is a solution of the external Dirichlet boundary – value problem for the Helmholtz equation. Evidently,

$$u(x_0) - u_N(x_0) = \sum_{j=1}^{N} \int\limits_{S_j^N} \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \left(\varphi_*(x_j) - z_j^*\right) dS_y +$$

$$\sum_{j=1}^{N} \int\limits_{S_j^N} \left(\frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} - \frac{\partial \Phi_k(x_0, x_j)}{\partial \vec{n}(x_j)} \right) \varphi_*(y)\, dS_y +$$

$$\sum_{j=1}^{N} \int\limits_{S_j^N} \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \left(\varphi_*(y) - \varphi_*(x_j)\right) dS_y +$$

$$\sum_{j=1}^{N} \int\limits_{S_j^N} \left(\frac{\partial \Phi_k(x_0, x_j)}{\partial \vec{n}(x_j)} - \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \right) \left(\varphi_*(x_j) - z_j^*\right) dS_y +$$

$$\sum_{j=1}^{N} \int\limits_{S_j^N} \left(\frac{\partial \Phi_k(x_0, x_j)}{\partial \vec{n}(x_j)} - \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \right) \left(\varphi_*(y) - \varphi_*(x_j)\right) dS_y \quad .$$

As $x_0 \notin S$, then

$$\left| \frac{\partial \Phi_k(x_0, x_j)}{\partial \vec{n}(x_j)} - \frac{\partial \Phi_k(x_0, y)}{\partial \vec{n}(y)} \right| \leq M\, R(N), \quad \forall\, y \in S_j^N .$$

As a result, taking into account Theorem 2.2, we obtain the proof of Theorem 2.3.

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Global Bifurcation of Solutions for the Problem of Population Modeling

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Abstract. We consider nonlinear Sturm-Liouville problem with indefinite weight function which arise from population modeling. We show the existence of two families of continua of solutions corresponding to the usual nodal properties and emanating from zero and infinity.

Key Words and Phrases: nonlinear Sturm-Liouville problem, indefinite weight, population modeling, bifurcation point, eigenvalue, oscillatory properties of eigen functions.

2010 Mathematics Subject Classifications: 34B15; 34B24; 34C10; 34C23; 34L15; 45C05; 47J10; 47J15

1. Introduction

We consider the following nonlinear Sturm-Liouville equation

$$(\ell y)(x) \equiv -(p(x)y'(x))' + q(x)y(x) = \lambda\rho(x)y(x) + g(x, y(x), y'(x), \lambda),\ x \in (0, 1), \tag{1}$$

subject to the boundary conditions

$$\alpha_0 y(0) - \beta_0 y'(0) = 0, \tag{2}$$

$$\alpha_1 y(1) + \beta_1 y'(1) = 0, \tag{3}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $p(x)$ is a positive and continuously differentiable function on $[0, 1]$, $q(x)$ and $\rho(x)$ are real-valued continuous functions on $[0, 1]$, α_i, β_i, $i = 0, 1$, are real constants such that $|\alpha_i| + |\beta_i| > 0$, $i = 0, 1$. We also assume that the nonlinear term g is continuous function on $[0, 1] \times \mathbb{R}^3$ satisfying the condition:

$$g(x, u, s, \lambda) = o(|u| + |s|),\ \text{as } |u| + |s| \to 0, \tag{4}$$

or

$$g(x, u, s, \lambda) = o(|u| + |s|),\ \text{as } |u| + |s| \to \infty, \tag{5}$$

uniformly in $x \in [0, 1]$ and $\lambda \in \Lambda$ for any bounded interval $\Lambda \subset \mathbb{R}$.

Nonlinear Sturm-Liouville eigenvalue problems arise in many applications, for example, the problem (1)-(3) with indefinite weight arise from population modeling. In this model, weight function ρ changes sign corresponding to the fact that the intrinsic population growth rate is positive at same points and is negative at others, for details, see [5, 7].

If condition (4) holds then we can consider bifurcation from zero, i.e., bifurcation of nontrivial solutions from the set of trivial solutions $\mathcal{R} = \mathbb{R} \times \{0\}$. Problem (1)-(3) in the case $\rho > 0$ has been considered in [10]. This paper prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1[0,1]$ emanating from bifurcation point (in $\mathcal{R}$) corresponding to the eigenvalues of the linear problem, obtained from (1)-(3) by setting $F \equiv 0$. Similar problems for the nonlinear eigenvalue problems of ordinary differential equations of second and fourth order with definite weight function have been considered in [1-4, 12].

