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Approximation of Hypersingular Integral Operators on Hölder Spaces

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Abstract. In the present paper, the hypersingular integral operator is approximated by a sequence of operators of the special form and is obtained the estimate of the convergence rate in Hölder spaces.

Key Words and Phrases: hypersingular integral, Hölder space, approximating operators, convergence rate.

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1. Introduction

An active development of numerical methods for solving hypersingular integral equations is of considerable interest in modern numerical analysis. This is due to the fact that hypersingular integral equations have numerous applications in acoustics, aerodynamics, fluid mechanics, electrodynamics, elasticity, fracture mechanics, geophysics and etc. (see [4, 5, 10, 13, 15, 20, 22, 23, 26, 27]). Therefore the construction and justification of numerical schemes for approximate solutions of hypersingular integral equations is a topical issue and numerous works [3-9,11,12,14,16-19,21-25, 27-31] are devoted to their development. In the present paper hypersingular integral operator In the present paper hypersingular integral operator

$$\left(S^{(0)}\varphi\right)(t) = \frac{1}{\pi i}\int_{\gamma_0} \frac{\varphi(\tau)}{|\tau - t|} d\tau$$

is approximated by a sequences of operators of the form

$$\left(S_n^{(0)}\varphi\right)(t) = \sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(\tau_k^{(t)}\right), \qquad t \in \gamma_0$$

in the unit circle $\gamma_0 = \{t \in C : |t| = 1\}$, where $\tau_k^{(t)} = e^{k\theta i} \cdot t$, $k = \overline{0,\, 2n}$, $\theta = \frac{\pi}{n}$, $n \in N$, $\alpha_k^{(n)}(t)$ – are continuous functions in γ_0, $k = \overline{0, 2n-1}$, $n \in N$.

It should be noted that, the determination of the inverse operator $\left[S_n^{(0)}\right]^{-1}$ is equivalent to the study of the equation

$$\sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(\tau_k^{(t)}\right) = f(t)\ , \quad t\in\gamma_0$$

at the points $\tau_0^{(t)}, \tau_1^{(t)}, ..., \tau_{2n-1}^{(t)}$, because solving the resulting system of linear algebraic equations with respect to $\left(\varphi\left(\tau_0^{(t)}\right),\ \varphi\left(\tau_1^{(t)}\right), ..., \varphi\left(\tau_{2n-1}^{(t)}\right)\right)$, we obtain the function $\varphi(t) = \varphi\left(\tau_0^{(t)}\right)$.

Note that, for the singular integral operators with Cauchy kernel and Hilbert kernel similar approximations and its applications to the singular integral equations are given in the papers [1] and [2], analogous approximations for hypersingular integral operators with Cauchy kernel are given in [3].

2. Hypersingular integral operator

Consider the following integral

$$\int_a^b \frac{g(x)}{|x-x_0|}dx\ , \quad x_0\in(a,b) \tag{1}$$

where the function $g(x)$ is defined in the interval $[a,b]$. If we define this integral similar to the Cauchy integral, even if $g\equiv 1$, we get the divergent integral:

$$\lim_{\varepsilon\to 0+}\left(\int_a^{x_0-\varepsilon}\frac{1}{|x-x_0|}dx + \int_{x_0+\varepsilon}^{b}\frac{1}{|x-x_0|}dx\right) = \lim_{\varepsilon\to 0+}\left(-2\ln\varepsilon + \ln(x_0-a)(b-x_0)\right) = \infty.$$

Therefore, using the idea of Hadamard finite part integral [15], we will define the integral (1) as follows:

Definition 1. *If a finite limit*

$$\lim_{\varepsilon\to 0+}\left(\int_a^{x_0-\varepsilon}\frac{g(x)\,dx}{|x-x_0|} + \int_{x_0+\varepsilon}^{b}\frac{g(x)\,dx}{|x-x_0|} + 2g(x_0)\ln\varepsilon\right)$$

exists, then the value of this limit is referred to as the hypersingular integral of the function $\frac{g(x)}{|x-x_0|}$, $x_0\in(a,b)$ *on* $[a,b]$ *and is denoted by* $\int_a^b \frac{g(x)}{|x-x_0|}dx$ *.*

Now consider the integral

$$\int_{\gamma_0}\frac{\varphi(\tau)\,d\tau}{|\tau-t|}, \quad t\in\gamma_0\,, \tag{2}$$

where the function $\varphi(t)$ *is defined in the unit circle*$\gamma_0 = \{t\in C:\ |t|=1\}$.

From the definition 1.1 for the hypersingular integral on interval, we define the integral (2) *as follows.*

Definition 2. *If a finite limit*

$$\lim_{\varepsilon\to 0+}\left(\int_{\gamma_\varepsilon}\frac{\varphi(\tau)\,d\tau}{|\tau-t|}+2it\varphi(t)\ln\varepsilon\right)$$

exists, then the value of this limit is referred to as the hypersingular integral of the function $\frac{\varphi(\tau)}{|\tau-t|}$, $t\in\gamma_0$ *on the circle*γ_0 *and is denoted by* $\int_{\gamma_0}\frac{\varphi(\tau)d\tau}{|\tau-t|}$, *where* $\gamma_\varepsilon=\{\tau\in\gamma_0:\,|\tau-t|>\varepsilon\}$.

From definitions 1.1 and 1.2, it follows that if $t=e^{ix_0}$, $x_0\in(-\pi,\pi)$, then

$$\int_{\gamma_0}\frac{\varphi(\tau)\,d\tau}{|\tau-t|}=\lim_{\varepsilon\to 0+}\left(\int_{\gamma_\varepsilon}\frac{\varphi(\tau)\,d\tau}{|\tau-t|}+2it\varphi(t)\ln\varepsilon\right)=$$

$$=\lim_{\varepsilon\to 0+}\left(\int_{x_0+\delta(\varepsilon)}^{x_0+2\pi-\delta(\varepsilon)}\frac{\varphi\left(e^{ix}\right)ie^{ix}dx}{|e^{ix}-e^{ix_0}|}+2ie^{ix_0}\varphi\left(e^{ix_0}\right)\ln\varepsilon\right)=$$

$$=\lim_{\varepsilon\to 0+}\left(\int_{[-\pi,\pi]/(x_0-\delta(\varepsilon),\,x_0+\delta(\varepsilon))}\frac{\varphi\left(e^{ix}\right)ie^{ix}}{|x-x_0|}\left|\frac{x-x_0}{e^{ix}-e^{ix_0}}\right|\cdot dx+2ie^{ix_0}\varphi\left(e^{ix_0}\right)\ln\varepsilon\right)=$$

$$=\int_{-\pi}^{\pi}\frac{\varphi\left(e^{ix}\right)ie^{ix}dx}{|e^{ix}-e^{ix_0}|}+2ie^{ix_0}\varphi\left(e^{ix_0}\right)\cdot\lim_{\varepsilon\to 0+}\left(\ln\varepsilon-\ln\delta(\varepsilon)\right)=\int_{-\pi}^{\pi}\frac{\varphi\left(e^{ix}\right)ie^{ix}dx}{|e^{ix}-e^{ix_0}|}\,,\qquad(3)$$

where $\delta(\varepsilon)=2\arcsin\frac{\varepsilon}{2}$.

Equation (3) shows that, by means of change of variables $t=e^{ix}$ the hypersingular integral on a circle is reduced to hypersingular integral on an interval.
We will calculate the hypersingular integral $\int_{\gamma_0}\frac{d\tau}{|\tau-t|}$, $t=e^{ix_0}\in\gamma_0$. We have

$$\int_{\gamma_0}\frac{d\tau}{|\tau-t|}=\lim_{\varepsilon\to 0+}\left(\int_{[x_0-\pi,x_0+\pi]/(x_0-\varepsilon,\,x_0+\varepsilon)}\frac{ie^{ix}}{|e^{ix}-e^{ix_0}|}dx+2it\ln\varepsilon\right)=$$

$$=\lim_{\varepsilon\to 0+}\left(\int_{[x_0-\pi,x_0+\pi]/(x_0-\varepsilon,\,x_0+\varepsilon)}\frac{ie^{ix}}{2\left|\sin\frac{x-x_0}{2}\right|}dx+2it\ln\varepsilon\right)=$$

$$=2it\cdot\lim_{\varepsilon\to 0+}\left(\int_{x_0+\varepsilon}^{x_0+\pi}\frac{\cos(x-x_0)}{2\sin\frac{x-x_0}{2}}+\ln\varepsilon\right)=2it\cdot(\ln 4-2)\,,\qquad(4)$$

where $\delta(\varepsilon)=2\arcsin\frac{\varepsilon}{2}\sim\varepsilon$ as $\varepsilon\to 0+$. From equation (4) it follows that,

$$\int_{\gamma_0}\frac{\varphi(\tau)\,d\tau}{|\tau-t|}=\int_{\gamma_0}\frac{\varphi(\tau)-\varphi(t)}{|\tau-t|}d\tau+(\ln 4-2)\,2it\varphi(t)\,.\qquad(5)$$

Let $H_\alpha(\gamma_0)$, $0<\alpha\le 1$ be the space of Hölder continuous functions with exponent α in γ_0 , i.e. the space of the functions which satisfies the following condition

$$\exists C>0\quad\forall t_1,\ t_2\in\gamma_0:\quad|\varphi(t_1)-\varphi(t_2)|\le C\cdot|t_1-t_2|^\alpha$$

with the norm

$$\|\varphi\|_{\alpha}=\|\varphi\|_{\infty}+H(\varphi;\alpha)$$

where

$$\|\varphi\|_{\infty}=\max_{t\in\gamma_0}|\varphi(t)|\,,H(\varphi;\alpha)=\sup\left\{\frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|}\,:\;t_1,t_2\in\gamma_0,\;t_1\neq t_2\right\}.$$

From equation (5) it follows that, if $\varphi\in H_{\alpha}(\gamma_0)$ then hypersingular integral $\int_{\gamma_0}\frac{\varphi(\tau)d\tau}{|\tau-t|}$ exists for all $t\in\gamma_0$

Consider the hypersingular integral operator:

$$\left(S^{(0)}\varphi\right)(t)=\frac{1}{\pi i}\int_{\gamma_0}\frac{\varphi(\tau)}{|\tau-t|}d\tau.$$

Theorem 1. *Hypersingular integral operator $S^{(0)}$ is bounded from the space $H_{\alpha}(\gamma_0)$ into the space $H_{\alpha-\varepsilon}(\gamma_0)$ for all $0<\alpha\leq 1$ and $0<\varepsilon<\alpha$.*

Proof. From equation (5) it follows that, it is sufficient to prove the stated theorem for the following operator:

$$(T\varphi)(t)=\frac{1}{\pi i}\int_{\gamma_0}\frac{\varphi(\tau)-\varphi(t)}{|\tau-t|}d\tau.$$

Let $\varphi\in H_{\alpha}(\gamma_0)$. Then

$$\|T\varphi\|_{\infty}=\max_{t\in\gamma_0}\left|\frac{1}{\pi i}\int_{\gamma_0}\frac{\varphi(\tau)-\varphi(t)}{|\tau-t|}d\tau\right|\leq\frac{1}{\pi}\left|\int_{\gamma_0}\frac{|\varphi(\tau)-\varphi(t)|}{|\tau-t|}|d\tau|\right|\leq$$

$$\leq\frac{1}{\pi}\left|\int_{\gamma_0}\frac{H(\varphi;\alpha)}{|\tau-t|^{1-\alpha}}|d\tau|\right|\leq C_1\cdot H(\varphi;\alpha)\leq C_1\cdot\|\varphi\|_{\alpha}\,, \tag{6}$$

where C_1 – constant which only depends on α.

Estimate the difference $(T\varphi)(t_1)-(T\varphi)(t_2)$ for any two points $t_1,\;t_2\in\gamma_0,\;t_1\neq t_2$. If $|t_1-t_2|\geq\frac{1}{2}$, then from inequality (6) it follows that,

$$|(T\varphi)(t_1)-(T\varphi)(t_2)|\leq 2C_1\cdot\|\varphi_{\alpha}\|\leq 4C_1\cdot\|\varphi_{\alpha}\|\cdot|t_1-t_2|\,. \tag{7}$$

Consider the case $|t_1-t_2|<\frac{1}{2}$. We plot the circle centered at the t_1 with radius $\delta=2\cdot|t_1-t_2|$. This circle and γ_0 intersect at two points, which we will denote by a and b. Denote by l the part of γ_0 which is inside of this circle .

Represent the difference $(T\varphi)(t_1)-(T\varphi)(t_2)$ as follows:

$$(T\varphi)(t_1)-(T\varphi)(t_2)=\frac{1}{\pi i}\int_{l}\frac{\varphi(\tau)-\varphi(t_1)}{|\tau-t_1|}d\tau-\frac{1}{\pi i}\int_{l}\frac{\varphi(\tau)-\varphi(t_2)}{|\tau-t_2|}d\tau+$$

$$+\frac{1}{\pi i}\int_{\gamma_0\setminus l}\left\{\frac{\varphi(\tau)-\varphi(t_1)}{|\tau-t_1|}-\frac{\varphi(\tau)-\varphi(t_2)}{|\tau-t_2|}\right\}d\tau=$$

$$= \frac{1}{\pi i}\int_l \frac{\varphi(\tau)-\varphi(t_1)}{|\tau-t_1|}d\tau - \frac{1}{\pi i}\int_l \frac{\varphi(\tau)-\varphi(t_2)}{|\tau-t_2|}d\tau + \frac{1}{\pi i}\int_{\gamma_0\backslash l} \frac{\varphi(t_2)-\varphi(t_1)}{|\tau-t_1|}d\tau +$$

$$+\frac{1}{\pi i}\int_{\gamma_0\backslash l}[\varphi(\tau)-\varphi(t_2)]\left[\frac{1}{|\tau-t_1|}-\frac{1}{|\tau-t_2|}\right]d\tau = J_1+J_2+J_3+J_4. \tag{8}$$

From the condition $\varphi \in H_\alpha(\gamma_0)$, $\delta = 2\cdot|t_1-t_2|$ we have the following estimate

$$|J_1| \le \frac{1}{\pi}\int_l \frac{|\varphi(\tau)-\varphi(t_1)|}{|\tau-t_1|}|d\tau| \le \frac{H(\varphi;\alpha)}{\pi}\int_l \frac{|d\tau|}{|\tau-t_1|^{1-\alpha}} \le$$

$$\le \frac{2H(\varphi;\alpha)}{\pi}\int_0^\delta \frac{dr}{(r/2)^{1-\alpha}} = \frac{4H(\varphi;\alpha)}{\pi\alpha}\cdot|t_1-t_2|^\alpha \le \frac{4\|\varphi\|_\alpha}{2^\varepsilon\pi\alpha}\cdot|t_1-t_2|^{\alpha-\varepsilon}.$$

Absolutely analogously

$$|J_2| \le \frac{1}{\pi}\int_l \frac{|\varphi(\tau)-\varphi(t_2)|}{|\tau-t_2|}|d\tau| \le \frac{H(\varphi;\alpha)}{\pi}\int_l \frac{|d\tau|}{|\tau-t_2|^{1-\alpha}} \le$$

$$\le \frac{2H(\varphi;\alpha)}{\pi}\int_0^{3\delta/2} \frac{dr}{(r/2)^{1-\alpha}} = \frac{6H(\varphi;\alpha)}{\pi\alpha}\cdot|t_1-t_2|^\alpha \le \frac{6\|\varphi\|_\alpha}{2^\varepsilon\pi\alpha}\cdot|t_1-t_2|^{\alpha-\varepsilon}.$$

We estimate the integral J_3as follows:

$$|J_3| \le \frac{|\varphi(t_2)-\varphi(t_1)|}{\pi}\left|\int_{\gamma_0\backslash l}\frac{d\tau}{|\tau-t_1|}\right| \le \frac{H(\varphi;\alpha)\cdot|t_1-t_2|^\alpha}{\pi}\left|\int_{\gamma_0\backslash l}\frac{d\tau}{|\tau-t_1|}\right| \le$$

$$\le 4H(\varphi;\alpha)\cdot|t_1-t_2|^\alpha\cdot\ln\frac{\pi}{|t_1-t_2|} \le C_2\cdot\|\varphi\|_\alpha\cdot|t_1-t_2|^{\alpha-\varepsilon},$$

where C_2 – constant which only depends on α and ε.

Now turn to the estimate of the integral J_4.

$$|J_4| = \frac{1}{\pi}\left|\int_{\gamma_0\backslash l}\frac{[\varphi(\tau)-\varphi(t_2)]\cdot[|\tau-t_2|-|\tau-t_1|]}{|\tau-t_1|\cdot|\tau-t_2|}d\tau\right| \le$$

$$\le \frac{H(\varphi;\alpha)}{\pi}\left|\int_{\gamma_0\backslash l}\frac{|\tau-t_2|-|\tau-t_1|}{|\tau-t_1|\cdot|\tau-t_2|^{1-\alpha}}d\tau\right| \le \frac{H(\varphi;\alpha)\cdot|t_1-t_2|}{\pi}\int_{\gamma_0\backslash l}\frac{|d\tau|}{|\tau-t_1|\cdot|\tau-t_2|^{1-\alpha}} =$$

$$= \frac{H(\varphi;\alpha)\cdot|t_1-t_2|}{\pi}\int_{\gamma_0\backslash l}|\tau-t_1|^{\alpha-2}\left|\frac{\tau-t_1}{\tau-t_2}\right|^{\alpha-1}|d\tau|.$$

Since for any $\tau\in\gamma_0\backslash l$, the following inequality is holds

$$|\tau-t_1| \le \frac{1}{3}|\tau-t_2|,$$

then we have

$$|J_4| \le \frac{H(\varphi;\alpha)\cdot|t_1-t_2|}{3^{1-\alpha}\pi}\int_{\gamma_0\setminus l}|\tau-t_1|^{\alpha-2}\,|d\tau| \le C_3\cdot\|\varphi\|_\alpha\cdot|t_1-t_2|^{\alpha-\varepsilon},$$

where C_3 - constant which only depends on α and ε.

Comparing obtained estimates for $J_1,\ J_2,\ J_3$ andJ_4, from equation (8) and inequality (7) it follows the validity of the theorem. This completes the proof of the theorem.

3. Approximation of hypersingular integral operator

Consider the sequences of operators

$$\left(S_n^{(0)}\varphi\right)(t) = \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left|\tau_{2k+1}^{(t)}-t\right|}\Delta\tau_{2k+1}^{(t)} \ +(\ln 4-2)\,2it\varphi(t)\,,\ t\in\gamma_0,\quad n=1,2,..,$$

where $\tau_k^{(t)}=e^{k\theta i}\cdot t,\ \Delta\tau_k^{(t)}=\left(\tau_{k+1}^{(t)}-\tau_{k-1}^{(t)}\right)\frac{\theta}{\sin\theta}=2ie^{k\theta i}\cdot t\cdot\theta\,,\ k=\overline{0,2n}\,,\ \theta=\frac{\pi}{n}.$

It is clear that, operators $S_n^{(0)}$, $n=1,2,..$ is bounded from the space $H_\alpha(\gamma_0)$ into the space $H_\alpha(\gamma_0)$ for all $0<\alpha\le 1$.

Theorem 2. *For any $\varphi\in H_\alpha(\gamma_0)$, $0<\alpha\le1$, the following estimate holds*

$$\left\|S_n^{(0)}\varphi-S^{(0)}\varphi\right\|_\infty \le \frac{C_4\ln(n+1)}{n^\alpha}\cdot H(\varphi;\alpha)\,,n=1,2,..,\tag{9}$$

where C_4 – constant which only depends on α.

Proof. From equation (5) it follows that, for all $t\in\gamma_0$

$$\left|\left(S_n^{(0)}\varphi\right)(t)-\left(S^{(0)}\varphi\right)(t)\right| = \left|\int_{\gamma_0}\frac{\varphi(\tau)-\varphi(t)}{|\tau-t|}d\tau-\sum_{k=0}^{n-1}\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left|\tau_{2k+1}^{(t)}-t\right|}\Delta\tau_{2k+1}^{(t)}\right| \le$$

$$\le \frac{1}{\pi}\sum_{k=0}^{n-1}\left|\int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}}\frac{\varphi(\tau)-\varphi(t)}{|\tau-t|}d\tau-\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left|\tau_{2k+1}^{(t)}-t\right|}\Delta\tau_{2k+1}^{(t)}\right| = \frac{1}{\pi}\sum_{k=0}^{n-1}I_k\,.\tag{10}$$

Estimate the difference I_k, $k=\overline{0,\,n-1}$. For the difference I_0we have

$$I_0 \le \int_{\tau_0^{(t)}\tau_2^{(t)}}\frac{|\varphi(\tau)-\varphi(t)|}{|\tau-t|}|d\tau| + \frac{\left|\varphi\left(\tau_1^{(t)}\right)-\varphi(t)\right|}{\left|\tau_1^{(t)}-t\right|}\left|\Delta\tau_1^{(t)}\right| \le$$

$$\le H(\varphi;\alpha)\cdot\left[\int_{\tau_0^{(t)}\tau_2^{(t)}}\frac{|d\tau|}{|\tau-t|^{1-\alpha}}+\frac{\left|\Delta\tau_1^{(t)}\right|}{\left|\tau_1^{(t)}-t\right|^{1-\alpha}}\right] \le \frac{C_5}{n^\alpha}\cdot H(\varphi;\alpha)\,,$$

where C_5 – constant which depends on α. Analogously it follows that,

$$I_{n-1} \le \frac{C_5}{n^\alpha} \cdot H(\varphi;\alpha).$$

For I_k, $k = \overline{1,\, n-2}$ we have

$$I_k \le \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\varphi\left(\tau_{2k+1}^{(t)}\right) - \varphi(t)}{|\tau - t|} d\tau - \frac{\varphi\left(\tau_{2k+1}^{(t)}\right) - \varphi(t)}{\left|\tau_{2k+1}^{(t)} - t\right|} \Delta\tau_{2k+1}^{(t)} \right| +$$

$$+ \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\varphi(\tau) - \varphi\left(\tau_{2k+1}^{(t)}\right)}{|\tau - t|} d\tau \right| = I_k^{(1)} + I_k^{(2)}.$$

Estimate for the difference $I_k^{(1)}$ as follows:

$$I_k^{(1)} \le \left|\varphi\left(\tau_{2k+1}^{(t)}\right) - \varphi(t)\right| \cdot \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{d\tau}{|\tau - t|} - \frac{\Delta\tau_{2k+1}^{(t)}}{\left|\tau_{2k+1}^{(t)} - t\right|} \right| \le$$

$$\le H(\varphi;\alpha) \cdot \left|\tau_{2k+1}^{(t)} - t\right|^\alpha \cdot \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \left[\frac{1}{|\tau - t|} - \frac{1}{\left|\tau_{2k+1}^{(t)} - t\right|} \right] d\tau \right| \le$$

$$\le H(\varphi;\alpha) \cdot \left|\tau_{2k+1}^{(t)} - t\right|^{\alpha-1} \cdot \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\left|\left|\tau_{2k+1}^{(t)} - t\right| - |\tau - t|\right|}{|\tau - t|} |d\tau|. \qquad (11)$$

Since for all $\tau \in \tau_{2k}^{(t)}\tau_{2k+2}^{(t)}$ the following inequality holds

$$\left|\left|\tau_{2k+1}^{(t)} - t\right| - |\tau - t|\right| \le \left|\tau - \tau_{2k+1}^{(t)}\right| \le \left|\tau_{2k}^{(t)} - \tau_{2k+1}^{(t)}\right| = 2\sin\frac{\theta}{2} \le \theta = \frac{\pi}{n}, |\tau - t| \ge \frac{1}{2}\left|\tau_{2k+1}^{(t)} - t\right|,$$

then from inequality (11) we get the following estimate:

$$I_k^{(1)} \le \frac{\pi^2}{2n^2} H(\varphi;\alpha) \cdot \left|\tau_{2k+1}^{(t)} - t\right|^{\alpha-2}.$$

Now turn to the estimate of the integral $I_k^{(2)}$.

$$I_k^{(2)} \le \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\varphi(\tau) - \varphi\left(\tau_{2k+1}^{(t)}\right)}{|\tau - t|} d\tau \right| \le H(\varphi;\alpha) \cdot \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\left|\tau - \tau_{2k+1}^{(t)}\right|^\alpha}{|\tau - t|} d\tau \right| \le$$

$$\le H(\varphi;\alpha) \cdot \left|\tau_{2k}^{(t)} - \tau_{2k+1}^{(t)}\right|^\alpha \cdot \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{|d\tau|}{|\tau - t|} \le H(\varphi;\alpha) \cdot \frac{\pi^\alpha}{n^\alpha} \cdot \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{|d\tau|}{|\tau - t|}.$$

Comparing obtained estimates for I_k, $k=\overline{0,\, n-1}$, from inequality (10) it follows the following inequality:

$$\left|\left(S_n^{(0)}\varphi\right)(t)-\left(S^{(0)}\varphi\right)(t)\right|\leq$$

$$\leq\frac{H(\varphi;\alpha)}{\pi}\left[\frac{2C_5}{n^\alpha}+\sum_{k=1}^{n-2}\left(\frac{\pi^2}{2n^2}\left|\tau_{2k+1}^{(t)}-t\right|^{\alpha-2}+\frac{\pi^\alpha}{n^\alpha}\int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}}\frac{|d\tau|}{|\tau-t|}\right)\right]. \qquad (12)$$

Since

$$\sum_{k=1}^{n-2}\left|\tau_{2k+1}^{(t)}-t\right|^{\alpha-2}=\sum_{k=1}^{n-2}\left|2\sin\frac{(2k+1)\pi}{2n}\right|^{\alpha-2}\leq 2\sum_{k=1}^{n}\left|\frac{2(2k+1)}{n}\right|^{\alpha-2}\leq\frac{C_6}{n^{\alpha-2}},$$

$$\sum_{k=1}^{n-2}\int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}}\frac{|d\tau|}{|\tau-t|}=\int_{\gamma_0\setminus\left(\tau_{-2}^{(t)}\tau_2^{(t)}\right)}\frac{|d\tau|}{|\tau-t|}\leq C_7\ln(n+1),$$

then from inequality (12) it follows the estimate (9). This completes the proof of the theorem.

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On Basicity of Perturbed Exponential System in Generalized Lebesgue Spaces

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Abstract. In the present work it is considered perturbed system of exponential functions with piecewise continuous phase. Special cases of this system form systems of eigenfunctios of model first order discontinuous ordinary differential operators. Sufficient conditions on the jumps of phase function, which guarantee the basisness of the system in generalized Lebesgue spaces are provided.

Key Words and Phrases: system of exponentials, basisness, variable exponent, generalized Lebesgue space.

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1. Introduction

When solving the PDEs of mixed type by Fourier method there frequently appear systems of sines and cosines of the following form

$$\{\cos (n + \alpha) t\}_{n \in Z_+}, \tag{1}$$

$$\{\sin (n + \alpha) t\}_{n \in N}, \tag{2}$$

where α is a real number (here, thereafter N is the set of all natural numbers, $Z_+ = \{0\} \bigcup N$). Justification of the Fourier method requires to study the basicity properties of such systems in some function spaces. Some examples of such equations and concrete systems of trigonometric-type functions that appear after applying Fourier method can be found, for example, in [1, 2, 3, 4]. The basicity properties of the systems (1) and (2) are well studied in Lebesgue and Sobolev spaces, as well as, in their weighted settings [5, 6, 7, 8, 9, 10, 11, 12, 27, 28, 29, 30, 31].

During the last two decades, non-standard function spaces became an extremely popular subject because of their appearance in modern problems of analysis and qualitative theory of PDEs. Introduction of Lebesgue spaces with variable exponents at the end of last century and variety of extraordinary results obtained therein were the main motivation

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and the inception of this new tendency in analysis. We only mention the monograph [13] and the comprehensive bibliography therein, where thoroughly treatment of these issues can be found.

In the present work it is considered the perturbed I system of exponentials with a piecewise continuous phase. Particular cases of these systems are eigenfunctions of model, discontinuous, ordinary differential operators of the first order. Sufficient conditions are obtained for phase jumps, in the course of which this system forms a basis in generalized Lebesgue spaces.

Notice that, similar problems for the double system of exponents with complex-valued coefficients in Lebesgue spaces with variable exponent were earlier studied in [15, 16, 17, 18, 19]. The basicity properties of the systems (1) and (2) in classical Lebesgue spaces were studied in [20, 24].

2. Preliminaries

We use the following standard denotations: $Z-$the set of all integers; $R-$the set of all real numbers; $C-$complex plane; $(\overline{\cdot})-$complex conjugate of $(\,.\,)$; $\delta_{nk}-$Kronecker delta; $\chi_A(\cdot)-$the indicator function of the set A. $\omega \equiv \{z \in C : |z| < 1\}-$ the unit disc; $\partial\omega \equiv \{z \in C : \ |z| = 1\}-$ the unit circle.

Let $p : [-\pi, \pi] \to [1, +\infty)-$be a Lebesgue measurable function. We denote by $\mathscr{L}_0$ the set of all Lebesgue measurable functions on $[-\pi, \pi]$. Set

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt$$

and

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

If $p^+ = \sup\limits_{[-\pi,\pi]} vrai\; p(t) < +\infty$, then $\mathscr{L}$ is a linear space with respect to pointwise linear operations. $\mathscr{L}$ is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

and we denote it by $L_{p(\cdot)}$. Set

$$WL \stackrel{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow$$

$$\Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\Big\}.$$

Throughout the paper $q(\cdot)$ denotes the conjugate function of $p(\cdot)$, that is, $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Let $p^- = \inf\limits_{[-\pi,\pi]} vrai\; p(t)$. The following generalized Hölder's inequality holds

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-;p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c\left(p^-;p^+\right) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

To get the main results we will need the following facts concerning the basicity in generalized Lebesgue spaces $L_{p(\cdot)}(0,\pi)$ of the following single-folded exponentional system, which were obtained in [32]:

$$v_n(t) \equiv a(t)e^{int} - b(t)e^{-int}, \ \ n \in N,$$

where $a(t) = |a(t)|\, e^{i\alpha(t)}$, $b(t) = |b(t)|\, e^{i\beta(t)}$are some complex-valued functions on $[0,\pi]$. It will be assumed that the functions $a(\cdot)$ and $b(\cdot)$ are subjected to the following conditions i)-iv):

i) $a^{\pm 1}(\cdot)$; $b^{\pm 1}(\cdot) \in L_\infty(0,\pi)$;

ii) $\alpha(\cdot)$; $\beta(\cdot)$ are piecewise continuous functions on $(0,\pi)$, with $\{t_k\}_{k\in N}$ and $\{\tau_k\}_{k\in N}$ as their jump points, respectively. Assume that the set $\{\tilde{s}_k\} \equiv \{t_k\} \bigcup \{\tau_k\}$ may have just one limit point $\tilde{s}_0 \in (0,\pi)$ and the function $\tilde{\theta}(t) \equiv \beta(t) - \alpha(t)$ has a finite left and right limits at the point $\tilde{s}_0$.

iii) $\sum_{k=1}^{\infty} |h(\tilde{s}_k)| < +\infty$, where $h(\tilde{s}_k) = \tilde{\theta}(\tilde{s}_k - 0) - \tilde{\theta}(\tilde{s}_k + 0)$ is the jump of the function $\tilde{\theta}(\cdot)$ at point $\tilde{s}_k$.

iv) The jumps $\left\{\tilde{h}_i\right\}$ satisfy $\left(\frac{\tilde{h}(\tilde{s}_i)}{2\pi} + \frac{1}{p(\tilde{s}_i)}\right) \notin$ Z, $\forall i \in N$.

From iii) it follows that there exists $r \in$ N, such that

$$-\frac{2\pi}{p(\tilde{s}_k)} < \tilde{h}(\tilde{s}_k) < \frac{2\pi}{q(\tilde{s}_k)}, k = \overline{r,\infty}.$$

Enumerate the elements of the set $\{\tilde{s}_i\}_1^r$ in increasing order and denote it by $\{s_i\}_1^r$:$0 < s_1 < ... < s_r < \pi$. Denote the jumps corresponding to them by $\{h(s_i)\}_1^r$:

$$h(s_i) = \beta(s_i - 0) - \beta(s_i + 0) + \alpha(s_i + 0) - \alpha(s_i - 0), i = \overline{1,r}.$$

Assume that for some n_0 it follows

$$\frac{1}{p(0)} + 2(n_0 - 1) < \frac{\beta(0) - \alpha(0)}{\pi} < \frac{1}{p(0)} + 2n_0. \tag{3}$$

By iv) define the integers $n_i, i = \overline{1,r}$ as follows

$$-\frac{1}{p(s_i)} < \frac{h(s_i)}{2\pi} + n_i - n_{i-1} < \frac{1}{q(s_i)}, i = \overline{1,r}. \tag{4}$$

We have the following main result:

Theorem 1. *Let the coefficient functions $a(\cdot)$ and $b(\cdot)$ satisfy i)-iv), the integers $\{n_i\}_1^r$ are defined as in* (3), (4). *Assume that*

$$\frac{\beta(\pi) - \alpha(\pi)}{2\pi} + \frac{1}{2p(\pi)} \notin Z. \tag{5}$$

If

$$-\frac{1}{p(\pi)}+2n_r<\frac{\beta(\pi)-\alpha(\pi)}{\pi}<-\frac{1}{p(\pi)}+2(n_r+1), \tag{6}$$

then the system $\{v_n\}_{n\in N}$ *forms a basis in* $L_{p(\cdot)}\left(0,\pi\right)$. *If*

$$\beta(\pi)-\alpha(\pi)<-\frac{\pi}{p(\pi)}+2n_r\pi,$$

then the system $\{v_n\}_{n\in \mathrm{N}}$ *is not complete but minimal in* $L_{p(\cdot)}\left(0,\pi\right)$; *If*

$$\beta(\pi)-\alpha(\pi)>-\frac{\pi}{p(\pi)}+2(n_r+1)\pi,$$

then the system $\{v_n\}_{n\in \mathrm{N}}$ *is complete but is not minimal in* $L_{p(\cdot)}\left(0,\pi\right)$.

3. Main Results

Consider the following system of exponentials:

$$\varphi_n\left(\theta\right)\equiv\exp\left[i\left(n\theta-sgnn\alpha\left(\theta\right)\right)\right],\quad n=\pm1,\pm2,..., \tag{7}$$

where $\alpha\left(\theta\right)$ is a piecewise continuous odd function on $\left[-\pi,\pi\right]$, that is $\alpha\left(-\theta\right)=-\alpha\left(\theta\right)$, $\forall\theta\in\left[-\pi,\pi\right]$. Let the set $\{t_k\}_1^\infty$ is the set of jump points of the function $\alpha\left(\theta\right)$ on $(0,\pi)$, which may have just one limit point $t_0\in(0,\pi)$. Assume that the function $\alpha\left(\theta\right)$ has finite left and right limits att_0. Furthermore, let

$$\sum_{k=1}^{\infty}\left|\alpha\left(t_k+0\right)-\alpha\left(t_k-0\right)\right|<+\infty. \tag{8}$$

Assume that

$$\frac{\alpha\left(t_i-0\right)-\alpha\left(t_i+0\right)}{\pi}\neq-\frac{1}{p\left(t_i\right)}+k,\qquad i=\overline{1,\infty}, \tag{9}$$

for any integer k.

Let for some integer n_0 it follows

$$\frac{\pi}{2p\left(0\right)}+\left(n_0-\frac{1}{2}\right)\pi<\alpha\left(0\right)<\frac{\pi}{2p\left(0\right)}+n_0\pi. \tag{10}$$

Denote by r the integer, for which

$$-\frac{\pi}{p\left(t_k\right)}<\alpha\left(t_k-0\right)-\alpha\left(t_k+0\right)<\frac{\pi}{q\left(t_k\right)},\ k=\overline{r,\infty}. \tag{11}$$

Enumerate the elements of the set $\{t_i\}$, $\quad i=\overline{1,r}$ in increasing order and denote the new set by $\{t_i\}_1^r:\ 0<t_1<...<t_r<\pi$. Define the integers n_i, $\quad i=\overline{1,r}$ as follows:

$$-\frac{1}{p\left(t_i\right)}<\frac{\alpha\left(t_i-0\right)-\alpha\left(t_i+0\right)}{\pi}+n_i-n_{i-1}<\frac{1}{q\left(t_i\right)},\quad i=\overline{1,r}. \tag{12}$$

Theorem 2. *Let $\alpha(t)$ be a real, piecewise continuous, odd function on $[-\pi,\pi]$, of which jumps satisfy* (8)-(10). *The integers n_i, $i=\overline{1,r}$ are defined as in* (10) - (12). *In addition, let*

$$\alpha(\pi)\neq-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi.$$

Then to be a basis of the system of exponentials (7) *in $L_{p(\cdot)}(-\pi,\pi)$ it is sufficient that*

$$-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+(n_r+1)\pi. \tag{13}$$

If $\alpha(\pi)<-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi$ then the system (7) *is not complete, but minimal in $L_{p(\cdot)}(-\pi,\pi)$; if $\alpha(\pi)\geq-\frac{\pi}{2p(\pi)}+(n_r+1)\pi$ then the system* (7) *is complete but is not minimal in $L_{p(\cdot)}(-\pi,\pi)$.*

Before proving the theorem we give some direct consequences of Theorem 1.

Let $\alpha(\cdot)$ be a piecewise continuous function on $[0,\pi]$ of which jumps satisfies the conditions (8), (9).

Corollary 1. *Let for some integer n_0*

$$\frac{\pi}{2p(0)}+(n_0-1)\pi<\alpha(0)<\frac{\pi}{2p(0)}+n_0\pi, \tag{14}$$

holds, the integer n_r is defined as in (14), (12) , *and it is assumed that $\alpha(\pi)\neq-\frac{\pi}{2p(\pi)}+n_r\pi$. If*

$$-\frac{\pi}{2p(\pi)}+n_r\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+(n_r+1)\pi,$$

then the system $\sin(nt-\alpha(t))$, $n=\overline{1,\infty}$, forms a basis in $L_{p(\cdot)}(0,\pi)$; if $\alpha(\pi)<-\frac{\pi}{2p(\pi)}+n_r\pi$, then the system $\sin(nt-\alpha(t))$, $n=\overline{1,\infty}$ is not complete, but minimal in $L_{p(\cdot)}(0,\pi)$; if $\alpha(\pi)\geq-\frac{\pi}{2p(\pi)}+(n_r+1)\pi$, then it is complete, but is not minimal in $L_{p(\cdot)}(0,\pi)$.

For the case of cosine system we have the following

Corollary 2. *Let $\alpha(\cdot)$ be a piecewise continuous function on $[0,\pi]$ of which jumps satisfy* (8), (9) , *and for some $n_0\in Z$ it holds*

$$\frac{\pi}{2p(0)}+\left(n_0-\frac{1}{2}\right)\pi<\alpha(0)<\frac{\pi}{2p(0)}+\left(n_0+\frac{1}{2}\right)\pi. \tag{15}$$

The integer n_r is defined as in (15), (12) *and it is assumed that $\alpha(\pi)\neq-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi$. If*

$$-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{3}{2}\right)\pi,$$

then the system $\cos(nt-\alpha(t))$, $n=\overline{1,\infty}$, forms a basis in $L_{p(\cdot)}(0,\pi)$; if $\alpha(\pi)<-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi$, then the system $\cos(nt-\alpha(t))$, $n=\overline{1,\infty}$ is not complete but minimal in $L_{p(\cdot)}(0,\pi)$; if $\alpha(\pi)>-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{3}{2}\right)\pi$, then it is complete but is not minimal in $L_{p(\cdot)}(0,\pi)$.

Proof of Theorem 3.1. Let us prove the sufficiency. First of all let us show that the system of exponentials (7) is complete in $L_{p(\cdot)}(-\pi,\pi)$ under the conditions of Theorem 2. Assume the contrary. Then there exists a nonzero function $f(\theta)\in L_{q(\cdot)}(-\pi,\pi)$, $\frac{1}{p(\cdot)}+\frac{1}{q(\cdot)}=1$, such that

$$\int_{-\pi}^{\pi} f(\theta)\exp\left[i\left(n\theta - sgnn\alpha(\theta)\right)\right]d\theta = 0,\quad n=\pm1,\pm2,... \tag{16}$$

From here we have

$$\begin{array}{l}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta-\alpha(\theta))\,d\theta + i\int_{-\pi}^{\pi} f(\theta)\sin(n\theta-\alpha(\theta))\,d\theta = 0\,,\\ \int_{-\pi}^{\pi} f(\theta)\cos(n\theta-\alpha(\theta))\,d\theta - i\int_{-\pi}^{\pi} f(\theta)\sin(n\theta-\alpha(\theta))\,d\theta = 0\,.\end{array}$$

By summing up we get

$$\begin{array}{l}0=\int_{-\pi}^{\pi} f(\theta)\cos(n\theta-\alpha(\theta))\,d\theta = \int_0^{\pi} f(\theta)\cos(n\theta-\alpha(\theta))\,d\theta +\\ +\int_0^{\pi} f(-\theta)\cos(-n\theta-\alpha(-\theta))\,d\theta = \int_0^{\pi}\left[f(\theta)+f(-\theta)\right]\,\cos(n\theta-\alpha(\theta))\,d\theta,\quad n=\overline{1,\infty}.\end{array}$$

Since under (13), as it follows from Corollary 2, the cosine system is complete in $L_{p(\cdot)}(0,\pi)$, we get that

$$f(\theta) = -f(-\theta)\,.$$

Since the function $f(\theta)$ is odd , by (16) we have

$$\begin{array}{l}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta-\alpha(\theta))\,d\theta = 0,\quad n=\overline{1,\infty}\,,\\ \int_0^{\pi} f(\theta)\sin(n\theta-\alpha(\theta))\,d\theta = 0,\quad n=\overline{1,\infty}\,.\end{array}$$

Under the condition (13), as it follows from Corollary 1, the sine system is complete in $L_{p(\cdot)}(0,\pi)$, from here we get that $f(\theta)\equiv 0$, which proves the completeness of the system of exponentials (7) in $L_{p(\cdot)}(-\pi,\pi)$.