If condition (5) holds then the problem (1)-(3) is said to be asymptotically linear (see [9]) and we consider bifurcation from infinity, i.e., the existence of solutions of problem (1)-(3) having arbitrarily large norm. In the case $\rho > 0$ the existence of solutions of problem (1)-(3) with large norm (bifurcating from infinity) is considered in [11] and [12]. In these papers the bifurcation problem from infinity is transformed to a problem involving bifurcation from zero for the eigenvalues of the corresponding linear problem and then the global bifurcation theorems from [11] is applied.

In the investigation of bifurcation from zero and infinity for the problem (1)-(3) with indefinite weight function ρ, the main difficulty is connected with the fact that the eigenfunctions of the linear problem corresponding to the positive and negative eigenvalues with the same serial numbers have same number of zeros. Therefore in this case the standard global bifurcation results from [10] and [11] are not directly applicable. However, by using the results of [6], [10] and [11], we shall establish the global bifurcation results from zero and infinity for the problem (1)-(3) with indefinite weight function ρ. We prove the existence of global continua of solutions bifurcating from zero and infinity which are similar to those obtained in [10], [11] and [12].

2. Preliminary

By (4) the linearization of problem (1)-(3) at $y = 0$ is the linear Sturm-Liouville problem

$$\begin{cases} -(p(x)y'(x))' + q(x)y(x) = \lambda\rho(x)y(x), \ x \in (0,\ 1), \\ y \in B.C., \end{cases} \tag{6}$$

where by $B.C.$ is the set of the boundary conditions (2)-(3). It is a classical result (see [8]) that the problem (6) in the case $\rho(x) > 0$, $x \in [0,1]$, possesses infinitely many real eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$, all of which are simple, and $\lim\limits_{k\to+\infty} \lambda_k = +\infty$. The eigenfunction $y_k(x)$ corresponding to eigenvalue λ_k, $k \in \mathbb{N}$, has exactly $k-1$ simple nodal zeros in the interval $(0,1)$ (by a nodal zero we mean the function changes sign at the zero and at a simple nodal zero, the derivative of the function is nonzero).

Let E be the Banach space of all continuously differentiable functions on $[0,1]$ which satisfy the conditions $B.C.$. E is equipped with its usual norm $||u||_1 = ||u||_\infty + ||u'||_\infty$, where $||u||_\infty = \max\limits_{x\in[0,1]} |u(x)|$. Let S_k^+ be the set of $u \in E$ which have exactly $k-1$ simple nodal zeroes on $(0,1)$ and which are positive for $0 \neq x$ near 0; then, $S_k^- = -S_k^+$ and $S_k = S_k^- \cup S_k^+$. The sets S_k^+, S_k^- and S_k are open in E. Moreover, if $u \in \partial S_k$, then u has at least one double zero in $[0,1]$.

Theorem 2.1. *(see [8; Ch. 10, §§ 10·6, 10·61]). If ρ changes sign in the interval $(0,1)$ (i.e. $\text{meas}\,\{x \in [0,1] : \sigma\rho(x) > 0\} > 0,\ \sigma \in \{+,-\}$), $q(x) \geq 0,\ x \in [0,1]$, and $\alpha_0\beta_0 \geq 0$, $\alpha_1\beta_1 \geq 0$, then the eigenvalues of problem (6) are all real and simple, and form a two sequences*

$$0 > \lambda_1^- > \lambda_2^- > ... > \lambda_k^- \mapsto -\infty \ \textit{and}\ 0 < \lambda_1^+ < \lambda_2^+ < ... < \lambda_k^+ \mapsto +\infty.$$

Moreover, for each $k \in \mathbb{N}$ and each $\sigma \in \{+,-\}$ the eigenfunction $y_k^\sigma(x)$ corresponding to eigenvalue λ_k^σ, has exactly $k-1$ simple nodal zeros in the interval $(0,1)$ (more precisely, $y_k^\sigma(x) \in S_k$).

Throughout the sequel we assume that the following conditions are satisfied:

$$\begin{array}{l} \text{meas}\,\{x \in [0,1] : \sigma\rho(x) > 0\} > 0,\ \sigma \in \{+,-\}, \\ q(x) \geq 0, x \in [0,1],\ \text{and}\ \alpha_i\beta_i \geq 0, i = 0,1. \end{array} \tag{7}$$

It follows from Theorem 2.1 that for each $k \in \mathbb{N}$ the eigenfunctions of $y_k^-(x)$ and $y_k^+(x)$, corresponding to the eigenvalues λ_k^- and λ_k^+, respectively, have exactly $k-1$ simple nodal zeros in the interval $(0,1)$. Hence, if the conditions (7) are satisfied, then first sight it seems that the continua which bifurcates from the point $(\lambda_k^+, 0)$ and is contained in $\mathbb{R} \times S_k$, will meet $(\lambda_k^-, 0)$ and this prevents the first alternative of [10, Theorem 1.3] occurring. But thanks to Dancer [6] we show that this is not happening.

3. Global bifurcation from zero for problem (1)-(3)

We denote by $\mathfrak{L}$ the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (1)-(3). The eigenfunction y_k^σ, $\sigma \in \{+,-\}$, corresponding to the eigenvalue λ_k^σ of problem (6) is made unique by requiring that $y_k^\sigma \in S_k^+$ and $||y_k^\sigma|| = 1$.

One of the main results is the following theorem.

Theorem 3.1. *For each $k \in \mathbb{N}$, each $\nu \in \{+,-\}$ and each $\sigma \in \{+,-\}$ there exists a continuum $(\mathfrak{L}_k^\sigma)^\nu$ of solutions of problem (1)-(3) in $(\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k^\sigma, 0)\}$ which meets $(\lambda_k^\sigma, 0)$ and ∞ in $\mathbb{R} \times E$.*

Proof. Let (λ, y) is a solution of problem (1)-(3) and $y \in \partial S_k^\nu$. Hence y has double zero in $[0,1]$. Then, using growth estimate on g near the double zero and linearity of ℓ and ρy and applying Gronwall's inequality we obtain that $y \equiv 0$ on $[0,1]$.

By (7) $\lambda = 0$ is not an eigenvalue of the spectral problem (6). Then using Green's function $h(x,t)$ of differential expression $\ell(y)$ together with the boundary conditions (2)-(3) problem (1)-(3) can be converted to the equivalent integral equation

$$y(x) = \lambda \int_0^\pi h(x,t)\,\rho(t)y(t)dt + \int_0^\pi h(x,t)\,g(t,y(t),y'(t),\lambda)\,dt. \tag{8}$$

Define $L : E \to E$ and $F : \mathbb{R} \times E \to E$ by

$$Ly(x) = \int_0^\pi h(x,t)\,\rho(t)y(t)dt \;\text{ and }\; F(\lambda, y(x)) = \int_0^\pi h(x,t)\,g(t,y(t),y'(t),\lambda)\,dt, \tag{9}$$

respectively.

Since $\rho(x)$ is continuous, it follows from the properties of $h(x,t)$ that $L : E \to E$ is a completely continuous operator. The operator G can be represented as the composition of the Fredholm operator L with $\rho(x) \equiv 1$ and the superposition operator $g(\lambda, y(x)) = g(x, y(x), y'(x), \lambda)$. Since g is continuous in $[0,l] \times \mathbb{R}^3$, it follows that $g : \mathbb{R} \times E \to C[0,1]$ is continuous. Hence $G : \mathbb{R} \times E \to E$ is completely continuous. By (4) we have

$$G(\lambda, y) = o(||y||) \;\text{ as } ||y|| \to 0, \tag{10}$$

uniformly with respect to $\lambda \in \Lambda$.

By virtue of (8)-(9) problem (1)-(3) can be written in the following equivalent form

$$y = \lambda Ly + G(\lambda, y), \tag{11}$$

and therefore, it is enough to investigate the structure of the set of solutions of (1)-(3) in $\mathbb{R} \times E$.

Note that problem (11) is of the form (0.1) of [10]. The linearization of this problem at $y = 0$ is the spectral problem

$$y = \lambda Ly. \tag{12}$$

Obviously, the problem (12) is equivalent to the spectral problem (6). Consequently, the eigenvalues of (6) are the characteristic values of (12) and are simple. Hence all eigenvalues λ_k^σ, $k \in \mathbb{N}$, $\sigma \in \{+,-\}$, satisfy the hypotheses of Theorem 1.3 from [10] and accordingly there exists a component $\mathfrak{L}_k^\sigma$ of $\mathfrak{L}$ with contains $(\lambda_k^\sigma, 0)$ and is either unbounded in $\mathbb{R} \times E$ or contains $(\lambda_j^\sigma, 0)$, where $j \neq k$. It follows from [10; Lemma 1.24] that if $(\lambda, y) \in \mathfrak{L}_k^\sigma$ and is near $(\lambda_k^\sigma, 0)$, then $y = \tau y_k^\sigma + w$ with $w = o(|\tau|)$. Since S_k is open in E and $y_k^\sigma \in S_k$, then

$$(\lambda, y) \in \mathbb{R} \times S_k \;\text{ and }\; ((\mathfrak{L}_k^\sigma \backslash \{(\lambda_k^\sigma, 0)\}) \cap B_\delta(\lambda_k^\sigma)) \subset \mathbb{R} \times S_k \tag{13}$$

for all $\delta > 0$ small, where $B_\delta(\lambda_k^\sigma)$ is a open ball in $\mathbb{R} \times E$ of radius δ centered at $(\lambda_k^\sigma, 0)$. By an above remark,

$$(\mathfrak{L}_k^\sigma \backslash \{(\lambda_k^\sigma, 0)\}) \cap (\mathbb{R} \times \partial S_k) = \emptyset. \tag{14}$$