Under the conditions of Theorem 2, as it follows from Corollary 1 and 2, the system $\sin(nt-\alpha(t))$ and $\cos(nt-\alpha(t))$ $n=\overline{1,\infty}$, forms a basis in $L_{p(\cdot)}(0,\pi)$. Let $h_m^s(t)$ and $h_m^c(t)$, $m=\overline{1,\infty}$, are biorthogonal with these systems, respectively:

$$\begin{array}{l}\int_0^{\pi}\sin(nt-\alpha(t))\,h_m^s(t)\,dt = \delta_{nm}\,,\\ \int_0^{\pi}\cos(nt-\alpha(t))\,h_m^c(t)\,dt = \delta_{nm},\end{array}$$

$n,m=\overline{1,\infty}$, δ_{nm} is Kronecker delta. Define the following system of functions:

$$h_n(\theta) = \frac{1}{4}\left[\hat{h}_{|n|}^c(\theta) - isgnn\hat{h}_{|n|}^s(\theta)\right]\,,\quad n=\pm1,\pm2,... \tag{17}$$

where

$$\hat{h}_{|n|}^c(\theta) = \begin{cases}\hat{h}_{|n|}^c(\theta)\,, & \theta\in(0,\pi)\,,\\ \hat{h}_{|n|}^c(-\theta)\,, & \theta\in(-\pi,0)\,,\end{cases}$$

$$\hat{h}_{|n|}^s(\theta) = \begin{cases} h_{|n|}^s(\theta)\,, & \theta\in(0,\pi)\,,\\ -h_{|n|}^s(-\theta)\,, & \theta\in(-\pi,0)\,.\end{cases}$$

Now we show that this system is biorthogonal with the system (7). Indeed, we have

$$(\varphi_m, h_n) =$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\left[\hat{h}^c_{|n|}(\theta) - isgnn\hat{h}^s_{|n|}(\theta)\right]\left[\cos(m\theta - sgnm\alpha(\theta)) + i\sin(m\theta - sgnm\alpha(\theta))\right]d\theta =$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\cos(m\theta - sgnm\alpha(\theta))\,\hat{h}^c_{|n|}(\theta)\,d\theta - \frac{i}{4}sgnn\int_{-\pi}^{\pi}\cos(m\theta - sgnm\alpha(\theta))\,\hat{h}^s_{|n|}(\theta)\,d\theta +$$

$$+\frac{i}{4}\int_{-\pi}^{\pi}\sin(m\theta - sgnm\alpha(\theta))\,\hat{h}^c_{|n|}(\theta)\,d\theta + \frac{sgnn}{4}\int_{-\pi}^{\pi}\sin(m\theta - sgnm\alpha(\theta))\,\hat{h}^s_{|n|}(\theta)\,d\theta =$$

$$= I_1(n,m) + I_2(n,m) + I_3(n,m) + I_4(n,m).$$

Since $\hat{h}^c_{|n|}(\theta)$, $\cos(m\theta - \alpha(\theta))$ are even functions, and $\hat{h}^c_{|n|}(\theta)$, $\sin(m\theta - \alpha(\theta))$ are odd functions on $(-\pi,\pi)$, from here we get that $I_2(n,m) = I_3(n,m) = 0, \quad n,m = \pm1,\pm2,...$ So, we have

$$(\varphi_m, h_n) = I_1(n,m) + I_4(n,m) = \frac{1}{2}\int_0^{\pi}\cos(m\theta - sgnm\alpha(\theta))\,h^c_{|n|}(\theta)\,d\theta +$$

$$+\frac{sgnn}{2}\int_0^{\pi}\sin(m\theta - sgnm\alpha(\theta))\,h^s_{|n|}(\theta)\,d\theta = \delta_{nm}.$$

It is clear that $h_n(\theta) \in L_{q(\cdot)}(-\pi,\pi)$, $\quad n = \pm1,\pm2,....$ Hence the minimality of the system (7) was proved.

Now take any function $\psi(t) \in L_{p(\cdot)}(-\pi,\pi)$. Consider the following series:

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\int_{-\pi}^{\pi}\psi(t)\,h_n(t)\,dt e^{i[n\theta - sgnn\alpha(\theta)]}. \tag{18}$$

We show that this series converges to the function $\psi(t)$ in $L_{p(\cdot)}(-\pi,\pi)$. Let $S_N(\theta)$ be truncated sum of the series (18). Then

$$\|\psi(\theta) - S_N(\theta)\|_{L_{p(\cdot)}} = \left\|\psi(\theta) - \sum_{\substack{n=-N \\ n\neq 0}}^{N}\int_{-\pi}^{\pi}\psi(t)\,h_n(t)\,dt\exp\left[i(n\theta - sgnn\alpha(\theta))\right]\right\|_{L_{p(\cdot)}} =$$

$$= \left\|\psi(\theta) - \sum_{\substack{n=-N \\ n\neq 0}}^{N}\frac{1}{4}\left\{\int_{-\pi}^{\pi}\psi(t)\,\hat{h}^c_{|n|}(t)\,dt\,\cos(n\theta - sgnn\alpha(\theta)) +\right.\right.$$

$$+i\int_{-\pi}^{\pi}\psi(t)\,\hat{h}^{c}_{|n|}(t)\,dt\sin(n\theta-sgnn\alpha(\theta))-isgnn\int_{-\pi}^{\pi}\psi(t)\,\hat{h}^{s}_{|n|}(t)\,dt\cos(n\theta-sgnn\alpha(\theta))+$$

$$+sgnn\int_{-\pi}^{\pi}\psi(t)\,\hat{h}^{s}_{|n|}(t)\,dt\sin(n\theta-sgnn\alpha(\theta))\Big\}\Big\|_{L_{p(\cdot)}}=$$

$$=\Bigg\|\psi(t)-\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{1}{4}\left\{\int_{-\pi}^{\pi}\psi(t)\,\hat{h}^{c}_{|n|}(t)\,dt\,\cos(n\theta-sgnn\alpha(\theta))+\right.$$

$$\left.+sgnn\int_{-\pi}^{\pi}\psi(t)\,\hat{h}^{s}_{|n|}(t)\,dt\sin(n\theta-sgnn\alpha(\theta))\right\}\Bigg\|_{L_{p(\cdot)}}=$$

$$=\Bigg\|\psi(\theta)-\sum_{n=1}^{N}\frac{1}{2}\left\{\int_{-\pi}^{\pi}\psi(t)\,\hat{h}^{c}_{n}(t)\,dt\,\cos(n\theta-\alpha(\theta))+\right.$$

$$\left.+\int_{-\pi}^{\pi}\psi(t)\,\hat{h}^{s}_{n}(t)\,dt\sin(n\theta-\alpha(\theta))\right\}\Bigg\|_{L_{p(\cdot)}}=\Bigg\|\psi(\theta)-\frac{1}{2}\psi(-\theta)+\frac{1}{2}\psi(-\theta)-$$

$$-\sum_{n=1}^{N}\frac{1}{2}\left\{\int_{0}^{\pi}(\psi(t)+\psi(-t))\,h^{c}_{n}(t)\,dt\,\cos(n\theta-\alpha(\theta))+\right.$$

$$\left.+\int_{0}^{\pi}(\psi(t)+\psi(-t))\,h^{s}_{|n|}(t)\,dt\,\sin(n\theta-\alpha(\theta))\right\}\Bigg\|_{L_{p(\cdot)}}\leq$$

$$\leq\left\|\frac{1}{2}(\psi(\theta)+\psi(-\theta))-\sum_{n=1}^{N}\int_{0}^{\pi}\frac{1}{2}(\psi(t)+\psi(-t))\,h^{c}_{n}(t)\,dt\cos(n\theta-\alpha(\theta))\right\|_{L_{p(\cdot)}(-\pi,\pi)}+$$

$$+\left\|\frac{1}{2}(\psi(\theta)-\psi(-\theta))-\sum_{n=1}^{N}\int_{0}^{\pi}\frac{1}{2}(\psi(t)-\psi(-t))\,h^{s}_{n}(t)\,dt\sin(n\theta-\alpha(\theta))\right\|_{L_{p(\cdot)}(-\pi,\pi)}=$$

$$=\left\|\psi(\theta)+\psi(-\theta)-\sum_{n=1}^{N}\int_{0}^{\pi}(\psi(t)+\psi(-t))\,h^{c}_{n}(t)\,dt\cos(n\theta-\alpha(\theta))\right\|_{L_{p(\cdot)}(0,\pi)}+$$

$$+\left\|\psi(\theta)-\psi(-\theta)-\sum_{n=1}^{N}\int_{0}^{\pi}(\psi(t)-\psi(-t))\,h^{s}_{n}(t)\,dt\sin(n\theta-\alpha(\theta))\right\|_{L_{p(\cdot)}(0,\pi)}\to 0,$$

as $N\to\infty$.

It proves that the series (18) converges to the function $\psi(\theta)$ in $L_{p(\cdot)}(-\pi,\pi)$. Hence, under the condition (13) the system (7) forms a basis in $L_{p(\cdot)}(-\pi,\pi)$.

Now let us prove the necessity part of the theorem. Consider the case $\alpha(\pi) < -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi$. As in this case the system $\cos(n\theta - \alpha(\theta))$ $n = \overline{1,\infty}$, is not complete in $L_{p(\cdot)}(0,\pi)$, there exists a nontrivial function $\psi(\theta) \in L_{q(\cdot)}(0,\pi)$, $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$, such that

$$\int_0^\pi \psi(\theta)\cos(n\theta - \alpha(\theta))\,d\theta = 0, \quad n = \overline{1,\infty}.$$

Introduce the following function:

$$f(\theta) = \begin{cases} \psi(\theta), & \theta \in (0,\pi), \\ \psi(-\theta), & \theta \in (-\pi, 0). \end{cases}$$

Since $\cos(nt - \alpha(t))$, is odd function on $(-\pi,\pi)$, but $f(\theta)$ is even, we get that

$$\int_{-\pi}^\pi f(\theta)\, e^{i[n\theta - sgnn\alpha(\theta)]}d\theta = 0, \;\; n = \pm1, \pm2, ...,$$

which shows that the system (7) is not complete in $L_{p(\cdot)}(-\pi,\pi)$.

Now, consider the case of $\alpha(\pi) > -\frac{\pi}{2p(\pi)} + (n_r + 1)\pi$. In this case by Corollary 1 the system $\sin(n\theta - \alpha(\theta))$, $n = \overline{1,\infty}$, is not minimal. We show that the system of exponentials (7) is not minimal as well. If it is not, there is a system of functions $h_n(\theta) \in L_q(-\pi,\pi)$, $n = \pm1, \pm2, ...$, such that

$$\int_{-\pi}^\pi h_m(\theta)\, e^{i(n\theta - sgnn\alpha(\theta))}d\theta = \delta_{nm}, \;\; n, m = \pm1, \pm2, ...$$

Define the following system of functions:

$$h_n^s(\theta) = \frac{2}{i}\left[h_n(-\theta) - h_n(\theta)\right], \quad n \geq 1.$$

We have for any integers $n, m \geq 1$:

$$I(n,m) = \int_0^\pi h_m^s(\theta)(n\theta - \alpha(\theta))\,d\theta = -\frac{1}{2i}\int_0^\pi h_m^s(\theta)\, e^{-i(n\theta - \alpha(\theta))}d\theta +$$

$$+\frac{1}{2i}\int_0^\pi h_m^s(\theta)\exp\left[i(n\theta - \alpha(\theta))\right]d\theta = -\frac{1}{2i}\int_{-\pi}^0 h_m^s(-\theta)\, e^{-i(n\theta - \alpha(-\theta))}d\theta +$$

$$+\frac{1}{2i}\int_0^\pi h_m^s(\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta = \int_{-\pi}^0 h_m(\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta - \int_{-\pi}^0 h_m(-\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta -$$

$$-\int_0^\pi h_m(-\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta + \int_0^\pi h_m(\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta = \int_{-\pi}^\pi h_m(\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta -$$

$$-\int_0^\pi h_m(-\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta = \delta_{nm} + \int_{-\pi}^\pi h_m(\theta)\cdot e^{-i(n\theta - \alpha(\theta))}d\theta = \delta_{nm}.$$

Hence we got a contradiction. This completes the proof of Theorem 2.

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A Mixed Problem for a Class of Nonlinear Tymoshenko Systems

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Abstract. In this paper a mixed problem for semilinear systems of equations describing the oscillations of a thin-walled bar is considered. Reducing the problem under consideration to a differential equation, a theorem on local solvability is proved.

Key Words and Phrases: system of equations of a bar vibration, mixed problem, local solvability.

2010 Mathematics Subject Classifications: 35B40, 35G15, 49K20

Let us consider the bars described by a system of two differential equations in the domain $Q = [0,T] \times [0,l]$

$$\left.\begin{array}{r} EIy_{xxxx} + \rho A y_{tt} - \rho A e \theta_{tt} = f_1(t,x,y,\theta) \\ EC_w \theta_{xxxx} - GC\theta_{xx} - \rho A e y_{tt} + \rho\left(I + Ae^2\right)\theta_{tt} = f_2\left(t,x,y,\theta\right) \end{array}\right\} \tag{1}$$

with boundary conditions

$$\left.\begin{array}{c} y\left(0,t\right) = 0,\ y\left(l,\ t\right) = 0,\ y_{xx}\left(0,\ t\right) = 0\ ,\ \ y_{xx}\left(l,\ t\right) = 0 \\ \theta\left(0,t\right) = 0\ ,\ \theta\left(l,\ t\right) = 0, \quad \theta_{xx}\left(0,\ t\right) = 0\ , \quad \theta_{xx}\left(l,\ t\right) = 0 \end{array}\right\} \tag{2}$$

with initial conditions

$$\left.\begin{array}{ll} y\left(x,0\right) = y_0\left(x\right), & y_t\left(x,0\right) = y_1\left(x\right) \\ \theta\left(x,0\right) = \theta_0\left(x\right), & \theta_t\left(x,0\right) = \theta_1\left(x\right) \end{array}\right\} \tag{3}$$

where $0 < x < l$, $0 < t < T$, $l > 0$, $T > 0$ are given numbers, $y\left(x,t\right)$ is a transverse displacement, $\theta\left(x,t\right)$ is an angle of cross-section of the bar, E is the Young's modulus, I is a polar moment of inertia of the cross section with respect to its center of gravity, ρ is a density of the material of the bar, A is a cross-sectional area, e is a distance from center of gravity to center of torsion, C_w is a sectorial moment of inertia of the cross section, G is a shear modulus, C is a geometric rigidity of free torsion, EC_w is a stiffness of bending torsion, GC is a stiffness of free torsion. Here, f_1 and f_2 are functions depending on t, x, y and θ (see e.g. [1, 2]).

The system of equations (1), (2) can be written as follows

$$Rw_{tt} + Sw + Nw = F(t,\ x,\ y,\ \theta), \tag{4}$$

$$w\,(0) = w_0, \qquad w_t\,(0) = w_1 \tag{5}$$

where

$$R = \begin{pmatrix} \rho A & -\rho Ae \\ -\rho Ae & \rho\left(I + Ae^2\right) \end{pmatrix}, S = \begin{pmatrix} EI\partial^4 & 0 \\ 0 & EC_w\partial^4 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 \\ 0 & -GC\partial^2 \end{pmatrix},$$

$$w = \begin{pmatrix} y \\ \theta \end{pmatrix}, w_0 = \begin{pmatrix} y_0 \\ \theta_0 \end{pmatrix}, w_1 = \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

Let us consider the functional space $\mathscr{H} = L_2\,(0,1) \times L_2(0,1)$ with a scalar product:

$$\left\langle w^1, w^2\right\rangle = \left\langle w^1, w^2\right\rangle_{\mathscr{H}} = \frac{I}{C_w}\left\langle y^1, y^2\right\rangle_{L_2(0,1)} + \left\langle \theta^1, \theta^2\right\rangle_{L_2(0,1)},$$

where

$$w^i = \left(y^i, \theta^i\right) \in \mathscr{H}, \quad i = 1, 2.$$

Let us define $\hat{H}_0^2$ and $\hat{H}_0^4$ in the following way:

$$\hat{H}_0^2 = \left\{u \ : \ u \in H^2,\ u(0) = u(l) = 0\right\},$$

$$\hat{H}_0^4 = \left\{u : \ u \in H^4,\ u(0) = u(1) = u_{xx}(0) = u_{xx}(l) = 0\right\}.$$

Denote by $\mathscr{H}_1$ the space $\widehat{H_0^2} \times \widehat{H_0^2}$, and by $\mathscr{H}_2$ the space $\widehat{H_0^4} \times \widehat{H_0^4}$.

Let the operator L be defined in the space $\mathscr{H}$:

$$D\,(L) = \mathscr{H}.$$

$$Lw = R^{-1}Sw = \begin{bmatrix} \frac{E(I+Ae^2)}{\rho A}\frac{\partial^4}{\partial x^4} & \frac{eEC_w}{\rho I}\frac{\partial^4}{\partial x^4} \\ \frac{eE}{\rho}\frac{\partial^4}{\partial x^4} & \frac{EC_w}{\rho I}\frac{\partial^4}{\partial x^4} \end{bmatrix} w, \quad where \quad w = \begin{pmatrix} y \\ \theta \end{pmatrix} \in D(L).$$

We also define the linear operator L_1 as follows:

$$D\,(L_1) = \mathscr{H}_1.$$

$$L_1w = R^{-1}Cw = \begin{bmatrix} 0 & -\frac{eGC}{\rho I}\frac{\partial^2}{\partial x^2} \\ 0 & -\frac{GC}{\rho I}\frac{\partial^2}{\partial x^2} \end{bmatrix} w, \quad where \quad w = \begin{pmatrix} y \\ \theta \end{pmatrix} \in D(L_1) \in \mathscr{H}_1.$$

We define the nonlinear operator $G(.)$ in the following way

$$G\,(t,\ w) = \begin{pmatrix} g_1(t,x,w) \\ g_2(t,x,w) \end{pmatrix},$$

where

$$g_1(t,x,w)=\frac{I+Ae^2}{\rho AI}f_1(t,x,y,\theta)+\frac{e}{\rho I}f_2(t,x,y,\theta),$$

$$g_2(t,x,w)=\frac{e}{\rho I}f_1(t,x,y,\theta)+\frac{1}{\rho I}f_2(t,x,y,\theta).$$

Then the problem (4), (5) can be written in the form

$$w_{tt}+Lw+L_1w=G(t,\ w), \tag{6}$$

$$w(0)=w_0,\ w'(0)=w_1. \tag{7}$$

Lemma 1. *L is a positive self-adjoint operator in $\mathscr{H}$.*

Proof. Let $w^i=(y^i,\theta^i)\in D(L)$.

$$Lw^1=\left(\frac{E\left(I+Ae^2\right)}{\rho A}y^1_{xxxx}+\frac{eEC_w}{\rho I}\theta^1_{xxxx},\ \frac{eE}{\rho}y^1_{xxxx}+\frac{EC_w}{\rho I}\theta^1_{xxxx}\right).$$

Hence we obtain that

$$\left\langle Lw^1,w^2\right\rangle=\frac{I}{C_w}\left\langle\frac{E\left(I+Ae^2\right)}{\rho A}y^1_{xxxx}+\frac{eEC_w}{\rho I}\theta^1_{xxxx},y^2\right\rangle_{L_2(0,1)}+$$

$$+\left\langle\frac{eE}{\rho}y^1_{xxxx}+\frac{EC_w}{\rho I}\theta^1_{xxxx},\theta^2\right\rangle_{L_2(0,1)}=$$

$$=\frac{E\left(I+Ae^2\right)}{\rho C_wA}\left\langle y^1_{xx},y^2_{xx}\right\rangle_{L_2(0,1)}+\frac{eE}{\rho}\left\langle\theta^1_{xx},y^2_{xx}\right\rangle_{L_2(0,1)}+$$

$$+\frac{eE}{\rho}\left\langle y^1_{xx},\theta^2_{xx}\right\rangle_{L_2(0,1)}+\frac{EC_w}{\rho I}\left\langle\theta^1_{xx},\theta^2_{xx}\right\rangle_{L_2(0,1)}. \tag{8}$$

Similarly we obtain that

$$Lw^2=\left(\frac{E\left(I+Ae^2\right)}{\rho A}y^2_{xxxx}+\frac{eEC_w}{\rho I}\theta^2_{xxxx},\ \frac{eE}{\rho}y^2_{xxxx}+\frac{EC_w}{\rho I}\theta^2_{xxxx}\right).$$

$$\left\langle w^1,Lw^2\right\rangle=\frac{I}{C_w}\left\langle u^1,\frac{E\left(I+Ae^2\right)}{\rho A}y^2_{xxxx}+\frac{eEC_w}{\rho I}\theta^2_{xxxx}\right\rangle_{L_2(0,1)}$$

$$+\left\langle v^1,\frac{eE}{\rho}y^2_{xxxx}+\frac{EC_w}{\rho I}\theta^2_{xxxx}\right\rangle_{L_2(0,1)}=$$

$$=\frac{E\left(I+Ae^2\right)}{\rho C_wA}\left\langle y^1_{xx},y^2_{xx}\right\rangle_{L_2(0,1)}+\frac{eE}{\rho}\left\langle y^1_{xx},\theta^2_{xx}\right\rangle_{L_2(0,1)}+$$

$$+\frac{eE}{\rho}\left\langle \theta_{xx}^{1},\, y_{xx}^{2}\right\rangle_{L_2(0,1)} + \frac{EC_w}{\rho I}\left\langle \theta_{xx}^{1},\, \theta_{xx}^{2}\right\rangle_{L_2(0,1)}. \tag{9}$$

Comparing (8) and (9), we obtain that

$$\left\langle Lw^1,\, w^2\right\rangle = \left\langle w^1,\, Lw^2\right\rangle.$$

On the other hand, the operator L is invertible.

Indeed, let $h = (h_1,\, h_2) \in \mathscr{H}$. Consider the equation

$$Lw = h, \qquad w = (y, \theta) \in D(L). \tag{10}$$

Equation (10) has the following form

$$\begin{cases} \frac{E(I+Ae^2)}{\rho A} y_{xxxx} + \frac{eEC_w}{\rho I}\theta_{xxxx} = h_1, \\ \frac{eE}{\rho} y_{xxxx} + \frac{EC_w}{\rho I}\theta_{xxxx} = h_2. \end{cases} \tag{11}$$

Hence we obtain that

$$\begin{cases} \frac{EI}{\rho A} u_{xxxx} = h_1 - eh_2, \\ y\,(0) = y\,(l) = y_{xx}\,(0) = y_{xx}\,(l) = 0. \end{cases} \tag{12}$$

The problem (11) has a unique solution $y \in \widehat{H_0^4}$. Similarly we obtain that the problem (11) has a unique solution

$$w = (y, \theta)\,, \text{ where } y,\ \theta \in \widehat{H_0^4}, \quad i.e.\ \ w \in \mathscr{H}.$$

From the definition of L and from the scalar product in $\mathscr{H}$, we get that

$$\langle Lw,\, w\rangle = \frac{EI\left(I + Ae^2\right)}{\rho C_w A}\left\|y_{xx}\right\|_{L_2(0,1)}^2 + \frac{2eE}{\rho}\left\langle y_{xx},\, \theta_{xx}\right\rangle_{L_2(0,1)} + \frac{EC_w}{\rho I}\left\|\theta_{xx}\right\|_{L_2(0,1)}^2. \tag{13}$$

Using the Holder's and Young's inequality, we obtain that

$$\left|2e\left\langle y_{xx}, \theta_{xx}\right\rangle\right| = 2\left|\left\langle e\sqrt{\frac{I}{C_w}}y_{xx}, \sqrt{\frac{C_w}{I}}\theta_{xx}\right\rangle\right| \le e^2\frac{I}{C_w}\left\|y_{xx}\right\|_{L_2}^2 + \frac{C_w}{I}\left\|\theta_{xx}\right\|_{L_2}^2. \tag{14}$$

From (13) and (14) we obtain that

$$\langle Lw,\, w\rangle \geq 0.$$

Thus, L is a positive self-adjoint operator.

Lemma 2. *Linear operator* L_1 *is subjected to the operator* $L^{\frac{1}{2}}$.

Proof. From the definition of L_1 it follows that

$$\|Lw\|^2_{\mathscr{H}} = \frac{(e+1)G^2C^2}{\rho^2 I^2} \int_0^{\partial} \left|\frac{\partial^2\theta}{\partial x^2}\right|^2 dx \le c\left\| L^{\frac{1}{2}} w\right\|^2_{\mathscr{H}},$$

i.e. L_1 is subjected to the operator $L^{\frac{1}{2}}$.

Applying the general theory of nonlinear hyperbolic differential equations, we obtain.

Theorem 1. *Let L be a positive self-adjoint operator and L_1 is subjected to the operator $L^{\frac{1}{2}}$. Suppose that $G(t,\ w)$ acts from $[0,T]\times\mathscr{H}_1$ to $\mathscr{H}$ and satisfies the local Lipschitz condition, i.e. if for any $t_1,t_2\in[0,\ T]$ and $w^1,w^2\in\mathscr{H}_1$*

$$\left\|G\left(t_1,\ w^1\right)-G\left(\ t_2,\ w^2\right)\right\|_{\mathscr{H}} \le c\left(\left\|w^1\right\|_{\mathscr{H}_1},\ \left\|w^2\right\|_{\mathscr{H}_1}\right)\times\left[|t_1-t_2|+\ \left\|w^1-w^2\right\|_{\mathscr{H}_1}\right].$$

Then for any $w_0\in\mathscr{H}_1$, $w_1\in\mathscr{H}$ there exists T', such that the problem (6), (7) has a unique solution

$$w\in C(\left[0,T'\right],\mathscr{H}_1)\cap C^1(\left[0,T'\right],\mathscr{H}).$$

If T_{max} is the length of the maximum interval of existence of solutions, then one of the following alternatives is fulfilled

i) $\lim_{t\to T_{max}-0}\left[\left\|w'(t)\right\|_{\mathscr{H}}+\ \left\|w(t)\right\|_{\mathscr{H}_1}\right]\ =+\infty$

or

ii) $T_{max}=T$.

Note that if $w_0\in\mathscr{H}_0$ and $w_1\in\mathscr{H}_1$, then

$$w\in C(\left[0,T'\right],\mathscr{H}_0)\cap C^1(\left[0,T'\right],\mathscr{H}_1)\cap C^2(\left[0,T'\right],\mathscr{H}).$$

Lemma 3. *Let*

$$f_i\left(t,x,y,\theta\right)\in C^1([0,T]\times[0,l]\times R^2).$$

Then $G\left(t,\ w\right)=\begin{pmatrix} g_1(t,x,w)\\ g_2(t,x,w)\end{pmatrix}$ *acts from $\mathscr{H}_1$ to $\mathscr{H}$ and satisfies the local Lipschitz condition.*

Proof. Let $t_i\in[0,T]$, $w^i=(y^i,\theta^i)\in\mathscr{H}$. Then

$$\left\|G\left(t_1,\ w^1\right)-G\left(\ t_2,\ w^2\right)\right\|^2_{\mathscr{H}}\le$$

$$\le c\left\|f_1\left(t_1,x,y^1,\theta^1\right)-f_2\left(t_2,x,y^2,\theta^2\right)\right\|^2_{L_2(0,l)}+\ c\left\|f_2\left(t_2,x,y^2,\theta^2\right)\right\|^2_{L_2(0,l)},$$

where $c=max\left\{\frac{I+Ae+Ae^2}{\rho AI},\frac{e+1}{\rho I}\right\}$, on the other hand

$$\left\|f_1\left(t_1,x,y^1,\theta^1\right)-f_2\left(t_2,x,y^2,\theta^2\right)\right\|^2_{L_2(0,l)}=$$

$$=\int_0^l\left|\int_0^1 f'_{1_t}\left(t_1+\tau\left(t_2-t_1\right),y^1+\tau\left(y^2-y^1\right),\theta^1+\tau\left(\theta^2-\theta^1\right)\right)d\tau\right|^2 dx\left|t_1-t_2\right|+$$

$$+\int_0^l \left|\int_0^1 f'_{1_u}\left(t_1+\tau\left(t_2-t_1\right), y^1+\tau\left(y^2-y^1\right), \theta^1+\tau\left(\theta^2-\theta^1\right)\right) d\tau\right|^2 \left|y^1-y^2\right|^2 dx+$$

$$+\int_0^l \left|\int_0^1 f'_{1_u}\left(t_1+\tau\left(t_2-t_1\right), y^1+\tau\left(y^2-y^1\right), \theta^1+\tau\left(\theta^2-\theta^1\right)\right) d\tau\right|^2 \left|\theta^1-\theta^2\right|^2 dx \le$$

$$\le \sup_{\substack{0\le t\le T\\ x\in[0,l]\\ |\xi|\le r_0\\ |\eta|\le r_1}} [\left|f_{1_t}\left(t_1,x,\xi,\eta\right)\right|+\left|f_{1_t}\left(t_1,x,\xi,\eta\right)\right|+\left|f_{1_t}\left(t_1,x,\xi,\eta\right)\right|]\times$$

$$\times[l\left|t_1-t_2\right|+\int_0^l\left|y^1(x)-y^2(x)\right|^2dx+\int_0^l\left|\theta^1(x)-\theta^2(x)\right|^2dx].$$

Hence we obtain that

$$\left\| f_1\left(t_1,x,y^1,\theta^1\right)-f_2\left(t_2,x,y^2,\theta^2\right)\right\|^2_{L_2(0,l)} \le$$

$$\le c\left(\left\|y^1\right\|_{\mathscr{H}_1},\left\|y^2\right\|_{\mathscr{H}_1},\left\|\theta^1\right\|_{\mathscr{H}_1},\left\|\theta^2\right\|_{\mathscr{H}_1}\right)\times[\left|t_1-t_2\right|+\left\| y^1-y^2\right\|^2_{L_2(0,l)}+\left\| \theta^1-\theta^2\right\|^2_{L_2(0,l)}] \le$$

$$\le c\left(\left\|w^1\right\|_{\mathscr{H}_1},\left\|w^2\right\|_{\mathscr{H}_1}\right)\cdot[\left|t_1-t_2\right|^2+\left\| w^1-w^2\right\|^2_{\mathscr{H}_1}],$$

where

$$r_0=\max_{x\in[0,l]}[\left|y^1(x)\right|+\left|y^2(x)\right|$$

$$r_1=\max_{x\in[0,l]}[\left|\theta^1(x)\right|+\left|\theta^2(x)\right|$$

Using Lemmas 1-3 from the Theorem 1, we obtain the following result:

Theorem 2. *Let*

$$f_i\left(t,x,y,\theta\right)\in C^1([0,T]\times[0,l]\times R^2).$$

Then for any $y_0,\ \theta_0\in\widehat{H_0^2},\quad y_1,\ \theta_1\in L_2\left(0,1\right)$ *there exists* $T'>0$ *, such that the problem* (1) -(3) *has a unique solution* (y,θ) *,* *where*

$$y,\theta\in C^1(\left[0,T'\right],L_2(0,1))\cap C(\left[0,T'\right],\widehat{H_0^2}).$$

Moreover, if T_{max} *is the length of the maximum interval of existence of solutions, then one of the following alternatives is fulfilled*

i)$\lim_{t\to T_{max}-0}\left[\left\| y_t(t,\cdot)\right\|^2_{L_2(0,l)}+\ \left\| \theta_t(t,\cdot)\right\|^2_{L_2(0,l)}+\left\| y(t,\cdot)\right\|^2_{\widehat{H_0^2}}+\left\| \theta(t,\cdot)\right\|^2_{\widehat{H_0^2}(0,l)}\right]\ =$ $+\infty$

or

ii)$T_{max}=T$.

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On Embedding Theorem in Variable Lebesgue Spaces with Mixed Norm

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Abstract. In this paper, we study theorem on continuously embedding between variable exponent Lebesgue spaces with mixed norm. In particular, we found a criterion characterizing the embedding between variable exponent Lebesgue spaces with mixed norm.

Key Words and Phrases: variable exponent Lebesgue spaces, variable exponent Lebesgue spaces with mixed norm, embedding theorem.

2010 Mathematics Subject Classifications: 46E30, 42B35

1. Introduction

It is well known that the variable Lebesgue space in the literature for the first time was studied by Orlicz [11] in 1931. In [11], Hölder's inequality for variable discrete Lebesgue space was proved. Orlicz also considered the variable Lebesgue space on the real line, and proved the Hölder inequality in this setting. However, this paper is essentially the only contribution of Orlicz to the study of the variable Lebesgue spaces (see also [8]). The next step in the development of the variable Lebesgue spaces came two decades later in the work of Nakano [9] and [10]. Somewhat later, a more explicit version of such spaces, namely modular function spaces, were investigated by Musielak and others Polish mathematicians (see [7]). In particular, the variable Lebesgue spaces were objects of interest during the last two decades(see [4, 5]). The further investigation of these spaces being undertaken in [6, 13, 14] and e.t.c. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [4, 12, 15]).

In this paper, we study theorem on continuously embedding between variable exponent Lebesgue spaces with mixed norm. In particular, we found a criterion characterizing the embedding between variable exponent Lebesgue spaces with mixed norm.

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. The main results are stated and proved in Section 3.

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2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x=(x_1,...,x_n)$ and let Ω be a Lebesgue measurable subset in $\mathbb{R}^n$. Suppose that $\mathbf{p}(x)=(p_1(x_1,\dots,x_n), p_2(0,x_2,\dots,x_n), \dots, p_n(0,\dots,0,x_n))$ is a vector function defined on $\mathbb{R}^n$ with Lebesgue measurable components $p_i\left(x^{(i)}\right)$, such that $1\le p_i(x^{(i)})<\infty$ and $x^{(i)}=(0,\dots,0,x_i,\dots,x_n)$ $(i=1,\dots,n)$. Further in this paper all sets and functions are supposed to be Lebesgue measurable and $x^{(1)}=x$, $x^{(n)}=(0,\dots,0,x_n)$. Throughout this paper $\underline{p}_i=\operatorname*{ess\,inf}_{x^{(i)}\in\mathbb{R}^n} p_i\left(x^{(i)}\right)$, $\overline{p}_i=\operatorname*{ess\,sup}_{x^{(i)}\in\mathbb{R}^n} p_i\left(x^{(i)}\right)$, $\underline{q}_i=\operatorname*{ess\,inf}_{x^{(i)}\in\mathbb{R}^n} q_i\left(x^{(i)}\right)$, $\overline{q}_i=\operatorname*{ess\,sup}_{x^{(i)}\in\mathbb{R}^n} q_i\left(x^{(i)}\right)$ and $p_n\left(x^{(n)}\right)=p_n(x_n)$. We denote by $\mathbf{p}'(x)=\left(p_1'(x), p_2'\left(x^{(2)}\right),\dots,p_n'\left(x^{(n)}\right)\right)$ the conjugate exponent vector-function defined by
$\dfrac{1}{\mathbf{p}(x)}+\dfrac{1}{\mathbf{p}'(x)}=1$, $x\in\mathbb{R}^n$, i.e. $\dfrac{1}{p_i\left(x^{(i)}\right)}+\dfrac{1}{p_i'\left(x^{(i)}\right)}=1$, $i=1,\dots,n$.

By $L_{p_1(x),x_1}(\mathbb{R}^n)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_1>0$

$$I_{1,p_1}f(x_2,\dots,x_n)=\int\limits_{\mathbb{R}}\left(\frac{|f(x)|}{\lambda_1}\right)^{p_1(x)}dx_1<\infty.$$

The expression

$$\|f\|_{L_{p_1(x),x_1}(\mathbb{R}^n)}=\|f\|_{p_1(\cdot),x_1}=\inf\left\{\lambda>0:\ \int\limits_{\mathbb{R}}\left(\frac{|f(x)|}{\lambda}\right)^{p_1(x)}dx_1\le 1\right\}$$

is the norm in $L_{p_1(x),x_1}(\mathbb{R}^n)$ with respect to the variable x_1. It is obvious that the result is a function of variables $x_2,\dots,x_n$, i.e. $\|f\|_{p_1(\cdot),x_1}=\|f\|_{p_1(\cdot),x_1}(x_2,\dots,x_n)$.

Further, by $L_{\left(p_1(x),p_2\left(x^{(2)}\right)\right),x_1,x_2}(\mathbb{R}^n)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_2>0$

$$I_{2,p_2}f(x_3,\dots,x_n)=\int\limits_{\mathbb{R}}\left(\frac{\|f\|_{p_1(\cdot),x_1}(x_2,\dots,x_n)}{\lambda_2}\right)^{p_2\left(x^{(2)}\right)}dx_2<\infty.$$

The expression

$$\|f\|_{L_{\left(p_1(x),p_2\left(x^{(2)}\right)\right),x_1,x_2}(\mathbb{R}^n)}=\left\|\|f\|_{p_1(\cdot),x_1}\right\|_{p_2(\cdot),x_2}$$

$$=\inf\left\{\mu>0:\ \int\limits_{\mathbb{R}}\left(\frac{\|f\|_{p_1(\cdot),x_1}(x_2,\dots,x_n)}{\mu}\right)^{p_2\left(x^{(2)}\right)}dx_2\le 1\right\}$$

is the norm in $L_{\left(p_1(x),p_2\left(x^{(2)}\right)\right),x_1,x_2}(\mathbb{R}^n)$. It is obvious that the result is a function of variables $x_3,\dots,x_n$.

Definition 1. *By* $L_{\mathbf{p}(x)}(\mathbb{R}^n) = L_{\left(p_1(x),\, p_2\left(x^{(2)}\right),\ldots,p_n(x_n)\right)}(\mathbb{R}^n)$ *we denote the space of measurable functions* f *on* $\mathbb{R}^n$ *such that for some* $\lambda_n > 0$

$$I_{n,p_n} f = \int\limits_{\mathbb{R}} \left(\frac{\left\| \cdots \left\| \|f\|_{p_1(\cdot),x_1} \right\|_{p_2(\cdot),\, x_2} \cdots \right\|_{p_{n-1}(\cdot),\, x_{n-1}} (x_n)}{\lambda_n} \right)^{p_n(x_n)} dx_n < \infty.$$

The expression

$$\|f\|_{L_{\mathbf{p}(x)}(\mathbb{R}^n)} = \left\| \cdots \left\| \|f\|_{p_1(\cdot),x_1} \right\|_{p_2(\cdot),\, x_2} \cdots \right\|_{p_n(\cdot),\, x_n}$$

$$= \inf \left\{ \nu > 0 : \int\limits_{\mathbb{R}} \left(\frac{\left\| \cdots \left\| \|f\|_{p_1(\cdot),x_1} \right\|_{p_2(\cdot),\, x_2} \cdots \right\|_{p_{n-1}(\cdot),\, x_{n-1}} (x_n)}{\nu} \right)^{p_n(x_n)} dx_n \le 1 \right\}$$

defines a norm in $L_{\mathbf{p}(x)}(\mathbb{R}^n)$.

Remark 1. *Let* $\mathbf{p}(x) = (p_1, \ldots, p_n) = \mathbf{p} \ge \mathbf{1}$, *i.e.* $1 \le p_i\left(x^{(i)}\right) = p_i = const$, $i = 1, \ldots, n$. *It is well known that usually Lebesgue spaces with mixed norm was introduced and studied in [3]. The variable Lebesgue spaces with mixed norm was introduced and studied in [1] and [2].*

Suppose that $\Omega \subset R^n$ is a measurable set and $f : \Omega \mapsto \mathbb{R}$. The norm in the space $L_{\mathbf{p}(x)}(\Omega)$ is defines as

$$\|f\|_{L_{\mathbf{p}(x)}(\Omega)} = \|f\chi\Omega\|_{L_{\mathbf{p}(x)}(\mathbb{R}^n)},$$

where $\chi_\Omega(x)$ is a characteristic function of a set Ω.

Remark 2. *Let* $\mathbf{p}(x) = (p_1, \ldots, p_n) = \mathbf{p} \ge \mathbf{1}$, *i.e.* $1 \le p_i\left(x^{(i)}\right) = p_i = const$, $i = 1, \ldots, n$. *Then* $L_{\mathbf{p}(x)}(\mathbb{R}^n)$ *coincides with the usual mixed norm Lebesgue spaces.*

Remark 3. *Let* $p_1(x_1, \ldots, x_n) = p_2\left(x^{(2)}\right) = \ldots = p_n\left(x^{(n)}\right) = p(x_n)$, *i.e.* $\mathbf{p}(x) = (p(x_n), \ldots, p(x_n))$. *Then* $L_{\mathbf{p}(x)}(\mathbb{R}^n) = L_{p(x_n)}(\mathbb{R}^n)$.

Now we introduce a analog of generalized Hölder inequality in variable Lebesgue space with mixed norm.

Lemma 1. *Let* $\mathbf{p}(x) = (p_1(x), \ldots, p_n(x_n))$, $\mathbf{q}(x) = (q_1(x), \ldots, q_n(x_n))$ *and* $\mathbf{r}(x) = (r_1(x), \ldots, r_n(x_n))$. *Suppose that* $1 \le \underline{p}_i \le p_i\left(x^{(i)}\right) \le q_i\left(x^{(i)}\right) \le \overline{q}_i < \infty$ *and*
$\dfrac{1}{r_i\left(x^{(i)}\right)} = \dfrac{1}{p_i\left(x^{(i)}\right)} - \dfrac{1}{q_i\left(x^{(i)}\right)}$, $i = 1, \ldots, n$.

Then the inequality

$$\|fg\|_{L_{\mathbf{p}(\cdot)}(\Omega)} \le \prod_{i=1}^{n} \left(A_i + B_i + \left\|\chi_{\Omega_{2,i}}\right\|_{L_\infty(\Omega)} \right)^{1/\underline{p}_i} \|f\|_{L_{\mathbf{q}(\cdot)}(\Omega)} \|g\|_{L_{\mathbf{r}(\cdot)}(\Omega)}$$

holds for any $f \in L_{\mathbf{q}(x)}(\Omega)$, $g \in L_{\mathbf{r}(x)}(\Omega)$, *where* $\Omega_{1,i} = \left\{x \in \Omega : p_i\left(x^{(i)}\right) < q_i\left(x^{(i)}\right)\right\}$, $\Omega_{2,i} = \left\{x \in \Omega : p_i\left(x^{(i)}\right) = q_i\left(x^{(i)}\right)\right\}$, $A_i = \sup\limits_{x\in\Omega_{1,i}} \dfrac{p_i\left(x^{(i)}\right)}{q_i\left(x^{(i)}\right)}$ *and*
$B_i = \sup\limits_{x\in\Omega_{1,i}} \dfrac{q_i\left(x^{(i)}\right) - p_i\left(x^{(i)}\right)}{q_i\left(x^{(i)}\right)}$.

Remark 4. *Note that in the case* $p_1(x) = p_2\left(x^{(2)}\right) = \ldots = p_n(x_n) = 1$, *Theorem Lemma 2.1 was proved in [1]. The proof of Lemma 2.1 is similar, but with some modifications (see [2]).*

3. Main results

Now we introduce an embedding theorem between variable Lebesgue spaces with mixed norm.

Theorem 1. *Let* $\mathbf{p}(x) = (p_1(x), \ldots, p_n(x_n))$, $\mathbf{q}(x) = (q_1(x), \ldots, q_n(x_n))$ *and* $\mathbf{r}(x) = (r_1(x), \ldots, r_n(x_n))$. *Suppose that* $1 \le \underline{p}_i \le p_i\left(x^{(i)}\right) \le q_i\left(x^{(i)}\right) \le \overline{q}_i < \infty$ *and satisfy condition* $\dfrac{1}{r_i\left(x^{(i)}\right)} = \dfrac{1}{p_i\left(x^{(i)}\right)} - \dfrac{1}{q_i\left(x^{(i)}\right)}$, $i = 1, \ldots, n$.

Then the following conditions are equivalent

a) $L_{\mathbf{q}(x)}(\Omega) \hookrightarrow L_{\mathbf{p}(x)}(\Omega)$;

b) $\|1\|_{L_{\mathbf{r}(\cdot)}(\Omega)} = \left\|\ldots\|\chi\Omega\|_{r_1(\cdot),x_1}\ldots\right\|_{r_n(\cdot),x_n} < \infty$.

Proof. The implication $b) \Rightarrow a)$ immediately implies from Lemma 2.1. Indeed, if we take $g = 1$ in Lemma 2.1 we proved this implication. The proof of implication $a) \Rightarrow b)$ is similar to the case $p_1(x_1, \ldots, x_n) = p_2\left(x^{(2)}\right) = \ldots = p_n\left(x^{(n)}\right) = p(x_n)$ (see [5]).

Now we introduce some particular case of Theorem 3.1.

Let $I = \{x \in \mathbb{R}^n : -\infty \le a_i \le x_i \le b_i \le \infty, i = 1, 2, \ldots, n\}$.