Next we decompose $\mathfrak{L}_k^\sigma$, $k \in \mathbb{N}$, $\sigma \in \{+,-\}$, into two subcontinua $(\mathfrak{L}_k^\sigma)^+$ and $(\mathfrak{L}_k^\sigma)^-$ in accordance with Dancer's construction (see [6, p. 1070-1071). Again writing $y = \tau y_k^\sigma + w$ for $(\lambda, y) \in (\mathfrak{L}_k^\sigma \backslash \{(\lambda_k^\sigma, 0)\})$ and near $(\lambda_k^\sigma, 0)$ we have $\tau y_k^\sigma \in \mathbb{R} \times S_k^\nu$ if $0 \neq \tau \in \mathbb{R}^\nu$. Therefore, by (13) we have

$$(((\mathfrak{L}_k^\sigma)^+ \backslash \{(\lambda_k^\sigma, 0)\}) \cap B_\delta(\lambda_k^\sigma)) \subset \mathbb{R} \times S_k^+ \text{ and } (((\mathfrak{L}_k^\sigma)^- \backslash \{(\lambda_k^\sigma, 0)\}) \cap B_\delta(\lambda_k^\sigma)) \subset \mathbb{R} \times S_k^-$$

for all $\delta > 0$ small. Moreover, it follows from (14) that

$$(\mathfrak{L}_k^\sigma)^+ \cap (\mathbb{R} \times \partial S_k^+) = \emptyset \text{ and } (\mathfrak{L}_k^\sigma)^- \cap (\mathbb{R} \times \partial S_k^-) = \emptyset.$$

It is clear from the last four relations that $(\mathfrak{L}_k^\sigma)^\nu \backslash \{(\lambda_k^\sigma, 0)\})$, $\nu \in \{+,-\}$, cannot leave $\mathbb{R} \times S_k^\nu$ outside of a neighborhood of $(\lambda_k^\sigma, 0)$. Since $S_k^+ \cap S_k^- = \emptyset$ it follows by remark to Theorem 2 from [6, p. 1073] that

$$((\mathfrak{L}_k^\sigma)^+ \backslash \{(\lambda_k^\sigma, 0)\}) \cap ((\mathfrak{L}_k^\sigma)^+ \backslash \{(\lambda_k^\sigma, 0)\}) = \emptyset.$$

Hence by [8, theorem 2] we have

$$(\mathfrak{L}_k^\sigma)^+ \subset ((\mathbb{R} \times S_k^+) \cup \{(\lambda_k^\sigma, 0)\}) \text{ and } (\mathfrak{L}_k^\sigma)^- \subset ((\mathbb{R} \times S_k^-) \cup \{(\lambda_k^\sigma, 0)\}),$$

and both are unbounded in $\mathbb{R} \times E$. The proof of this theorem is complete.

Remark 3.1. If the nonlinear term g has the form $g(x, y, s, \lambda) = \lambda\, g_1(x, y, s, \lambda)$, where g_1 is continuous function on $[0,\, 1] \times \mathbb{R}^3$ satisfying the condition (4), then the problem (1)-(3) does not have a solution of the form $(0, u)$ (this follows from the fact that 0 is not an eigenvalue of the corresponding linear problem (6)). In this case it is obvious that for each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\, -\}$ the continua $(\mathfrak{L}_k^+)^\nu$ and $(\mathfrak{L}_k^-)^\nu$ do not intersect and this again confirms the validity of Theorem 3.1

4. Global bifurcation from infinity for problem (1)-(3)

Now we consider problem (1)-(3) under condition (5). We say (λ, ∞) is a bifurcation point for (1)-(3) if every neighborhood of (λ, ∞) contains solutions of (1)-(3), i.e. there exists a sequence $\{(\lambda_n, u_n,)\}_{n=1}^\infty$ of solutions of this problem such that $\lambda_n \to \lambda$ and $u_n \to \infty$ as $n \to \infty$.