Corollary 1. *[2] Let* $x \in I$, *and let* $\mathbf{p}(x) = (p_1(x), \ldots, p_n(x_n))$ *and* $\mathbf{q}(x) = (q_1(x), \ldots, q_n(x_n))$ *be a vector-functions such that* $1 \le \underline{p}_i \le p_i\left(x^{(i)}\right) \le q_i\left(x^{(i)}\right) \le \overline{q}_i < \infty$. *Suppose that satisfy the following conditions*

$$A_i = \sup_{x^{(i+1)}} \int_{a_i}^{b_i} \left(q_i\left(x^{(i)}\right) - p_i\left(x^{(i)}\right)\right) dx_i < \infty, \quad i = 1, \ldots, n-1,$$

$$A_n = \int_{a_n}^{b_n} (q_n(x_n) - p_i(x_n))) \, dx_n < \infty.$$

Then $L_{\mathbf{q}(x)}(I) \hookrightarrow L_{\mathbf{p}(x)}(I)$ *and the inequality*

$$\|f\|_{L_{\mathbf{p}(\cdot)}(I)} \le \prod_{i=1}^{n} \left[B_i(p_i, q_i)\right]^{\gamma_i} \|f\|_{L_{\mathbf{q}(\cdot)}(I)}$$

holds, where $B_i(p_i,q_i)=\dfrac{1}{s_i}+\dfrac{A_i}{\underline{q}_i}$, $s_i=\operatorname*{ess\,inf}_{x^{(i)}}\dfrac{q_i\left(x^{(i)}\right)}{p_i\left(x^{(i)}\right)}$ *and* $\gamma_i=\begin{cases}\frac{1}{\underline{p}_i}, & \text{for } B_i(p_i,q_i)\geq 1\\ \frac{1}{\overline{p}_i}, & \text{for } B_i(p_i,q_i)\leq 1.\end{cases}$

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Global Bifurcation for Half-linearizable Sturm-Liouville Problems with Spectral Parameter in the Boundary Condition

G.M. Mamedova

Abstract. We consider half-linearizable Sturm-Liouville problems with spectral parameter in the boundary condition. We study the structure of the set of bifurcation points and the behaviour of global sets of solutions of this problem bifurcating from the points of the line of trivial solutions.

Key Words and Phrases: half-linearizable Sturm-Liouville problems, half-eigenvalue, half-eigenfunction, bifurcation point, global sets of solutions.

2010 Mathematics Subject Classifications: 34B15, 34B24, 34C23, 47J10, 47J15

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1. Introduction

In the present paper, we continue the study [2] of the boundary value problem

$$\ell(y) \equiv -\left(p(x)y'\right)' + q(x)y = \lambda r(x)y + h(x, y, y', \lambda),\ x \in (0, \pi), \tag{1}$$

$$b_0 y(0) = d_0 y'(0), \tag{2}$$

$$(a_1\lambda + b_1)y(\pi) = (c_1\lambda + d_1)y'(\pi), \tag{3}$$

where λ is a real parameter, the functions $p \in C^1[0, \pi],\ q, r \in C^0[0, \pi]$, and $b_0,\ d_0,\ a_1,\ b_1,\ c_1,\ d_1$ are real numbers such that $|b_0| + |d_0| > 0$ and

$$a_1 d_1 - b_1 c_1 > 0. \tag{4}$$

We also assume that p and r are strictly positive on $[0, \pi]$. The nonlinear term h has a representation $h = \alpha y^+ + \beta y^- + g$, where α, β are the continuous functions on $[0, \pi]$, $y^+ = \max\{y, 0\}$, $y^- = \max\{-y, 0\}$, and g is a continuous function on $[0, \pi] \times \mathbb{R}^3$, satisfying the condition:

$$g(x, u, s, \lambda) = o(|u| + |s|), \tag{5}$$

near $(u, s) = (0, 0)$, uniformly in $x \in [0, \pi]$ and in $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

The purpose of this paper is to study the structure of the set of bifurcation points on real axis and more accurately describe the structure and behaviour of bifurcation branches of solutions of problem (1)-(3).

In nonlinear analysis an important role is played bifurcation theory of nonlinear eigenvalue problems. The study of nonlinear eigenvalue problems has an applied interest since problems of this type arise in the theory of vibrations, thermal convection theory, hydrodynamics, the theory of critical modes of operation of nuclear and chemical reactors, the theory of critical loads and the theory of elasticity (see, for example, [7, 9, 10]).

Bifurcation problems for nonlinear Sturm-Liouville problems when the spectral parameter is not involved in the boundary conditions was considered by many authors (see [2, 3, 6, 12, 13, 15]). In these papers prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1$ corresponding to the usual nodal properties and emanating from bifurcation points or bifurcation intervals (in $\mathbb{R} \times \{0\}$ which we identify with $\mathbb{R}$) surrounding the eigenvalues of the corresponding linear problem. It should be noted that in a recent paper [1] of the first author obtained similar results for nonlinear eigenvalue problems for ordinary differential equations of fourth order.

In [3] was also studied problem (1)-(3) in the case of $a_1 = c_1 = 0$ where shown that for this problem possessing different linearizations as $y \to 0^+$ and $y \to 0^-$, the half-eigenvalues of the half-linear problem (1)-(3) with $a_1 = c_1 = 0$ and $g \equiv 0$ correspond to bifurcation points in a global sense.

Problem (1)-(3) in a more general case (i.e. when the nonlinear term h is of the form $h = f + g$, f being continuous and satisfying the condition $|f(x, u, s, \lambda)| < M|u|$ in a neighborhood of $u = s = 0$, uniformly in $x \in [0, \pi]$ and in $\lambda \in \Lambda$) was considered in [2]. In this paper prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1$ emanating from bifurcation intervals surrounding the eigenvalues of the linear problem obtained from (1)-(3) by setting $h \equiv 0$.

In [5] Browne uses the Prüfer angle techniques for half-eigenvalue problem (1)-(3) with $g \equiv 0$ obtain the existence of two sequences of half-eigenvalues which are different according to the sign of the corresponding half-eigenfunctions in a neighborhood of 0. He studies also oscillatory properties of the corresponding half-eigenfunctions, but in the case $c_1 \neq 0$ could not accurately determine the serial numbers of the half-eigenfunctions which have the same number of zeros in the interval $(0, 1)$. And it is prevents to detailed study of global bifurcation of solutions of problem (1)-(3) in the case $c_1 \neq 0$.

By applying the results of works [2, 5, 8, 11] (in the case of $c_1 \neq 0$ for additional restrictions on the functions $\alpha(x)$ and $\beta(x)$) we update the oscillatory properties of corresponding half-linear problem (1)-(3) with $g \equiv 0$, and show that the set of bifurcation points of problem (1)-(3) consists of all half-eigenvalues of problem (1)-(3) with $g \equiv 0$. Using the approximation technique from [3] and combining it with the global bifurcation results in [2, 8, 11] we prove the existence of global sets of solutions emanating from bifurcation points which are similar to those obtained in [3].

2. Preliminary

Along with problem (1)-(3), we consider the following boundary value problem

$$\begin{cases} \ell(y) = \lambda r(x)y + \alpha(x)y^+ + \beta(x)y^-, \; x \in (0, \pi), \\ b_0 y(0) = d_0 y'(0), \\ (a_1\lambda + b_1)y(\pi) = (c_1\lambda + d_1)y'(\pi), \end{cases} \tag{6}$$

The problem (6) is non-linear, but is positively homogeneous (in the sense that if y is a solution of this problem, then αy is also a solution for all $\alpha > 0$) and linear in the cones $y > 0$ and $y < 0$. Hence nonlinear eigenvalue problems of this type called "half-linear" by Berestycki [3].

The following definitions are given by Berestycki [3] (see also [14]). We say that λ is a half-eigenvalue of problem (6) if there exists a nontrivial solution (λ, y_λ) of this problem; the function y_λ be called a half-eigenfunction. In this situation, the set $\{(\lambda, ty_\lambda) : t > 0\}$ is a half-line of nontrivial solutions of problem (6). The number λ is said to be simple if all solutions (λ, v) of (6) with v and y_λ, having the same sign in a deleted neighborhood of 0, are on this half-line. There may exist another half-line of solutions $\{(\lambda, v_\lambda) : t > 0\}$, but then we say that λ is simple if v_λ and y_λ have different signs in a deleted neighborhood of 0, and all solutions (λ, v) of problem (6) lie on these two half-lines.

From now on ν will denote an element of $\{+, -\}$ that is, either $\nu = +$ or $\nu = -$.

Half-linear problem (6) in the case $|a_1| + |c_1| > 0$ was investigated in [5], where the author shows that for each ν there exists infinitely increasing sequence $\{\lambda_k^\nu\}_{k=1}^\infty$, of real and simple half-eigenvalues of this problem. The corresponding half-eigenfunctions $y_k^\nu(x)$, $k = 1\,2, \ldots,$ have the following properties:

(i) $\nu y_k^\nu > 0$ in a deleted neighborhood of 0;

(i) if $c_1 = 0$, then function $y_k^\nu(x)$, $k \in \mathbb{N}$, has exactly $k-1$ simple nodal zeros in $(0, \pi)$;

(iii) if $c_1 \neq 0$, then $y_k^\nu(x)$ has exactly $k-1$ simple nodal zeros for $k \leq N_0^\nu$, and exactly $k-2$ simple nodal zeros for $k > N_0^\nu$ in the interval $(0, \pi)$, where a positive integer N_0^ν is determined from the inequality $\mu_{N(0)-1}^\nu < -\frac{d_1}{c_1} \leq \mu_{N(0)}^\nu$; μ_k^ν, $k \in \mathbb{N}$, is the kth half-eigenvalue of equation (1) with the boundary conditions (2) and $y(\pi) = 0$ (by a nodal zero we mean the function changes sign at the zero and at a simple nodal zero, the derivative of the function is nonzero).

For $c_1 \neq 0$ let N_0 be an integer such that $\tau_{N_0-1} < -\frac{d_1}{c_1} \leq \tau_{N_0}$, where τ_k, $k \in \mathbb{N}$, is the kth eigenvalue of the Sturm-Liouville equation $\ell(y) = \lambda r(x)y$, $x \in (0, \pi)$, with the boundary conditions (2) and $y(\pi) = 0$; here we take $\tau_0 = -\infty$.

We also consider the following eigenvalue problem

$$\begin{cases} \ell(y) = \lambda r(x)y, \; x \in (0, \pi), \\ b_0 y(0) = d_0 y'(0), \\ (a_1\lambda + b_1)y(\pi) = (c_1\lambda + d_1)y'(\pi). \end{cases} \tag{7}$$

It is known [4] that the eigenvalues of problem (7) are real, simple, and form an infinitely increasing sequence $\{\mu_k\}_{k=1}^\infty$. The corresponding eigenfunctions $v_k(x)$, $k = 1, 2, \ldots,$ have the following oscillation properties : (a) if $c_1 = 0$, then $v_k(x)$, $k \in \mathbb{N}$, has exactly $k-1$ simple nodal zeros in $(0,\pi)$; (b) if $c_1 \neq 0$, then $v_k(x)$ has exactly $k-1$ simple nodal zeros for $k \leq N_0$, and exactly $k-2$ simple nodal zeros for $k > N_0$ in the interval $(0,\pi)$.

Should be noted that in [5] for the case $c_1 \neq 0$ is not given the connection between the natural numbers N_0^ν, $\nu \in \{+,-\}$ and N_0. Therefore, in this case, applying the global bifurcation result from [2] (see [2, Theorem 3.4]), it is impossible to study the structure of all the bifurcation branches of the solutions of problem (1)-(3).

3. Global bifurcation of solutions of problem (1)-(3)

Let E be the Banach space of all continuously differentiable functions on $[0,\pi]$ which satisfy the boundary condition (2). E is equipped with its usual norm $||y||_1 = \max_{x\in[0,\pi]} |y(x)| + \max_{x\in[0,\pi]} |y'(x)|$. Let S_k^+ be the set of functions $y \in E$ which have exactly $k-1$ simple nodal zeros in $(0,\pi)$ and which are positive near $x=0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. The sets S_k^+ and S_k^- are disjoint and open in E. We say that $(\lambda, 0)$ is a bifurcation point of (1)-(3) with respect to the set $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, if in every small neighborhood of this point there is a solution to this problem which is contained in $\mathbb{R} \times S_k^\nu$.

Let $J_k = [\lambda_k - \frac{M}{r_0}, \lambda_k + \frac{M}{r_0}]$, $k \in \mathbb{N}$. For $c_1 = 0$ let $I_k = J_k,\ k \in \mathbb{N}$, and for $c_1 \neq 0$ let

$$I_k = \begin{cases} \tilde{J}_k, & \text{if } k \neq N_0, \\ [\lambda_{N_0} - \frac{M}{r_0}, \lambda_{N_0+1} + \frac{M}{r_0}], & \text{if } k = N_0, \end{cases}$$

where $M = \max_{x\in[0,\pi]} \{|\alpha(x)| + |\beta(x)|\}$, $r_0 = \min_{x\in[0,\pi]} r(x)$ and $\tilde{J}_k = \begin{cases} J_k, & \text{if } k < N_0, \\ J_{k+1}, & \text{if } k > N_0. \end{cases}$

It is obvious that

$$|\alpha(x)y^+(x) + \beta(x)y^-(x)| \leq M|y(x)|,\ x \in [0,\pi].$$

Hence for the boundary value problem (1)-(3) the assertions of section 3 of the work [2] is true. Therefore, for this problem have the following results.

Lemma 1. *The set of bifurcation points of problem* (1)-(3) *is nonempty.*

Lemma 2. *If* $(\lambda, 0)$ *is a bifurcation point of* (1)-(3)*, then* λ *is an half-eigenvalue of problem* (6)*.*

Proof. Let $(\lambda_n, y_n) \in \mathbb{R} \times E$, $y_n \neq 0$, be a sequence of solutions of problem (1)-(3) converging to $(\lambda, 0)$. Let $v_n = \frac{y_n}{||y_n||_1}$. Then dividing (1)-(3) by $||y_n||_1$ and setting

$$v_n(x) = \frac{y_n(x)}{||y_n||_1} \quad \text{and} \quad g_n(x) = \frac{g(x, y_n(x), y_n'(x), \lambda_n)}{||y_n||_1}$$

we have

$$\begin{cases} \ell(v_n)(x) = \lambda_n v_n(x) + \alpha(x) v_n^+(x) + \beta(x) v_n^-(x) + g_n(x), x \in (0,\pi), \\ v_n \in BC_\lambda \ , \end{cases} \tag{8}$$

where denote by BC_λ the set of boundary conditions (2)-(3). Since $\{v_n\}_{n=1}^\infty$ is bounded in $C^1[0,\pi]$, α, β are bounded in $C^0[0,\pi]$, and $g_n \to 0$ in $C^0[0,\pi]$ (by (5)), it follows from (8) that $\{v_n\}_{n=1}^\infty$ is bounded in $C^2[0,\pi]$. Therefore, by the Arzela-Ascoli theorem, we may assume that $v_n \to v$ in $C^1[0,\pi]$, $||v||_1 = 1$, and thus also $v_n \to v$ in $C^2[0,\pi]$ by equation (8). Consequently, by passing to the limit as $n \to \infty$ in (8) we obtain

$$\begin{cases} \ell(v)(x) = \lambda v(x) + \alpha(x) v^+(x) + \beta(x) v^-(x), x \in (0,\pi), \\ v \in BC_\lambda \ . \end{cases}$$

The proof of this lemma is complete.

As an immediate consequence of Lemmas 3.1, 3.2 and [2, Corollary 3.1], we obtain the following result.

Lemma 3. *The set of bifurcation points of problem* (1)-(3) *with respect to* $\mathbb{R}\times S_k^\nu$ *nonempty.*

Lemma 4. *If* $(\lambda, 0)$ *is a bifurcation point of* (1)-(3) *with respect to* $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, *then* $\lambda \in I_k$; *moreover,* $\lambda = \lambda_k^\nu$ *if* $k < N_0$, $\lambda = \lambda_{k+1}^\nu$ *if* $k > N_0$, *and either* $\lambda = \lambda_{N_0}^\nu$ *or* $\lambda = \lambda_{N_0+1}^\nu$ *if* $k = N_0$.

We define the positive numbers γ_k, $k \in \mathbb{N} \cup \{0\}$, as follows:

$$\gamma_k = \lambda_{k+1} - \lambda_k, \ k \in \mathbb{N}, \ \gamma_0 = \min\{\gamma_k \ : \ k \in \mathbb{N}\}.$$

It is known (see [4]) that $\lim\limits_{k\to\infty} \gamma_k = +\infty$.

Throughout what follows, for $c_1 \neq 0$ we shall assume that the following condition fulfilled:

$$M < \frac{1}{2} r_0 \gamma_0. \tag{9}$$

Then for any $k, m \in \mathbb{N}$ $(k \neq m)$, we have

$$J_k \cap J_m = \emptyset. \tag{10}$$

Hence it follows by Lemmas 3.3, 3.4 and [2, Theorem 3.5] that

Lemma 5. *For each* $k \in \mathbb{N}$ *the following relation hold:*

$$\lambda_k^\nu \in J_k.$$

Corollary 1. *If* $k' > k \geq 1$, *then*

$$\lambda_{k'}^{\nu'} > \lambda_k^\nu \ \ for \ \ each \ \ \nu', \nu \in \{+\,, -\}.$$

Now we introduce the approximate problem

$$\begin{cases} \ell(y) = \lambda r(x)y + \alpha(x)||y||_1^\varepsilon y^+ + \beta(x)||y||_1^\varepsilon y^- + \\ \qquad\qquad\qquad + g(x, y, y', \lambda), \ x \in (0, \pi), \\ (\lambda, y) \in BC_\lambda, \end{cases} \tag{11}$$

where $\varepsilon \in (0, 1]$. This type of problem has been considered in [1-3, 6, 11, 13].

For each $y \in E$ we define the function $\tilde{g}(y) \in C[0, \pi]$ as follows:

$$\tilde{g}(y)(x) = \alpha(x)y^+(x) + \beta(x)y^-(x), \ x \in [0, \pi].$$

Since $\alpha(x), \beta(x) \in C[0, \pi]$, the map $\tilde{g} : E \to C[0, \pi]$ is continuous and satisfies the condition

$$||\tilde{g}(y)||_\infty \leq M||y||_1. \tag{12}$$

Problem (11) can be rewritten in the following equivalent form:

$$\begin{cases} \ell(y) = \lambda r(x)y + \tilde{g}(||y||_1^\varepsilon y) + g(x, y, y', \lambda), \ x \in (0, \pi), \\ (\lambda, y) \in BC_\lambda. \end{cases} \tag{13}$$

By (12), for each fixed $\varepsilon \in (0, 1]$

$$||\tilde{g}(||y||_1^\varepsilon y)||_\infty = o(||y||_1) \ \text{ as } \ ||y||_1 \to 0.$$

Hence the assertion of [2, Theorem 2.2] holds for (13) (that is, for problem (11)): for each $k \in \mathbb{N}$ and each ν there exists an unbounded continuum $C_{k,\varepsilon}^\nu$ of solutions of problem (13) such that

$$(\lambda_k, 0) \in C_{k,\varepsilon}^\nu \subset (\mathbb{R} \times T_k^\nu) \cup \{(\lambda_k, 0)\}. \tag{14}$$

We define the positive numbers $\tilde{\gamma}_0$ and δ_0 as follows:

$$\tilde{\gamma}_0 = \lambda_{N_0+1} - \lambda_{N_0}, \ \delta_0 = \frac{\gamma_0}{4} - \frac{M}{2r_0}.$$

Lemma 6. *There exits $\sigma_0 \in (0, 1)$ such that for any $\varepsilon \in (0, 1)$ the problem (11) has no solution (λ, w) satisfying the conditions $\delta_0 < \operatorname{dist}\{\lambda, J_k\} < 2\delta_0$, $k \in \{N_0, N_0+1\}$, $w \in S_{N_0}^\nu$ and $||w||_1 < \sigma_0$.*

Proof. To prove this statement, assume the contrary. Then for any $\sigma \in (0, 1)$ there exit $\varepsilon_\sigma \in (0, 1)$ such that problem (13) with $\varepsilon = \varepsilon_\sigma$ has a nontrivial solution $(\lambda_\sigma, v_\sigma)$ satisfying the conditions

$$\delta_0 < \operatorname{dist}\{\lambda_\sigma, J_k\} < 2\delta_0, \ k \in \{N_0, N_0 + 1\}, \ v_\sigma \in S_{N_0}^\nu \text{ and } ||v_\sigma||_1 < \sigma.$$

Let $\{\sigma_n\}_{n=1}^\infty \subset (0, 1)$ be a sequence such that $\lim\limits_{n\to\infty} \sigma_n = 0$. Then for each $n \in \mathbb{N}$ problem (13) with $\varepsilon = \varepsilon_n$ $(\varepsilon_n = \varepsilon_{\sigma_n})$ has a solution $(\lambda_n, v_n) = (\lambda_{\sigma_n}, v_{\sigma_n})$ such that

$$2\delta_0 < \operatorname{dist}\{\lambda_n, J_k\} < 2\delta_0, \ k \in \{N_0, N_0 + 1\}, \ v_n \in S_{N_0}^\nu \text{ and } ||v_n||_1 < \sigma_n.$$

Let $w_n(x) = \frac{v_n(x)}{||v_n||_1}$. Then by (13) we have

$$\left\{ \begin{array}{l} \ell(w_n) = \lambda r(x) w_n + \tilde{g}(||v_n||_1^{\varepsilon_n} w_n) + \frac{g(x, v_n, v_n', \lambda_n)}{||v_n||_1}, \ x \in (0, \pi), \\ (\lambda_n, v_n) \in BC_\lambda. \end{array} \right. \tag{15}$$

Hence it follows from (15) that the sequence $\{(\lambda_n, w_n)\}_{n=1}^{\infty}$ is bounded in $\mathbb{R} \times C^2[0, \pi]$. Then there exists a subsequence $\{(\lambda_{n_s}, w_{n_s})\}_{s=1}^{\infty}$ converging to $(\tilde{\lambda}, \tilde{w})$ in $\mathbb{R} \times E$. Moreover, we may assume that $||v_{n_s}||_1^{\varepsilon_{n_s}} \to \tilde{\tau}$ as $s \to \infty$ for some $\tilde{\tau} \in [0, 1]$. By (15) some subsequence $\{(\lambda_{n_s}, w_{n_s})\}_{s=1}^{\infty}$ also converges to $(\tilde{\lambda}, \tilde{w})$ in $\mathbb{R} \times C^2[0, \pi]$. In addition, $\delta_0 < \text{dist}\{\tilde{\lambda}, J_k\} < 2\delta_0$, $k \in \{N_0, N_0 + 1\}$, $||\tilde{w}|| = 1$, $\tilde{w} \in \overline{S_{N_0}^{\nu}}$ and

$$\left\{ \begin{array}{l} \ell(\tilde{w}) = \tilde{\lambda} r(x) \tilde{w} + \tilde{\tau}\, \alpha(x) \tilde{w}^+ + \tilde{\tau}\, \beta(x) \tilde{w}^-, \ x \in (0, \pi), \\ (\tilde{\lambda}, \tilde{w}) \in BC_\lambda. \end{array} \right. \tag{16}$$

Since $\overline{S_{N_0}^{\nu}} = S_{N_0}^{\nu} \cup \partial S_{N_0}^{\nu}$ and $||\tilde{w}|| = 1$ it follows from the proof of Lemma 1 in [3, p. 379] that $\tilde{w} \in S_{N_0}^{\nu}$. Problem (16) is of the same form as (6) so Lemma 3.5 shows that $\tilde{\lambda} \in J_{N_0} \cup J_{N_0+1}$ in contradiction with the inequality $\delta_0 < \text{dist}\{\tilde{\lambda}, J_k\} < 2\delta_0$, $k \in \{N_0, N_0 + 1\}$. The proof is complete.

Lemma 7. *For each ν the points $(\lambda_{N_0}^{\nu}, 0)$ and $(\lambda_{N_0+1}^{\nu}, 0)$ are bifurcation points of* (1)-(3) *with respect to the set* $\mathbb{R} \times S_{N_0}^{\nu}$.

Proof. Let $\sigma \in (0, \sigma_0)$ is an arbitrary fixed number. Since $C_{N_0, \varepsilon}^{\nu}$ is connected, it follows by (14) and Lemma 3.6 that for each $\varepsilon \in (0, 1)$ problem (11) has a solution $(\lambda_\varepsilon, y_\varepsilon)$ such that $\lambda_\varepsilon \in [\lambda_{N_0} - \frac{M}{r_0} - \delta_0, \lambda_{N_0} + \frac{M}{r_0} + \delta_0]$, $y_\varepsilon \in S_{N_0}^{\nu}$ and $||y_\varepsilon||_1 = \sigma$.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence such that $\lim\limits_{n \to \infty} \varepsilon_n = 0$. Then for each $n \in \mathbb{N}$ problem (11) with $\varepsilon = \varepsilon_n$ has a solution $(\lambda_{\varepsilon_n}, y_{\varepsilon_n})$ such that $\lambda_{\varepsilon_n} \in \Lambda_0 \equiv [\lambda_{N_0} - \frac{M}{r_0} - \delta_0, \lambda_{N_0} + \frac{M}{r_0} + \delta_0]$, $y_{\varepsilon_n} \in S_{N_0}^{\nu}$ and $||y_{\varepsilon_n}||_1 = \sigma$. Using the above argument we can show that there exists a subsequence $\{\lambda_{\varepsilon_{n_s}}, y_{\varepsilon_{n_s}}\}_{s=1}^{\infty}$ converging to $(\lambda_\sigma, y_\sigma)$ in $\mathbb{R} \times C^2[0, \pi]$. Note that $||y_{\varepsilon_{n_s}}||_1^{\varepsilon_{n_s}} \to 1$ as $s \to \infty$. In addition, $\lambda_\sigma \in \Lambda_0$, $||y_\sigma||_1 = \sigma$, $y_\sigma \in S_{N_0}^{\nu}$ and

$$\left\{ \begin{array}{l} \ell(y_\sigma) = \lambda_\sigma y_\sigma + \alpha(x) y_\sigma^+ + \beta(x) y_\sigma^- + g(x, y_\sigma, y_\sigma', \lambda_\sigma), \ x \in (0, \pi), \\ (\lambda_\sigma, y_\sigma) \in BC_\lambda\,. \end{array} \right.$$

which implies that $(\lambda_\sigma, y_\sigma)$ solves (1)-(3).

Thus we have shown that for each ν and each σ, $0 < \sigma < \sigma_0$, problem (1)-(3) has a solution $(\lambda_\sigma, y_\sigma)$ such that $||y_\sigma||_1 = \sigma$, $\lambda_\sigma \in \Lambda_0$ and $y_\sigma \in S_{N_0}^{\nu}$. Hence it follows that interval $\Lambda_0 \times \{0\}$ contains at least one bifurcation point of problem (1)-(3) with respect to $\mathbb{R} \times S_{N_0}^{\nu}$. Therefore, by virtue of Lemmas 3.2, 3.4 and 3.5, $(\lambda_{N_0}, 0)$ is the unique bifurcation point in $J_{N_0} \times \{0\} \subset \Lambda_0 \times \{0\}$ of (1)-(3) with respect to $\mathbb{R} \times S_{N_0}^{\nu}$.

In a similar way, one can show that $(\lambda_{N_0+1}, 0)$ is the unique bifurcation point in $J_{N_0+1} \times \{0\}$ of (1)-(3) with respect to $\mathbb{R} \times S_{N_0}^{\nu}$. The proof is complete.

If $c_1 = 0$ let $T_k^{\nu} = S_k^{\nu}$, $k \in \mathbb{N}$, if $c_1 \neq 0$ let

$$T_k^{\nu} = \left\{ \begin{array}{ll} S_k^{\nu}, & \text{if } k \leq N_0, \\ S_{k-1}^{\nu}, & \text{if } k > N_0. \end{array} \right.$$

We denote by $\mathfrak{L}$ the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions of problem (1)-(3) and by $\mathfrak{L}_k^\nu$ the closure in $\mathbb{R}\times E$ of the set of all solutions (λ, y) of (1)-(3) with $y \in T_k^\nu$.

The next result describes the structure and behaviour of global sets of solutions of problem (1)-(3) bifurcating from the line of trivial solutions.

Theorem 1. *For each $k \in \mathbb{N}$ and each ν there exists an unbounded continuum of solutions of problem* (1)-(3), $\mathfrak{D}_k^\nu$ *such that* $(\lambda_k^\nu, 0) \in \mathfrak{D}_k^\nu \subset (\mathbb{R}\times T_k^\nu) \cup \{(\lambda_k^\nu \times 0)\}$.

Proof. For each $k \in \mathbb{N}$ and each ν let denote by $\tilde{\mathcal{D}}_k^\nu$ the union of all components $\tilde{\mathcal{D}}_{k,\lambda}^\nu$ of $\mathfrak{L}$ emanating from the bifurcation points $(\lambda, 0) \in I_k \times \{0\}$ of problem (1)-(3) with respect to $\mathbb{R}\times S_k^\nu$. Let $\mathcal{D}_k^\nu = \tilde{\mathcal{D}}_k^\nu \cup (I_k \times 0)$. Note that the set $\mathcal{D}_k^\nu$ is connected in $\mathbb{R}\times E$, but $\tilde{\mathcal{D}}_k^\nu$ my not be connected. Then it follows by [2, Theorem 3.4] that for each $k \in \mathbb{N}$ and each ν, the set $\mathcal{D}_k^\nu$ is unbounded in $\mathbb{R}\times E$ and lies in $(\mathbb{R}\times S_k^\nu) \cup (I_k \times 0)$.

Let $c_1 = 0$. Then by Lemma 3.2 we have $\mathfrak{L}_k^\nu \cap (\mathbb{R}\times \{0\}) \subset \{(\lambda_k^\nu, 0)\}$. We define $\mathfrak{D}_k^\nu = \mathcal{D}_k^\nu \cap \mathfrak{L}_k^\nu$. Then it follows by the above argument that $\mathfrak{D}_k^\nu$ is an unbounded component of $\mathfrak{L}_k^\nu$ and $(\lambda_k^\nu, 0) \in \mathfrak{D}_k^\nu \subset (\mathbb{R}\times S_k^\nu) \cup \{(\lambda_k^\nu \times 0)\}$.

Let $c_1 \neq 0$. Then it follows from Lemma 3.2 that

$$\mathfrak{L}_k^\nu \cap (\mathbb{R}\times\{0\}) \subset \{(\lambda_k^\nu, 0)\}, \quad \text{if } \ k < N_0,$$

$$\mathfrak{L}_k^\nu \cap (\mathbb{R}\times\{0\}) \subset \{(\lambda_{k+1}^\nu, 0)\}, \ \text{if } k > N_0,$$

$$\mathfrak{L}_{N_0}^\nu \cap (\mathbb{R}\times\{0\}) \subset \{(\lambda_{N_0}^\nu, 0), (\lambda_{N_0+1}^\nu, 0)\}\,.$$

We define $\tilde{\mathfrak{D}}_k^\nu = \mathcal{D}_k^\nu \cap \mathfrak{L}_k^\nu$, $k \in \mathbb{N}$. By Lemma 3.7 the set $\tilde{\mathfrak{D}}_{N_0}^\nu$ has the representation $\tilde{\mathfrak{D}}_{N_0}^\nu = \tilde{\mathfrak{D}}_{N_0,1}^\nu \cup \tilde{\mathfrak{D}}_{N_0,2}^\nu$ such that $(\lambda_{N_0}^\nu, 0) \in \tilde{\mathfrak{D}}_{N_0,1}^\nu$ and $(\lambda_{N_0+1}^\nu, 0) \in \tilde{\mathfrak{D}}_{N_0,2}^\nu$.

Now we define the set $\mathfrak{D}_k^\nu$, $k \in \mathbb{N}$, as follows:

$$\mathfrak{D}_k^\nu = \begin{cases} \tilde{\mathfrak{D}}_k^\nu, & \text{if } k < N_0, \\ \tilde{\mathfrak{D}}_{N_0,1}^\nu, & \text{if } k = N_0, \\ \tilde{\mathfrak{D}}_{N_0,2}^\nu, & \text{if } k = N_0 + 1, \\ \tilde{\mathfrak{D}}_{k-1}^\nu, & \text{if } k > N_0 + 1. \end{cases}$$

It is then readily verified that $\mathfrak{D}_k^\nu$ for $k \neq N_0, N_0 + 1$, is an unbounded component of $\mathfrak{L}_k^\nu$ and $(\lambda_k^\nu, 0) \in \mathfrak{D}_k^\nu \subset (\mathbb{R}\times T_k^\nu) \cup \{(\lambda_k^\nu, 0)\}$. Moreover, it follows by [8, Theorem 2.1], [11, Theorem 1] that the set $\mathfrak{D}_{N_0}^\nu$ contains $(\lambda_{N_0}^\nu, 0)$ and is either unbounded in $\mathbb{R}\times E$ or meets $(\lambda_{N_0+1}^\nu, 0)$ through $\mathbb{R}\times S_{N_0}^\nu$. (It also shows that a similar result holds for $\mathfrak{D}_{N_0+1}^\nu$.) Hence it follows that, if $\mathfrak{D}_{N_0}^\nu$ is bounded in $\mathbb{R}\times E$, then $\mathfrak{D}_{N_0+1}^\nu$ will also be bounded in $\mathbb{R}\times E$, which contradicts the unboundedness of the set $\tilde{\mathfrak{D}}_{N_0}^\nu = \mathcal{D}_{N_0}^\nu \cap \mathfrak{L}_{N_0}^\nu = \mathfrak{D}_{N_0}^\nu \cup \mathfrak{D}_{N_0+1}^\nu$. Therefore, $\mathfrak{D}_{N_0}^\nu$ and $\mathfrak{D}_{N_0+1}^\nu$ are both unbounded in $\mathbb{R}\times E$. The proof is complete.

If in the case $c_1 \neq 0$ not satisfied condition (9), then it follows from the relation $\lim\limits_{k\to\infty} \gamma_k = +\infty$ that there exists $\tilde{k}_0 \in \mathbb{N}$ such that $\gamma_k > \frac{2M}{r_0}$ for $k > \tilde{k}_0$. Now we define the number $k_0 \in \mathbb{N}$ as follows: $k_0 = \max\{N_0^-, N_0^+, \tilde{k}_0\}$. Then from Theorem 3.1 implies the folllowing result which describes the bifurcation structure of problem (1)-(3) for $k > k_0$.

Theorem 2. *If $c_1 \neq 0$, then for each $k > k_0$ and each ν there exists an unbounded continuum of solutions of problem* (1)-(3)*, $\mathfrak{D}_k^\nu$ such that $(\lambda_k^\nu, 0) \in \mathfrak{D}_k^\nu \subset (\mathbb{R} \times S_{k-1}^\nu) \cup \{(\lambda_k^\nu, 0)\}$.*

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On Strong Solvability of the Dirichlet Problem for a Class of no Uniformly Degenerated Elliptic Equations of Second Order

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Abstract. In this paper it has been considered the Dirichlet problem for a class of no uniformly degenerated elliptic equations of second order. For that, it has been proved main a prior estimate, which helps to prove a strong solvability and uniqueness result in the anisotropic weighted Sobolev spaces.
Key Words and Phrases: Dirichlet problem, solvability, elliptic equation.
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1. Introduction

Let E_n be n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$, $n \geq 3$, $D-$ be a bounded domain lying in E_n, $\partial D-$ be the boundary of domain D, wherein $\partial D \in C^2 u$, $0 \in \bar{D}$. Let us consider in D the first boundary value problem

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{ij} + \sum_{i,j=1}^{n} b_i(x) u_i + c(x) u = f(x), x \in D, \tag{1}$$

$$u|_{\partial D} = 0, \tag{2}$$

with $\|a_{ij}(x)\|-$ be real and symmetric matrix having measurable elements defined in D, such that for any $x \in D$, $\zeta \in E_n$ it holds the condition

$$\gamma \sum_{i=1}^{n} \lambda_i(x) \zeta_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \zeta_i \zeta_j \leq \gamma^{-1} \sum_{i=1}^{n} \lambda_i(x) \zeta_i^2. \tag{3}$$

Where $\gamma \in (0,1]$ is a constant, $u_i = \frac{\partial u}{\partial x_i}, u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $\lambda_i(x) = g_i(\rho(x))$, $\rho(x) = \sum_{i=1}^{n} \omega(|x_i|)$, $g_i(t) = \left(\omega_i^{-1}(t)/t\right)^2, i = 1, ..., n$, $\omega_i(t)$ are positive and continues functions, monotony increasing on $[0, diamD]$, and $\omega_i(0) = 0, \omega_i^{-1}(t)-$ are the inverse functions to $\omega_i(t)$. The functions $\frac{\omega_i(t)}{t}$ are decreasing for $t > 0$ and the constants $\alpha, \beta, \eta \in (0, \infty)$ there exist such that

$$\alpha \omega_i(t) \leq \omega_i(\eta t) \leq \beta \omega_i(t), t \in (0, diamD). \tag{4}$$

Further more, we assume that $\lambda_i(x)$ −are positive and finite a.e. in D, such that the coefficients and right hand side terms in (1) are measurable functions in$D$.

Also the condition

$$h_{ij}(x) = \frac{a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}} \in C(\bar{D}), \ \ i,j = 1,...,n. \tag{5}$$

will be assumed.

From the condition (5) it follows that there is a positive and continuous function $\omega(t)$ on $[0, diamD]$ such that $\omega(0) = 0$ and

$$|h_{ij}(x) - h_{ij}(y)| \le \omega(|x-y|), \quad x,y \in \bar{D}, \ \ i,j = 1,...,n. \tag{6}$$

Concerning the little term coefficients of operator L the following conditions

$$b_i(x) \in L_m(D), m = n+2; c(x) \in L_\mu(D), \mu = \frac{n+2}{2}, \ c(x) \le 0$$

$$\text{for a. e. } x \in D, \text{ and } f(x) \in L_q(D), q > \frac{n}{3} \ \text{ is assumed.} \tag{7}$$

Let $x^0 \in E_n$, $R > 0, K > 0$, for $\prod_{R:K}(x^0)$ −being the parallelepiped $\left\{x : \left|x_i - x_i^0\right| < K \cdot \omega_i^{-1}(R)\right\}$, and $\Theta_{R:K}(x^0)$ is the ellipsoid $\left\{x : \sum_{i=1}^n \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2} < K^2\right\}$, $B_R(x^0)$ is a ball $\left\{x : \left|x - x^0\right| < R\right\}$.

Let $x' \in \partial\Theta_{R:1+r/2}(0)$, $\Theta_r = \Theta_r(x') = \Theta_{R:r}(x')$ and $\bar{\Theta}_{R:1+r}(0) \subset D$, where R−is an arbitrary fixed number in $(0,1]$, and $r \in \left(0, \frac{1}{2}\right]$ that will be specified latter.

We say that $u \in C_0^\infty(\Theta_r)$ if a compact set $K_u \subset Y'$ exists such that, $\sup pu(x) \subset K_u, u(x) \in C^\infty(\bar{\Theta}_r)$. We call the number

$$\varphi_{f:p}(\sigma) = \left(\sup_{\substack{E \subset D \\ mesE \le \sigma}} \int_E |\varphi(x)|^p \, dx\right)^{1/p},$$

AC− modulo of continuity of the function $|\varphi|^p$ for $\varphi(x) \in L_p(D)$, $1 < p < \infty$.

Denote by $W_{2,\lambda}^2(D)$ a Banach space of functions $u(x)$ in D, such that the norm

$$\|U\|_{w_{2,\lambda}^2(D)} = \left(\int_D \left(U^2 + \sum_{i=1}^n \lambda_i(x) U_i^2 + \sum_{i,j=1}^n \lambda_i(x)\lambda_j(x) U_{ij}^2\right) dx\right)^{1/2}$$

is finite. Let $\overset{o}{W}{}_{2,\lambda}^2(D)$ − be close of the functions class $u(x) \in C^\infty(\bar{D})$, $u|_{\partial D} = 0$ on the norm $W_{2,\lambda}^2(D)$.

The aim of this paper is to prove the unique strong solvability of the problem (1), (2) in weighted Sobolev's spaces. Notice, the proof for an analogous result in the case of uniformly elliptic equations may be found in [1-3]. As to uniformly degenerated elliptic equations we refer to [4-5]. The elliptic equations having weak degeneration (logarithmic) the strong solvability and uniqueness results have been proved in [6]. We refer to [7-9], for a study of the strong solvability and uniqueness results of the first boundary value problem in the case of no uniformly degeneration of power function type degeneration in a fixed point. We refer also to the results in [10-12] on strong solvability.

2. Auxiliary integral estimates

Lemma 1. *Let $x \in \Theta_r$, then it holds the estimates*

$$C_1(n)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2 \le \lambda_i(x) \le C_2(n)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2, \quad i = 1, ..., n. \tag{8}$$

Proof. Always in the feature, by $C(.,.,.)$ we denote the different positive constants, the value of which depends on the content in the bracket.

Let $x \in \Theta_r$, then using the Minkowsky inequality it follows

$$\left(\sum_{i=1}^n \frac{x_i^2}{\left(\omega_i^{-1}(R)\right)}\right)^{1/2} \le \left(\sum_{i=1}^n \frac{(x_i - x_i')^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} + \left(\sum_{i=1}^n \frac{(x_i')^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \le$$

$$\le r + 1 + \frac{r}{2} = 1 + \frac{3r}{2} \le 1 + \frac{3}{4} = \frac{7}{4}.$$

Therefore, for $i = 1, ..., n$

$$|x_i| < \frac{7}{4}\omega_i^{-1}(R).$$

From condition (4) we get

$$\rho(x) = \sum_{i=1}^n \omega_i(|x_i|) \le \beta n R.$$

On other hand,

$$\left(\sum_{i=1}^n \frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \ge \left(\sum_{i=1}^n \frac{(x_i')^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} - \left(\sum_{i=1}^n \frac{(x_i - x_i')^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \ge$$

$$\le 1 + \frac{r}{2} - r = 1 - \frac{r}{2} = 1 - \frac{1}{4} = \frac{3}{4}.$$

Then it will be found an i_0, $1 \le i_0 \le n$ such that,

$$|x_i| \ge \frac{3}{4\sqrt{n}}\omega_i^{-1}(R)$$

therefore

$$\rho(x) = \sum_{i=1}^n \omega_i(|x_i|) \ge \sum_{i=1}^n \omega_i\left(\frac{3}{4\sqrt{n}}\omega_i^{-1}(R)\right) \ge \alpha n R$$

Hence

$$\left(\frac{\omega_i^{-1}(\alpha nR)}{\beta nR}\right)^2 \leq \lambda_i(x) \leq \left(\frac{\omega_i^{-1}(\beta nR)}{\alpha nR}\right)^2$$

that completes the proof of Lemma 1.

Let us to consider the operator with a constant coefficients

$$L_0 = \sum_{i=1}^{n} \lambda_i(x') \frac{\partial^2}{\partial x_i^2}, \quad \lambda_i(x') = const.$$

Lemma 2. *Let $u(x) \in C_0^\infty(\Theta_r)$. Then it holds an inequality*

$$\int_{\Theta_r} \sum_{i,j=1}^{n} \lambda_i(x) \lambda_j(x) u_{ij}^2 dx \leq C_3(n) \int_{\Theta_r} (L_0 u)^2 dx. \tag{9}$$

Proof. Apply a change of coordinate variables $y_i = x_i / \omega_i^{-1}(R)$, $i = 1, ...n$. Let $\tilde{u}(y)$ and $\tilde{\Theta}_r-$ be the image of the function $u(x)$ and the ellipsoid Θ_r, respectively. It is clear that the operator L_0 will be transformed to

$$\tilde{L}_0 = \sum_{i=1}^{n} \lambda_i(x') \cdot \frac{1}{\left(\omega_i^{-1}(R)\right)^2} \frac{\partial^2}{\partial y_i^2} \tag{10}$$

According to Lemma 1 for any $\zeta \in E_n$ it holds

$$C_1(n) R^{-2} |\zeta|^2 \leq \sum_{i=1}^{n} \lambda_i(x') \frac{1}{\left(\omega_i^{-1}(R)\right)^2} \zeta_i^2 \leq C_2(n) R^{-2} |\zeta|^2, \tag{11}$$

i.e. $\tilde{L}_0$ is a uniformly elliptic operator in $\tilde{\Theta}_r$.

We have

$$\int_{\tilde{\Theta}_r} \left(\tilde{L}_0 \tilde{u}\right) dy = \int_{\tilde{\Theta}_r} \left(\sum_{i=1}^{n} \frac{\lambda_i(x')}{\left(\omega_i^{-1}(R)\right)^2} \tilde{u}_{ii}\right)^2 dy = \int_{\tilde{\Theta}_r} \sum_{i,j=1}^{n} \frac{\lambda_i(x') \lambda_j(x')}{\left(\omega_i^{-1}(R)\right)^2 \left(\omega_j^{-1}(R)\right)^2} \tilde{u}_{ii} \cdot \tilde{u}_{jj} =$$

$$= \sum_{i,j=1}^{n} \int_{\tilde{\Theta}_r} \frac{\lambda_i(x') \lambda_j(x')}{\left(\omega_i^{-1}(R)\right)^2 \left(\omega_j^{-1}(R)\right)^2} \tilde{u}_{ij}^2 dy \geq \frac{C_1^2}{R^4} \int_{\tilde{\Theta}_r} \sum_{i,j=1}^{n} \tilde{u}_{ij}^2 dy. \tag{12}$$

Coming back, the preceding variables$x$, we infer that

$$\frac{C_1^2}{R^4} \int_{\Theta_r} \sum_{i,j=1}^{n} [\omega_i^{-1}(R), \omega_j^{-1}(R)]^2 u_{ij}^2 dx \leq \int_{\Theta_r} (L_0 u)^2 dx.$$

Now it suffices to apply Lemma 1 in order to get the estimate (9) and to complete the proof of Lemma 2.

Corollary 1. *Let*

$$L_a = \sum_{i,j=1}^{n} a_{ij}\left(x'\right) \frac{\partial^2}{\partial x_i \partial x_j}, \quad a_{ij}\left(x'\right) = const$$

Then for the function $u\left(x\right) \in C_0^\infty\left(\Theta_r\right)$ satisfying condition (3) *it holds an estimate*

$$\int_{\Theta_r} \sum_{i,j=1}^{n} \lambda_i\left(x\right) \lambda_j\left(x\right) u_{ij}^2 dx \leq C_4\left(\gamma, n\right) \int_{\Theta_r} \left(L_a u\right)^2 dx.$$

First, consider the operator without little terms

$$L' = \sum_{i,j=1}^{n} a_{ij}\left(x\right) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Lemma 3. *Let the conditions* (3) *and* (6) *be satisfied for the coefficients of operator L'. Then the estimate*

$$\int_{\Theta_r} \sum_{i,j=1}^{n} \lambda_i\left(x\right) \lambda_j\left(x\right) u_{ij}^2 dx \leq C_5\left(\gamma, n\right) \int_{\Theta_r} \left(L' u\right)^2 dx. \tag{13}$$

holds for a function $u\left(x\right) \in C_0^\infty\left(\Theta_r\right)$ as $r \leq r_0\left(L', n\right)$.

Proof. Assume that $r_0 \leq \frac{1}{2}$. We have

$$\left(L_a u\right)^2 \leq 2\left(L' u\right)^2 + 2\left(\left(L' - L_a\right) u\right)^2. \tag{14}$$

On other hand

$$\left(L' - L_a\right) u = \sum_{i,j=1}^{n} \left[\frac{a_{ij}\left(x\right)}{\sqrt{\lambda_i\left(x\right) \lambda_j\left(x\right)}} - \frac{a_{ij}\left(x'\right)}{\sqrt{\lambda_i\left(x\right) \lambda_j\left(x\right)}}\right] \sqrt{\lambda_i\left(x\right) \lambda_j\left(x\right)} \cdot u_{ij}\left(x\right).$$

Therefore

$$\left|\left(L' - L_a\right) u\right| \leq \sum_{i,j=1}^{n} \left| h_{ij}\left(x\right) - h_{ij}\left(x'\right) + \frac{a_{ij}\left(x'\right)}{\sqrt{\lambda_i\left(x'\right) \lambda_j\left(x'\right)}} - \frac{a_{ij}\left(x'\right)}{\sqrt{\lambda_i\left(x\right) \lambda_j\left(x\right)}} \right| \cdot$$

$$\cdot \sqrt{\lambda_i\left(x\right) \lambda_j\left(x\right)} \cdot \left|u_{ij}\right| \leq \sum_{i,j=1}^{n} \left| h_{ij}\left(x\right) - h_{ij}\left(x'\right) \left| \sqrt{\lambda_i\left(x\right) \lambda_j\left(x\right)} \right| u_{ij} \right| +$$

$$+ \sum_{i,j=1}^{n} \left|h_{ij}\left(x'\right)\right| \left| 1 - \sqrt{\frac{\lambda_i\left(x'\right) \lambda_j\left(x'\right)}{\lambda_i\left(x\right) \lambda_j\left(x\right)}} \right| \sqrt{\lambda_i\left(x\right) \lambda_j\left(x\right)} \left|u_{ij}\right| = j_1 + j_2. \tag{15}$$

Further, we have

$$j_1 \leq \omega\left(\left|x - x'\right|\right) \sum_{i,j=1}^{n} \sqrt{\lambda_i\left(x\right) \lambda_j\left(x\right)} \left|u_{ij}\right|. \tag{16}$$

for $x \in \Theta_r$ and $i = 1, ..., n$ $\left|x_i - x_i'\right| \leq r\omega_i^{-1}(R)$.

Inserting $\zeta_i = \frac{\eta_i}{\sqrt{\lambda_i(x)}}$, $i = 1, ..., n$ in condition (3) for $x \in D$, we get

$$\gamma \left|\eta\right|^2 \leq \sum_{i,j}^{n} h_{ij}(x)\, \eta_i \eta_j \leq \gamma^{-1} \left|\eta\right|^2,$$

where $\eta \in E_n$.

From this it follows that $\gamma \leq h_{ii}(x) \leq \gamma^{-1}$ and $2\gamma \leq h_{ii}(x) + h_{jj}(x) + 2h_{ij}(x) \leq 2\gamma^{-1}$ for $i \neq j, x \in D, i, j = 1, ..., n$. Thus it follows that $\left|h_{ij}(x)\right| \leq h_0(\gamma)$, $i, j = 1, ..., n$ for $x \in D$.

Therefore

$$j_2 \leq h_0 \sum_{i,j=1}^{n} \left|1 - \sqrt{\frac{\lambda_i(x')\, \lambda_j(x')}{\lambda_i(x)\, \lambda_j(x)}}\right| \sqrt{\lambda_i(x)\, \lambda_j(x)}\, \left|u_{ij}\right|.$$

On other hand

$$\left|1 - \sqrt{\frac{\lambda_i(x')\, \lambda_j(x')}{\lambda_i(x)\, \lambda_j(x)}}\right| \leq \left|1 - \sqrt{\frac{\lambda_j(x')}{\lambda_j(x)}}\right| + \sqrt{\frac{\lambda_j(x')}{\lambda_j(x)}} \left|1 - \sqrt{\frac{\lambda_i(x')}{\lambda_i(x)}}\right| \leq$$

$$\leq \left|1 - \sqrt{\frac{\lambda_j(x')}{\lambda_j(x)}}\right| + \sqrt{\frac{c_2}{c_1}} \left|1 - \sqrt{\frac{\lambda_j(x')}{\lambda_j(x)}}\right| = \left(1 + \sqrt{\frac{c_2}{c_1}}\right) K_i,$$

$$i = 1, ..., n.$$

Therefore

$$K_i = \left|1 - \sqrt{\frac{\lambda_i(x')}{\lambda_i(x)}}\right| \leq \frac{\left|\lambda_i(x) - \lambda_i(x')\right|}{\lambda_i(x)} \leq C_6(n) \tag{17}$$

Thus

$$j_2 \leq C_7(n)\, h_0 \sum_{i,j=1}^{n} \sqrt{\lambda_i(x)\, \lambda_j(x)}\, \left|u_{ij}\right| \leq$$

$$\leq C_7 h_0 n \left(\sum_{i,j=1}^{n} \lambda_i(x)\, \lambda_j(x)\, u_{ij}^2\right)^{1/2}. \tag{18}$$

Inserting conditions (16) and (18) in (15), we get

$$\left|\left(L' - L_a\right) u\right|^2 \leq n^2 \left[\omega\left(r_0 \sqrt{n}\right) + C_7 h_0\right]^2 \sum_{i,j}^{n} \lambda_i(x)\, \lambda_j(x)\, u_{ij}^2. \tag{19}$$

Now, from (14), (19) and Lemma 2 it follows that

$$\int_{\Theta_r} \sum_{i,j=1}^{n} \lambda_i(x)\, \lambda_j(x)\, u_{ij}^2 dx \leq 2C_4 \int_{\Theta_r} \left(L'u\right)^2 dx +$$

$$+2C_4n^2\left[\omega\left(r_0\sqrt{n}\right)+C_7h_0\right]^2\cdot\int_{\Theta_r}\sum_{i,j=1}^{n}\lambda_i\left(x\right)\lambda_j\left(x\right)u_{ij}^2dx.$$

Now it suffices to choose r_0 from the condition

$$\omega\left(r_0\sqrt{n}\right)+C_7h_0\leq\frac{1}{2n\sqrt{C_4}}$$

in order to get the estimate (13).

In the feature, we assume that $r=r_0$ no reminding about.

Lemma 4. *Let be satisfied all conditions of preceding Lemma. Then for a function* $u\left(x\right)\in C_0^\infty\left(\Theta_r\right)$ *it holds the estimate*

$$\|u\|_{W_{2,\lambda}^2(\Theta_r)}\leq C_8\left(\gamma,n\right)\left\|L'u\right\|_{L_2(\Theta_r)} \tag{20}$$

Proof. It suffices to show that for a function $u\left(x\right)\in C_0^\infty\left(\Theta_r\right)$ it is satisfied the inequalities

$$\int_{\Theta_r}u^2dx\leq C_9\left(n\right)\int_{\Theta_r}\sum_{i=1}^{n}\lambda_i\left(x\right)u_i^2dx,$$

$$\int_{\Theta_r}\sum_{i=1}^{n}\lambda_i\left(x\right)u_i^2dx\leq C_{10}\left(n\right)\int_{\Theta_r}\sum_{i,j=1}^{n}\lambda_i\left(x\right)\lambda_j\left(x\right)u_{ij}^2dx.$$

To proof this, apply the change of coordinate axes as that was carry out in Lemma 2. Let $\Theta_r^0=\{y:|y_i-y_i'|<r\}$, $i=1,...,n$, where y' is image of the point x'. Continue the function $\tilde{u}\left(y\right)$ on $\bar{\Theta}_r^0\backslash\tilde{\Theta}_r$ inserting zero in it and denote it again as $\tilde{u}\left(y\right)$. Let $y''=\left(y_2,...,y_n\right)$, $y_1\in\left(y_1'-r,y_1'+r\right)$. We have

$$\tilde{u}\left(y_1,y''\right)=\int_{y_1'-r}^{y_1}\tilde{u}_1\left(z,y''\right)dz,\quad\text{i.e.}$$

$$\tilde{u}^2\left(y_1,y''\right)=\left(\int_{y_1'-r}^{y_1}\tilde{u}_1\left(z,y''\right)dz\right)^2\leq\left(\int_{y_1'-r}^{y_1}1^2dz\right)\left(\int_{y_1'-r}^{y_1}\tilde{u}_1^2dz\right)=$$

$$=\left(y_1-y_1'+r\right)\int_{y_1'-r}^{y_1}\tilde{u}_1^2dz\leq 2r\int_{y_1'-r}^{y_1'+r}\tilde{u}_1^2dz.$$

After integration the last inequality over Θ_r^0, we get

$$\int_{\Theta_r^0}\tilde{u}^2dy\leq 4r^2\int_{\Theta_r^0}\tilde{u}_1^2dy\leq 4r^2\int_{\Theta_r^0}\sum_{i=1}^{n}\tilde{u}_i^2dy.$$

Therefore

$$\int_{\Theta_r^0}\tilde{u}^2dy\leq 4r^2\int_{\tilde{\Theta}_r}\sum_{i=1}^{n}\tilde{u}_i^2dy.$$

By the analogy, we can derive

$$\int_{\tilde{\Theta}_r} \sum_{i=1}^{n} \tilde{u}_i^2 dy \leq 4r^2 \int_{\tilde{\Theta}_r} \sum_{i,j=1}^{n} \tilde{u}_{ij}^2 dy.$$

Coming back to the first variables x, we get

$$\int_{\Theta_r} u^2 dx \leq 4r^2 \int_{\Theta_r} \sum_{i=1}^{n} \left[\omega_i^{-1}(R)\right]^2 u_i^2 dx \leq$$

$$\leq 16r^2 \int_{\Theta_r} \sum_{i,j=1}^{n} \left[\omega_i^{-1}(R)\,\omega_j^{-1}(R)\right]^2 u_{ij}^2 dx.$$

Now it suffices to apply Lemma 1 in order to complete the proof of estimate (20).

Let $\Theta'_r = \Theta_{r/2}(x') = \Theta_{R:\frac{r}{2}}(x')$.

Lemma 5. *Let be satisfied all conditions of Lemma 3, then it holds an estimate for any function* $u(x) \in C^{\infty}\left(\bar{\Theta}_r\right)$ *and* $\varepsilon > 0$ *:*

$$\|U\|_{W^2_{2,\lambda}(\Theta'_r)} \leq C_8 \left\|L'u\right\|_{L_2(\Theta_r)} + \varepsilon \|u\|_{W^2_{2,\lambda}(\Theta_r)} + \frac{C_{11}(\gamma, n)}{\varepsilon r^2 R^2} \|u\|_{L_2(\Theta_r)}. \tag{21}$$

Proof. Fix arbitrary $\varepsilon' > 0$. Let $\zeta(x) \in C_0^{\infty}(\Theta_{R:r}(x'))$, $0 \leq \zeta(x) \leq 1$, $\zeta(x) = 1$ in $\Theta_{R:\frac{r}{2}}(x')$, and $\zeta(x) = 0$ on the complement of $\Theta_{R:\frac{3r}{4}}(x')$, moreover

$$|\zeta_i| \leq \frac{C_{12}(n)}{r\omega_i^{-1}(R)}, \quad |\zeta_{ij}| \leq \frac{C_{12}(n)}{r^2\omega_i^{-1}(R)\,\omega_j^{-1}(R)}, \qquad i,j = 1,...,n. \tag{22}$$

It is clear that, $u(x) \cdot \zeta(x) \in C_0^{\infty}(\Theta_r)$. Applying Lemma 4 for this function, we get

$$\|U\|_{W^2_{2,\lambda}(\Theta'_r)} \leq C_8 \left\|L'(u(x) \cdot \zeta(x))\right\|_{L_2(\Theta_r)} \tag{23}$$

On the other hand

$$L'(u \cdot \zeta) = \zeta \cdot L'u + 2\sum_{i,j=1}^{n} a_{ij}(x)\, u_i \zeta_j + u \cdot L'\zeta.$$

Therefore, and using (22), it follows

$$\left|L'(u \cdot \zeta)\right| \leq \left|L'u\right| + 2\left|\sum_{i,j=1}^{n} a_{ij}(x)\, u_i \zeta_j\right| + |u| \cdot \left|L'\zeta\right| \leq$$

$$\leq \left|L'u\right| + 2\left(\sum_{i,j=1}^{n} a_{ij}(x)\, u_i u_j\right)^{1/2} \cdot \left(\sum_{i,j=1}^{n} a_{ij}(x)\, \zeta_i \zeta_j\right)^{1/2} +$$

$$+\left|u\right|\cdot\left|\sum_{i,j=1}^{n}a_{ij}\left(x\right)\zeta_{ij}\right|\leq\left|L'u\right|+2\gamma^{-1}\left|\sum_{i=1}^{n}\lambda_{i}\left(x\right)u_{i}^{2}\right|\times\left(\sum_{i=1}^{n}\lambda_{i}\left(x\right)\zeta_{i}^{2}\right)^{1/2}+$$

$$+\left|u\right|\cdot\sum_{i,j=1}^{n}a_{ij}\left(x\right)\cdot\frac{C_{12}}{r^{2}\omega_{i}^{-1}\left(R\right)\omega_{j}^{-1}\left(R\right)}\leq\left|L'u\right|+2\gamma^{-1}\left(\sum_{i=1}^{n}\lambda_{i}\left(x\right)u_{i}^{2}\right)^{1/2}\cdot$$

$$\cdot\left[\sum_{i,=1}^{n}C_{2}\left(\frac{\omega_{i}^{-1}\left(R\right)}{R}\right)^{2}\cdot\frac{C_{12}^{2}}{r^{2}\left(\omega_{i}^{-1}\left(R\right)\right)^{2}}\right]^{1/2}+\left|u\right|\cdot\frac{C_{12}}{\gamma r^{2}}\cdot\sum_{i=1}^{n}\lambda_{i}\left(x\right)\cdot\frac{1}{\left(\omega_{i}^{-1}\left(R\right)\right)^{2}}\leq\left|L'u\right|+$$

$$+\frac{2C_{12}\sqrt{nC_{2}}}{\gamma rR}\left(\sum_{i=1}^{n}\lambda_{i}\left(x\right)u_{i}^{2}\right)^{1/2}+\left|u\right|\frac{nC_{2}C_{12}}{\gamma r^{2}R^{2}}. \tag{24}$$

Taking into the account this inequality it follows from (23) that

$$\left\|u\right\|_{W_{2,\lambda}^{2}\left(\Theta_{r}'\right)}\leq C_{8}\left\|L'u\right\|_{L_{2}\left(\Theta_{r}\right)}+\frac{C_{13}\left(\gamma,n\right)}{r^{2}R^{2}}\left\|u\right\|_{L_{2}\left(\Theta_{r}\right)}+$$

$$+\frac{C_{14}\left(\gamma,n\right)}{rR}\left\|\sqrt{\sum_{i=1}^{n}\lambda_{i}\left(x\right)u_{i}^{2}}\right\|_{L_{2}\left(\Theta_{r}\right)}. \tag{25}$$

On other hand

$$J^{2}=\left\|\sqrt{\sum_{i=1}^{n}\lambda_{i}\left(x\right)u_{i}^{2}}\right\|_{L_{2}\left(\Theta_{r}\right)}^{2}=\int_{\Theta_{r}}\sum_{i=1}^{n}\lambda_{i}\left(x\right)u_{i}^{2}dx\leq$$

$$\leq\frac{C_{2}}{R_{2}}\int_{\Theta_{r}}\sum_{i=1}^{n}\left(\omega_{i}^{-1}\left(R\right)\right)u_{i}^{2}dx=\frac{C_{2}}{R_{2}}\sum_{i=1}^{n}\left\|\omega_{i}^{-1}\left(R\right)u_{i}\right\|_{L_{2}\left(\Theta_{r}\right)}^{2}.$$

Therefore

$$J\leq\frac{\sqrt{C_{2}}}{R_{2}}\left\|\sum_{i=1}^{n}\left(\omega_{i}^{-1}\left(R\right)\right)u_{i}\right\|_{L_{2}\left(\Theta_{r}\right)}=\frac{\sqrt{C_{2}}}{R}\sum_{i=1}^{n}\left\|\tilde{u}_{i}\right\|_{L_{2}\left(\tilde{\Theta}_{r}\right)}.$$

According to the interpolation inequality from [13], for any $\varepsilon'>0$ there exists a constant $C_{15}\left(n\right)$ such that

$$\sum_{i=1}^{n}\left\|\tilde{u}_{i}\right\|_{L_{2}\left(\Theta_{r}\right)}\leq\varepsilon'\sum_{i,j=1}^{n}\left\|\tilde{u}_{ij}\right\|_{L_{2}\left(\tilde{\Theta}_{r}\right)}+\frac{C_{15}}{\varepsilon'}\left\|\tilde{u}\right\|_{L_{2}\left(\tilde{\Theta}_{r}\right)}.$$

Coming back to the variables x and using Lemma 1, it follows

$$\frac{C_{14}}{r^{R}}\cdot J\leq\frac{C_{14}}{r^{R}}\cdot\frac{\sqrt{C_{2}}}{R}\cdot\varepsilon'\cdot\sum_{i,j=1}^{n}\left\|\tilde{u}_{ij}\right\|_{L_{2}\left(\tilde{\Theta}_{r}\right)}+$$

$$+\frac{C_{14}}{r^R}\cdot\frac{\sqrt{C_2}}{R}\cdot\frac{C_{15}}{\varepsilon'}\cdot\|\tilde{u}\|_{L_2(\tilde{\Theta}_r)}=\frac{\varepsilon'\cdot C_{14}\sqrt{C_2}}{rR^2}\sum_{i,j=1}^{n}\left\|\omega_i^{-1}(R)\,\omega_j^{-1}(R)\,u_{ij}\right\|_{L_2(\Theta_r)}+$$

$$+\frac{C_{14}C_{15}\sqrt{C_2}}{\varepsilon' rR^2}\cdot\|u\|_{L_2(\Theta_r)}\leq\frac{\varepsilon'\cdot C_{14}\sqrt{C_2}\cdot R^2}{rR^2\cdot C_1}\|u\|_{W^2_{2,\lambda}(\Theta_r)}+\frac{C_{14}C_{15}\sqrt{C_2}}{\varepsilon' rR^2}\cdot\|u\|_{L_2(\Theta_r)}. \quad (26)$$

Finally, choose $\varepsilon'=\frac{\varepsilon C_1\cdot r}{C_{14}\sqrt{C_2}}$ in order to get the needed estimate (21) using (25) and (26).

3. Main estimation on coercivity

Lemma 6. *Let* $A_R=\Theta_{R:1+\frac{r}{2}+\frac{r^2}{16}}(0)\backslash\overline{\Theta}_{R:1+\frac{r}{2}-\frac{r^2}{16}}(0)$. *Then for the countable ellipsoids system*

$$\Theta'_r(x^\nu)=\Theta_{R:\frac{r}{2}}(x^\nu),\quad x^\nu\in\partial\Theta_{R:1+\frac{r}{2}}(0),\quad \nu=1,2,...$$

there exists a covering for the set A_R.

Proof. Let $x\in A_R$. Without losing the generality, we may assume that $x_1\neq 0$. Choose α_1 so that a point $x^\nu=(\alpha_1,x_2,...,x_n)$ belongs to $\partial\Theta_{R:1+\frac{r}{2}}(0)$:

$$\left(\sum_{i=2}^{n}\frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}+\frac{\alpha_1^2}{\left(\omega_1^{-1}(R)\right)^2}\right)^{1/2}=1+\frac{r}{2},$$

then

$$1+\frac{r}{2}-\frac{r^2}{16}<\left(\sum_{i=1}^{n}\frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}<1+\frac{r}{2}+\frac{r^2}{16},$$

where from it follows that there exists such an α_1. Assume that $sign x_1=sign\alpha_1$. Let for the convenience, $|x_1|\geq|\alpha_1|$, then

$$\frac{x_1^2-\alpha_1^2}{\left(\omega_1^{-1}(R)\right)^2}=\sum_{i=1}^{n}\frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}-\left(\sum_{i=2}^{n}\frac{x_i^2}{\left(\omega_1^{-1}(R)\right)^2}+\frac{\alpha_1^2}{\left(\omega_i^{-1}(R)\right)^2}\right)\leq$$

$$\leq\left(1+\frac{r}{2}+\frac{r^2}{16}\right)^2-\left(1+\frac{r}{2}\right)^2=\left(1+\frac{r}{2}\right)^2+2\left(1+\frac{r}{2}\right)\cdot\frac{r^2}{16}+\frac{r^4}{256}-\left(1+\frac{r}{2}\right)^2=$$

On other hand $x_1^2-\alpha_1^2\geq(x_1-\alpha_1)^2$. Therefore,

$$\left(\sum_{i=1}^{n}\frac{(x_i-x_i^\nu)}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}=\left(\frac{(x_1-\alpha_1)^2}{\left(\omega_1^{-1}(R)\right)^2}\right)^{1/2}\leq\left(\frac{\left(x_1^2-\alpha_1^2\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}<\frac{r}{2},$$

which completes the proof of Lemma.

Lemma 7. *Let $\bar{A}_{R_0} \subset D$ and for $m = 1, 2, ...$ it is $R_m = R_0 \cdot a^m$, where the number a is so that*

$$\frac{\alpha}{\beta} \leq a < 1.$$

Therefore

$$\Theta_{R_0:1+\frac{r}{2}+\frac{r^2}{16}}(0) \setminus \{0\} \subset \bigcup_{m=0}^{\infty} A_{R_m}$$

Proof. For $m = 1, 2, ...$ it suffices to set

$$\Theta_{R_{m+1}:1+\frac{r}{2}+\frac{r^2}{16}}(0) \supset \bar{\Theta}_{R_m:1+\frac{r}{2}-\frac{r^2}{16}}(0). \tag{27}$$

The inclusion (27) is equivalently to that of

$$\left(1 + \frac{r}{2} - \frac{r^2}{16}\right) \omega_i^{-1}(R_m) \leq \left(1 + \frac{r}{2} + \frac{r^2}{16}\right) \omega_i^{-1}(R_{m+1}).$$

for $m = 1, 2, ...,\ \ i = 1, ..., n$. It follows from (4) that

$$\alpha R_m \leq \beta R_{m+1}$$

i.e.

$$\frac{\alpha}{\beta} \leq \frac{R_{m+1}}{R_m} = a < 1.$$

This completes the lemma.

Remark 1. *It holds an inclusion*

$$\bigcup_{\nu=1}^{\infty} \Theta_r(x^\nu) \subset B_R = \Theta_{R:1+\frac{3r}{2}}(0) \supset \bar{\Theta}_{R_m:1-\frac{r}{2}}(0),$$

where is a cover with ellipsoids $\Theta_r(x^\nu) = \Theta_{R:k}(x^\nu)$ has a finite multiplicity $N_1(n, r)$ and $x^\nu \in \partial\Theta_{R:1+\frac{r}{2}}(0)$.

Remark 2. *It holds an inclusion*

$$\bigcup_{m=0}^{\infty} B_{R_m} \setminus \Theta_{R:1+\frac{3r}{2}}(0),$$

where is a cover with spherical layers B_{R_m} has a finite multiplicity $N_2(n, r)$.

Let

$$\Theta^1_{R_0}(\bar{x}) = \Theta_{R_0:1+\frac{r}{2}+\frac{r^2}{16}}(\bar{x}), \Theta^1_{R_0}(0) = \Theta^1_{R_0}, \quad \Theta^2_{R_0}(\bar{x}) = \Theta_{R_0:1+\frac{3r}{2}}(\bar{x}), \Theta^2_{R_0}(0) = \Theta^2_{R_0}.$$

It is easy to see that

$$\Theta^1_{R_0} \subset \Theta^2_{R_0},\ \Theta^1_{R_0} = \bigcup_{m=0}^{\infty} A_{R_m},\ \ \Theta^2_{R_0} = \bigcup_{m=0}^{\infty} B_{R_m}. \tag{28}$$

Lemma 8. *Let be satisfied the conditions* (3) *and* (5), *then for a function* $u(x) \in C^{\infty}\left(\Theta^2_{R_0}\right)$ *it holds an estimate for any* $\varepsilon > 0$ *:*

$$\|u\|_{W^2_{2,\lambda}\left(\Theta^1_{R_0}\right)} \leq C_{16}\left(L',n\right)\left\|L'u\right\|_{L_2\left(\Theta^2_{R_0}\right)} + \varepsilon\|u\|_{W^2_{2,\lambda}\left(\Theta_{R^2_0}\right)} + \frac{C_{17}\left(L',n\right)}{\varepsilon}\sup_{\Theta^2_{R_0}}\|u\|. \quad (29)$$

Proof. Fix arbitrary $\varepsilon > 0$. It follows from Lemma 5 that for any $\varepsilon' > 0$ and $\nu = 1, 2, ...$ it holds the estimate

$$\|u\|^2_{W^2_{2,\lambda}(\Theta'_r(x^\nu))} \leq C_{18}\left(L',n\right)\left\|L'u\right\|_{L_2(\Theta_r(x^v))} + \left(\varepsilon'\right)\|u\|^2_{W^2_{2,\lambda}(\Theta_r(x^\nu))} + \frac{C_{19}(\gamma,n)}{\left(\varepsilon'\right)^2 r^4 R^4}\|u\|^2_{L_2(\Theta_r(x^\nu))}, \quad (30)$$

on the ellipsoids Θ'_r and Θ_r where is $R = R_m$, $m = 0, 1, 2, \ldots$.

Summing all inequalities (30) over ν and using Lemma 6 with help of Remark 1 to Lemma 7, we infer

$$\|u\|^2_{W^2_{2,\lambda}(A_{R_m})} \leq C_{20}\left(L',n\right)\left\|L'u\right\|^2_{L_2(B_{R_m})} + N_1\left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}(B_{R_m})} + \frac{C_{21}\left(L',n\right)}{\left(\varepsilon'\right)^2 r^4 R^4_m}\|u\|^2_{L_2(B_{R_m})}$$

On other hand, it is

$$\|u\|^2_{L_2(B_{R_m})} = \int_{B_{R_m}} u^2 dx \leq \left(\sup_{\Theta^2_{R_0}}|u|\right)^2 \cdot mes\Theta^2_{R_0}.$$

Thus

$$\|u\|^2_{W^2_{2,\lambda}(A_{R_m})} \leq C_{20}\left\|L'u\right\|^2_{L_2(B_{R_m})} + N_1\left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}(B_{R_m})} + \frac{C_{22}\left(L',n\right)}{\left(\varepsilon'\right)^2}\left(\sup_{\Theta^2_{R_0}}|u|\right)^2. \quad (31)$$

After summing all inequalities (31) over m beginning from zero to infinity and applying Lemma 7, Remark 2 on it, we come to the inequality

$$\|u\|^2_{W^2_{2,\lambda}\left(\Theta^1_{R_0}\right)} \leq C_{23}\left\|L'u\right\|^2_{L_2\left(\Theta^2_{R_0}\right)} + N_1 N_2 \cdot \left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}\left(\Theta^2_{R_0}\right)} + \frac{C_{24}\left(L',n\right)}{\left(\varepsilon'\right)^2}\left(\sup_{\Theta^2_{R_0}}|u|\right)^2$$

Finally, choosing $\varepsilon' = \frac{\varepsilon}{\sqrt{N_1 N_2}}$ we complete the proof on needed estimation (29).

Remark 3. *Since the operator* L' *degenerates on a point 0, the estimate* (29) *takes place in the ellipsoids* $\Theta^1_{R_0}(\bar{x})$ *and* $\Theta^2_{R_0}(\bar{x})$ *provided that* $\bar{\Theta}^2_{R_0}(\bar{x}) \subset D$, $\Theta^2_{R_0}(x) \bigcap \Theta_{R_0:1}(0) = \emptyset$. *Also, the mentioned estimate takes place for any* $R \in (0, R_0]$.

Let $D(\rho) = \left\{x : x \in D, \Theta^2_\rho(x) \subset D\right\}$, for $\rho > 0$, and $\Theta^2_\rho(x) = \Theta_{\rho:1+\frac{3r}{2}}(x)$, $\Theta^1_\rho(x) = \Theta_{\rho:1+\frac{r}{2}+\frac{r^2}{16}}(x)$.

Lemma 9. *For a function $u(x) \in C^{\infty}\left(\Theta^2_{R_0}\right)$, a number $\varepsilon > 0$, and sufficiently small $\rho > 0$ it holds an estimate*

$$\|u\|_{W^2_{2,\lambda}(D(\rho))} \le C_{25}\left(L', n, \rho, D\right)\left\|L'u\right\|_{L_2(D)} + \varepsilon\|u\|_{W^2_{2,\lambda}(D)} + \\ + \frac{C_{26}\left(L', n, \rho, D\right)}{\varepsilon}\sup_D |u| \tag{32}$$

Proof. Fix a number $\varepsilon > 0$ and sufficiently small $\rho > 0$. Cover the set $\overline{D(\rho)}$ with finite $N_3(n, \rho, D)$ number ellipsoids $\left\{\Theta^1_\rho(x^v)\right\}$. According to the Lemma 8, for a $\varepsilon' > 0$ it holds

$$\|u\|^2_{W^2_{2,\lambda}\left(\Theta^1_\rho(x^v)\right)} \le C_{27}\left(L', n\right)\left\|L'u\right\|^2_{L_2\left(\Theta^2_\rho(x^v)\right)} + \left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}\left(\Theta^2_{R_0}(x^v)\right)} + \\ + \frac{C_{26}}{\left(\varepsilon'\right)^2}\left(\sup_D |u|\right)^2, \quad \nu = 1, ..., N_3. \tag{33}$$

Summing all inequalities (33) over the ν from 1 to N_3, we get

$$\|u\|^2_{W^2_{2,\lambda}(D(\rho))} \le C_{27}\cdot N_3\left\|L'u\right\|^2_{L_2(D)} + N_3\left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}(D)} + \\ + \frac{C_{28}\left(L', n, \rho, D\right)}{\left(\varepsilon'\right)^2}\cdot N_3\left(\sup_D |u|\right)^2.$$

Now it suffices to set $\varepsilon' = \frac{\varepsilon}{\sqrt{N_3}}$ in order to get the estimate (32).

For a $\rho > 0$ set $D_\rho = \{x : x \in D,\ dist\,(x, \partial D) > \rho\}$.

Lemma 10. *Let the conditions* (3) *and* (5) *be satisfied. Then for a function $u(x) \in W^2_{2,\lambda}(D)$ it holds an estimate for any $\varepsilon > 0$ and $\rho > 0$:*

$$\|u\|^2_{W^2_{2,\lambda}(D_\rho)} \le C_{29}\left(n, \rho, D, \gamma\right)\left(\int_D \left(L'u\right)^2 dx + \varepsilon\|u\|^2_{W^2_{2,\lambda}(D)} + \frac{1}{\varepsilon}\left(\sup_D |u|\right)^2\right). \tag{34}$$

Proof. Fix a number $\varepsilon > 0$ and arbitrary small $\rho > 0$. Cover $\bar{D}_\rho$ with finite $N_4(n, \rho, D, \gamma)$ number ellipsoids $\Theta^1_\rho(x^v)$ applying Lemma 8 in the everyone.

Lemma 11. *Let the conditions* (3) *and* (5) *be satisfied. Then for a function $u(x) \in \overset{0}{W}{}^2_{2,\lambda}(D)$ it holds the estimate for a $\rho > 0$:*

$$\|u\|^2_{W^2_{2,\lambda}(D\setminus D_\rho)} \le C_{30}\left(n, \gamma, \rho, D\right)\left(\int_D \left|L'u\right|^2 dx + \left(\sup_D |u|\right)^2\right). \tag{35}$$

Proof. Since $\partial D \subset C^2$, according to [2], for sufficiently small ρ such that $(D \backslash D_\rho) \bigcap \Theta_{\rho:1}(0) = \emptyset$, it holds that

$$\|u\|^2_{W_2^2(D\backslash D_\rho)} \leq C_{31}(n,\gamma,\rho,D)\left(\int_D |L'u|^2\,dx + \int_D |u|^2\,dx\right)$$

Now, in order to complete the proof of the following Theorem, it suffices to apply

$$\|u\|_{W_2^2(D\backslash D_\rho)} \leq C_{32}(n,\rho)\cdot\|u\|_{W_2^2(D\backslash D_\rho)}$$

and the inequality

$$\int_D |u|^2\,dx \leq mesD\cdot\left(\sup_D |u|\right)^2.$$

Theorem 1. *Let the coefficients of operator L' satisfy the conditions* (3) *and* (5), *then for a function $u(x) \in \overset{0}{W}{}^2_{2,\lambda}(D)$ the estimate*

$$\|u\|^2_{W^2_{2,\lambda}(D)} \leq C_{33}(n,\gamma,D)\left(||L'u||^2_{L_2(D)} + \left(\sup_D |u|\right)^2\right) \tag{36}$$

takes place.

Proof. Fix the sufficiently small number $\rho > 0$ and sum the inequalities (34) and (35). We get

$$\|u\|^2_{W^2_{2,\lambda}(D)} \leq (C_{29}+C_{30})\int_D |L'u|^2\,dx + C_{29}\cdot\varepsilon\cdot\|u\|^2_{W^2_{2,\lambda}(D)} +$$
$$+\left(\frac{C_{29}}{\varepsilon}+C_{30}\right)\left(\sup_D |u|\right)^2.$$

Now, it suffices to set $\varepsilon = \frac{1}{2C_{29}}$ and $C_{33} = \max\left\{2(C_{29}+C_{30});2\left(2C_{29}^2+C_{30}\right)\right\}$ in order to complete the proof.

For proving the estimate (36) for operator L, we need the following imbedding assertion from [2].

Theorem 2. *For a function $u(x) \in C^\infty(D)$, with $u|_{\partial D} = 0$ it holds the estimate*

$$\sum_{i=1}^n \|u_i\|_{L_p(D)} \leq C_{34}(p,q,n)\sum_{i,j=1}^n \|u_{ij}\|_{L_q(D)},$$

provided that

$$p \geq q \geq 1,\ \ 1-\left(\frac{1}{q}-\frac{1}{p}\right)\cdot(n+2) \geq 0, \tag{37}$$

and

$$\|u\|_{L_{p_1}(D)} \leq C_{35}\left(p_1, q_1, n\right) \sum_{i,j=1}^{n} \|u_{ij}\|_{L_{q_1}(D)},$$

provided that

$$p_1 \geq q_1 \geq 1, \;\; 2-\left(\frac{1}{q_1}-\frac{1}{p1}\right)(n+2) \geq 0. \tag{38}$$

Theorem 3. *Let the coefficients of operator* L *satisfy conditions* (3)-(7)*, then for a function* $u(x) \in C^{\infty}\left(\bar{D}\right)$, *with* $u|_{\partial D}=0$ *takes place the estimate*

$$\|u\|_{W^2_{2,\lambda}(D)} \leq C_{36}\left(L, n, D\right)\left(\|Lu\|_{L_2(D)} + \sup_D |u|\right). \tag{39}$$

Proof. First, prove that

$$\sum_{i=1}^{n} \|b_i u_i\|_{L_2(D)} \leq C_{37}\left(L, n, D\right) \varphi_{B:m}\left(\sigma\right) \sum_{i,j=1}^{n} \|u\|_{W_{2,\lambda}(D)}, \tag{40}$$

where $b_i(x) \in L_m(D)$, $m=n+2$, $i=1,...,n$, and $\varphi_{B:M}(\sigma)=\max\limits_{1\leq i\leq n} \varphi_{b_i:m}$, $\sigma = mesD$.

Evidently, (37) takes place for $q=\frac{p(n+2)}{p+n+2}$. Using Holder's inequality, we have

$$W_1 = \sum_{i=1}^{n} \|b_i u_i\|_{L_2(D)} = \sum_{i=1}^{n}\left(\int_D b_i^2 u_i^2 dx\right)^{1/2} \leq$$

$$\leq \sum_{i=1}^{n}\left(\int_D |b_i|^m dx\right)^{1/m} \cdot \left(\int_D |u_i|^{\frac{2m}{m-2}} dx\right)^{\frac{m-2}{2m}} \leq$$

$$\leq \varphi_{B:m}\left(\sigma\right) \cdot \sum_{i=1}^{n} ||u_i||_{L_{\frac{2m}{m-2}}(D)} \leq C_{34} \varphi_{B:m}\left(\sigma\right) \cdot \sum_{i,j=1}^{n} ||u_{ij}||_{L_q(D)},$$

where $p=\frac{2m}{m-2}$ and $q=\frac{2m(n+2)}{m(n+4)-2(n+2)}$ (see [2]). Since $m=n+2$, it is $p=\frac{2(n+2)}{n}$ and $q=2$. Therefore

$$W_1 \leq C_{34}\varphi_{B:m}\left(\sigma\right) \cdot \sum_{i,j=1}^{n} \|u_{ij}\|_{L_2(D)} \leq C_{34}\varphi_{B:m}\left(\sigma\right) \|u_{ij}\|_{W_2^2(D)} \leq C_{37}\varphi_{B:m}\left(\sigma\right) \|u\|_{W^2_{2,\lambda}(D)}. \tag{41}$$

Show that, for a $c(x) \in L_{\mu}(D)$ and $\mu = \frac{n+2}{2}$, it holds

$$\|Cu\|_{L_2(D)} \leq C_{38}\left(L, n, D\right) \varphi_{C:\mu}\left(\sigma\right) \|u\|_{W^2_{2,\lambda}(D)}. \tag{42}$$

Evidently, the estimate (38) holds for $q_1 = \frac{p_1(n+2)}{2p_1+n+2}$.

We have

$$W_2 = \|Cu\|_{L_2(D)} = \left(\int_D C^2 u^2 dx\right)^{1/2} \leq \left(\int_D C^\mu dx\right)^{1/\mu} \cdot \left(\int_D |u|^{\frac{2\mu}{\mu-2}} dx\right)^{\frac{\mu-2}{2\mu}} =$$

$$= \|C\|_{L_\mu(D)} \cdot \|u\|_{L_{\frac{2\mu}{\mu-2}}(D)} \leq C_{35}\varphi_{C:\mu}(\sigma) \cdot \|u_{ij}\|_{L_{q_1}(D)},$$

According to Theorem 2 with $p_1 = \frac{2\mu}{\mu-2}$ and $q_1 = \frac{2\mu(n+2)}{\mu(n+6)-2(n+2)}$. Since, $\mu = \frac{n+2}{2}$, it is $p_1 = \frac{2(n+2)}{n-2}$ and $q_1 = 2$. Therefore,

$$W_2 \leq C_{35}\varphi_{C:\mu}(\sigma) \|u_{ij}\|_{L_2(D)} \leq C_{35}\varphi_{C:\mu}(\sigma) \|u\|_{W_2^2(D)} \leq C_{38}\varphi_{C:\mu}(\sigma) \|u\|_{W_{2,\lambda}^2(D)} \tag{43}$$

From Theorem 1 it follows that

$$\|u\|_{W_{2,\lambda}^2(D)} \leq C_{33}\left(\|Lu\|_{L_2(D)} + \sum_{i=1}^n \|b_i u_i\|_{L_2(D)} + \|Cu\|_{L_2(D)} + \sup_D |u|\right) \leq$$

$$\leq C_{33}\left(\|Lu\|_{L_2(D)} + (C_{37}\varphi_{B:m}(\sigma) + C_{38}\varphi_{C:\mu}(\sigma)) \|u\|_{W_{2,\lambda}^2(D)} + \sup_D |u|\right).$$

Now, it suffices to set

$$C_{37}\varphi_{B:m}(\sigma) + C_{38}\varphi_{C:\mu}(\sigma) \leq \frac{1}{2C_{33}},$$

in order to get the estimate (39).