Theorem 4.1. *If* (5) *holds then, for each $k \in \mathbb{N}$ and each $\sigma \in \{+\,,\, -\}$ there exists an unbounded component $\mathfrak{D}_k^\sigma$ of $\mathfrak{L} \cup (\lambda_k^\sigma \times \{\infty\})$, containing $I_k \times \{\infty\}$. Moreover, if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap \{\lambda_k^\sigma\}_{k=1}^\infty = \lambda_k^\sigma$ and $\mathcal{M}$ is a neighborhood of $I_k^\sigma \times \{\infty\}$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from* 0, *then either*

1^o. *$\mathfrak{D}_k^\sigma \backslash \mathcal{M}$ is bounded in $\mathbb{R} \times E$ in which case $D_k \backslash M$ meets $\mathcal{R} = \{(\lambda, 0) \,|\, \lambda \in \mathbb{R}\}$ or*

2^o. *$D_k \backslash \mathcal{M}$ is unbounded.*

If 2^o occurs and $\mathfrak{D}_k \backslash \mathcal{M}$ has a bounded projection on $\mathbb{R}$, then $\mathfrak{D}_k \backslash \mathcal{M}$ meets $\lambda_j^\sigma \times \{\infty\}$ for some $j \neq k$.

The proof of this theorem is similar to that of [11, Theorems 1.6 and 2.4] with the use of Theorem 3.1.

By using Theorems 3.1, 3.4 and [11, Corollary 1.8 and Theorem 2.4] we can prove the following theorem.

Theorem 4.2. *If* (5) *holds, then for each $k \in \mathbb{N}$ and each $\sigma \in \{+\,, -\}$ there are two subcontinua $(\mathfrak{D}_k^\sigma)^+$ and $(\mathfrak{D}_k^\sigma)^-$, consisting of the bifurcation branch $\mathfrak{D}_k^\sigma$, which satisfy the alternates of Theorem 4.1. Moreover, there exists a neighborhood $\mathcal{N} \subset \mathcal{M}$ of $\lambda_k^\sigma \times \{\infty\}$ such that $((\mathfrak{D}_k^\sigma)^\nu \cap \mathcal{N}) \subset ((\mathbb{R} \times S_k^\nu) \cup (\lambda_k^\sigma \times \{\infty\}))$ for each $\nu \in \{+\,, -\}$.*

Next, if conditions (4) and (5) both hold then we can improve Theorems 3.1, 4.1 and 4.2 as follows.

Theorem 4.3. *If* (4) *and* (5) *hold then, for each $k \in \mathbb{N}$, each $\sigma \in \{+\, -\}$ and each $\nu \in \{+\,, -\}$, we have $(\mathfrak{D}_k^\sigma)^\nu \subset \mathbb{R} \times S_k^\nu$, and alternative 2^o of Theorem 4.1 cannot hold. Furthermore, if $(\mathfrak{D}_k^\sigma)^\nu$ meets $\mathcal{R}$ for some λ, then $\lambda = \lambda_k^\sigma$. Similarly, if $(\mathfrak{L}_k^\sigma)^\nu$ meets $\{(\lambda, \infty) \,:\, \lambda \in \mathbb{R}\}$, then $\lambda = \lambda_k^\sigma$.*

Proof. If (4) holds, then it follows from the proof of Theorem 3.1 that $\mathfrak{L} \cap (\mathbb{R} \times \partial S_k^\nu) = \emptyset$. Hence the sets $\mathfrak{L} \cap (\mathbb{R} \times S_k^\nu)$ and $\mathfrak{L} \backslash (\mathbb{R} \times S_k^\nu)$ are mutually separated in $\mathbb{R} \times E$. Then by virtue of [13, Corollary 26.6] every component of $\mathfrak{L}$ must be a subset of one or another of these sets. Since for each $\sigma \in \{+\,, -\}$ the set $(\mathfrak{D}_k^\sigma)^\nu$ is the component of $\mathfrak{L}$ which intersect $\mathbb{R} \times S_k^\nu$, this component must be a subset of $\mathbb{R} \times S_k^\nu$, i.e. $(\mathfrak{D}_k^\sigma)^\nu \subset (\mathbb{R} \times S_k^\nu)$. Hence by the second assertion of Theorem 4.2 it follows that alternative 2^0 of Theorem 4.1 cannot hold. Then from Theorem 2.1 and 3.1 implies that $(\mathfrak{D}_k^\sigma)^\nu$ can only meet $\mathcal{R}$ if $\lambda = \lambda_k^\sigma$. In a similar way, by Theorems 4.1 and 4.2, $(\mathfrak{L}_k^\sigma)^\nu$ can only meet (λ, ∞) if $\lambda = \lambda_k^\sigma$. The proof of this theorem is complete.

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