Theorem 4. *Let the conditions* (3)-(7) *be satisfied for the coefficients of operator* L*. Then for a function* $u(x) \in \overset{o}{W}{}_{2,\lambda}^2(D)$, *it holds the estimate too*

$$\|u\|_{W_{2,\lambda}^2(D)} \leq C_{39}(n, L, D) \|Lu\|_{L_q(D)}. \tag{44}$$

Proof. By assumptions, $c(x) \leq 0$ and therefore the Aleksandrov's inequality [14] takes place

$$\sup_D |u| \leq C_{40}(n, D) \left\| \frac{f}{\sqrt[n]{\det(a_{ij})}} \right\|_{L_n(D)} \cdot F_n\left(\left\| \frac{b}{\sqrt[n]{\det(a_{ij})}} \right\|_{L_n(D)} \right), \tag{45}$$

where $F_n(z) = l^{\frac{1}{n\omega_n}\left(\frac{z}{n}\right)^n + \varphi_n(z)}$, moreover φ_n are bounded and $\varphi_n(0) = 0$ (in particular, $\varphi_1 = 0$), and ω_n- is volume of unit $n-$dimensional ball

$$\|b\|_{L_n(D)} = \left\| \sqrt{\sum_{i=1}^n b_i^2} \right\|_{L_n(D)}.$$

Evidently

$$\det(a_{ij}(x)) \geq C_{41}(n, D) \prod_{i=2}^n \lambda_i(x) \geq C_{41} \prod_{i=1}^n \left[\frac{\omega_i^{-1}\left(\sum_{i=1}^n \omega_i(|x_i|)\right)}{\sum_{\varepsilon=1}^n \omega_i(|x_i|)} \right]^2.$$

By assumptions, the function $\frac{\omega_i(t)}{t}$, decreases on t in $(0,\infty)$ for any $i=1,...,n$. Therefore, the function $\frac{\omega_i^{-1}(t)}{t}$ will be increasing on $(0,\infty)$. On base of inequality $\rho(x)=\sum_{i=1}^{n}\omega_i(|x_i|)\geq\omega_i(|x_i|)$ and that the function $\frac{\omega_i^{-1}(t)}{t}$ is increasing, we get

$$\det(a_{ij}(x))\geq C_{41}\prod_{i=1}^{n}\left[\frac{\omega_i^{-1}(\omega_i(|x_i|))}{\omega_i(|x_i|)}\right]^2=C_{41}\prod_{i=1}^{n}\left(\frac{|x_i|}{\omega_i(|x_i|)}\right)^2.$$

We have

$$\left\|\frac{f}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}=\left(\int_D\frac{|f|^n}{\det(a_{ij})}dx\right)^{1/n}\leq\left(\int_D|f|^{nS}dx\right)^{1/nS}\cdot\left(\int_D\frac{dx}{(\det(a_{ij}))^{S'}}\right)^{1/S'n},$$

where $\frac{1}{S}+\frac{1}{S'}=1$.

Let $q=nS$, then $S=\frac{q}{n}$, $S'=\frac{S}{S-1}=\frac{q}{q-n}$ and

$$\left\|\frac{f}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}\leq\frac{1}{\sqrt[n]{C_{41}}}\|f\|_{L_q(D)}\left(\int_D\prod_{i=1}^{n}\left(\frac{\omega_i(|x_i|)}{(|x_i|)}\right)^{\frac{2q}{q-n}}dx\right)^{\frac{q-n}{qn}}. \tag{46}$$

Here the condition

$$\frac{2q}{q-n}>-1,$$

is needed in order to get the finiteness of the integral in the right hand side. That integral is finite, since $q>\frac{n}{3}$.

Now, prove that the multiplier in the right hand side (45) is finite. Indeed

$$\left\|\frac{b}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}=\left\|\sqrt{\sum_{i=1}^{n}\left(\frac{b}{\sqrt[n]{\det(a_{ij})}}\right)^2}\right\|_{L_n(D)}=$$

$$=\left[\int_D\left(\sqrt{\sum_{i=1}^{n}\left(\frac{b_i}{\sqrt[n]{\det(a_{ij})}}\right)^2}\right)dx\right]^{1/n}\leq C_{42}(n)\left(\sum_{i=1}^{n}\int_D\frac{|b_i|^n}{\det(a_{ij})}dx\right)^{1/n}.$$

Therefore, we have

$$\int_D\frac{|b_i|^n}{\det(a_{ij})}dx\leq\left(\int_D|b_i|^m dx\right)^{\frac{n}{m}}\cdot\left(\int_D\frac{dx}{(\det(a_{ij}))^{\frac{m}{m-n}}}\right)^{\frac{m-n}{m}}\leq C_{43}(n,\gamma,D)\|b_i\|^n_{L_m^{(D)}}\cdot$$

$$\cdot\left(\int_D\prod_{i=1}^{n}\left(\frac{\omega_i(|x_i|)}{(|x_i|)}\right)^{\frac{2m}{m-n}}dx\right)^{\frac{m-n}{m}}=C_{43}\|b_i\|^n_{L_m^{(D)}}\cdot\left(\int_D\prod_{i=1}^{n}\left(\frac{\omega_i(|x_i|)}{(|x_i|)}\right)^{n+2}dx\right)^{\frac{2}{n+2}}=$$

$$= C_{43} \|b_i\|^n_{L_m^{(D)}} \cdot \left\| \prod_{i=1}^{n} \left(\frac{\omega_i\left(|x_i|\right)}{\left(|x_i|\right)} \right) \right\|^2 . \tag{47}$$

Then according to (7), the right hand side (47) is finite. Thus

$$\sup_D |u| \leq C_{44}\left(n, \gamma, q, D\right) \|Lu\|_{L_q(D)} . \tag{48}$$

On other hand

$$\|Lu\|_{L_2(D)} = \left(\int_D |Lu|^2 \, dx \right)^{1/2} \leq \left(\int_D |Lu|^q \, dx \right)^{1/q} \cdot \left(\int_D 1^{\frac{q}{q-2}} dx \right)^{\frac{q-2}{2q}} =$$

$$= (mesD)^{\frac{q-2}{2q}} \cdot \|Lu\|_{L_q(D)} . \tag{49}$$

From (39), (45) and (49), we infer

$$\|u\|_{W^2_{2,\lambda}(D)} \leq C_{36} \left((mesD)^{\frac{q-2}{2q}} + C_{44} \right) \|Lu\|_{L_q(D)} ,$$

Therefore, the estimate (44) has been proved.

4. Strong solvability of the first boundary value problem

Consider the first boundary value problem (1)-(2) in the domain $D \subset \Re^n$. A function $u\left(x\right) \in \overset{o}{W}{}^2_{2,\lambda}\left(D\right),$ is called the strong solution of this problem if that satisfies (1) almost everywhere in D.

Theorem 5. *Let the coefficients of operator L are defined in D and it is satisfied the conditions* (3)-(7). *Then for $q > \frac{n}{3}$, the first boundary value problem* (1)-(2) *uniquely solvable in space $\overset{o}{W}{}^2_{2,\lambda}\left(D\right)$ for any function $f\left(x\right) \in L_q\left(D\right)$. Moreover, the function $u\left(x\right)$ satisfies to the inequality*

$$\|u\|_{W^2_{2,\lambda}(D)} \leq C_{39} \|f\|_{L_q(D)} . \tag{50}$$

Proof. Assume first the littler terms coefficients of equation (1) and the right hand side $f\left(x\right)$ be infinitely differentiable in $\bar{D}$. Introduce the integer numbers $s \in \aleph$, $D^+\left(s\right) = \left\{x : x \in D, \, \rho\left(x\right) < \frac{1}{s}\right\}$; and $i, j = 1, .., n$

$$\lambda_i^{(S)}\left(x\right) = \begin{cases} \lambda_i\left(x\right), & if \;\; x \in \bar{D} \backslash D^+\left(s\right), \\ \left[\frac{\omega_i^{-1}\left(\frac{1}{s}\right)}{\frac{1}{s}} \right]^2, & if \;\; x \in D^+\left(s\right); \end{cases}$$

$a_{ij}^{(s)}\left(x\right) = a_{ij}\left(x\right)$, for $x \in \bar{D} \backslash D^+\left(\frac{s}{2}\right)$, $a_{ij}^{((s)}\left(x\right)$ are extending over $D^+\left(\frac{s}{2}\right)$ such that, $a_{ij}^{(s)}\left(x\right) \in C\left(\bar{D}\right)$ and for any $x \in D$ and $\zeta \in E_n$ it satisfies

$$\bar{\gamma} \sum_{i=1}^{n} \lambda_i^{(s)}\left(x\right) \zeta_i^2 \leq \sum_{i,j=1}^{n} a_{ij}^{(s)}\left(x\right) \zeta_i \zeta_j \leq \bar{\gamma}^{-1} \sum_{i=1}^{n} \lambda_i^{(s)}\left(x\right) \zeta_i^2,$$

where $\bar{\gamma} = \gamma/2$, and $\gamma-$ is a constant from (3).

Set

$$L^{(S)} = \sum_{i,j=1}^{n} a_{ij}^{(s)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x).$$

It is clear to see that for an integer s the operator $L(s)$ is uniformly elliptic in D. Let $u^{(s)}(x)-$ be a strong solution of the first boundary value problem

$$L^{(S)} u^{(S)} = f(x), \ \ x \in D; \ \ u^{(S)}\Big|_{\partial D} = 0. \tag{51}$$

Since $a_{ij}^{(s)}(x) \in C\left(\bar{D}\right)$, according to [2] there exists a strong solution for the problem (51) that is unique and belongs to the space $\overset{o}{W}{}_p^2(D)$ for any $p \in (1,\infty)$. Where $\overset{o}{W}{}_p^2(D)$ denotes the closure of all functions $u(x) \in C^\infty\left(\bar{D}\right)$ with $u|_{\partial D} = 0$ in the norm

$$\|u\|_{W_p^2(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^{n} |u_i|^p + \sum_{i,j=1}^{n} |u_{ij}|^p\right) dx\right)^{1/p}.$$

Show that $u^{(s)}(x) \in \overset{o}{W}{}_{2,\lambda}^2(D)$. Let $p > 2-$ be a real number. For $i,j = 1,...,n$, we have

$$\int_D \lambda_i(x)\lambda_j(x)\left(u_{ij}^S\right)^2 dx \le \left(\int_D \left|u_{ij}^{(S)}\right|^p dx\right)^{2/p} \cdot \left(\int_D \left[\lambda_i(x)\lambda_j(x)\right]^{\frac{p}{p-2}} dx\right)^{\frac{p}{p-2}}$$

By using Lemma 1, there exists a large number p, such that

$$\int_D \left[\lambda_i(x)\lambda_j(x)\right]^{\frac{p}{p-2}} dx < \infty, \ \ i,j = 1,...,n. \tag{52}$$

Evidently

$$\lambda_i(x) \cdot \lambda_j(x) = \left[\frac{\omega_i^{-1}(\rho(x))}{\rho(x)}\right]^2 \cdot \left[\frac{\omega_j^{-1}(\rho(x))}{\rho(x)}\right]^2.$$

From this according to Theorem 4, we infer

$$\left\|u^{(S)}\right\|_{W_{2,\lambda}^2(D)} \le C_{39} \|f\|_{L_q(D)}. \tag{53}$$

It follows from the strong boundedness of the sequence $\left\{u^{(s)}(x)\right\}$ in $\overset{o}{W}{}_{2,\lambda}^2(D)$ that this is a weakly compact sequence in this space. Therefore, there exists a function $u'(x) \in \overset{o}{W}{}_{2,\lambda}^2(D)$ and a subsequence of integer numbers $\{s_k\}$ such that

$$\lim_{k\to\infty}\left(Lu^{(s_k)}, \psi\right) = \left(Lu', \psi\right), \tag{54}$$

for a function $\psi(x) \in C^{\infty}\left(\bar{D}\right)$ as $k \to \infty$. Where $(g,\psi) = \int_D g(x) \cdot \psi(x)\,dx$. On other hand

$$\left(Lu^{(s_k)}, \psi\right) = \left(L^{(s_k)} u^{(s_k)}, \psi\right) + \left(\left(L - L^{s_k}\right) u^{(s_k)}, \psi\right) =$$
$$= (f,\psi) + \left(\left(L - L^{(s_k)}\right) u^{(s_k)}, \psi\right) = (f,\psi) + i_k,$$

which together with (54) means that

$$(f,\psi) + \lim_{k\to\infty} i_k = \left(Lu', \psi\right). \tag{55}$$

Further, we have

$$|i_k| \le \sum_{i,j=1}^{n} \int_{D^+(S_k/2)} \frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\,\lambda_j(x)}} \sqrt{\lambda_i(x)\,\lambda_j(x)} \left|u_{ij}^{(S_k)}\right| \cdot |\psi|\,dx +$$

$$+ \sum_{i,j=1}^{n} \int_{D^+(S_k/2)} \frac{\left|a_{ij}^{(S_k)}(x)\right|}{\sqrt{\lambda_i(x)\,\lambda_j(x)}} \sqrt{\lambda_i(x)\,\lambda_j(x)} \left|u_{ij}^{(S_k)}\right| \cdot |\psi|\,dx = i_k^1 + i_k^2. \tag{56}$$

From conditions (5)-(6) it follows that $|h_{ij}(x)| \le h_0(L)$ for $x \in D$ and $i,j = 1,...,n$. Therefore, and using (53), we get

$$i_k^1 \le h_0 \cdot \|u\|_{W^2_{2,\lambda}(D^+(S_k/2))} \cdot \|\psi\|_{L_2(D)}, \quad \text{i.e. } \lim_{k\to\infty} i_k^1 = 0. \tag{57}$$

Arguing by the analogy with preceding it follows that for the equality

$$\lim_{k\to\infty} i_k^2 = 0 \tag{58}$$

it suffices that

$$\int_D \frac{dx}{\lambda_i(x)\,\lambda_j(x)} < \infty, \; i,j = 1,...,n. \tag{59}$$

Indeed, by using Lemma 1,

$$\int_D \frac{dx}{\lambda_i(x)\,\lambda_j(x)} \le \int_D \left(\frac{\omega_i(|x_i|)}{|x_i|}\right)^2 dx < \infty,$$

therefore, the inequality satisfied. From (55)-(58) it follows that for a function $\psi(x) \in C^{\infty}\left(\bar{D}\right)$ it holds the equality

$$\left(Lu', \psi\right) = (f,\psi),$$

therefore, $Lu' = f(x)$ almost everywhere in D.

Consider the general situation. Let O_1 be $n-$ dimensional ball of unit radii and center in the coordinate center, a function $\vartheta_1(x) \subset C_0^{\infty}(E_n)$ be such that $\vartheta_1(x) \ge 0$, $\vartheta_1(x) = 0$ everywhere outside O_1 and $\int_{E_n} \vartheta_1(x)\,dx = 1$.

Set $\vartheta_\varepsilon(x) = \frac{1}{\varepsilon^n} \vartheta_1\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon > 0$.

For a locally integrable function $\psi(x)$ in E_n denote $\psi^{\varepsilon}(x)=\int_{E_n}\vartheta_{\varepsilon}(x-y)\,\psi(y)\,dy$ the Frederiche's average of $\psi(x)$ with parameter ε.

Let for $i=1,...,n$ the functions $b_i^{[l]}(x)$, $C^{[l]}(x)$ and $f^{[l]}(x)$ are mollifies of the proper functions with parameter $\frac{1}{l}$

$$L^{[l]}=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i^{[l]}(x)\frac{\partial}{\partial x_i}+C^{[l]}(x),$$

and $u^{[l]}(x)$ be a strong solution from $\overset{o}{W}{}^2_{2,\lambda}(D)$ of the first boundary value problem

$$L^{[l]}u^{[l]}=f^{[l]}(x),\ \ x\in D;\ \ u^{[l]}\Big|_{\partial D}=0.$$

According to the preceding results, such a solution exists, moreover the Theorem 4 conforms on the estimate

$$\left\|u^{[l]}\right\|_{W^2_{2,\lambda}(D)}\le C_{39}\left\|f^{[l]}\right\|_{L_q(D)}\le C_{45}(L,n,q,D,f). \tag{60}$$

Therefore, there exists a solution $u(x)\in\overset{o}{W}{}^2_{2,\lambda}(D)$ and a subsequence of natural numbers $\{l_k\}$ such that

$$\lim_{k\to\infty}\left(Lu^{[l_k]},\psi\right)=(Lu,\psi) \tag{61}$$

as $k\to\infty$ for a function $\psi(x)\in C^{\infty}\left(\bar{D}\right)$.

On other hand

$$\left(Lu^{[l_k]},\psi\right)=\left(L^{[l_k]}u^{[l_k]},\psi\right)+\left(\left(L-L^{[l_k]}\right)u^{[l_k]},\psi\right)=$$

$$=\left(f^{[l_k]},\psi\right)+\left(L-L^{[l_k]}u^{[l_k]},\psi\right)=\left(f^{[l_k]},\psi\right)+j_k.$$

The (61) and the limit expression

$$\lim_{k\to\infty}\left(f^{[l_k]},\psi\right)=(f,\psi)$$

yields

$$(f,\psi)+\lim_{k\to\infty}j_k=(Lu,\psi). \tag{62}$$

Further, we have

$$|j_k|\le\sum_{i=1}^{n}\int_D\left|b_i(x)-b_i^{[l_k]}(x)\right|\cdot\left|u_i^{[l_k]}\right|\cdot|\psi|\,dx+$$

$$+\int_D\left|C(x)-C^{[l_k]}(x)\right|\cdot\left|u^{[l_k]}(x)\right|\cdot|\psi(x)|\,dx=j_k^1+j_k^2. \tag{63}$$

According to (40) and (42), we infer

$$j_k^1 \leq \|\psi\|_{L_2(D)} \cdot \sum_{i=1}^{n} \int_D \left\| \left(b_i - b_i^{[l_k]} \right) u_i^{[l_k]} \right\|_{L_2(D)} \leq$$

$$\leq C_{46}\left(L,n,D\right) \|\psi\|_{L_2(D)} \cdot \max \left\| b_i - b_i^{[l_k]} \right\|_{L_m(D)} \cdot \left\| u^{[l_k]} \right\|_{W_{2,\lambda}^2(D)};$$

$$j_k^2 \leq \|\psi\|_{L_2(D)} \cdot \left\| \left(C - C^{[l_k]} \right) u^{[l_k]} \right\|_{L_2(D)} \leq$$

$$\leq C_{47}\left(L,n,D\right) \|\psi\|_{L_2(D)} \cdot \left\| C - C^{[l_k]} \right\|_{L_\mu(D)} \cdot \left\| u^{[l_k]} \right\|_{W_{2,\lambda}^2(D)},$$

where the constants m and μ have the meaning as in the condition (7).

Using (60), we get

$$\lim_{k\to\infty} j_k^1 = \lim_{k\to\infty} j_k^2 = 0,$$

which together with (62) and (63) give

$$(Lu,\psi) = (f,\psi)\,. \tag{64}$$

Since the equality (64) is true for a function $\psi\left(x\right) \in C^\infty\left(\bar{D}\right)$, then $Lu = f\left(x\right)$ for almost everywhere D. Therefore, it has been proved the existence of strong solution of the boundary value problem (1)-(2) and the estimate (50) follows from the Corollary of Theorem 4.

Prove now the uniqueness of the boundary value problem (1)-(2). Let $u_1\left(x\right)$ and $u_2\left(x\right)$ be different solutions of that problem. Set $\vartheta\left(x\right) = u_1\left(x\right) - u_2\left(x\right)$, then the function $\vartheta\left(x\right)$ will be a generalized solution of the problem (1)-(2) with $f\left(x\right) \equiv 0$. According to (50) $\vartheta\left(x\right) \equiv 0$ almost everywhere in D, i.e. $u_1\left(x\right) \equiv u_2\left(x\right)$ a.e. in D.

This completes the proof of Theorem 5.

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On the Influence of the Short- and Open-circuit Conditions on Stability loss of the PZT/Metal/PZT Sandwich Circular Plate-disc Condition

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Abstract. The axisymmetric stability loss of the PZT/Metal/PZT sandwich circular plate is investigated simultaneously within the scope of the open-circuit and short-circuit electrical conditions. It is assumed that these conditions satisfy on the upper and lower face-planes of the piezoelectric layers. Moreover, it is assumed that on the lateral-boundary cylindrical surfaces of the piezoelectric face layers the short-circuitconditions satisfy. The 3D linearized stability loss theory for piezoelectric materials is employed for investigation of the corresponding eigenvalue problem. Concrete numerical results are obtained by utilizing FEM for various piezoelectric face and metal core layers and the main attention is focused on the influence of the piezoelectricity on the values of the compressional critical stress and the influence of the aforementioned two type electrical boundary conditions on these stresses. According to the comparison of the results, it is made conclusions on the significance of the influence of the electrical boundary conditions on the values of the absolute values of the critical stresses. In particular, it is established that in the case where the open-circuit boundary conditions satisfy the influence of the piezoelectricity of the face layers materials on the critical stresses is more significant than that in the case where the short-circuit boundary conditions satisfy.

Key Words and Phrases: Piezoelectric material, open-circuit, short-circuit, circular sandwich plate, stability loss, critical stress
2010 Mathematics Subject Classifications: 74H55

1. Introduction

Investigations of stability loss of plate type element of constructions made of piezoelectric materials (shortly PZT) or made of layered composites containing PZT layers has a great significance not only in the theoretical, but also in the practical sense. Researchers such as [11], Jerom and Ganesan (2010) and many others listed therein can be taken as examples for such investigations in which it was established that the piezoelectricity of the plate or beam materials causes an increase in the values of the mechanical critical forces

Now we consider a brief review of the related recent investigations and first note the paper by [10] in which static analysis of the simply supported rectangular plate made

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of functionally graded piezoelectric material is studied with the use of the refined plate theories. The "open- and closed- circuit" conditions on the upper and lower face surfaces are considered. The paper by [4] deals with the study the response of the bi-layered circular plate made of functionally-graded piezoelectric material and resting on a Winkler-Pasternak foundation.

[7] studies the buckling of the sandwich circular plate with piezoelectric face and porous middle layers under radial compression within the scope of the Kirchhoff-Love plate theory and therefore results obtained in this paper are acceptable for very thin plates. The analytical expression for the critical force is obtained and according to this expression the influence of the problem parameters, as well as of the piezoelectricity of the covering layer material is discussed.

The foregoing brief review shows that all the foregoing investigations have been made within the scope of the approximate plate theories, the accuracy of which depends significantly on the geometrical and electro-mechanical properties. It is obvious that the order of the accuracy of these results can be estimated with the use of the corresponding results obtained within the scope of the 3D linearized exact stability loss theories the present level of which has been detailed in the monograph by [5] who made many fundamental contributions to creating this theory. At the same time we note that the 3D linearized stability loss theories for the elements of constructions made of time-dependent materials was developed in the monograph by [1].

In the foregoing sense, in the paper by [2] the first attempt with respect to the stability loss problems related to the system comprising elastic and piezoelectric constituents was made. At the same time, it should be noted that the study of stability loss of elements of constructions made of piezoelectric materials by employing 3D linearized stability loss theories just is beginning.

One of the main question in the theory of piezoelectricity, as well as in the investigations of a stability loss of element of constructions made of these materials, is the study the influence of the "open-circuit" and "short-circuit" type electrical boundary conditions on the electro-mechanical behavior of these constructions. Taking the this statement into consideration in the present paper the aforementioned influence is studied for the circular sandwich PZT/Metal/PZT plate within the scope of the 3D linearized stability loss theory. Under this study it is assumed that the plate is compressed in the radial inward direction by uniformly distributed rotationally symmetric normal forces.

We recall that the corresponding 3D stability loss problems for the circular plate consisting of elastic and viscoelastic constituents are made in the papers by [3] and [9] the results of which are also detailed in the monograph by [1].

2. Formulation of the problem

We consider a circular sandwich plate whose geometry is shown in Fig. 1 and assume that the materials of the upper and lower face layer are the same and PZT.

We associate with the lower face layer of the plate the cylindrical coordinate system $Or\theta z$(Fig. 1) and the position of the points of the plate we determine through the Lagrange coordinates in this system. Thus, according to Fig.1, in the selected coordinate system, the plate occupies the region $\{0 \le r \le \ell/2; 0 \le \theta \le 2\pi; 0 \le z \le h\}$. Investigate the axisymmetris (rotationally symmetric) stability loss of the mentioned plate under compression of that in the inward radial direction by uniformly distributed rotationally symmetric normal forces with intensity p acting on the lateral boundary-surface.

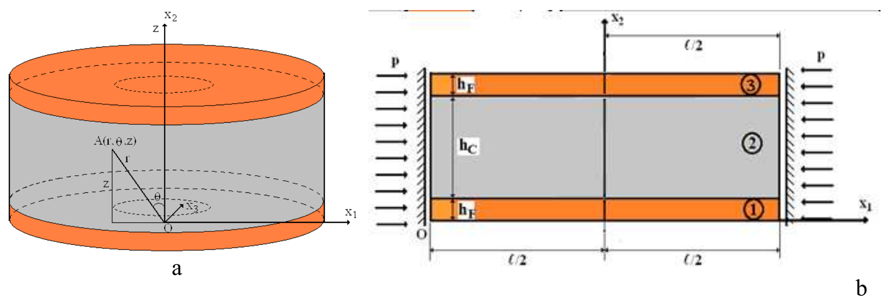

Fig. 1. The geometry of the considered circular plate (a) and the cross section of this plate with loading condition and some geometric values (b).

Below we will denote the values related to the upper and lower face layers by upper indices (3) and (1) respectively, whereas the values related to the core layer are denoted by (2). Moreover, the values related to the pre-critical stress-strain state are denoted by additional upper index 0.

Under investigations we will consider two type boundary conditions on the upper and lower face planes of the PZT layers. The first type of these conditions are the "open-circuit" ones, according to which it is assumed that $D_z^1 = 0$ at $z = 0$ and h_F, and $D_z^3 = 0$ at $z = h_c + h_F$ and $h_c + 2h_F$, $D_z^{(k)}$ is a normal component of the electric displacement. The second type conditions are the "short-circuit" ones, according to which, $\phi^1 = 0$ at $z = 0$ and h_F, and $\phi^3 = 0$ at $z = h_c + h_F$ and $h_c + 2h_F$, where $\phi^{(k)}$ is a potential of an electric field and $E_r^{(k)} = -\partial\phi^{(k)}/\partial r$, $E_z^{(k)} = -\partial\phi^{(k)}/\partial z$. Here $E_r^{(k)}$ and $E_z^{(k)}$are the radial and normal component of the electric field vector.

In the case where the "open-circuit" conditions take place the pre-critical stress state is determined according to the following expressions:

$$\sigma_{zz}^{(k),0} = 0, \sigma_{rz}^{(k),0} = 0, s_{zz}^{(k),0} = 0, D_z^{(k),0} = D_r^{(k),0} = 0, k = 1,2,3.$$

$$E_r^{(k),0} = a_1^{(k)} s_{rr}^{(k),0} + b_1^{(k)} s_{zz}^{(k),0}, E_z^{(k),0} = d_1^{(k)} s_{rr}^{(k),0} + c_1^{(k)} s_{zz}^{(k),0},$$

$$a_1^{(k)} = \frac{\varepsilon_{13}^{(k)}(e_{31}^{(k)} + e_{32}^{(k)}) - \varepsilon_{33}^{(k)}(e_{11}^{(k)} + e_{22}^{(k)})}{\varepsilon_{11}^{(k)}\varepsilon_{33}^{(k)} - \varepsilon_{13}^{(k)}\varepsilon_{31}^{(k)}}, b_1^{(k)} = \frac{\varepsilon_{13}^{(k)} e_{33}^{(k)} - \varepsilon_{33}^{(k)} e_{13}^{(k)}}{\varepsilon_{11}^{(k)}\varepsilon_{33}^{(k)} - \varepsilon_{13}^{(k)}\varepsilon_{31}^{(k)}},$$

$$d_1^{(k)} = \frac{\varepsilon_{11}^{(k)}(e_{31}^{(k)} + e_{32}^{(k)}) - \varepsilon_{31}^{(k)}(e_{11}^{(k)} + e_{12}^{(k)})}{\varepsilon_{13}^{(k)}\varepsilon_{31}^{(k)} - \varepsilon_{11}^{(k)}\varepsilon_{33}^{(k)}}, c_1^{(k)} = \frac{\varepsilon_{11}^{(k)} e_{33}^{(k)} - \varepsilon_{31}^{(k)} e_{13}^{(k)}}{\varepsilon_{13}^{(k)}\varepsilon_{31}^{(k)} - \varepsilon_{11}^{(k)}\varepsilon_{33}^{(k)}}.$$

$$s_{zz}^{(k),0} = a_{zr}^{(k)} s_{rr}^{(k),0}, a_{zr}^{(k)} = \frac{c_{31}^{(k)} + c_{32}^{(k)} - e_{13}^{(k)} a_1^{(k)} - e_{33}^{(k)} d_1^{(k)}}{c_{33}^{(k)} - e_{13}^{(k)} b_1^{(k)} - e_{33}^{(k)} c_1^{(k)}}.$$

$$\sigma_{rr}^{(k),0} = A_r^{(k)} s_{rr}^{(k),0}, A_r^{(k)} = c_{11}^{(k)} + c_{12}^{(k)} - e_{11}^{(k)} a_1^{(k)} +$$
$$+ a_{zr}^{(k)} c_{13}^{(k)} - a_{zr}^{(k)} e_{11}^{(k)} b_1^{(k)} - a_{zr}^{(k)} e_{31}^{(k)} c_1^{(k)}.$$

$$s_{rr}^{(1),0} = s_{rr}^{(2),0}, 2h_f \sigma_{rr}^{(1),0} + h_C \sigma_{rr}^{(2),0} = hp, \sigma_{rr}^{(1),0} = p \left(2\frac{h_F}{h} + \frac{h_C}{h} \frac{A_r^{(2)}}{A_r^{(1)}} \right)^{-1}. \tag{1}$$

In (1) $\sigma_{rr}^{(k),0}, \ldots,$ and $s_{rr}^{(k),0}, \ldots,$ are the components of the stress and Green strain tensors, respectively, $u_r^{(k),0}$ and $u_z^{(k),0}$ are components of the displacement vector, $D_r^{(k),0}$ and $D_z^{(k),0}$ are the components of the electrical displacement vector, $E_r^{(k),0}$ and $E_z^{(k),0}$ are the components of the electric field vector, and $c_{ij}^{(k)}$, $e_{ij}^{(k)}$ and $\varepsilon_{nj}^{(k)}$ are the elastic, piezoelectric and dielectric constants, respectively.

In the case where the aforementioned "short-circuit" conditions are satisfied, the precritical state is determined according to the following expressions.

$$\phi^{(k),0} = 0, E_r^{(k),0} = E_z^{(k),0} = 0, \sigma_{zz}^{(k),0} = \sigma_{rz}^{(k),0} = 0, \sigma_{rr}^{(k),0} = \sigma_{\theta\theta}^{(k),0},$$

$$s_{rr}^{(k),0} = s_{\theta\theta}^{(k),0} = (c_{11}^{(k)} + c_{12}^{(k)})^{-1} \sigma_{rr}^{(k),0}, \sigma_{rr}^{(2),0} = \sigma_{rr}^{(1),0} (c_{11}^{(1)} + c_{12}^{(1)})^{-1} (c_{11}^{(2)} + c_{12}^{(2)}),$$

$$\sigma_{rr}^{(1),0} = p \left(2\frac{h_F}{h} + \frac{h_C}{h} \frac{(c_{11}^{(2)} + c_{12}^{(2)})}{(c_{11}^{(1)} + c_{12}^{(1)})} \right)^{-1},$$

$$D_r^{(k),0} = \left(e_{11}^{(k)} + e_{12}^{(k)} - \frac{c_{31}^{(k)} + c_{32}^{(k)}}{c_{33}^{(k)}} e_{13}^{(k)} \right) s_{rr}^{(k),0},$$

$$D_z^{(k),0} = \left(e_{31}^{(k)} + e_{32}^{(k)} - \frac{c_{31}^{(k)} + c_{32}^{(k)}}{c_{33}^{(k)}} e_{33}^{(k)} \right) s_{rr}^{(k),0}. \tag{2}$$

Note that the expressions in (1) and (2) are approximate in the near vicinity of the lateral boundary surface on which the external compressional radial forces act. Nevertheless, as we will consider the cases where $h/\ell \sim 10^{-1}$ (where $h = 2h_F + h_C$ (Fig. 1), therefore the influence of the mentioned proximity on the values of the critical parameters can be taken as insignificant one. Moreover, note that the expressions in (1) are obtained within the scope of the "open-circuit" condition satisfied on the face layers upper and lower plane-boundaries, according to which, the normal component of the electrical displacement vector on these planes is equal to zero.

Thus, within the scope of the foregoing assumptions, according to [5], [11], [2] the 3D linearized stability loss equations and relations for the case under consideration are obtained as follows:

3D linearized stability loss equations

$$\frac{\partial t_{rr}^{(k)}}{\partial r}+\frac{\partial t_{zr}^{(k)}}{\partial z}+\frac{1}{r}(t_{rr}^{(k)}-t_{\theta\theta}^{(k)})=0, \frac{\partial t_{rz}^{(k)}}{\partial r}+\frac{\partial t_{zz}^{(k)}}{\partial z}+\frac{1}{r}t_{rz}^{(k)}=0,$$

$$\frac{\partial D_R^{(k)}}{\partial r}+\frac{1}{r}D_R^{(k)}+\frac{\partial D_Z^{(k)}}{\partial z}=0.$$

$$t_{rr}^{(k)}=\sigma_{rr}^{(k)}+\sigma_{rr}^{(k),0}\frac{\partial u_r^{(k)}}{\partial r}+M_{rr}^{(k)}, t_{\theta\theta}^{(k)}=\sigma_{\theta\theta}^{(k)}+\sigma_{\theta\theta}^{(k),0}\frac{u_r^{(k)}}{r}+M_{\theta\theta}^{(k)},$$

$$t_{zr}^{(k)}=\sigma_{zr}^{(k)}+M_{zr}^{(k)}, t_{rz}^{(k)}=\sigma_{rz}^{(k)}+\sigma_{rr}^{(k),0}\frac{\partial u_z^{(k)}}{\partial r}+M_{rz}^{(k)}, t_{zz}^{(k)}=\sigma_{zz}^{(k)}+M_{zz}^{(k)},$$

$$M_{rr}^{(k)}=E_r^{(k),0}E_r^{(k)}-E_\theta^{(k),0}E_\theta^{(k)}-E_z^{(k),0}E_z^{(k)},$$

$$M_{\theta\theta}^{(k)}=E_\theta^{(k),0}E_\theta^{(k)}-E_r^{(k),0}E_r^{(k)}-E_z^{(k),0}E_z^{(k)},$$

$$M_{zz}^{(k)}=E_z^{(k),0}E_z^{(k)}-E_\theta^{(k),0}E_\theta^{(k)}-E_r^{(k),0}E_r^{(k)}. \tag{3}$$

Linearized strain-displacement relations

$$s_{rr}^{(k)}=\frac{\partial u_r^{(k)}}{\partial r}, s_{\theta\theta}^{(k)}=\frac{u_r^{(k)}}{r}, s_{zz}^{(k)}=\frac{\partial u_z^{(k)}}{\partial z}, s_{rz}^{(k)}=\frac{1}{2}\left(\frac{\partial u_r^{(k)}}{\partial z}+\frac{\partial u_z^{(k)}}{\partial r}\right), \tag{4}$$

Linearized electro-mechanical relations

$$\sigma_{rr}^{(k)}=c_{11}^{(k)}s_{rr}^{(k)}+c_{12}^{(k)}s_{\theta\theta}^{(k)}+c_{13}^{(k)}s_{zz}^{(k)}-e_{11}^{(k)}E_r^{(k)}-e_{31}^{(k)}E_z^{(k)},$$

$$\sigma_{\theta\theta}^{(k)}=c_{12}^{(k)}s_{rr}^{(k)}+c_{22}^{(k)}s_{\theta\theta}^{(k)}+c_{23}^{(k)}s_{zz}^{(k)}-e_{12}^{(k)}E_r^{(k)}-e_{32}^{(k)}E_z^{(k)},$$

$$\sigma_{zz}^{(k)}=c_{31}^{(k)}s_{rr}^{(k)}+c_{32}^{(k)}s_{\theta\theta}^{(k)}+c_{33}^{(k)}s_{zz}^{(k)}-e_{13}^{(k)}E_r^{(k)}-e_{33}^{(k)}E_z^{(k)},$$

$$\sigma_{rz}^{(k)}=c_{15}^{(k)}s_{rz}^{(k)}-e_{15}^{(k)}E_r^{(k)}-e_{35}^{(k)}E_z^{(k)},$$

$$D_r^{(k)}=e_{11}^{(k)}s_{rr}^{(k)}+e_{12}^{(k)}s_{\theta\theta}^{(k)}+e_{13}^{(k)}s_{zz}^{(k)}+\varepsilon_{11}^{(k)}E_r^{(k)}+\varepsilon_{13}^{(k)}E_z^{(k)},$$

$$D_z^{(k)}=e_{31}^{(k)}s_{rr}^{(k)}+e_{32}^{(k)}s_{\theta\theta}^{(k)}+e_{33}^{(k)}s_{zz}^{(k)}+\varepsilon_{31}^{(k)}E_r^{(k)}+\varepsilon_{33}^{(k)}E_z^{(k)},$$

$$E_r^{(k)}=-\frac{\partial\phi^{(k)}}{\partial r}, E_z^{(k)}=-\frac{\partial\phi^{(k)}}{\partial z}. \tag{5}$$

Note that the relations in (1) - (3) are written for the piezoelectric materials and supposing that $e_{ij}^{(k)}=0$ and $\varepsilon_{ij}^{(k)}=0$ we can obtain the mechanical relations for the elastic materials. Moreover, note that under writing of the relations in (5) it is assumed that the polled direction of the piezoelectric material is the Oz axis direction (Fig. 1). At the same time, we assume that the contact and boundary conditions given below satisfy.

$$t_{zz}^{(3)}\Big|_{z=h_F+h_C} = t_{zz}^{(2)}\Big|_{z=h_F+h_C}, \; t_{zr}^{(3)}\Big|_{z=h_F+h_C} = t_{zr}^{(2)}\Big|_{z=h_F+h_C},$$

$$u_z^{(3)}\Big|_{z=h_F+h_C} = u_z^{(2)}\Big|_{z=h_F+h_C},$$

$$u_r^{(3)}\Big|_{z=h_F+h_C} = u_r^{(2)}\Big|_{z=h_F+h_C}, \; t_{zz}^{(2)}\Big|_{z=h_F} = t_{zz}^{(1)}\Big|_{z=h_F}, \; t_{zr}^{(2)}\Big|_{z=h_F} = t_{zr}^{(1)}\Big|_{z=h_F},$$

$$u_z^{(2)}\Big|_{z=h_F} = u_z^{(1)}\Big|_{z=h_F}, \; u_r^{(2)}\Big|_{z=h_F} = u_r^{(1)}\Big|_{z=h_F},$$

$$t_{zz}^{3}\big|_{z=2h_F+h_C} = 0, \; t_{zr}^{3}\big|_{z=2h_F+h_C} = 0, \; t_{zz}^{3}\big|_{z=0} = 0, \; t_{zr}^{1}\big|_{z=0} = 0 \text{ for } 0 \le r \le l/2,$$

$$t_{rr}^{(k)}\Big|_{r=l/2} = 0, \; u_z^{(k)}\Big|_{r=l/2} = 0, \text{ for } k = 1,2,3 \,\text{under}\, r = l/2 \, 0 \le z \le 2h_F + h_C. \quad (6)$$

Note that the conditions given in (6) relate to the mechanical quantities and the corresponding conditions for the electrical quantities are given for the components of the electrical displacements $D_z^{(k)}$and $D_r^{(k)}$, or for the electric potential $\phi^{(k)}$. In the case where we assume that the "open-circuit" conditions satisfy the following relations take place

$$D_z^{(3)}\Big|_{z=2h_F+h_C} = 0, \, D_z^{(3)}\Big|_{z=h_F+h_C} = 0, \, D_z^{(1)}\Big|_{z=0} = 0, \, D_z^{(1)}\Big|_{z=h_F} = 0, \quad (7)$$

$$\phi^{(3)}\Big|_{r=l/2} = 0, \, \phi^{(1)}\Big|_{r=l/2} = 0. \quad (8)$$

However in the case where we assume that the "short-circuit" conditions satisfy the relations in (7) are replaced with the following ones.

$$\phi^{(3)}\Big|_{z=2h_F+h_C} = 0, \, \phi^{(3)}\Big|_{z=h_F+h_C} = 0, \, \phi^{(1)}\Big|_{z=0} = 0, \, \phi^{(1)}\Big|_{z=h_F} = 0. \quad (9)$$

Consequently, in the present investigation we consider simultaneously two cases determined by conditions in (7) and (9).

This completes the formulation of the problem, according to which, the determination of the critical values of the pre-critical quantities is reduced to the solution of the eigenvalue problem (1), (3) – (8) for the "open-circuit" case, and (2),(3) – (6), (8) and (9) for the "short-circuit" case.

3. FEM modelling of the problem

We attempt to solve to the problem formulated in the previous section by employing FEM and for this purpose, according to [5], [11], [2] and others, we introduce the following functional.

$$\Pi(u_r^{(1)}, u_r^{(2)}, u_r^{(3)}, u_z^{(1)}, u_z^{(2)}, u_z^{(3)}, \phi^{(1)}, \phi^{(2)}, \phi^{(3)}) =$$

$$\frac{1}{2}\sum_{k=1}^{3}\iint_{\Omega^{(k)}}\left[t_{rr}^{(k)}\frac{\partial u_r^{(k)}}{\partial r}+t_{\theta\theta}^{(k)}\frac{u_r^{(k)}}{r}+t_{rz}^{(k)}\frac{\partial u_z^{(k)}}{\partial r}+t_{zr}^{(k)}\frac{\partial u_r^{(k)}}{\partial z}+\right.$$

$$\left.+t_{zz}^{(k)}\frac{\partial u_z^{(k)}}{\partial z}+E_r^{(k)}D_r^{(k)}+E_z^{(k)}D_z^{(k)}\right]rdrdz, \tag{10}$$

where

$$\Omega^{(1)} = \{0 \le r \le \ell/2; 0 \le z \le h_F\},$$

$$\Omega^{(2)} = \{0 \le r \le \ell/2; h_F \le z \le h_F + h_C\},$$

$$\Omega^{(3)} = \{0 \le r \le \ell/2; h_F + h_C \le z \le 2h_F + h_C\}. \tag{11}$$

From equating to zero the first variation of the functional (7), i.e. from the relation

$$\delta\Pi = \sum_{k=1}^{3}\frac{\partial\Pi}{\partial u_r^{(k)}}\delta u_r^{(k)} + \sum_{k=1}^{3}\frac{\partial\Pi}{\partial u_z^{(k)}}\delta u_z^{(k)} + \sum_{k=1}^{3}\frac{\partial\Pi}{\partial\phi^{(k)}}\delta\phi^{(k)} = 0, \tag{12}$$

and after well-known mathematical manipulations we obtain the first three equations in (3). The boundary and contact conditions in (6) and (7) are given with respect to the forces and electrical displacements. In this way it is proven that the first three equations in (3) are the Euler equations for the functional (10) and the boundary and contact conditions in (6) and (7) which are given with respect to the forces and electrical displacements, are the related natural boundary and contact conditions.

According to FEM modelling, the solution domains indicated in (11) are divided into a finite number of finite elements. For the considered problem each of the finite elements is selected as a standard rectangular Lagrange family quadratic finite element (i.e. with nine nodes) and each node has three degrees of freedom, i.e. radial displacement $u_r^{(k)}$, transverse displacement $u_z^{(k)}$ and electric potential $\phi^{(k)}$. Employing the standard Ritz technique detailed in many references, for instance, in the book by [13], we determine the displacements and electrical potential at the selected nodes. After this determination, from the equation

$$\det(K) = 0 \tag{13}$$

the values of the critical compressional forces are determined, where $K$is a corresponding stiffness matrix. The solution procedure of the equation (13) is made according to the well-know “bi-section” method which basis on the sign change of the $\det(K)$.

Note that in the "open-circuit" case under FEM modeling the nodes on the planes $z = 0$, h_F, $h_F + h_C$ and $2h_F + h_C$the electrical potentials $\phi^{(1)}$ and $\phi^{(3)}$ are taken as unknown ones, however in the "short-circuit" case these potentials are taken as known ones and are equated to zero. Namely with these the FEM modeling in the "short-circuit" case is distinguished with that in the "open-circuit" case.

This completes the consideration of the method of solution.

4. Numerical results and discussions

Note that in the present paper, the piezoelectric materials PZT -5H, PZT -4 and BaTiO3 are taken as the face layer materials, however the metal materials - aluminum (Al) and steel (St) are taken as the core layer materials. The values of the elastic, piezoelectric and dielectric constants of the selected piezoelectric materials and the references used are given in Table 1.

Table 1. The values of the mechanical, piezoelectrical and dielectrical constants of the selected piezoelectric materials

Materials (Source Ref)	$c_{11}^{(r_1)}$	$c_{12}^{(r_1)}$	$c_{13}^{(r_1)}$	$c_{33}^{(r_1)}$	$c_{44}^{(r_1)}$	$c_{66}^{(r_1)}$	$e_{31}^{(r_1)}$	$e_{33}^{(r_1)}$	$e_{15}^{(r_1)}$	$\varepsilon_{11}^{(r_1)}$	$\varepsilon_{33}^{(r_1)}$
PZT-4 [11]	13.9	7.78	7.40	11.5	2.56	3.06	-5.2	15.1	12.7	0.646	.562
PZT-5H [11]	12.6	7.91	8.39	11.7	2.30	2.35	-6.5	23.3	17.0	1.505	1.302
BaTiO3 [8]	16.6	7.66	7.75	16.2	4.29	4.29	-4.4	18.6	11.6	1.434	1.182
	$\times 10^{10} N/m^2$						C/m^2			$\times 10^{10} C/Vm$	

Table 2. The values of the critical dimensionless stresses σ_{cr}^1, σ_{cr}^2 and $\bar{p}_{cr}$ obtained for the case where the material of the core layer is Steel in the cases where the piezoelectric constants of PZT are equated to zero (upper number), the "short-circuit" (middle number) and the "open-circuit" conditions (lower number) satisfy and the piezoelectric and dielectric constants are equal to the corresponding data given in Table 1

h_F/l	Crit. Param.	Materials of the face layers		
		PZT-5H	PZT-4	BaTiO$_3$
1/40	$\sigma_{cr}^{(2.1)}$	0.1015 0.1017 0.1925	0.1399 0.1401 0.2244	0.1249 0.1249 0.1418
	$\sigma_{cr}^{(2.2)}$	0.3246 0.3251 0.3578	0.3120 0.3123 0.3375	0.2010 0.2010 0.2053
	$\bar{p}_{cr}$	0.2689 0.2693 0.3165	0.2690 0.2693 0.3093	0.1820 0.1820 0.1895
1/30	$\sigma_{cr}^{(2.1)}$	0.0946 0.0950 0.1858	0.1337 0.1342 0.2194	0.1222 0.1222 0.1394
	$\sigma_{cr}^{(2.2)}$	0.3026 0.3037 0.3453	0.2982 0.2991 0.3302	0.1965 0.1967 0.2019
	$\bar{p}_{cr}$	0.2333 0.2342 0.2922	0.2434 0.2442 0.2933	0.1718 0.1719 0.1811
1/24	σ_{cr}^{1}	0.0896 0.0904 0.1816	0.1296 0.1304 0.2165	0.1206 0.1208 0.1381
	$\sigma_{cr}^{(2.2)}$	0.2866 0.2890 0.3377	0.2888 0.2908 0.3258	0.1940 0.1944 0.2000
	$\bar{p}_{cr}$	0.2046 0.2063 0.2727	0.2225 0.2240 0.2803	0.1635 0.1638 0.1743
1/20	σ_{cr}^{1}	0.0867 0.0881 0.1800	0.1274 0.1290 0.2156	0.1202 0.1206 0.1379
	σ_{cr}^{2}	0.2774 0.2818 0.3345	0.2841 0.2875 0.3243	0.1933 0.1939 0.1998
	$\bar{p}_{cr}$	0.1821 0.1850 0.2573	0.2058 0.2083 0.2700	0.1568 0.1573 0.1689

According to [6], the values of Lame's constants of the core layer material is selected as follows: for the Al: $\lambda = 48.1\,GPa$ and $\mu = 27.1\,GPa$; for the St: $\lambda = 92.6\,GPa$ and $\mu = 77.5\,GPa$.

Under FEM modelling using the symmetry with respect to the plane $z = h_F + h_C/2$ and the axial symmetry with respect to the Oz (Fig. 1a) axis of the mechanical and geometrical properties of the plate, we consider only the region $\{0 \le r \le \ell/2;\ \ 0 \le z \le h_F + h_C\}$ and this region is divided into 40 finite elements along the radial direction and 12 finite elements along the plate's thickness direction, resulting in 31022 NDOF. Such selection of the finite elements numbers is established according to the convergence of the numerical results. All the corresponding PC programs are composed by the authors of the paper.

The algorithm and programs employed in the present investigations are some modifications and development of the corresponding algorithm and programs used and testing in the many investigations and discussed in the monograph by [1]. Consequently, the validity and trustiness of the used in the present investigations PC programs and algorithm cause no doubt.

For simplification of the consideration, we introduce the following notation for the dimensionless critical radial stresses and critical compressive forces:

$$\sigma_{cr}^{(1)} = \sigma_{rr.cr}^{(1),0} \Big/ c_{44}^{(1)}, \sigma_{cr}^{(2)} = \sigma_{rr.cr}^{(2),0} \Big/ c_{44}^{(1)}, \bar{p}_{cr} = p/c_{44}^{(1)}. \tag{14}$$

Thus, according to (14), we estimate the work carrying capacity of the plate under consideration with respect to the stability loss by simultaneous use of the values of three dimensionless critical parameters which are the dimensionless radial compressive stress $\sigma_{cr}^{(1)}$ in the face piezoelectric layer, the dimensionless radial compressive stress $\sigma_{cr}^{(2)}$ in the core metal layer and the dimensionless intensity $\bar{p}_{cr}$ of the external compressive force. Such an approach for estimation of the buckling delamination allows us to have more precise information on the influence of the problem parameters such as the piezoelectricity of the face layers' materials, the face layers' thickness and the mechanical properties of the layers' materials.

Thus, we consider the numerical results obtained for the critical parameters indicated in (14) and detailed above. Note that these results are given in Tables 2 and 3 which are obtained for the cases where the material of the core layer is St and Al respectively. Moreover, note that these results are obtained for the cases where face layers materials are PZT-5H, PZT-4 and $BaTiO_3$. For estimation of the influence of the face layers' piezoelectricity on the values of the critical stresses in the tables, three types of results are presented simultaneously, the first of which (upper number) relates to the case where the values of the piezoelectric and dielectric constants of the face layer materials are equated to zero, i.e. coupling of the mechanical and electrical fields is not taken into consideration. However, under obtaining the second (third type) of results indicated by the middle numbers (by the lower numbers) the values of the piezoelectric and dielectric constants are taken into consideration as given in Table 1 and the coupling effect between the electrical and mechanical fields is taken into consideration completely and the "short-circuit" (9) (the "open-circuit" (7)) condition is satisfied.

Analysis of the results shows that for all the cases under consideration the piezoelectricity of the face layers causes an increase in the values of the dimensionless critical stressesσ_{cr}^{1}. $\sigma_{cr}^{2}$and $\bar{p}_{cr}$. However this increase is more significant for the PZT-5H and PZT-4 than that for the $BaTiO_3$. At the same time, this increase is very significant for the "open-circuit" case than that for the "short-circuit" case.

Table 3. The values of the critical dimensionless stresses σ_{cr}^{1}, $\sigma_{cr}^{(2)}$ and $\bar{p}_{cr}$ obtained for the case where the material of the core layer is Aluminum in the cases where the piezoelectric constants of PZT are equated to zero (upper number), the "short-circuit" (middle number) and the "open-circuit" conditions (lower number) satisfy and the piezoelectric and dielectric constants are equal to the corresponding data given in Table 1

h_F/l	Crit. Param.	Materials of the face layers		
		PZT-5H	PZT-4	$BaTiO_3$
1/40	$\sigma_{cr}^{(1)}$	$\frac{\frac{0.1264}{0.1268}}{0.2430}$	$\frac{\frac{0.1772}{0.1776}}{0.2828}$	$\frac{\frac{0.1586}{0.1587}}{0.1793}$
	$\sigma_{cr}^{(2)}$	$\frac{\frac{0.1491}{0.1495}}{0.1667}$	$\frac{\frac{0.1458}{0.1461}}{0.1570}$	$\frac{\frac{0.0941}{0.0942}}{0.0958}$
	$\bar{p}_{cr}$	$\frac{\frac{0.1435}{0.1439}}{0.1858}$	$\frac{\frac{0.1537}{0.1540}}{0.1884}$	$\frac{\frac{0.1103}{0.1104}}{0.1167}$
1/30	$\sigma_{cr}^{(1)}$	$\frac{\frac{0.1259}{0.1267}}{0.2434}$	$\frac{\frac{0.1777}{0.1786}}{0.2823}$	$\frac{\frac{0.1590}{0.1593}}{0.1792}$
	$\sigma_{cr}^{(2)}$	$\frac{\frac{0.1485}{0.1494}}{0.1670}$	$\frac{\frac{0.1461}{0.1469}}{0.1567}$	$\frac{\frac{0.0944}{0.0946}}{0.0957}$
	$\bar{p}_{cr}$	$\frac{\frac{0.1410}{0.1419}}{0.1925}$	$\frac{\frac{0.1567}{0.1575}}{0.1986}$	$\frac{\frac{0.1160}{0.1162}}{0.1236}$
1/24	$\sigma_{cr}^{(1)}$	$\frac{\frac{0.1259}{0.1274}}{0.2422}$	$\frac{\frac{0.1773}{0.1790}}{0.2795}$	$\frac{\frac{0.1580}{0.1584}}{0.1775}$
	$\sigma_{cr}^{(2)}$	$\frac{\frac{0.1485}{0.1503}}{0.1662}$	$\frac{\frac{0.1458}{0.1472}}{0.1552}$	$\frac{\frac{0.0938}{0.0940}}{0.0948}$
	$\bar{p}_{cr}$	$\frac{\frac{0.1391}{0.1408}}{0.1979}$	$\frac{\frac{0.1590}{0.1605}}{0.2070}$	$\frac{\frac{0.1206}{0.1209}}{0.1293}$
1/25	σ_{cr}^{1}	$\frac{\frac{0.1261}{0.1288}}{0.2402}$	$\frac{\frac{0.1762}{0.1791}}{0.2750}$	$\frac{\frac{0.1559}{0.1567}}{0.1746}$
	σ_{cr}^{2}	$\frac{\frac{0.1488}{0.1519}}{0.1647}$	$\frac{\frac{0.1449}{0.1474}}{0.1527}$	$\frac{\frac{0.0926}{0.0930}}{0.0933}$
	$\bar{p}_{cr}$	$\frac{\frac{0.1375}{0.1404}}{0.2205}$	$\frac{\frac{0.1606}{0.1633}}{0.2139}$	$\frac{\frac{0.1243}{0.1249}}{0.1340}$

The discussed above character of the influence of the piezoelectricity of the face layers materials on the values of the dimensionless critical stresses can be explained with the so-called "piezoelectric stiffening" effect of the piezoelectric materials, i.e. with the increase of the material stiffness as a result of the piezoelectricity of that. The mentioned "piezoelectric stiffening" effect in the "open-circuit" case is more significant than that in the "short-circuit" case.

Consequently, the fact that an increase of the thickness of the face layers also increases the stiffness of the piezoelectric layers. However, under fixed h/l (=0.2) thickness of the plate an increase h_F/l cases a decrease of the h_C/l as a result of which the whole stiffness of the plate depends on the ratio of stiffnesses of the core and face layers materials. Under explanation of the results discussed above it is also necessary to take into consideration of the complicate character of the dependence between the selected dimensionless critical stress and the ratio of the stiffnesses of the layers.

Namely with the foregoing statements it can be explained the character of the influence of the change h_F/l on the values of the critical stresses. According to the results given in Tables 2 and 3 this character can be formulated as follows:

1. For the pairs of materials consisting of PZT + St an increase in the values of h_F/l causes to decrease in all the values of the critical stresses under consideration;

2. For the pairs of materials consisting of PZT + Al the values of $\bar{p}_{cr}$ increase with h_F/l, however dependence among σ_{cr}^1, σ_{cr}^2 and h_F/l has non-monotonic character;

3. The foregoing conclusions take place not only in the “open-circuit” case but also in the “short-circuit” case.

This completes the discussions of the obtained numerical results.

5. Conclusions

Thus, in the present paper within the scope of 3D linearized theory of stability for piezoelectric materials, the axisymmetric stability loss of the PZT/Metal/PZT sandwich circular plate has been investigated. The cases where “open-circuit” and “short-circuit” conditions with respect to the electrical displacement and electric potential respectively on the upper and lower surfaces, and short-circuit conditions with respect to the electrical potential on the lateral surface of the face layers are satisfied, are considered. The corresponding eigenvalue problem is solved numerically by employing FEM. Numerical results are presented in Tables 2 and 3 for the PZT-5H/Al/PZT-5H, PZT-4/Al/PZT-4, BaTiO3/Al/BaTiO3, PZT-5H/St/PZT-5H, PZT-4/St/PZT-4 and BaTiO3/St/ BaTiO3 plates, respectively. These results illustrate simultaneously the values of the critical dimensionless radial compressive stress $\sigma_{cr}^{(1)}$ acting in the face piezoelectric layer, the values of the dimensionless critical compressive radial stress $\sigma_{cr}^{(2)}$ acting in the core-metal layer and the values of the dimensionless critical stress of the intensity $\bar{p}_{cr}$ of the external compressive forces obtained in the case where the piezoelectricity, i.e. the coupling effect, are taken into consideration (middle and lower numbers in the tables) and in the case where the coupling effect is not taken into consideration (upper number in the tables). According to these results, the concrete conclusions on the influence of the electro-mechanical and geometrical parameters of the sandwich circular plate under consideration on the values of the dimensionless stresses are made. Note that these conclusions are formulated in the text of the previous section and the main of them is the increase of the critical stresses as a result of the piezoelectricity of the face layers materials and the great magnitude of this increase in the “open-circuit” case than that in the “short-circuit” case.

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Development of a Modified Asymmetric McEliece Crypto-code System Elongated on Elliptic Truncated Codes

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Abstract. Offers mathematical model of asymmetric crypto-code system based on McEliece theoretical-code scheme, practical algorithms of cryptogram/codegram encryption/encoding and decryption/decoding, analyze the expenses on software implementation of the information protection crypto-code means based on McEliece TCS.

Key Words and Phrases: asymmetric crypto-code system, theoretical-code system, modified error-correcting codes.

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1. Introduction and analysis of the literature

Development of telecommunication systems and technologies, the rapid growth of computer technology put forward new requirements for the basic quality of customer service criteria (authorized users). The main indicators for the results of the analysis of standards in this area are ensuring authenticity of (reliability) transmitting data and ensuring the security of the entire processing cycle and data storage [1, 2, 3]. To provide the authenticity are used mechanisms of error-correcting coding, and to provide security - cryptographic mechanisms based on the methods of symmetrical and asymmetrical cryptography. Perspective direction, in our opinion, is the use of asymmetric cryptosystems based on McEliece theoretical - code schemes, which provide integrated (one mechanism) authenticity of indicators at the level of 2^9– 2^{12} and cryptographic strength - $2^{30} - 2^{35}$of group operations while it build over GF(2^{10}). Given cryptosystem has been widely used with the development of computing capabilities and communication devices and their software. In [4], the authors propose to use a cryptosystem McEliece Sequitur software, which allows integrated to solve performance problems and security in the transmission of confidential information. In [5, 6, 7] McEliece cryptosystem is offered to use for provide basic security services: confidentiality and integrity in stegasystem based on MPEG Layer-III or MP3 audio files, to ensure accessibility and digital signature while transferring confidential medical information. At the same time, carried out in [8] analysis of program realization of asymmetric crypto-code system on the Niederreiter TCS showed significant implementation complexity that makes it difficult to use theoretical coding schemes for the construction of asymmetric cryptographic systems. In [9] are considered the new approaches to breaking the McEliece cryptosystem based on randomized concatenated codes.

To provide the required indicators of cryptographic strength and increase volume of transmitted data by the authors is proposed McEliece modified asymmetric crypto-code system (MACCS) on the elongated elliptical codes, which is a promising direction in solving this scientific and technical problems.

2. The aims and tasks of the research

The purpose of work is to consider the mathematical model of McEliece MACCS, algorithms, encryption / decryption information MACCS, study their implementation complexity, analysis of program realization costs of MACCS on modified (elongated) elliptic codes.

To achieve this goal the following tasks were set:

– develop a method for masking McEliece MACCS on the elongated elliptical codes;

– consider the mathematical model and basic algorithms to transform information McEliece MACCS on the elongated codes;

– carry out a study of tasks complexity encoding/decoding and encryption/ decryption codegram/cryptogram when implemented in different levels of cryptographic resistance;

– analyze the costs of software implementation of the crypto-code means of information security based on McEliece TCS.

3. Developing a method of masking McEliece elliptic codes MACCS using curve parameters as secret data

Known methods for the modification of linear block codes more fully discussed in [10 – 14]. Fig. 1 shows the most common modification methods.

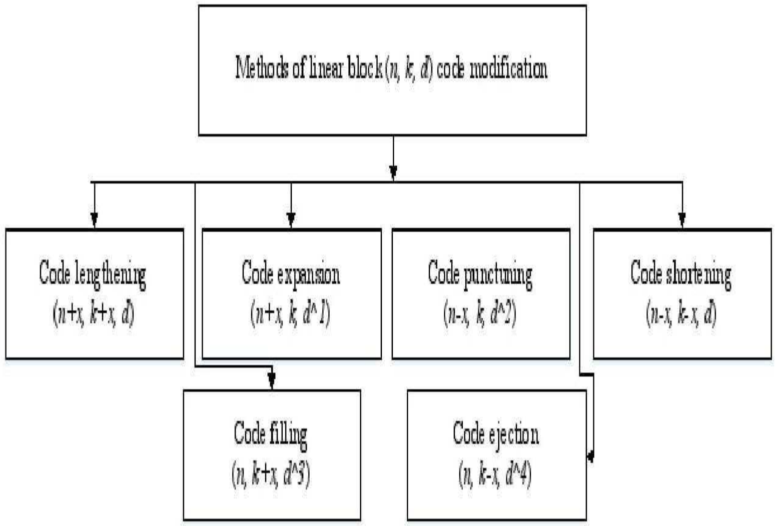

Fig. 1. Means of linear block codes modification

Lengthening of (n, k, d) linear block code is to increase the length of the $n + x$ by adding new information symbols $k + x$. *Expansion* of $(n,\ k,\ d)$ linear block code is to increase the length of the $n + x$ by adding new check symbols $r + x$. *Punctuning* $(n,\ k,\ d)$ of linear block code is to reduce the length of the n - x by decreasing of check symbols r - x. *Shortening (n, k, d)* of linear block code is to reduce the length of the n - x by decreasing of check symbols k – x. *Filling* (n, k, d) of linear block code is to increase the length of the $k + x$ information symbols without increasing the code length. *Ejection* (n, k, d) of linear block code is to reduce the k – x information symbols without code length increasing.

Potential resistance of theoretical code schemes defined by the complexity of decoding the random (n, k, d) block code. Hence, for the construction of a potentially persistent theoretical code schemes should be used modification techniques that do not allow reducing the minimum code distance. Methods of lengthening and shortening of the linear block codes do not change the minimum distance and, therefore, allow us to construct asymmetric crypto-code systems resistant to breaking [15].

Using the definition of elliptic codes [15, 16], we have the following properties:

Property 1. Elliptic (n, k, d) code over $GF(q)$, built through projection $\varphi : EC \rightarrow P^{k-1}$, connected with characteristics $k + d \geq n$, where: $n \leq 2\sqrt{q}+q+1$, $k \geq \alpha$, $d \geq n - \alpha$, $\alpha = 3\ .\ degF$.

Property 2. Elliptic (n, k, d) code over $GF(q)$, built through projection $\varphi : EC \rightarrow P^{r-1}$, connected with characteristics $k + d \geq n$, where $n \leq 2\sqrt{q} + q + 1$, $k \geq n - \alpha$, $d \geq \alpha$, $\alpha = 3\ .\ degF$.

Suppose A - generating matrix of elliptic $(n,\ k,\ d)$ code over *GF(q)* dimension of $M \times n$, $M = \alpha$, α*=3. degF.*

$$A = \begin{pmatrix} F_0(P_0) & F_0(P_1) & ... & F_0(P_{n-1}) \\ F_1(P_0) & F_1(P_1) & ... & F_1(P_{n-1}) \\ ... & ... & ... & ... \\ F_{M-1}(P_0) & F_{M-1}(P_1) & ... & F_{M-1}(P_{n-1}) \end{pmatrix} = \|F_j\,(P_i)\|_{n,M}\,.$$

To reduce the amount of key data in code-theoretic scheme on elliptic codes use the following features of the matrix A construction.

The generating matrix A is formed as a result of displaying elliptic curve points by basis of generating functions. The generating matrix of the elliptic code is built on curve

$$y^2z + a_1xy + a_3yz^2 = x^3 + a_2x^2z + a_4xz + a_6z^3,$$

$a_i \in$ *GF(q),* with the polynomial coefficients, which uniquely define the form of the curve and, accordingly, multiplicity of projective points which construct elliptic code (its generating matrix). Following statement is true.

Statement 1. [15] Elliptic *(n, k, d)* code over *GF(q)* is uniquely defined by multiplicity $a_1 \ldots\ a_6, \forall a_i \in$*GF(q).*

Proof. Consider an elliptic generating matrix of elliptic *(n, k, d)* code over *GF(q):*

$$A = \begin{pmatrix} F_0(P_0) & F_0(P_1) & ... & F_0(P_{n-1}) \\ F_1(P_0) & F_1(P_1) & ... & F_1(P_{n-1}) \\ ... & ... & ... & ... \\ F_{M-1}(P_0) & F_{M-1}(P_1) & ... & F_{M-1}(P_{n-1}) \end{pmatrix}.$$

Each character of generating matrix is formed by calculating the value of the generating function Fj in the point Pi of elliptic curve. The number M of generating functions is determined by the design characteristics of an elliptic $(n,\ k,\ d)$ code. Kind of functions Fj is determined by degree a of curve points projection and, therefore, defined by code design parameters.

Thus, if design $(n,\ k,\ d)$ elliptic code characteristics is given, the uniqueness of the generator matrix defines a multiplicity of points *P1, P2, ..., Pn*, which are computed generator functions values. A specific multiplicity of points from space P^2 is uniquely determined by polynomial curve view i.e. multiplicity of coefficients $a_1 \ldots\ a_6, \forall a_i \in$*GF(q).*

Corollary 1. The volume of private key (in bits) in motivated crypto-code system based on the theoretical - code McEliece scheme built on elliptical *(n, k, d)* code over *GF(2^m)* is determined by the sum of matrix elements *X, P, D* (in bits), and is given by

$$l_{K_+} = 5 \times n^2 \times k^2 \times m. \tag{1}$$

Proof. Indeed, secret key in McEliece scheme - generating matrix A (generating code matrix) and masking matrix *X, P, D.* In order to determine private key (in bits) of an elliptic *(n, k, d)* code over *GF(2^m)*, according to 1, it is sufficient to define multiplicity of coefficients $a_1 \ldots\ a_6$, $\forall a_i \in GF(2^m)$, and elements of masking matrixes. Total must be stored $l_{K+} = 5 \times n^2 \times\ k^2 \times\ m$ bits of secret key information.

Expression (1) enables to evaluate the amount of secret key data in motivated crypto-code system based on McEliece theoretical-code scheme with elliptical codes. Fig. 2 shows the dependence of the volume of key data on the dimension of $GF(q^m)$ field for a different $q = 2,\ 4,\ 16,\ 32$. The figure also shows the time required for exhaustive search of key data while performing of 10^{15} searches in second.

Thus, the proposed method of masking based on construction of the modified theoretical-code schemes on elliptic codes, in which use the parameters of the elliptic curve as secret data, can significantly reduce the amount of key data in compare to the classical McEliece scheme. At the same time as a potentially resistant, are considered scheme with $l_{K\ +} > 80$ bits. As follows from the above in Fig. 2 dependencies for building a theoretical code - scheme should be used elliptical codes with code word length> 220 bits.

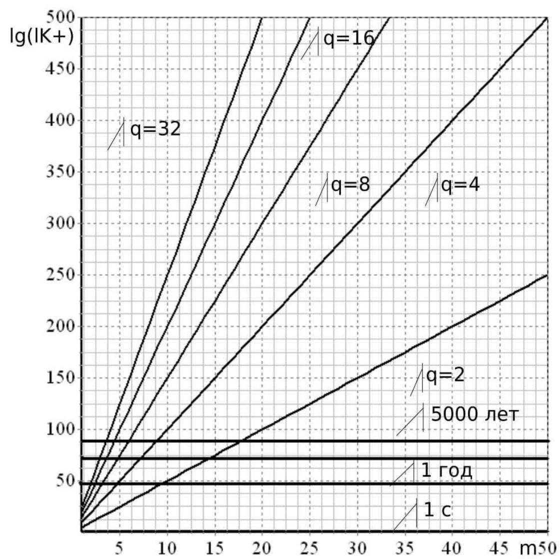

Fig. 2. The dependencies of the volume of secret key data in McEliece MACCS

The most simple and convenient method for modifying a linear block code, which stores the minimum code distance and increases the amount of data transmitted is the elongation of its length after forming initialization vector, by reducing the information symbols. Let $I = (I_1,\ I_2,\ \ldots,\ I_k)$ - information vector of $(n,\ k,\ d)$ block code. Choose a subset h of the information symbols, $|h|\ = x$, $x \leq \frac{1}{2}\ k$ and form *initialization vector*.

We place an information vector I in a subset of zeros h, i.e. $I_i\ = 0,\ \forall\ I_i\ \in h$. On the other positions of the vector I put the information symbols. After in position of initialization vector add information symbols. For the modification (lengthening) elliptic codes will use reduction of the curve points multiplicity. The following statement is true.

Statement 2. Let EC - elliptic curve over $GF\ (q)$, $g = g\ (EC)$ - curve genus, $EC\ (GF\ (q))$ - multiplicity of its points over a finite field, $N = EC\ (GF\ (q))$ - their number.

Fix a subset $h_1 \subseteq h$, $|h_1| = x_1$. Let an elliptic *(n, k, d)* code over *GF(q)* built through a mapping in the form φ*:* $X \to P^{k-1}$ is given. Then the parameters of the elongate on x_1 symbols from *GF(q)* elliptic code built through mapping φ*:(*$X \cup h_1$*)* $\to P^{k\ -1}$, are related as follows : $k \geq \alpha - x + x_1$, $d \geq n - \alpha$, $\alpha = 3 \ . \ degF$.

Proof. If $x_1 < x$, then the lengthening code on x_1 is equivalent to shortening the source code on the x - x_1. Having substituted these parameters in the expression (1), we obtain the result of corollary 1.

Corollary 2. If you know the type of elliptic curve (multiplicity $a_1 \ldots\ a_6$, $\forall a_i \in GF(q)$), the subset of h and *h1* are completely determine the modified elliptical (n, k, d) codes over *GF(q)*, built through the mapping of the form:φ*:*$X \to P^{k\ -1}$ and φ*:(*$X \cup h_1$*)* $\to P^{k-1}$.

Proof. Multiplicity of coefficients $a_1 \ldots\ a_6$, $\forall a_i \in GF(q)$ is uniquely defined form of the elliptic curve, and, accordingly, multiplicity of its points *EC(GF(q))*. Using a mapping in the form of φ*: EC* $\to P^M$ and the results of statements 1-2, construct the elliptical (n, k, d) code over *GF(q)*. If you know the elongating symbols, then we construct the elongated codes.

According to the statement 3, it are symbols from multiplicity *h1*, which completely determine the modified elliptical *(n, k, d)* code over *GF(q)*.

Statement 3. Fix a subset $h_1 \subseteq h$, $|h_1\ | = x_1$. Let an elliptic (n, k, d) code over *GF(q)*, built through a mapping of the form φ*:* $X \to P^{r-1}$ is given. Then the elliptic code parameters of the elongated on x_1 characters from *GF(q)*, built by mapping of the form φ*:(*$X \cup h_1$*)* $\to P^{r-1}$, will be connected by the relations: $n = 2\sqrt{q} + q + 1 - x + x_1$, $k \geq n - \alpha$, $d \geq \alpha$, $\alpha = 3 \ . \ degF$.

Corollary 3. If you know the form of an elliptic curve (multiplicity $a_1 \ldots\ a_6$, $\forall a_i \in GF(q)$), the subset of h and *h1* completely determine the modified elliptical (n, k, d) codes over *GF(q)*, built through the mapping of the form: φ*:*$X \to P^{r-1}$ and φ*:(*$X \cup h_1$*)* $\to P^{r-1}$.

Proof. The multiplicity of coefficients $a_1 \ldots\ a_6$, $\forall a_i \in GF(q)$ uniquely defines form of an elliptic curve, and, accordingly, multiplicity of its points *EC(GF(q))*. Using a mapping of the form φ*: EC* $\to P^?$ and results of statements 1 - 2, construct an elliptic (n, k, d) code over *GF(q)*. If you know the lengthening symbols, then we construct the elongated codes. According to the statement 3, the symbols of the multiplicities h and *h1*, which completely determine the modified elliptical (n, k, d) code over *GF(q)*.

Results of statements 2, 3, and their corollaries allow us to construct modified (elongated) elliptical (n, k, d) codes over *GF(q)*. Define the following algorithm for constructing modified elliptic codes.

Algorithm for constructing elongated elliptic codes.

Step 1. Fix an elliptic curve over *GF(q)*. Find a lot of simple points of the curve *EC(GF(q))*: $(P_1, P_2, \ldots, P_N)$. Construct a shortened (n, k, d) code over *GF(q)* as a result of mapping φ*:* $X \to P^M$.

Step 2. Fix a subset of points of the curve h_1*(GF(q))*: $(P_{x1}, P_{x2}, \ldots, P_{xx1})$, $h_1 \subseteq h$, $|h_1| = x_1$.

Step 3. Construct a mapping φ*: (*$X \cup h_1$*)* $\to P^M$. If $M = k$, we obtain an elongated elliptical (n, k, d) code over *GF(q)* with the parameters, $n = 2\sqrt{q} + q + 1 - x + x_1$, $k \geq \alpha - x + x_1$, $d \geq n - \alpha$, $\alpha = 3 \ . \ degF$. (see Corollary of Statement 4). If $M = r$, we obtain an elongated elliptical $(n,\ k,\ d)$ code over *GF(q)* with the following parameters: $n = 2\sqrt{q} + q + 1 - x + x_1$, $k \geq n - \alpha$, $d \geq \alpha$, $\alpha = 3 \cdot degF$ (see Corollary of Statement 3).

Using the result of Statement 2 and its corollaries, define a theoretical-code scheme on the modified elliptic codes built by mapping of the form $\varphi{:}X \to P^{k-1}$ and $\varphi{:}(X \cup h_1) \to P^{k-1}$. The following statement is true.

Statement 4. The elongated elliptical $(n,\ k,\ d)$ code over $GF(2^m)$, built through the mapping of the form $\varphi{:}(X \cup h_1) \to P^{k-1}$, determines the modified theoretic-code scheme with parameters:

- the dimension of secret key (in bits):

$$l_{K+} = (x - x_1) \cdot |\log_2 (2\sqrt{q} + q + 1)|; \tag{2}$$

- the dimension of information vector (in bits):

$$l_I = (\alpha - x + x_1) \cdot m; \tag{3}$$

- the dimension of cryptogram (in bits):

$$l_S = (2\sqrt{q} + q + 1 - x + x_1) \cdot m; \tag{4}$$

- relative transmission rate:

$$R = (\alpha - x + x_1) / (2\sqrt{q} + q + 1 - x + x_1). \tag{5}$$

Proof. According to the result of Statement 1, a modified crypto-code system based on McEliece theoretical-code scheme built using the generating matrix of algebraic $(n,\ k,\ d)$ block of code over $GF(2^m)$, has the following parameters: the size of the secret key $k \times n$ symbols from $GF(2^m)$; a vector of length k of information symbols from $GF(2^m)$; length of codegram - n symbols from $GF(2^m)$; relative transmission rate - R = $k\ /\ n$.

Enumerate all the points of the curve. Number of them $N \leq 2\sqrt{q} + q + 1$. Consequently, to enumerate the curve points it is necessary $|\log_2 \left(2\sqrt{q} + q + 1\right)|$ bits. If the subset power of shortening symbols is $|\ h| = x$, then to denote all shortening symbols is needed $x \cdot \log_2 \left(2\sqrt{q} + q + 1\right)$ bit. These symbols are held in secret and set the amount of key data - the expression (2). If the subset power of lengthening symbols is $|\ h_1\ | = x_1$, then to denote all modifications symbols is required $(x - x_1) \cdot |\log_2 \left(2\sqrt{q} + q + 1\right)|$ bit. These symbols are held in secret and set the amount of key data - the expression (2).

Using the result of Statement 3 and its corollaries, define theoretical – code scheme on the modified elliptic codes built by mapping in the form $\varphi{:}X \to P^{r-1}$ and $\varphi{:}(X \cup h_1) \to P^{r-1}$. The following statement is true

Statement 5. The elongated elliptical $(n,\ k,\ d)$ code over $GF(2^m)$, built through the mapping of the form: $\varphi{:}(X \cup h_1) \to P^{r-1}$ specifies the modified theoretic - code scheme with parameters:

- the dimension of the secret key is defined by expression (2);
- the dimension of information vector (in bits):

$$l_I = (2\sqrt{q} + q + 1 - \alpha) \cdot m; \tag{6}$$

- the dimension of codegram is defined by expression (3);
- the relative transmission rate:

$$R = (2\sqrt{q} + q + 1 - \alpha) \,/\, (2\sqrt{q} + q + 1 - x + x_1)\,. \tag{7}$$

Proof. According to the result of Statement 1, theoretical - code scheme is constructed using the check matrix of algebraic block $(n,\ k,\ d)$ code over $GF(2^m)$, has the following parameters: an information vector of length k characters from $GF(2^m)$; codegram length - n symbols from $GF(2^m)$; relative transmission rate - $R = k\,/\,n$. Substitute the parameters of modified (shortened and elongated) elliptic $(n,\ k,\ d)$ codes over $GF(q)$, built through the mapping of the form $\varphi{:}X \rightarrow P^{r-1}$ and $\varphi{:}(X \cup h_1) \rightarrow P^{r-1}$(see statement 3) obtain, accordingly, the expression (6), (7).

Thus, the results of statements 2, 3 and their corollaries allow to build a modified elongated elliptical $(n,\ k,\ d)$ codes over $GF(q)$. Statements 4 and 5 allow you to specify a modified asymmetric crypto-code system on McEliece TCS on modified elliptic codes, thereby providing the required cryptographic resistance.

Consider the formal description of a modified asymmetric crypto-code system of information protection based on the use of modification methods.

4. Mathematical model and basic algorithms of information converting in the proposed McEliece system on elongated codes

Mathematical model of modified asymmetric crypto-code information protection system using algebraic block codes based on McEliece theoretic -code scheme based on elongation (information symbols increassng) is formally defined by combination of the following elements:

- multiplicity of plaintexts
$M = \{M_1, M_2, ..., M_{q^k}\}$, where $M_i = \{I_0, I_{h_{r_1}}, ..\, I_{h_{r_j}}, I_{k-1}\}, \forall I_j \in GF\,(q)$, h_j –information symbols equal to zero, $|h| \;\; = \frac{1}{2}k$, i.e. $I_i\ =\ 0,\ \forall\ I_i\ \in\ h$; h_r –information symbols of lengthening k, $|h| \;\; = \frac{1}{2}k$;

- multiplicity of closed texts *(codegrams)*
$C = \{C_1, C_2, ..., C_{q^k}\}$, where $C_i = (c^*_{X_0}, c^*_{h_{r_1}}, ..., c^*_{h_{r_j}}, c^*_{X_{n-1}})$, $\forall c^*_{X_j} \in GF(q)$;

- multiplicity of straight mappings (based on the use of generating matrix public key)
$\varphi = \{\varphi_1{,}\varphi_2, ...{,}\varphi_s\}$, where $\varphi_i : M \rightarrow C_{h_r},\ i = 1, 2, ..., s$;

- multiplicity of reverse mappings (based on the use of masking matrix private key)
$\varphi^{-1} = \{\varphi_1^{-1}{,}\varphi_2^{-1}, ...{,}\varphi_s^{-1}\}$, where $\varphi_i^{-1} : C_{h_r} \rightarrow M,\ i = 1, 2, ..., s$;

- multiplicity of keys, parametrizing straight mapping (the public key of an authorized user)

$$K_{a_1} = \{K_{1_{a_1}}, K_{2_{a_1}}, ..., K_{1_{s_1}}\} = \{G_{X_{a_1}}^{EC_1}, G_{X_{a_1}}^{EC_2}, ..., G_{X_{a_1}}^{EC_s}\},$$

where $G_{X_{a_1}}^{EC_i}$-generating $n \times k$ matrix masked as a random albebra-geometric block (n, k, d) code with elrments from $GF(q)$, i.e. $\varphi_i : M \xrightarrow{K_{i_{a_1}}} C_{h_r}, i = 1, 2, ..., s$.

a_i – multiplicity of coefficients of the polynomial curve $a_1 \dots a_6$, $\forall a_i \in GF(q)$, uniquely defining a specific set of curve points from the space P^2.

- multiplicity of keys, parameterizing reverse mappings (personal (private) key of authorized user)

$$K^* = \{K_1^*, K_2^*, ..., K_s^*\} = \{\{X, P, D\}_1, \{X, P, D\}_2, ..., \{X, P, D\}_s\},$$

$$\{X, P, D\}_i = \{X^i, P^i, D^i\},$$

where X^i – masking nondegenerate randomly equiprobably formed by source of keys matrix $k \times k$ with elements from $GF(q)$; P^i – permutational randomly equiprobably formed by source of keys matrix $n \times n$ with elements from $GF(q)$; D^i – diagonal formed by source of keys matrix $n \times n$ with elements from $GF(q)$, i.e.

$$\varphi_i^{-1} : C \xrightarrow{K_i^*} M, i = 1, 2, ..., s.$$

Complexity of performing reverse mapping φ_i^{-1} without knowledge a key $K_i^* \in K^*$ associated with solution of theoretic complexity problems in random code decoding (generic position code).

Initial data in the description of the considered asymmetric crypto-code information protection systems are the parameters described in the previous model.

In asymmetric crypto-code system based on McEliece TCS modified (elongated) algebrogeometric *(n, k, d)* code C_{h_r} with rapid decoding algorithm is masking random *(n, k, d)* code $C_{h_r}{}^*$ by multiplying generating matrix G^{EC} of C_{k-h_j} code on the secret masking matrices X^u, P^u and D^u, what provide formation of open key for authorized user:

$$G_X^{ECu} = X^u \cdot G^{EC} \cdot P^u \cdot D^u, u \in \{1, 2, ..., s\},$$

where G^{EC} – generating $n \times k$ matrix of algebrogeometric (n, k, d) code with elements from $GF(q)$, built on the basis of using the polynomial curve coefficients $a_1 \dots a_6$, $\forall a_i \in GF(q)$, chose by user, uniquely defining a specific set of curve points from the space P^2.

Forming secret text $C_j \in C_{h_r}$ by the entered plaintext $M_i \in M$ and given public key $G{X_{a_1}^{EC_u}}$, $u \in \{1, 2, ..., s\}$ is performed by forming of shortened code word and then elongation of masked code with adding to its randomly formed vector $e = (e_0, e_1, ..., e_{n-1})$:

$$C_j = \varphi_u\left(M_i, G_X^u\right) = M_i \cdot \left(G_X^u\right)^T + e.$$

For each formed secret text $C_j \in C_{h_r}$ the appropriate vector $e = (e_0, e_1, ..., e_{n-1})$ acts as a single session key, i.e. for specific E_j, vector e is formed randomly, equiprobably and independently of the other secret texts.

The channel receives $C_j^* = C_j - C_{k-h_j} + C_{h_r}$.

On the receiving side, an authorized user who knows the rule of masking, the number and location of zero information symbols can take advantage of a fast decoding algorithm of algebrogeometric code (with polynomial complexity) to recover the plaintext:

$$M_i = \varphi_u^{-1}\left(C_j^*, \{X, P, D\}_u\right).$$

To recover the plaintext an authorized user replaces lengthening symbols on non-zero information symbols

$$C_j^* = C_{h_r} \to C_{k-h_j},$$

from recovered secret text C_j reduces the effect of the secret of permutational and diagonal matrices P^u and D^u:

$$\begin{aligned}
&C = C_j^* \cdot (D^u)^{-1} \cdot (P^u)^{-1} = \Big(M_i \cdot (G_X^u)^T + e\Big) \cdot (D^u)^{-1} \cdot (P^u)^{-1} = \\
&= \Big(M_i \cdot (X^u \cdot G \cdot P^u \cdot D^u)^T + e\Big) \cdot (D^u)^{-1} \cdot (P^u)^{-1} = \\
&= M_i \cdot (X^u)^T \cdot (G)^T \cdot (P^u)^T \cdot (D^u)^T \cdot (D^u)^{-1} \cdot (P^u)^{-1} + e \cdot (D^u)^{-1} \cdot (P^u)^{-1} = \\
&= M_i \cdot (X^u)^T \cdot (G)^T + e \cdot (D^u)^{-1} \cdot (P^u)^{-1},
\end{aligned}$$

decodes received vector with Berlekamp-Massey algorithm [10 – 14]:

$$C = M_i \cdot (X^u)^T \cdot \left(G^{EC}\right)^T + e \cdot (D^u)^{-1} \cdot (P^u)^{-1},$$

i.e. get rid of the second term and from the multiplier $\left(G^{EC}\right)^T$ in the first term at right side of equation, and then reduces the effect of masking matrix X^u.

Received result of decoding M_i^* is need to be multiplied by $(X^u)^{-1}$:

$$M_i^* \cdot (X^u)^{-1} = M_i.$$

Received solution is plaintext M_i, to which are added lengthening symbols: $M_j = M_i + h_r$ – the essence of sent message.

Consider the practical algorithms of codegram forming and decoding, and a block diagram of communication protocol in a real time at developed McEliece ACCS.

The algorithm of codegram formation in modified McEliece asymmetric crypto-code system with shortened modified code define by sequence of the following steps:

Step 1. Fix a definite field *GF(q)*. Fix an elliptic curve $y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz + a_6z^3$ and set of it points $EC(GF(q))$:$(P_1, P_2, \ldots, P_N)$ over *GF(q)*. Fix subset of points $h(GF(q))$: $(P_{x1}, P_{x2}, \ldots, P_{xx})$, $h \subseteq EC(GF(q))$, $|h|=x$ and keep it in secret.

Step 2. Form initialization vector $IV=EC–h_j$, h_j–information symbols equal to zero, $|h| =\frac{1}{2}k$, i.e. $I_i = 0$, $\forall\ I_i \in h$;

Step 3. By entering information vector I form the code word c. If *(n, k, d)* code over *GF(q)* is given by its generating matrix in such case $c = IG$.

Step 4. Form the random vector of error e such, as $w(e)\leq t$, $t = \lfloor (d-1)/2 \rfloor$. Add formed vector to code word, receive the code word: $c^*=c+e$.

Step 5. Form the codegram by initialization vector symbols deleting (shortening): $c_X^* = c^* - IV$.

Fig. 3 shows algorithm of encoding in McEliece MACCS.

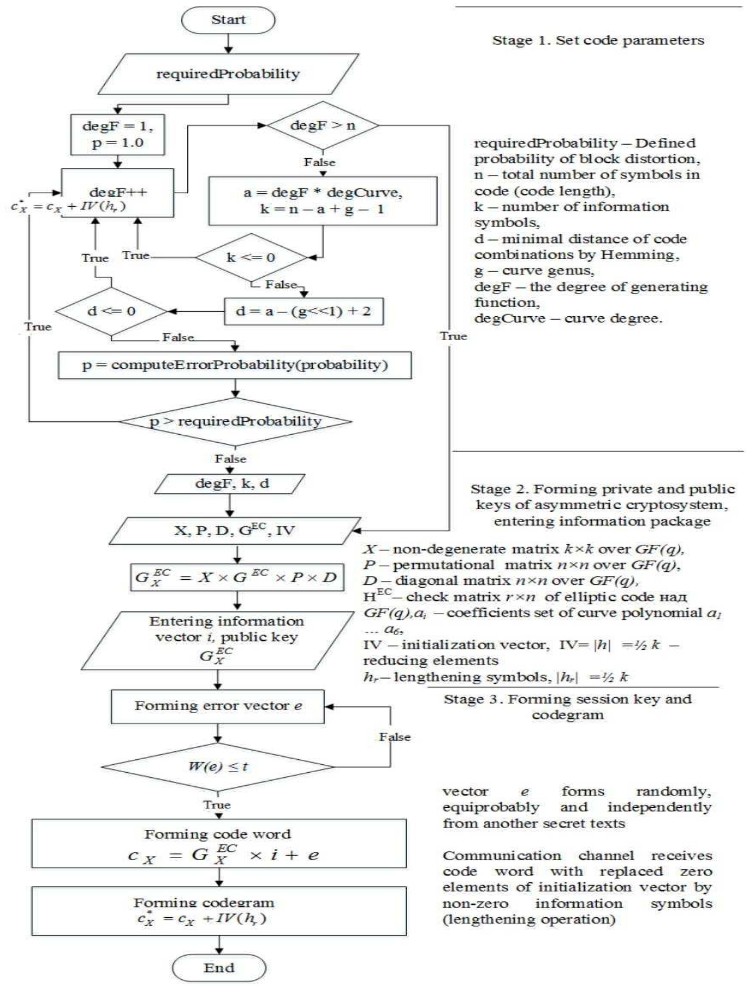

Fig. 3. Algorithm of codegram formation in McEliece MACCS

Algorithm of codegram decoding in modified theoretical-code schemas on elliptic codes define by sequence of the following steps:

Step 1. Entering codegram to be decoding. Entering the private key - generating and / or the elliptic code check matrix.

Step 2. Codegram - a code word with elliptic code errors. Error vector weight $w(e) \leq t$. Decoding codegram – find error vector.

Step 3. Form needed information vector.

Step 4. Add to information vector symbols of information packages from initialization vector position.

Offered decoding algorithm on McEliece MACCS is shown on fig.4.

$c_X^* = c^* + IV\,(h_r)$.

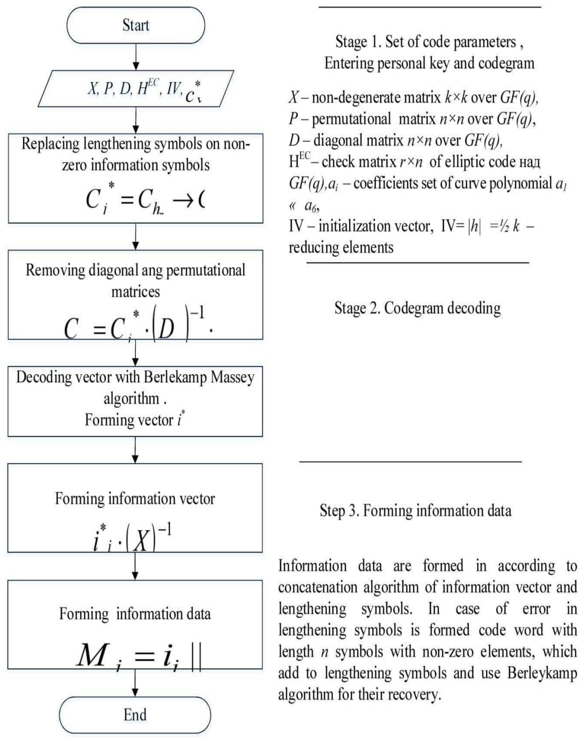

Fig. 4. Algorithm of codegram decoding in McEliece MACCS

Block diagram of information exchange protocol in a real time mode with the use of asymmetric cryptosystems based on a modified McEliece TCS with modified (elongated) elliptical codes is shown in Fig. 5.

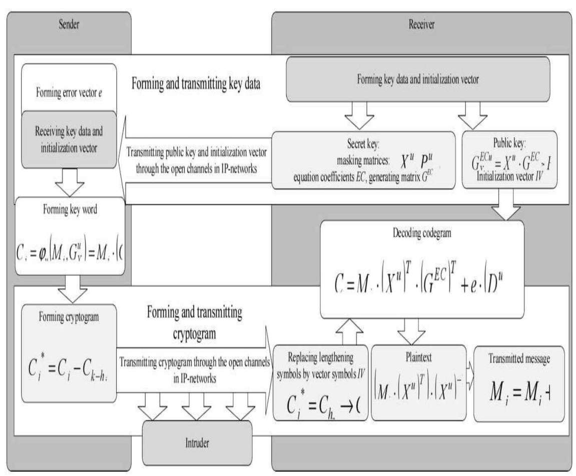

Fig. 5. Protocol of information exchange in a real time mode with use of modified McEliece TCS with elongated EC

5. Evaluation of energy costs for program implementation and the complexity of the proposed McEliece MACCS code transformation

To estimate time and speed parameters is common to use the unit of measurement *cpb* where cpb (*cycles per byte*) - the number of processor cycles, which should be spent to process 1 byte of incoming information. Algorithm complexity calculates from expression:

$$Per = Utl * CPU\_clock/Rate$$

where Utl– utilization of the CPU core (%);

$Rate$ – algorithm bandwidth (bytes/sec).

In table. 1 are shown dependency research results of code length sequence of algebro-geometric code in McEliece and Niederraiter TCS from number of processor cycles due to executing elementary operations in program realization of crypto-code systems.

Table 1

Research results according to the length of the code sequence in McEliece ACCS in dependency of CPU cycles number

Code sequence length		McEliece on elongated codes			McEliece		
		10	100	1000	10	100	1000
The number of function calls realizing elementary operations	Symbol reading	11 432 131	33 460 317	82 473 442	11 018 042	30 800 328	80 859 933
	String comparing	3 673 756	12 119 867	29 469 389	3 663 356	10 199 898	26 364 634
	String concatenation	1 947 681	6 114 478	14 456 729	1 834 983	5 125 564	13 415 329
Sum		17 053 568	51 694 662	126 399 560	16 516 381	46 125 790	120 639 896
Duration of executing functions in processor cycles*	Symbol reading	300 479	843 705	2 745 148	297 487	831 609	2 183 218
	String comparing	213 478	561 754	1 739 170	197 821	550 794	1 423 690
	String concatenation	578 174	1 647 638	4 007 883	544 990	1 522 293	3 984 353
Sum		109 157	1 092 131	3 053 097	1 040 298	2 904 696	7 591 261
Executing duration** in msec		0,56	1,55	4,1	0,55	1,53	4

Note:

** duration of 1000 operations in processor cycles: symbol reading – 27 cycles, string comparing – 54 cycles, string concatenation – 297 cycles;*

*** for calculating is taken processor with a clock speed 2 GHz taking into account operating system loading 5 %*

Table 2 shows the investigation results for evaluating time and speed parameters of procedures of forming and decoding information in the non-symmetric crypto-code systems based on McEliece ACCS and MCCS.

Table 2

Investigation results for evaluating time and speed parameters of procedures of forming and decoding information

Crypto-code systems	Code sequence length	Algorithm bandwidth, Rate (bytes / sec)	CPU utilization (%)	Algorithm complexity, Per (cpb)
McEliece ACCS	100	46 125 790	56	61,5
	1000	120 639 896	56	62,0
McEliece MCCS	100	51 694 662	56	61,7
	1000	126 399 560	56	62,2

Analysis of table 1.2 shows that the use of modified (elongated) elliptic codes allows to save the volume of transmitted in McEliece a crypto-code system data, but at the same time provide the required level of cryptographic resistance during the implementation over smaller field $GF(2^6 - 2^8)$ through the use of entropy of initialization vector h_r.

Research information reliability and secrecy, which can be provided by modified crypto-code systems on elliptic curves. Fix $(n,\ k,\ d)$ elliptic code over $GF(q)$. Define modified crypto-code scheme on the basis of McEliece TCS on modified (elongated) codes. Define the session key e – error vector, which adds to code word during codegram formation. Let $w(e) \leq t,\ t = \lfloor (d-1)/2 \rfloor$. Denote share of error vector weight e by symbol $\rho = w(e)\ /\ t$. Then potential resistance of theoretical-code scheme with elliptic codes, will be determined by $\rho{\times}t$, interference resistance of transmitted codegrams by $(1 - \rho) \times t$. The complexity of hacking the proposed modified system define by the expression of the random code decoding analysis complexity with commutation decoder:

$$I_{K+} = N_{nokp} nr, \text{where } N_{nokp} \geq \frac{C_n^t}{C_{n-k}^t} = \frac{n\,(n-1)\ldots(n-t-1)}{(n-k)\,(n-k-1)\ldots(n-k-t-1)}$$

Interference resistance is defined by minimal ratio signal/noise, needed for providing the required reliability. Fix the ratio signal/noise and modulation type. Suppose that digital message transmission is carried out through discrete channel without memory, i.e. errors in sequently transmitted code symbols happen independently with probability P_o. Then the probability of the error multiplicity i on the block length is equal [10 – 14]:

$$P_i = C_n^i P_o^i \left(1 - P_o\right)^{n-i}.$$

If the decoding procedure allows correcting $t = \lfloor (d-1)/2 \rfloor$ errors, the probability of an incorrect decoding is:

$$P_{ou} = \sum_{i=t+1}^{n} P_i = \sum_{i=t+1}^{n} C_n^i P_o^i \left(1 - P_o\right)^{n-i}.$$

At the integrated solution of problems of reliability and information secrecy of data transmitting, modified crypto-code system will be correct $(1 - \rho) \times t$ happened errors, hence:

$$P_{ou} = \sum_{i=(1-\rho)t+1}^{n} P_i = \sum_{i=(1-\rho)t+1}^{n} C_n^i P_o^i (1 - P_o)^{n-i} .$$

Fix $GF(2^{10})$ and $P_o = 10^{-3}$.

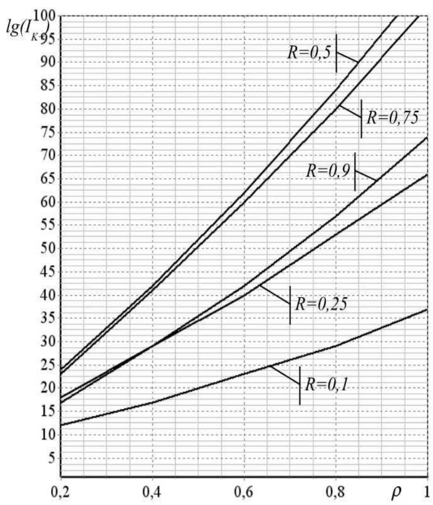

Fig. 6. Dependency of hacking complexity $I_{K+}(\rho)$ over $GF(2^{10})$

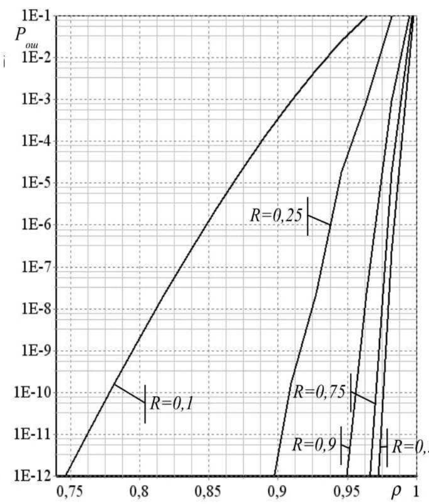

Fig. 7. Dependency of error decoding probability $P_{ou}(\rho)$ over $GF(2^{10})$

Fig. 6 shows dependencies of theoretical-code scheme hacking complexity with permutational decoder $I_{K+}(\rho)$ while use of elliptic codes with relative speed R. Fig. 7 shows dependencies of error decoding probability $P_{ou}(\rho)$ with an integrated solution of problems of reliability and information secrecy.

As it is seen from the dependences shown in Fig. 6, 7, modified crypto-code system based on McEliece TCS have high indexes of reliability and information secrecy. Increasing index ρ leads on the one hand to increasing of circuit resistance and on the other side reduce its noise resistance. Research integrated increasing of reliability and information secrecy of data transmission with use of offered systems.

Fig. 8 summarizes dependencies of error decoding probabilities and complexity of hacking theoretical-code scheme with elliptic codes under different R and $\rho = 0,9$.

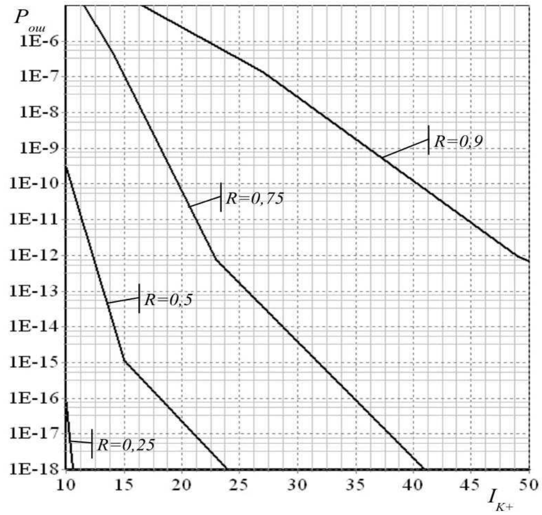

Fig. 8. Summarized dependencies of error decoding probability and hacking complexity $P_{ош}(I_{K+})$ for $\rho = 0,9$

As it is seen from the dependences shown in Fig. 8 proposed modified crypto-code systems based on McEliece TCS provide high resistance and reliability indicators of the processed and transmitted information. Their use will enable use open channels of IP-networks for transmitting confidential (commercial) information in the real-time mode thus providing required indexes of reliability and safety.

6. Conclusions

In a result of conducted researches:

1. Analyzed overall structure of the asymmetrical crypto-code systems construction based on McEliece TCS enabling to provide integrated (with single device) the required indicators for reliability, efficiency and data security. A major shortcoming of ACCS based on McEliece TCS is big volume of key data, that constricts their use in different communication system areas (today cryptographic resistance on the level of provable resistance model is provided while building ACCS in Galua field $GF(2^{13})$). Using modified elongated elliptic codes allows reducing the volume of key data while keeping the cryptographic resistance requirements and transmission of big volume of information.

2. Offered mathematical model, practical algorithms of codegram encoding/decoding in developed McEliece MACCS enable to implement high-speed information processing at the real-time mode. The complexity of codegram formation and decoding is defined by encoding/decoding complexity of modified (elongated) elliptic codes and a polynomially depends on the code length and it correcting dependence.

3. Transferring the key sequence using a modified McEliece ACCS based on the shortened codes allows using open communication channels of communication systems and significantly reducing the volume of the key sequences that are stored by users of the

system. Evaluation software implementation complexity of information protection crypto-code means based on McEliece TCS confirms the assumption if reducing the computing costs to calculate cryptogram/codegram, necessity to store key data (public key) by authorized user.

Performed researches of error vector ρ usage enable on the basis of the main indexes of telecommunication system channels to enhance one of the integrated mechanisms indicator – reliability or safety.

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Free Vibrations of Fluid-containing Spheres

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Abstract. In the paper free vibrations of a spherical shell containing compressed fluid are studied. Its natural frequencies of vibrations are determined under some values of the parameters of the system, influence of geometrical and physical parameters of the system "spherical shell-fluid" on free vibrations of the sphere is studied.

Key Words and Phrases: spherical shell, frequency of free vibrations, potential motion, density

1. Introduction

Shells as elements of machines and constructions are widely used in aircraft and shipbuilding, etc. Therefore, recently the researchers are interested in the issues associated with dynamic behavior of thin-shelled constructions that in working conditions are in contact with external medium. The problems of free vibrations of elastic thin shells contacting with elastic medium and fluid, occupy important place among dynamical contact problems of shell theory. Filled shells may be used in practice for storage and transportation of products. As the problems of strength and life of the shells of tanks are very actual in connection with oil and gas recovery, necessity of storage, transportation and processing of different chemical mixtures. Furthermore, the Earth may be considered as a special shells with a filler.

Frequencies and forms of free vibrations of spherical and cylindrical shells contacting with elastic and liquid medium are studied in [1]-[3]. Approximate simple formulas for calculating frequency and determination of vibration forms of the systems under consideration that restricts the use of the obtained results, as in a number of important cases it excludes the possibility of conducting qualitative analysis of the studied processes, are obtained by approximate methods. These investigations are connected with great difficulties as it is necessary to solve transcendental system of equations.

Free vibrations of a thin-walled shell containing compressible fluid, are studied in [4]-[6]. Under some values of the parameters of the system, its eigenvalues of the frequencies of vibrations were determined, influence of geometrical and physical parameters of the system "cylindrical shell-fluid" on free vibrations of the cylinder is studied.

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In [7], a problem of free vibrations of a thin-walled elastic spherical shell containing an elastic medium with different properties, usually with modulus of elasticity that is significantly less than the elasticity modulus of the shell material, is studied.

Analysis of vibrations of fluid-containing sphere with regard to finite thickness differs from the analysis of a very thin sphere with the fact that loads are not introduced into the equation of motion, and in the equations of motion the terms containing derivatives along the radius, are not ignored. The external load on the shell enters into the boundary conditions. The results may be used when analyzing the tanks subjected to seismic impacts, at transportation and also when studying the Earth vibrations.

In this connection, in this paper we consider free vibrations of a finitely-thickened sphere of radius r_1 and r_2, respectively and filled with compressible fluid. The equation of motion of a spherical shell is disconnected into two parts: the system describing the potential motion, and the equation describing the vortex motion [8].
The first system is of the form:

$$\frac{2(1-\nu)}{1-2\nu}\left(\frac{\partial^2 w}{\partial r^2}+\frac{2}{r}\frac{\partial w}{\partial r}-\frac{2}{r^2}\omega\right)+$$

$$+\frac{1}{r^2}\Delta_0 w+\frac{1}{1-2\nu}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{4\nu-3}{r^2}\right)\Delta_0\phi+\lambda^2 w=0$$

$$\frac{1}{1-2\nu}\frac{1}{r}\left(\frac{\partial w}{\partial r}+\frac{4-4\nu}{r}w\right)+\frac{2(1-\nu)}{1-2\nu}\frac{1}{r^2}\Delta_0\phi+\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\lambda^2 w=0. \tag{1}$$

In the case under consideration, the conditions on the boundary are:

$$\frac{2G}{1-2\nu}\left|[1-\nu)\frac{\partial w}{\partial r}+\frac{\nu}{r}(2w+\Delta_0\phi]\right|_{r=r_1}=p$$

$$\left|(1-\nu)\frac{\partial w}{\partial r}+\frac{\nu}{r}(2w+\Delta_0\phi)\right|_{r=r_2}=0$$

$$\left|\frac{1}{r}w+\frac{\partial\phi}{\partial r}-\frac{1}{r}\phi\right|_{r=r_1}=0 \tag{2}$$

$$\left|\frac{1}{r}w+\frac{\partial\phi}{\partial r}-\frac{1}{r}\phi\right|_{r=r_2}=0.$$

Here r is the distance from the center of the sphere, w is radial displacement, ϕ is displacement potential, G is shear modulus, ν is Poisson's ratio, q is density of the shell's material.

$$\lambda^2=\frac{q}{G}\omega^2,$$

ω is the frequency of vibrations.

p is the pressure on the inner boundary. Δ_0 is an operator:

$$\Delta_0=\frac{\partial^2}{\partial\theta^2}+ctg\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{3}$$

According to the problem under consideration the solutions are represented by means of spherical harmonics Y_n:

$$w = \omega_n Y_n, \quad \phi = \phi_n Y_n, \;\; p = p_n Y_n. \tag{4}$$

Then

$$\Delta_0 w = -n\,(n+1)\,w, \quad \Delta_0 \phi = -n\,(n+1)\,\phi, \;\; (n = 0, 1, 2)\,,$$

equations (1) and (2) take the form:

$$\left(w_n^{'} + \frac{2}{r} w_n\right)^{'} + \frac{1-2\nu}{2\,(1-2\nu)}\left[\lambda^2 - \frac{n\,(n+1)}{r^2}\right] w_n -$$

$$-\frac{1}{2\,(1-\nu)} \frac{n(n+1)}{r}\left(\phi_n^{'} + \frac{4\nu-3}{r}\phi_n\right) = 0$$

$$\frac{1}{2\,(1-\nu)}\left(\frac{1}{r} w_n^1 + \frac{4-4\nu}{r^2} w_n\right) + \frac{1-2\nu}{2\,(1-\nu)}\Phi_n'' + \frac{1}{r}\Phi_n' \frac{1-2\nu}{(1-\nu)} +$$

$$+\left[\frac{1-2\nu}{2\,(1-\nu)}\lambda^2 - \frac{n\,(n+1)}{r^2}\right] = 0$$

$$\left[(1-\nu)\,w_n' + \frac{2\lambda}{r} w_n - \frac{\nu}{r} n\,(n+1)\,\phi_n\right]_{r=r_1} = p_n \frac{(1-2\nu)}{2G} \tag{5}$$

$$\left[(1-\nu)\,w_n' + \frac{2\nu}{r} w_n - \frac{\nu}{r} n\,(n+1)\,\phi_n\right]_{r=r_2} = 0$$

$$\left(\frac{1}{r} w_n + \phi_n^1 - \frac{1}{r}\phi_n\right)_{r=r_1} = 0$$

$$\left(\frac{1}{r} w_n + \phi_n^1 - \frac{1}{r}\phi_n\right)_{r=r_2} = 0.$$

For the case of potential motion, the pressure of the compressed fluid is determined as follows [10]:

$$p = -\rho \frac{\partial \Pi}{\partial t} \tag{6}$$

where ρ is the fluid density, Π is velocity potential satisfying the equation:

$$a^2 \Delta \Pi = \partial^2 \Pi / \partial t^2 \tag{7}$$

Δ is the Laplace operator, a is the velocity of perturbation propagation.

Radial velocity of the shell and potential of fluid's velocity on the contact surface are connected with the relations:

$$\frac{\partial w}{\partial t} = \frac{\partial \Pi}{\partial r}, \tag{8}$$

where:

$$\Pi = \Pi_\omega i e^{\omega i t}, \;\; w = w_\omega e^{i\omega t}, \;\; p = p_\omega e^{i\omega t}.$$

Taking into account that under vibrations the relations (6) and (8) take the form:

$$\omega w_\omega = \frac{\partial \Pi_\omega}{\partial r} \tag{9}$$

$$p_\omega = \rho\omega\Pi_\omega .$$

Equation (7) turns into the Helmholtz equation. Then the solution of the problem under consideration will have the form:

$$\Pi_{\omega n} = D_n j_n\left(\frac{\omega r}{a}\right), \tag{10}$$

where $j_n\left(\frac{\omega r}{a}\right)$ is Bessel's first kind spherical function. (9), (10) and (4) yield

$$\omega w_n = D_n \frac{\omega}{a} j'\left(\frac{\omega r}{a}\right) \tag{11}$$

$$p_{\omega n} = \rho\omega a j_n\left(\frac{\omega r}{a}\right) w_n / j_n'\left(\frac{\omega r}{a}\right) \tag{12}$$

here $n = 2$, then

$$\frac{j}{j'} = \frac{z \sin z\left(z^2-2\right) + 2z^2\cos z}{z\cos z\left(z^2-6\right) - 3\sin z\left(z^2-2\right)}.$$

Having integrated the first two equations in (5) within r_1 and r_2 and assuming that the thickness of the solid body of the sphere is small compared with the radius, we get:

$$w_n^1\Big|_{\substack{r=r_2\\ r=r_1}} + \frac{2}{r}w_n\Big|_1^2 + \frac{1-2\nu}{2(1-\nu)}\left[\lambda^2 - \frac{n(n+1)}{r^2}\right]w_n h -$$

$$-\frac{1}{2(1-\nu)}\frac{n(n+1)}{r}\left(\phi_n|_1^2 + \frac{4\nu-3}{r}\phi_n h\right) = 0$$

$$\frac{1}{2(1-\nu)}\left(\frac{1}{r}\omega_n\Big|_1^2 + \frac{4-4\nu}{r^2}\omega h\right) +$$

$$+\frac{1-2\nu}{2(1-\nu)}\phi_n'\Big|_1^2 + \frac{1-2\nu}{(1-\nu)r}\phi_n'\Big|_1^2 + \left[\frac{1-2\nu}{2(1-\nu)}\lambda^2 - \frac{n(n+1)}{r^2}\right]\phi_n h = 0. \tag{13}$$

Here the values of quantities without indication of the limit, are average.

Connect the deformation in radial direction with inner pressure p, assuming the layer as centrally-symmetric static. Then preserving in the first equation of the system (5) two terms, we have:

$$w_{2n}^1 - w_{1n}^1 + \frac{2}{r}\left(w_{2n} - w_{1n}\right) = 0 \tag{14}$$

or

$$\varepsilon_{2n}^r - \varepsilon_{1n}^r + \frac{2}{r}\left(w_{2n} - w_{1n}\right) = 0 \tag{15}$$

here ε_{in} is deformation.

From the third and fourth equations of the system (5) it follows:

$$(1-\nu)\left(\varepsilon_{2n}^{r}-\varepsilon_{1n}^{r}\right)+\frac{2\nu}{r}\left(w_{2n}-w_{1n}\right)-\frac{\nu}{r}n\left(n+1\right)\left(\phi_{2n}-\phi_{1n}\right)=-\phi_{\omega n}\frac{1-2\nu}{2G} \tag{16}$$

Or, taking into account (15), we get

$$\frac{4\nu-2}{r}\left(w_{2n}-w_{1n}\right)-\frac{\nu}{r}n\left(n+1\right)\left(\phi_{2n}-\phi_{1n}\right)=-\phi_{\omega n}\frac{1-2\nu}{2G} \tag{17}$$

Under the conditions stipulated above, the second equation of the system (5) gives:

$$w_{2n}-w_{1n}+\frac{4-4\nu}{r}w_{n}h=0. \tag{18}$$

From (14) and (18) we have:

$$w_{2}'-w_{1}'=\frac{8\left(1-\nu\right)}{r^{2}}w_{n}h. \tag{19}$$

Substituting (18) in (17), we get:

$$\phi_{2n}-\phi_{1n}=\frac{r}{\nu n\left(n+1\right)}\left[\frac{8\left(1-2\nu\right)\left(1-\nu\right)}{r^{2}}\omega h+\phi_{n}\frac{1-2\nu}{2G}\right]. \tag{20}$$

From the fifth and sixth equation of the system (5) we have

$$\phi_{2n}^{'}-\phi_{1n}^{'}=\frac{1}{r}\left(\phi_{2n}-\phi_{1n}-w_{2n}-w_{1n}\right).$$

Or, using (18) and (20), we get

$$\phi_{2n}^{'}-\phi_{1n}^{'}=\frac{4\left(1-\nu\right)}{r^{2}}\left[\frac{2\left(1-2\nu\right)}{\nu n\left(n+1\right)}+1\right]w_{n}h+\frac{1-2\nu}{\nu n\left(n+1\right)}\frac{P_{\omega n}}{2G}. \tag{21}$$

Substituting (18), (20) and (21) in (13) we get

$$\frac{8\left(1-\nu\right)}{r^{2}}w_{n}h-\frac{8\left(1-\nu\right)}{r^{2}}w_{n}h+\frac{1-2\nu}{2\left(1-\nu\right)}\left[\lambda^{2}-\frac{n\left(n+1\right)}{r^{2}}\right]w_{n}h-$$

$$-\frac{n\left(n+1\right)}{2\left(1-\nu\right)r}\left\{\frac{r}{\nu n\left(n+1\right)}\left[\frac{8\left(1-2\nu\right)\left(1-\nu\right)}{r^{2}}w_{n}h+\frac{1-2\nu}{2G}\phi_{n}\right]+\frac{4\nu-3}{r}P_{\omega n}h\right\}=0$$

$$\frac{1}{2\left(1-\nu\right)}\left[-\frac{4}{r^{2}}\left(1-\nu\right)w_{n}h+4\frac{1-\nu}{r^{2}}w_{n}h\right]+$$

$$+\frac{1-2\nu}{2\left(1-\nu\right)}\left\{\frac{1}{\nu n\left(n+1\right)}\left[8\frac{\left(1-2\nu\right)\left(1-\nu\right)}{r^{2}}w_{n}h+\right.\right.$$

$$\left.\left.+p_{\omega n}\frac{1-2\nu}{2G}\right]+\frac{4\left(1-\nu\right)}{r^{2}}w_{n}h\right\}+\frac{1-\nu}{r}\frac{r}{\nu n\left(n+1\right)}\left[8\frac{\left(1-2\nu\right)\left(1-\nu\right)}{r^{2}}w_{n}h+\right.$$

$$+\frac{1-2\nu}{2G}p_{\omega n}h\right]+\left[\frac{1-2\nu}{2\left(1-\nu\right)}\lambda^2-\frac{n\left(n+1\right)}{r^2}\right]p_{\omega n}h=0.$$

or

$$\frac{1-2\nu}{2\left(1-\nu\right)}\lambda^2 w_n h-\frac{1-2\nu}{2\left(1-\nu\right)}\frac{n\left(n+1\right)}{r^2}w_n h-\frac{4\left(1-2\nu\right)}{\nu r^2}w_n h-$$

$$-\frac{n\left(n+1\right)}{\nu 2\left(1-\nu\right)}-\frac{1-2\nu}{2G}p_{\omega n}+\frac{n\left(n+1\right)}{2\left(1-\nu\right)}\frac{4\nu-3}{r^2}p_{\omega n}h=0.$$

$$\left\{8\frac{\left(1-2\nu\right)}{\nu n\left(n+1\right)}\left[\frac{1-2\nu}{2\left(1-\nu\right)}+1-\nu\right]+2\left(1-2\nu\right)\right\}\frac{w_n h}{r^2}+$$

$$+\left[\frac{1-2\nu}{2\left(1-\nu\right)}+1-\nu\right]\frac{1-2\nu}{\nu n\left(n+1\right)2G}p_{\omega n}+\left(\frac{1-2\nu}{2\left(1-\nu\right)}\lambda^2-\frac{n\left(n+1\right)}{r^2}\right)p_{\omega n}h=0. \quad (22)$$

Substituting the expression $p_{\omega n}=\frac{\rho\omega a\,j}{j'}$ from (12) to (22) , we get

$$\left\{\frac{1-2\nu}{2\left(1-\nu\right)}\lambda^2-\frac{1-2\nu}{r^2}\left[\frac{n\left(n+1\right)}{2\left(1-\nu\right)}+4\right]\right\}w_n h+$$

$$+\frac{n\left(n+1\right)\left(4\nu-3\right)}{2\left(1-\nu\right)}\phi_n h-\frac{\left(1-2\nu\right)}{2\left(1-\nu\right)2G}\frac{\rho\omega aj}{j'}w_n+$$

$$+\left\{4\frac{\left(1-2\nu\right)\left(1-\nu\right)}{\nu n\left(n+1\right)}\left[\frac{1-2\nu}{2\left(1-\nu\right)}+1-\nu\right]+1-2\nu\right\}2\frac{w_n h}{r^2}+$$

$$+\left[\frac{1-2\nu}{2\left(1-\nu\right)}+1-\nu\right]\frac{\left(1-2\nu\right)\rho\omega aj}{\nu n\left(n+1\right)2Gj'}+\left[\frac{1-2\nu}{2\left(1-\nu\right)}\lambda^2-\frac{n\left(n+1\right)}{r^2}\right]p_{\omega n}h$$

$$\left\{\frac{1-2\nu}{2\left(1-\nu\right)}h\lambda^2-\frac{1-2\nu}{r^2}\left[\frac{n\left(n+1\right)}{2\left(1-\nu\right)}+4\right]h-\frac{\left(1-2\nu\right)}{2\left(1-\nu\right)2G}\frac{\rho\omega aj}{j'}\right\}w_n+$$

$$+\frac{n\left(n+1\right)\left(4\nu-3\right)}{2\left(1-\nu\right)}hp_{n\omega}=0$$

$$\left(2\left\{4\frac{\left(1-2\nu\right)\left(1-\nu\right)}{\nu n\left(n+1\right)}\left[\frac{1-2\nu}{2\left(1-\nu\right)}+1-\nu\right]+1-2\nu\right\}\frac{h}{r^2}+\right.$$

$$\left.+\left[\frac{1-2\nu}{2\left(1-\nu\right)}+1-\nu\right]\frac{\left(1-2\nu\right)}{\nu n\left(n+1\right)}\frac{\rho\omega aj}{2Gj'}\right)w_n+$$

$$+\left[\frac{1-2\nu}{2\left(1-\nu\right)}\lambda^2-\frac{n\left(n+1\right)}{r^2}\right]hp_{\omega n}=0. \quad (23)$$

Accept the following denotation

$$\alpha=\frac{1-2\nu}{2\left(1-\nu\right)}h$$

$$\beta_1=-\frac{1-2\nu}{r^2}\left[\frac{n\left(n+1\right)}{2\left(1-\nu\right)}+4\right]h-\frac{n\left(n+1\right)\left(1-2\nu\right)}{4\left(1-\nu\right)G}\frac{\rho\,\omega\,a\,j}{j'}$$

$$\gamma_1 = \frac{n(n+1)(4\nu-3)}{2(1-\nu)}h$$

$$\beta_2 = 2\left\{4\frac{(1-2\nu)(1-\nu)}{\nu\cdot n(n+1)}\left[\frac{1-2\nu}{2(1-\nu)}+1-\nu\right]+1-2\nu\right\}\frac{h}{r^2}+$$
$$+\left[\frac{1-2\nu}{2(1-\nu)}+1-\nu\right]\frac{(1-2\nu)}{\nu n(n+1)}\frac{\rho\,\omega\,a\,j}{2Gj^1}$$

$$\gamma_1 = \frac{n(n+1)}{r^2}h.$$

Then (23) has the following form:

$$\begin{cases} \left(\alpha\lambda^2-\beta_1\right)w_n+\gamma_1\phi_n=0 \\ \beta_2 w_n+\left(\alpha\lambda^2-\frac{n(n+1)}{r^2}\right)\phi_n=0 \end{cases} \tag{24}$$

The equation of the system (24) is a system of homogeneous linear equation with respect to variables w_n and ϕ_n. For nontrivial solution, its determinant should equal zero. Then the frequency equation has the form:

$$\alpha^2\lambda^4-(\gamma_2+\beta_1)\,\alpha\lambda^2+\beta_1\gamma_2-\beta_2\gamma_1=0. \tag{25}$$

Here $\gamma_2 = \frac{n(n+1)}{r^2}h$.
The solution of the last equation with respect to λ has the form:

$$\lambda^2 = \frac{\alpha(\gamma_2+\beta_1)+\sqrt{\alpha(\gamma_2+\beta_1)^2+4(\beta_2\gamma_1-\beta_1\gamma_2)\,\alpha^2}}{2\alpha^2}. \tag{26}$$

In the case when the sphere is not filled and having denoted by $\lambda=\lambda_0$, $\beta_1=\beta_1^0$, $\beta_2=\beta_2^0$ we get the following dependence:

$$\lambda_0^2 = \lambda^2\frac{\alpha\left(\gamma_2+\beta_1^0\right)+\sqrt{\alpha^2\left(\gamma_2+\beta_1^0\right)^2+4\left(\beta_2^0\gamma_1-\beta_1^0\gamma_2\right)\alpha^2}}{\alpha(\gamma_2+\beta_1)+\sqrt{\alpha^2\left(\gamma_2+\beta_1^0\right)^2+4(\beta_2\gamma_1-\beta_1\gamma_2)\,\alpha^2}} \tag{27}$$

$$\beta_1^0 = \beta_1|_{\rho=0};\quad \beta_2^0 = \beta_2|_{\rho=0}.$$

In this case we use the following denotation:

$$z(\omega) = \frac{\omega\cdot r}{a}$$

$$\zeta(\omega) = \frac{z(\omega)\cdot\left[(z(\omega))^2-2\right]\cdot\sin(z(\omega))+2\,(z(\omega))^2\cos(z(\omega))}{\left[(z(\omega))^2-6\right]\cdot z(\omega)\cdot\cos(z(\omega))-3\left[(z(\omega))^2-2\right]\cdot\sin(z(\omega))}$$

$$\beta_1 = L(\omega) = \frac{-(1-2\nu)}{r^2}\cdot\left[\frac{n\cdot(n+1)}{2\cdot(1-\nu)}-4\right]\cdot h+\omega\cdot\frac{\rho\cdot a\cdot(n+1)\cdot(1-2\cdot\nu)}{4\cdot(1-\nu)\cdot G}\cdot\zeta(\omega)$$

$$\beta_2 = g(\omega) = \left[\frac{-4 (1 - 2 \cdot \nu)^2 \cdot (3 - 2\nu)}{\nu \cdot n \cdot (n+1)} + 1 - \nu \right] \cdot \frac{h}{r^2} +$$

$$+ \frac{(3 - 2\nu) \cdot (1 - 2\nu)}{4 \cdot (1 - \nu) \cdot [n (n+1)] \cdot \nu \cdot G} \cdot \rho \, \omega \, a \, \zeta(\omega)$$

$$\beta_1^0 = f(\omega) = \frac{-(1 - 2 \cdot \nu) \cdot h}{r^2} \cdot \left[\frac{n \cdot (n+1)}{2 \cdot (1 - \nu)} + 4 \right] - \frac{n \cdot (n+1) \cdot (1 - 2 \cdot \nu) \, \rho \, \omega \, a \, \zeta(\omega)}{4 \cdot (1 - \nu) \cdot G}$$

$$\beta_2^0 = j(\omega) = \left[\frac{-4 (1 - 2 \cdot \nu)^2 \cdot (3 - 2\nu)}{\nu \cdot n \cdot (n+1)} + 1 - \nu \right] \cdot \frac{h}{r^2} +$$

$$+ \frac{(3 - 4\nu) \cdot (1 - 2 \cdot \nu)}{4 \cdot (1 - \nu) \cdot n \cdot (n+1) \cdot \nu \cdot G} \cdot \rho \, \omega \, a \, \zeta(\omega) .$$

Then (27) takes the form:

$$M(\omega) = \sqrt{\omega^2 \cdot \frac{\alpha (\gamma_2 + f(\omega)) + \sqrt{[\alpha (\gamma_2 + f(\omega))]^2 + 4 \cdot \alpha^2 (j(\omega) \cdot \gamma_1 - f(\omega) \cdot \gamma_2)}}{(\gamma_2 + L(\omega)) + \sqrt{\alpha^2 (\gamma_2 + L(\omega))^2 + 4 \cdot \alpha^2 (g(\omega) \cdot \gamma_1 - L(\omega) \cdot \gamma_2)}}}. \quad (28)$$

This expression shows dependence of $m\omega$ frequency of unfilled sphere on the ω frequency of the system.

The graphs of dependences were constructed for different values of parameters. Different parameters of the sphere's thickness were taken into account (fig.1., fig. 2., fig. 3., fig. 4.)

When calculating, the following parameters were taken into account:

$$\gamma = \omega = 0,3; \;\; n = 2; \;\; r = 100 \; m; \;\; \rho = 1000 \frac{kg}{m^3}; \;\; a = 1400 \frac{m}{sec.}.$$

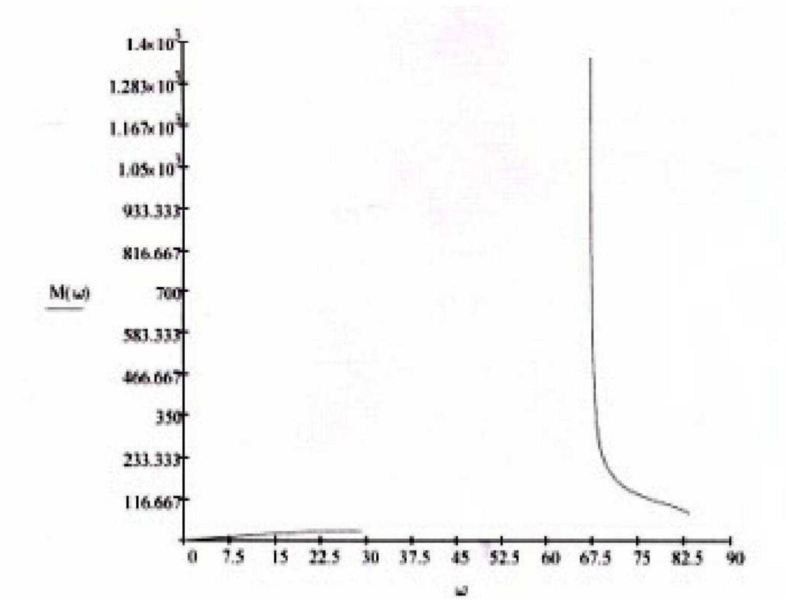

Fig. 1. $h = 0,2m$

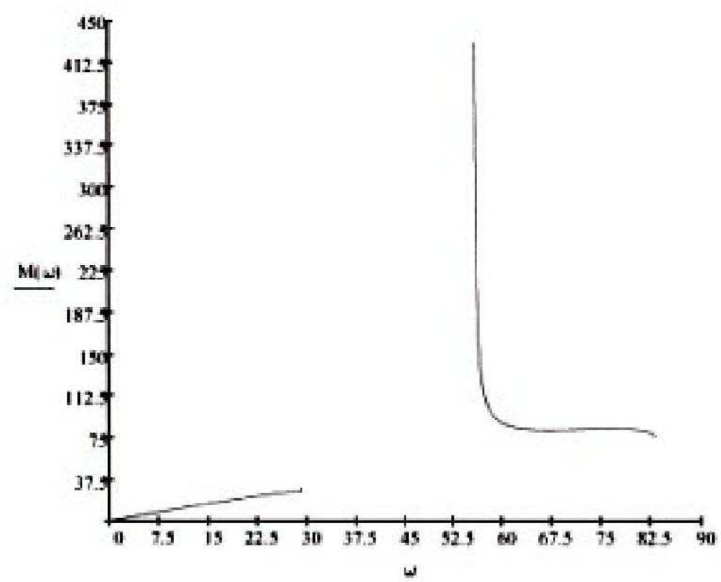

Fig. 2. $h = 0,5m$

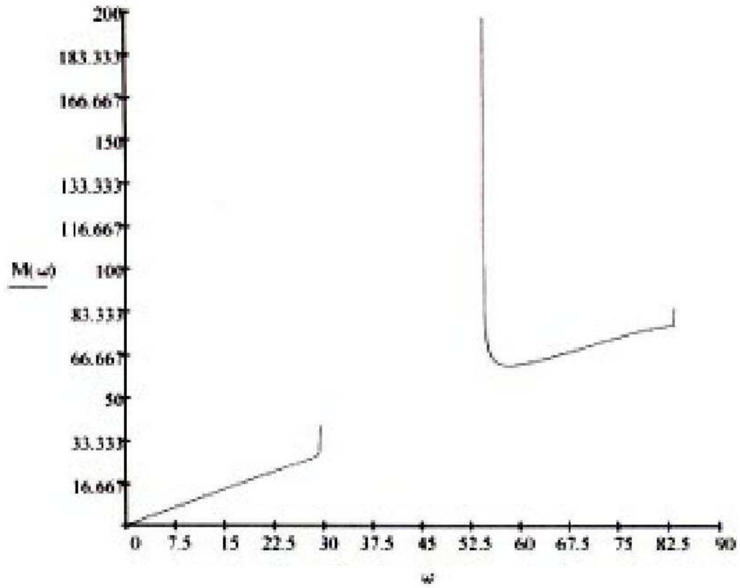

Fig. 3. $h = 1m$

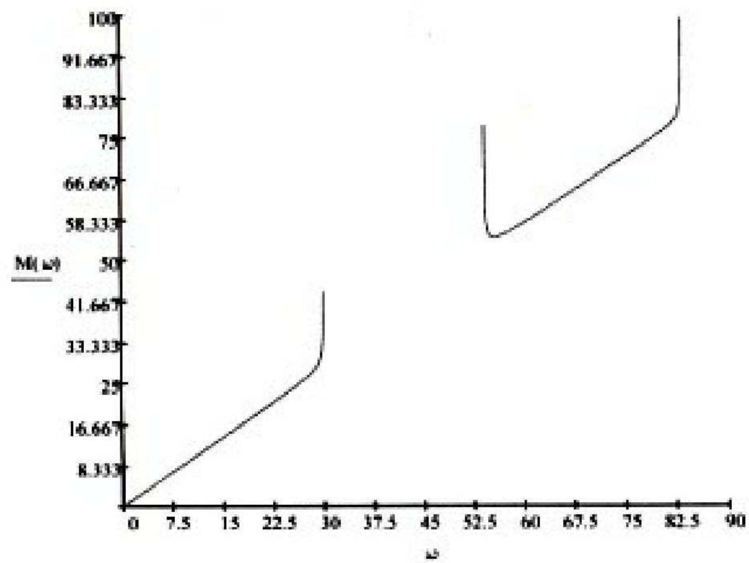

Fig. 4. $h = 5m$

As is seen from the graphs, the frequency of the system for the first mode is linearly connected with the frequency of the empty shell. The system's frequency reaches approximately 30 hertz.

However, for different thicknesses h of the shell for greater thickness, the frequency of the empty shell has the least value (for $h = 0,3m$, $\omega_0 = 37 hertz$, for $h = 5m$,

$\omega_0 = 27hertz$). At the end of the mentioned interval, the system's frequency asymptotically approaches to the constant value. Passage to the second mode is accompanied by the "failure" 30 hertz. Then with the same interval the picture of the first mode is repeated. At the ends of the second interval, the system's frequency passes to the constant value.

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(L_p, L_q)-boundedness of the Fractional Integral Operator with Rough Kernels on Heisenberg Groups

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Abstract. Let Ω is an homogeneous of degree zero function on Heisenberg group $\mathbb{H}_n$, integrable to a power $s > 1$ on the unit sphere generated by the corresponding Heisenberg metric. We study $L_p(\mathbb{H}_n)$-boundedness of the maximal operator M_Ω with rough kernels Ω in Heisenberg groups and the $\big(L_p(\mathbb{H}_n), L_q(\mathbb{H}_n)\big)$-boundedness of the fractional maximal and integral operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$, $0 < \alpha < Q$ with rough kernels.

Key Words and Phrases: fractional maximal function, fractional integral, Heisenberg group.

2010 Mathematics Subject Classifications: Primary: 42B25, 42B35, 43A15.

1. Introduction

The Heisenberg group [3, 4, 7, 9] appears in quantum physics and many fields of mathematics, including harmonic analysis, the theory of several complex variables and geometry. In this paper, we establish the norm inequalities for the maximal operator on the Heisenberg group in Lebesgue spaces. We begin with some basic notation. The Heisenberg group $\mathbb{H}_n$ a non-commutative nilpotent Lie group with the product spaces $\mathbb{R}^{2n+1}$ that have the multiplication

$$xy = \Big(x' + y', x_{2n+1} + y_{2n+1} + 2\sum_{k=1}^{n} x_k y_{n+k} - x_{n+k} y_k\Big),$$

where $x = (x', x_{2n+1})$, and $y = (y', y_{2n+1})$. By the definition, the identity element on $\mathbb{H}_n$ is $0 \in \mathbb{R}^{2n+1}$, while the inverse element of $x = (x', t)$ is $x^{-1} = (-x', -t)$.

The corresponding Lie algebra is generated by the left-invariant vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n,$$
$$X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n,$$

$$X_{2n+1} = \frac{\partial}{\partial x_{2n+1}}.$$

The only non-trivial commutator relations are

$$\big[X_j, X_{n+j}\big] = -4X_{2n+1}, \quad j = 1, \dots, n.$$

The non-isotropic dilation on $\mathbb{H}_n$ is defined as $\delta_t(x', x_{2n+1}) = (tx', t^2 x_{2n+1})$ for $t > 0$. The Haar measure dx on this group coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. It is easy to check that

$$d\big(\delta_t x\big) = r^Q dx.$$

In the above, $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}_n$.

The norm of $x = (x', x_{2n+1}) \in \mathbb{H}_n$ is given by

$$|x|_h = (|x'|^4 + x_{2n+1}^2)^{1/4},$$

where $|x'|^2 = \sum\limits_{k=1}^{2n} x_k^2$. The norm satisfies the triangle inequality and leads to the left-invariant distance $d(x, y) = |xy^{-1}|_h$. With this norm we define the Heisenberg ball,

$$B(x, r) = \{y \in \mathbb{H}_n : |xy^{-1}| < r\},$$

where x is the center and r is the radius. The volume of $B(x, r)$ is $C_n r^{2n+2}$, where C_n is the volume of the unit ball $B_1 \equiv B(e, 1)$, i.e.,

$$C_n = \frac{2\pi^{n+\frac{1}{2}}\Gamma\big(\frac{1}{2}\big)}{(n+1)\Gamma(n)\Gamma\big(\frac{n+1}{2}\big)}.$$

Let $S_H = \{x \in \mathbb{H}_n : |x|_h = 1\}$ be the unit sphere in $\mathbb{H}_n$ equipped with the normalized Haar surface measure $d\sigma$ and Ω be δ_t-homogeneous of degree zero, i.e. $\Omega(\delta_t x) \equiv \Omega(x)$, $x \in \mathbb{H}_n$, $t > 0$. The fractional maximal function $M_{\Omega,\alpha} f$ and the fractional integral $I_{\Omega,\alpha} f$ by with rough kernels, $0 < \alpha < Q$ of a function $f \in L_1^{\mathrm{loc}}(\mathbb{H}_n)$ are defined by

$$M_{\Omega,\alpha} f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{Q}} \int_{B(x,t)} |\Omega(y^{-1}x)|\ |f(y)| dy,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y^{-1}x)\ f(y)}{\big|y^{-1}x\big|_h^{Q-\alpha}} dy.$$

If $\Omega \equiv 1$, then $M_\alpha \equiv M_{1,\alpha}$ and $I_\alpha \equiv I_{1,\alpha}$ are the fractional maximal operator and the fractional integral operator, respectively. If $\alpha = 0$, then $M_\Omega \equiv M_{\Omega,0}$ is the maximal operator with rough kernel. It is well known that the fractional maximal operator on Heisenberg groups play an important role in harmonic analysis (see [4, 8]).

The boundedness of classical operators of the real analysis, such as the maximal operator and singular integral operators etc, from one Lebesgue space to another one is well studied by now, and there are well known various applications of such results in partial differential equations. In this paper, we study the L_p-boundedness of the maximal operator with rough kernels in Heisenberg groups, including also the case of weak boundedness. Also we obtain $\big(L_p(\mathbb{H}_n), L_q(\mathbb{H}_n)\big)$-boundedness of the fractional maximal and integral operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$, $0 < \alpha < Q$ with rough kernels.

Throughout the paper, for a measurable set E, $|E|$ denotes the normalized Haar measure of E, i.e., $|B_1| = \int_{B_1} dx = 1$. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. For a number p, p' denotes the conjugate exponent of p. d B are equivalent.

2. Boundedness of the fractional integral operators in the spaces $L_p(\mathbb{H}_n)$

In this section we prove the $L_p(\mathbb{H}_n)$-boundedness of the operator M_Ω and the $\big(L_p(\mathbb{H}_n), L_q(\mathbb{H}_n)\big)$-boundedness of the operators $I_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$.

Theorem 1. *Let $\Omega \in L_s(S_H)$, $1 < s \le \infty$, be δ_t-homogeneous of degree zero. Then the operator M_Ω is bounded in the space $L_p(\mathbb{H}_n)$, $p > s'$.*

Proof.

In the case $s = \infty$ the statement of Theorem 1 is known and may be found in [2] and [8]. So we assume that $1 < s < \infty$.

Note that

$$\begin{aligned}\|\Omega(\cdot^{-1}x)\|_{L_s(B(x,t))} &= \Big(\int_{B(0,t)} |\Omega(y)|^s dy\Big)^{1/s} \\ &= \Big(\int_0^t r^{Q-1} dr \int_{S_H} |\Omega(\omega)|^s d\sigma(\omega)\Big)^{1/s} \\ &= c_0\ \|\Omega\|_{L_s(S_H)}\ |B(x,t)|^{1/s},\end{aligned} \tag{1}$$

where $c_0 = \big(Qv_H\big)^{-1/s}$ and $v_H = |B(0,1)|$.

The case $p = \infty$ is easy. Indeed, making use of (1), we get

$$\begin{aligned}\|M_\Omega f\|_{L_\infty(\mathbb{H}_n)} &\le \|f\|_{L_\infty(\mathbb{H}_n)} \sup_{t>0} |B(x,t)|^{-1+\frac{1}{s'}} \|\Omega(\cdot^{-1}x)\|_{L_s(B(x,t))} \\ &\le c_0\ \|\Omega\|_{L_s(S_H)}\ \|f\|_{L_\infty(\mathbb{H}_n)}.\end{aligned}$$

So we assume that $s' < p < \infty$. Applying Hölder's inequality, we get

$$M_\Omega f(x) \le \sup_{t>0} |B(x,t)|^{-1} \|\Omega(\cdot^{-1}x)\|_{L_s(B(x,t))}\ \|f\|_{L_{s'}(B(x,t))}. \tag{2}$$

Then from (2) and (1) we have

$$M_\Omega f(x) \le c_0\ \|\Omega\|_{L_s(S_H)} \sup_{t>0} |B(x,t)|^{-1+1/s}\ \|f\|_{L_{s'}(B(x,t))}$$

$$= c_0 \ \|\Omega\|_{L_s(S_H)} \Big(\sup_{t>0} |B(x,t)|^{-1} \| |f|^{s'} \|_{L_1(B(x,t))} \Big)^{1/s'}$$

$$= c_0 \ \|\Omega\|_{L_s(S_H)} \Big(M(|f|^{s'})(x) \Big)^{1/s'}. \tag{3}$$

Therefore, from (3) for $1 \le s' < p < \infty$ we get

$$\|M_\Omega f\|_{L_p(\mathbb{H}_n)} \le c_0 \ \|\Omega\|_{L_s(S_H)} \big\| \big(M(|f|^{s'})(\cdot) \big)^{1/s'} \big\|_{L_p(\mathbb{H}_n)}$$
$$= c_0 \ \|\Omega\|_{L_s(S_H)} \big\| M\big(|f|^{s'}\big) \big\|^{1/s'}_{L_{p/s'}(\mathbb{H}_n)} \lesssim \| |f|^{s'} \|^{1/s'}_{L_{p/s'}(\mathbb{H}_n)} = \|f\|_{L_p(\mathbb{H}_n)}.$$

We prove the boundedness of the fractional maximal and integral operators $M_{\Omega,\alpha}$, $I_{\Omega,\alpha}$ with rough kernel from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/Q$, and from the space $L_1(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$, $1 \le q < \infty$, $1 - 1/q = \alpha/Q$.

Theorem 2. *Suppose that $0 < \alpha < Q$ and the function $\Omega \in L_{\frac{Q}{Q-\alpha}}(S_H)$ is δ_t-homogeneous of degree zero. Let $1 \le p < \frac{Q}{\alpha}$ and $1/p - 1/q = \alpha/Q$. Then the fractional integration operator $I_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ for $p > 1$ and from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$ for $p = 1$.*

Proof. We denote

$$K(x) := \frac{\Omega(x)}{|x|_h^{Q-\alpha}}$$

for brevity, and may assume that $K(x) \ge 0$. We have

$$\Big| \{x \in \mathbb{H}_n : I_{\Omega,\alpha} f(x) > \lambda\} \Big| \le \Big| \{x \in \mathbb{H}_n : I_{\Omega,\alpha} f(x) > C_{Q,\alpha}^{-1} \lambda\} \Big| \le I_1 + I_2,$$

where

$$I_1 := \Big| \Big\{ x \in \mathbb{H}_n : |K_\mu^1 * f(x)| > \frac{\lambda}{2} \Big\} \Big|, \quad I_2 := \Big| \Big\{ x \in \mathbb{H}_n : |K_\mu^2 * f(x)| > \frac{\lambda}{2} \Big\} \Big|,$$

$$K_\mu^1(x) = (K(x) - \mu)\chi_{E(\mu)}(x) \ \text{ and } \ K_\mu^2(x) = K(x) - K_\mu^1(x),$$

$\mu > 0$ and $E(\mu) = \{x \in \mathbb{H}_n : |K(x)| > \mu\}$. Note that

$$|E(\mu)| \le B\mu^{\frac{Q}{Q-\alpha}}. \tag{4}$$

where $B = \frac{1}{\alpha} \|\Omega\|^{\frac{Q}{Q-\alpha}}_{L_{\frac{Q}{Q-\alpha}}(S_H)}$ as seen from the following estimation:

$$|E(\mu)| \le \frac{1}{\mu} \int_{E(\mu)} \frac{|\Omega(x)|}{|x|_h^{Q-\alpha}} dx$$
$$= \frac{1}{\mu} \int_{S_H} \Omega(x') d\sigma(x') \int_0^{\left(\frac{|\Omega(x')|}{\mu}\right)^{\frac{1}{Q-\alpha}}} r^{\alpha-1} dr = B\mu^{\frac{Q}{Q-\alpha}}.$$

By means of (4) we can prove the estimate

$$\|K_\mu^2\|_{L_{p'}(\mathbb{H}_n)} \le \Big(\frac{Q-\alpha}{Q}Bq\Big)^{\frac{1}{p'}} \mu^{\frac{Q}{(Q-\alpha)\,q}}, \quad 1 \le p < \frac{Q}{\alpha}.$$

For $p=1$ it easily follows from (4), and for $p>1$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |K_\mu^2(x)|^{p'} dx &= p' \int_0^\mu t^{p'-1} |E(t)| dt \\ &\le p' B \int_0^\mu t^{p'-1-\frac{Q}{Q-\alpha}} dt \\ &= \frac{Q-\alpha}{Q} B q \mu^{\frac{Q}{Q-\alpha}\frac{p'}{q}}.\end{aligned}$$

Then by the Young inequality we obtain

$$\|K_\mu^2 * f\|_{L_\infty(\mathbb{H}_n)} \le \|K_\mu^2\|_{L_{p'}} \|f\|_{L_p(\mathbb{H}_n)} \le \Big(\frac{Q-\alpha}{Q}Bq\Big)^{\frac{1}{p'}} \mu^{\frac{Q}{(Q-\alpha)\,q}} \ \|f\|_{L_p(\mathbb{H}_n)}.$$

Now for a $\lambda > 0$, we choose μ such that

$$\Big(\frac{Q-\alpha}{Q}Bq\Big)^{\frac{1}{p'}} \mu^{\frac{Q}{(Q-\alpha)\,q}} \ \|f\|_{L_p(\mathbb{H}_n)} = \frac{\lambda}{2},$$

then

$$\Big|\Big\{x \in \mathbb{H}_n : |K_\mu^2 * f(x)| > \frac{\lambda}{2}\Big\}\Big| = 0.$$

Thus

$$\begin{aligned}\Big|\{x \in \mathbb{H}_n : I_{\Omega,\alpha} f(x) > \lambda\}\Big| &\le \Big|\Big\{x \in \mathbb{H}_n : |K_\mu^1 * f(x)| > \frac{\lambda}{2}\Big\}\Big| \\ &\le \Big(\frac{2}{\lambda}\ \|K_\mu^1 * f\|_{L_p(\mathbb{H}_n)}\Big)^p. \end{aligned} \tag{5}$$

The following estimations take (4) into account:

$$\begin{aligned}\int_{\mathbb{H}_n} |K_\mu^1(x)| dx &= \int_{E(\mu)} \big(|K(x)| - \mu\big) dx \\ &\le \int_0^\infty |E(t+\mu)| dt \\ &\le B \int_\mu^\infty t^{-\frac{Q}{Q-\alpha}} dt \\ &= \frac{\alpha B}{Q-\alpha}\ \mu^{-\frac{\alpha}{Q-\alpha}}.\end{aligned} \tag{6}$$

For all $f \in L_\infty(\mathbb{H}_n)$ and $x \in \mathbb{H}_n$, from (6) it follows that

$$|K^1_\mu * f(x)| \leq \|f\|_{L_\infty(\mathbb{H}_n)} \int_{\mathbb{H}_n} |K^1_\mu(x)|dx \leq \frac{\alpha B}{Q-\alpha}\ \mu^{-\frac{\alpha}{Q-\alpha}}\ \|f\|_{L_\infty(\mathbb{H}_n)}. \tag{7}$$

For all $f \in L_1(\mathbb{H}_n)$, from (6) follows

$$\begin{aligned} \|K^1_\mu * f\|_{L_1(\mathbb{H}_n)} &\leq \int_{\mathbb{H}_n}\int_{\mathbb{H}_n} |K^1_\mu(x-y)||f(y)|dxdy \\ &\leq \frac{\alpha B}{Q-\alpha}\ \mu^{-\frac{\alpha}{Q-\alpha}}\ \|f\|_{L_1(\mathbb{H}_n)}. \end{aligned} \tag{8}$$

Thus from (7) and (8) follows that the operator $T_1 : f \to K^1_\mu * f$ is of (∞,∞) and $(1,1)$-type. Then by the Riesz-Thorin theorem the operator T_1 is also of (p,p)-type, $1 < p < \infty$, and

$$\|T_1 f\|_{L_p(\mathbb{H}_n)} \leq \frac{\alpha B}{Q-\alpha}\ \mu^{-\frac{\alpha}{Q-\alpha}}\ \|f\|_{L_p(\mathbb{H}_n)}. \tag{9}$$

From (5) and (9) we get

$$\begin{aligned} \Big|\{x \in \mathbb{H}_n : I_{\Omega,\alpha}f(x) > \lambda\}\Big| &\leq \Big(\frac{2}{\lambda}\ \|K^1_\mu * f\|_{L_p(\mathbb{H}_n)}\Big)^p \\ &\leq C\Big(\frac{1}{\lambda}\ \|f\|_{L_p(\mathbb{H}_n)}\Big)^q, \end{aligned} \tag{10}$$

where C is independent of λ and f.

To finish the proof, i.e. prove that the operator $I_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ for $1 < p < \frac{Q}{\alpha}$ and $1/p - 1/q = \alpha/Q$, observe that the inequality (10) tells us that $I_{\Omega,\alpha}$ is bounded from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$ with $1 - 1/q = \alpha/Q$. We choose any p_0 such that $p < p_0 < \frac{Q}{\alpha}$, and put $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{Q}$. By (10) the operator $I_{\Omega,\alpha}$ is of weak (p_0, q_0)-type. Since it is also of weak $(1,q)$-type by the Marcinkiewicz interpolation theorem, we conclude that $I_{\Omega,\alpha}$ is of $\big(L_p, L_q\big)$-type.

Corollary 1. *Under the assumptions of Theorem 2, the fractional maximal operator $M_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ for $p > 1$ and from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$ for $p = 1$.*

Proof. It suffices to refer to the known fact that

$$M_{\Omega,\alpha}f(x) \leq C_{Q,\alpha} I_{\Omega,\alpha}f(x), \qquad C_{Q,\alpha} = |B(0,1)|^{\frac{Q-\alpha}{Q}},$$

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Global Bifurcation from Zero and Infinity in Nonlinear Beam Equation with Indefinite Weight

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Abstract. We consider a nonlinear eigenvalue problem for the beam equation with an indefinite weight function. We investigate the bifurcation from zero and infinity for this problem and prove the existence of unbounded continua bifurcating from the principal eigenvalues of the corresponding linear problem contained in the classes of positive and negative functions.

Key Words and Phrases: nonlinear eigenvalue problem, bifurcation point, principal eigenvalues, global continua, indefinite weight.
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1. Introduction

We consider the following fourth order boundary value problem

$$(\ell u)(t) \equiv (p(t)u''(t))''(t) - (q(t)u'(t))' = \lambda g(t)f(u(t)),\ t \in (0,1), \tag{1}$$

$$\begin{aligned} &u'(0)\cos\alpha - (pu'')(0)\sin\alpha = 0, \\ &u(0)\cos\beta + Tu(0)\sin\beta = 0, \\ &u'(1)\cos\gamma + (pu'')(1)\sin\gamma = 0, \\ &u(1)\cos\delta - Tu(1)\sin\delta = 0, \end{aligned} \tag{2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $Ty \equiv (pu'')' - qu'$, $p \in C^2[0,1]$ with $p(t) > 0$, $t \in [0,1]$, $q \in C^1[0,1]$ with $q(t) \geq 0$, $t \in [0,1]$, $g \in C[0,l]$ is a sign-changing weight function (i.e. $\text{meas}\{t \in (0,1) : \sigma u(t) > 0\} > 0$ for each $\sigma \in \{+,-\}$) and $\alpha, \beta, \gamma, \delta \in [0, \frac{\pi}{2}]$ with except the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$. The nonlinear term $f \in C(\mathbb{R};\mathbb{R})$ and satisfies the conditions: $tf(t) > 0$ for $t \in \mathbb{R}\backslash\{0\}$ and there exist $f_0, f_\infty \in (0,+\infty)$ such that

$$f_0 = \lim_{|t|\to 0} \frac{f(t)}{t},\ \ f_\infty = \lim_{|t|\to\infty} \frac{f(t)}{t}\ . \tag{3}$$

It is well known that fourth-order problems arise in many applications (see [8, 24] and the references therein); problem (1)-(2) in particular, is often used to describe the deformation of an elastic beam, which is subject to axial forces (see [8]). Problems with sign-changing weight arise from population modeling. In this model, weight function g changes sign corresponding to the fact that the intrinsic population growth rate is positive at same points and is negative at others, for details, see [10, 15].

The purpose of this work is to study the global bifurcation of solutions of problem (1)-(2) in the classes of positive and negative functions, emanating from the zero and infinity.

It should be noted that the nonlinear problem (1)-(2) is closely related to the following linear eigenvalue problem

$$\begin{array}{l} (p(t)u''(t))''(t) - (q(t)u'(t))' = \lambda g(t)u(t), \ t \in (0,1), \\ u \in B.C., \end{array} \tag{4}$$

where by $B.C.$ we denote the set of boundary conditions (2). The nonlinear problem (1)-(2) and linear problem (4) in the case $p \equiv 1$, $q \equiv 0$ and $\alpha = \gamma = \frac{\pi}{2}$, $\beta = \delta = 0$ was previously considered in [23] the results of which contain gaps.

The problems (4) and (1)-(2) in the case when the first condition in (3) is satisfied are studied in [18], where, in particular, it was shown that there exist two positive and negative principal eigenvalues, λ_1 and λ_{-1}, respectively, of the linear problem (4) and the corresponding eigenfunctions have no zeros in $(0,1)$; moreover, also proved that for each $k \in \{1, -1\}$ and each $\nu \in \{+, -\}$ there exists a continuum (connected closed set) $\mathfrak{L}_k^\nu$ of solutions of problem (1)-(2) bifurcating from the point $(\lambda_k, 0)$, which is unbounded in $\mathbb{R} \times C^3[0,1]$, and $\nu \operatorname{sgn} y(x) = 1$, $x \in (0,1)$ for each $(\lambda, y) \in \mathfrak{L}_k^\nu$. Note that, similar problems have been considered before in, for example, [10] and [30], but the results of these works are not true (see [4]).

In Section 2, a family of sets to exploit oscillatory properties of eigenfunctions of problem (4) and their derivatives is introduced. The existence of global continua of solutions of the problem (1)-(2) bifurcating simultaneously from the zero and infinity, and contained in these sets is proved in Section 3. Here we give the application of global bifurcation technique to the study of positive or negative solutions for the some nonlinear boundary value problems.

2. Preliminary

In [23] the authors note that there are few papers discussing the existence and multiplicity of positive solutions to (4), the main reason of which is that the spectrum of the linear eigenvalue problem is not clear. They showed that the problem (4) has exactly two principal eigenvalues, one positive and one negative, and the corresponding eigenfunctions do not change its sign on $(0,1)$. But it should be noted that in the proof of this fact, the authors did not give a correct reference to the work [16]. However until recently there no results on the multiplicities of the first m ($m > 2$) (for the definition of m, see [19, 21]) eigenvalues and on the oscillatory properties for the corresponding eigenfunctions of the following problem

$$\begin{aligned} &(p(t)u''(t))''(t) - (q(t)u'(t))' + h(t)u(t) = \mu u(t), \ t \in (0,1), \\ &u \in B.C., \end{aligned} \tag{5}$$

where $h \in C([0,1];\mathbb{R})$. In [19, 21] it was shown that, in the case of $h(t)$ not identically vanishing on any subinterval of $[0,1]$, the eigenvalues of problem (5) are real, and simple, except, possibly, the first m eigenvalues, and the corresponding eigenfunctions with numbers larger than m have the Sturm oscillation properties, i.e. the eigenfunction has only simple nodal zeros and the number of zeros of the eigenfunction is equal to the serial number of the corresponding eigenvalue increased by 1. But, in [23], the authors in proving Theorem 2.1 recall the work [16] and claim that the eigenfunction, corresponding to the first eigenvalue of the problem (5), has no zeros in the interval $(0,1)$. Unfortunately in [16] oscillatory properties of eigenfunctions of the problem (4) were not studied. Recently, in [3] (see also [5, 6]) it was established that all eigenvalues of the problem (5) are simple and the corresponding eigenfunctions have the Sturm oscillation properties.

For the linear eigenvalue problem (4) we have the following result.

Theorem 1. [18, Theorem 2.1] . *The spectral problem* (4) *has two sequences of real eigenvalues*

$$0 < \lambda_1^+ \le \lambda_2^+ \le ... \le \lambda_k^+ \mapsto +\infty,$$

and

$$0 > \lambda_1^- \ge \lambda_2^- \ge ... \ge \lambda_k^- \mapsto -\infty$$

and no other eigenvalues. Moreover, λ_1^+ and λ_1^- are simple principal eigenvalues, i.e. the corresponding eigenfunctions $u_1^+(t)$ and $u_1^-(t)$ have no zeros in the interval $(0,1)$.

Similar problems have been considered in [9, 13, 14, 17, 22].

Let E be the Banach space of all continuously three times differentiable functions on $[0,1]$ which satisfy the conditions $B.C.$ and is equipped with its usual norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$, where $||u||_\infty = \max\limits_{t\in[0,1]} |u(t)|$.

Let

$$S = S_1 \cup S_2,$$

where

$$S_1 = \{u \in E : u^{(i)}(t) \neq 0, Tu(t) \neq 0, \ t \in [0,1], \ i = 0, \ 1, \ 2\}$$

and

$$\begin{aligned} S_2 = \{&u \in E: \text{ there exists } i_0 \in \{0,\ 1,\ 2\} \text{ and } t_0 \in (0,1) \text{ such that } u^{(i_0)}(t_0) = 0, \\ &\text{or } Tu(t_0) = 0 \text{ and if } u(t_0)u''(t_0) = 0, \text{ then } u'(t)Tu(t) < 0 \text{ in a neighborhood of } t_0, \\ &\text{and if } u'(t_0)Tu(t_0) = 0, \text{ then } u(t)u''(t) < 0 \text{ in a neighborhood of } t_0\}. \end{aligned}$$

Note that if $u \in S$ then the Jacobian $J = \rho^3 \cos\psi \sin\psi$ (see [1-3, 5, 6, 20]) of the Prüfer-type transformation

$$\begin{cases} u(x) = \rho(x) \sin\psi(x) \cos\theta(x), \\ u'(x) = \rho(x) \cos\psi(x) \sin\varphi(x), \\ (pu'')(x) = \rho(x) \cos\psi(x) \cos\varphi(x), \\ Tu(x) = \rho(x) \sin\psi(x) \sin\theta(x), \end{cases} \tag{6}$$

does not vanish on $(0,1)$.

For each $u \in S$ we define $\rho(u,t)$, $\theta(u,t)$, $\varphi(u,t)$, $w(u,t)$ to be the continuous functions on $[0,1]$ satisfying

$$\rho(u,t) = u^2(t) + u'^2(t) + (p(t)u''(t))^2 + (Tu(t))^2,$$

$$\theta(u,t) = \operatorname{arctg}\frac{Tu(t)}{u(t)},\ \theta(u,0) = \beta - \pi/2\,,$$

$$\varphi(u,t) = \operatorname{arctg}\frac{u'(t)}{(pu'')(t)},\ \varphi(u,0) = \alpha\,,$$

$$w(u,t) = \operatorname{ctg}\psi(u,t) = \frac{u'(t)\cos\theta(u,t)}{u(t)\sin\varphi(u,t)},\ w(u,0) = \frac{u'(0)\sin\beta}{u(0)\sin\alpha}\,,$$

and $\psi(u,t) \in (0,\pi/2)$, $t \in (0,1)$, in the cases of $u(0)u'(0) > 0$; $u(0) = 0$; $u'(0) = 0$ and $u(0)u''(0) > 0$, $\psi(u,t) \in (\pi/2,\pi)$, $t \in (0,1)$, in the cases $u(0)u'(0) < 0$; $u'(0) = 0$ and $u(0)u''(0) < 0$; $u'(0) = u''(0) = 0$, $\beta = \pi/2$ in the case $\psi(u,0) = 0$ and $\alpha = 0$ in the case $\psi(u,0) = \pi/2$.

It is apparent that ρ, θ, $\varphi, w : S \times [0,1] \to \mathbb{R}$ are continuous.

Remark 3.1. By (7) for each $u \in S$ the function $w(u,t)$ can be determined by one of the following relations

$$\text{a) } w(u,x) = \operatorname{ctg}\psi(u,x) = \frac{(pu'')(x)\cos\theta(u,x)}{u(x)\cos\varphi(u,x)},\ w(u,0) = \frac{(pu'')(0)\sin\beta}{u(0)\cos\alpha}\,,$$

$$\text{b) } w(u,x) = \operatorname{ctg}\psi(u,x) = \frac{(pu'')(x)\sin\theta(u,x)}{Tu(x)\cos\varphi(u,x)},\ w(u,0) = -\frac{(pu'')(0)\cos\beta}{Tu(0)\cos\alpha}\,,$$

$$\text{c) } w(u,x) = \operatorname{ctg}\psi(u,x) = \frac{u'(x)\sin\theta(u,x)}{Tu(x)\sin\varphi(u,x)},\ w(u,0) = -\frac{u'(0)\cos\beta}{Tu(0)\sin\alpha}\,.$$

For each $\nu \in \{+\,,\,-\}$ let S_1^ν denotes the subset of such $u \in S$ that:

1) $\theta(u,1) = \pi/2 - \delta$, where $\delta = \pi/2$ in the case $\psi(u,1) = 0$;

2) $\varphi(u,1) = 2\pi - \gamma$ or $\varphi(u,1) = \pi - \gamma$ in the case $\psi(u,0) \in [0,\pi/2)$; $\varphi(u,1) = \pi - \gamma$ in the case $\psi(y,0) \in [\pi/2,\pi)$, where $\gamma = 0$ in the case $\psi(y,l) = \pi/2$;

3) for fixed u, as t increases from 0 to 1, the function $\theta(u,t)$ ($\varphi(u,t)$) strictly increasingly takes values of $m\pi/2$, $m \in \{-1,\,0,\,1\}$ ($s\pi$, $s \in \{0,\,1,\,2\}$); as t decreases from 1 to 0, the function $\theta(u,t)$ ($\varphi(u,t)$), strictly decreasing takes values of $m\pi/2$, $m \in \{-1,\,0,\,1\}$ ($s\pi$, $s \in \{0,\,1,\,2\}$);

4) the function $\nu u(t)$ is positive in a neighborhood of $t=0$.

By [2; Theorem 4.4], [6; Theorem 1.1], [7; Lemma 2.2, Theorems 5.1, 5.2, 6.1, 6.3-6.5] and Theorem 2.1 we have $u_1^+,\, u_1^- \in S_1$, i.e the sets S_1^+ and S_1^- are nonempty. It immediately follows from the definition of these sets that they are disjoint and open in E. Moreover, by [2; Lemma 2.2] if $u(t)\in \partial S_1^\nu \cap C^4[0,1]$, $\nu\in\{+\,,\,-\}$, then $u(t)$ has at least one zero of multiplicity 4 in $(0,1)$.

Let $u_{1,+}^+$ $(u_{1,+}^-)$ denote the unique eigenfunction of (4) corresponding to the eigenvalue λ_k^+ (λ_k^-) such that $u_{1,+}^+\in S_1^+$ $(u_{1,+}^-\in S_1^+)$ and $||u_{1,+}^+||_3=1$ $(||u_{1,+}^-||_3=1)$.

Lemma 1. (see [1, 2]) *If* $(\lambda,u)\in\mathbb{R}\times E$ *is a solution of* (1)-(2) *and* $u\in\partial S_1^\nu$, $\nu\in\{+\,,\,-\}$, *then* $u\equiv 0$.

3. Global bifurcation from zero and infinity for the problem (1)-(2)

It should be noted that in order to prove the existence of at least one solution of the problem (1)-(2) in the class of positive functions, in [23], the authors used global bifurcation results (see [23, p. 6598]) which also contains gaps. This result is similar to that for the nonlinear Sturm-Liouville problems which has been obtained by Rabinowitz [26]. In the nonlinear Sturm-Liouville problem considered in [26] nodal properties are preserved on the continuous branch of nontrivial solutions emanating from bifurcation points and this prevents the first alternative in part (ii) of [29; Lemma 2.6] occurring. But for the nonlinear fourth order eigenvalue problem nodal properties need not be preserved, so we must considered this alternative. Therefore, in the study of nonlinear fourth-order eigenvalue problem there is a need to study the following questions: to construct the classes of functions that preserve the oscillation properties of eigenfunctions of the linear problem (4) and their derivatives, such that if the solution of the nonlinear problem is contained on the boundary of this set, then this must be identically zero (if means that continuous branch of solutions can not go from the boundary of this set). This question was solved in a recent paper [3] (see also [2]), in which global bifurcation from zero of solutions of the nonlinear eigenvalue problems for ordinary differential equations of fourth order was studied.

Let $\mathfrak{L}$ denotes the closure of the set of nontrivial solutions of (1)-(2).

Theorem 2. *For each* $k\in\{-1,\,1\}$ *and each* $\nu\in\{-\,,\,+\}$ *there exists a continuum* $\mathfrak{L}_k^\nu$ *of solutions of problem* (1)-(2) *in* $(\mathbb{R}\times S_1)\cup\left\{\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0},0\right)\right\}\cup\left\{\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty},\infty\right)\right\}$ *which meets* $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0},0\right)$ *and* $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty},\infty\right)$ *in* $R^{\mathrm{sgn}k}\times E$, *where* $R^{\mathrm{sgn}k}=\{\chi\in\mathbb{R}\,:\,0<\chi\mathrm{sgn}k\le+\infty\}$.

Proof. By virtue of (3) there exists the functions $\tau\in C(\mathbb{R},\mathbb{R})$ and $\varepsilon\in C(\mathbb{R},\mathbb{R})$ such that

$$f(u)=f_0u+\tau(u),\;\; f(u)=f_\infty u+\varepsilon(u), \tag{7}$$

and

$$\lim_{|u|\to 0}\frac{\tau(u)}{u}=0,\quad \lim_{|u|\to +\infty}\frac{\varepsilon(u)}{u}=0. \tag{8}$$

It follows from (7) that the problem (1)-(2) can be rewritten in the following form

$$\begin{array}{l}(\ell u)(t)=\lambda f_0 g(t)u(t)+\lambda g(t)\tau(u(t)),\ t\in(0,1),\\ u\in B.C.\end{array} \tag{9}$$

or

$$\begin{array}{l}(\ell u)(t)=\lambda f_\infty g(t)u(t)+\lambda g(t)\varepsilon(u(t)),\ t\in(0,1),\\ u\in B.C..\end{array} \tag{10}$$

Since $\lambda=0$ is not eigenvalue of the linear problem (5) for $h\equiv 0$ it follows that the problems (9) and (10) are equivalent to the following integral equations

$$u(t)=\lambda f_0\int_0^1 K(t,s)g(s)u(s)ds+\lambda\int_0^1 K(t,s)\,g(s)\,\tau(u(s))\,ds, \tag{11}$$

$$u(t)=\lambda f_\infty\int_0^1 K(t,s)g(s)u(s)ds+\lambda\int_0^1 K(t,s)\,g(s)\,\varepsilon(u(s))\,ds, \tag{12}$$

respectively, where $K(t,s)$ is a Green's function of differential expression $\ell(u)$ with respect to the $B.C.$.

Define $\mathcal{L}:E\to E$ by

$$(\mathcal{L}u)(t)=\int_0^1 K(t,s)\,g(s)\,u(s)\,ds$$

$\mathcal{F}:\mathbb{R}\times E\to E$ by

$$(\mathcal{F}(u))(t)=\int_0^1 K(t,s)\,g(s)\,\tau(u(s))\,ds.$$

and $\mathcal{G}:\mathbb{R}\times E\to E$ by

$$(\mathcal{G}(u))(t)=\int_0^1 K(t,s)\,g(s)\,\varepsilon(u(s))\,ds.$$

It is easily seen that the operator $\mathcal{L}$ is compact in E and the operators $\mathcal{F}:\mathbb{R}\times E\to E$ and $\mathcal{G}:\mathbb{R}\times E\to E$ are completely continuous. Thus problems (11) (or (9)) and (12) (or (10)) can be rewritten in the following equivalent forms

$$u=\lambda f_0\mathcal{L}u+\lambda\mathcal{F}(u) \tag{13}$$

and

$$u = \lambda f_\infty \mathcal{L}u + \lambda \mathcal{G}(u). \tag{14}$$

By (3) we have

$$\mathcal{F}(u) = o(||u||_3) \text{ as } ||u||_3 \to 0, \tag{15}$$

and

$$\mathcal{G}(u) = o(||u||_3) \text{ as } ||u||_3 \to +\infty\,. \tag{16}$$

By virtue of (15) and (16) the linearization of (13) at $u = 0$ and of (14) at $u = \infty$ are spectral problems

$$u = \lambda f_0 \mathcal{L}u \tag{17}$$

and

$$u = \lambda f_\infty \mathcal{L}u, \tag{18}$$

respectively. Obviously, the problem (17) and (18) are equivalent to the spectral problems

$$\begin{aligned} &\ell u(t) = \lambda f_0 g(t)u(t),\ t \in (0,1),\\ &u \in B.C. \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\ell u(t) = \lambda f_\infty g(t)u(t),\ t \in (0,1),\\ &u \in B.C.\,, \end{aligned} \tag{20}$$

respectively.

The principal eigenvalues $\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}$, $k \in \{-1,\ 1\}$, of problem (19) are the characteristic values of problem (17) and are simple. Hence all the conditions of Theorem 1.3 from [26] are satisfied and there exists a continua $\mathfrak{L}_{\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}} \equiv \mathfrak{L}_k$, $k \in \{-1,\ 1\}$, of the set of solutions of problem (13), as in Theorem 1.3 in [26]. By virtue of [3, Theorem 1.1] (see also [12, Theorem 2]) continua $\mathfrak{L}_k$, $k \in \{-1,\ 1\}$, decomposes into two subcontinua $\mathfrak{L}_k^-$ and $\mathfrak{L}_k^+$ with meets $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)$, are contained in $(\mathbb{R}\times S_1^-) \cup \left\{\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)\right\}$ and $(\mathbb{R}\times S_1^+) \cup \left\{\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)\right\}$, respectively, and both are unbounded in $\mathbb{R}^{\mathrm{sgn}k} \times E$.

On the other hand, since the principal eigenvalues $\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}$, $k \in \{-1,\ 1\}$, of problem (20) are the characteristic values of problem (18) and are simple, by the discussion above and [25; Theorem 2.4] (see also [27, 28]) for each $k \in \{-1, 1\}$ there exists an unbounded component $\mathcal{D}_{\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}} \equiv \mathcal{D}_k \subset \mathbb{R}^{\mathrm{sgn}k} \times E$ of $\mathfrak{L}$ which contains $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right)$. In addition, if $\Lambda \subset \mathbb{R}^{\mathrm{sgn}k}$ is an interval such that $\Lambda \cap \sigma(L,g) = \frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}$ ($\sigma(L,g)$ is a set of eigenvalues of problem (4)) and $\mathcal{M}$ is a neighborhood of $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{rf_\infty}, \infty\right)$ whose projection on $\mathbb{R}^{\mathrm{sgn}k}$ lies in Λ and whose projection on E is bounded away from 0, then either

(i) $\mathcal{D}_k\backslash\mathcal{M}$ is bounded in $\mathbb{R}^{\mathrm{sgn}k} \times E$, in which case $\mathcal{D}_k\backslash\mathcal{M}$ meets $\mathbb{R}^{\mathrm{sgn}k} \times \{0\}$, or

(ii) $\mathcal{D}_k \backslash \mathcal{M}$ is unbounded; if additionally $\mathcal{D}_k \backslash \mathcal{M}$ has a bounded projection on $\mathbb{R}^{\mathrm{sgn}k}$, then $\mathcal{D}_k \backslash \mathcal{M}$ contains $\left(\frac{\lambda_m^{\mathrm{sgn}k}}{f_\infty}, \infty\right)$, where $m \in \mathbb{N}$ and $m > 1$.

Moreover, $\mathcal{D}_k$, $k \in \{-,+\}$, can be decomposed into two subcontinua $\mathcal{D}_k^-$, $\mathcal{D}_k^+$ and there exists a neighborhood $Q \subset \mathcal{M}$ of $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right)$ such that $(\lambda, u) \in \mathcal{D}_k^-(\mathcal{D}_k^+) \cap Q$ and $(\lambda, u) \neq \left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right)$ implies

$$(\lambda, u) = (\lambda, s u_{1,+}^{\mathrm{sgn}k} + w),$$

where

$$s < 0 \ (s > 0) \text{ and } |\lambda - \lambda_1^k| = o(1), \ w = o(|s|) \text{ at } |s| = \infty.$$

Consequently,

$$\text{if } (\lambda, u) \in \mathcal{D}_k^\nu \backslash Q, \text{ then } (\lambda, u) \in \mathbb{R}^{\mathrm{sgn}k} \times S_1^\nu. \tag{21}$$

Let

$$(\lambda_n, u_n) \in \mathfrak{L}_k^\nu \text{ and } |\lambda_n| + ||u_n||_3 \to \infty \text{ as } n \to \infty.$$

We note that $\lambda_n \mathrm{sgn}k > 0$ for all $n \in \mathbb{N}$, since $\mathfrak{L} \cap (\{0\} \times E \backslash \{0\}) = \emptyset$. As in the proof of Theorem 1.1 from [23] we can prove that there exists a positive constant M such that

$$|\lambda_n| \leq M, \ n \in \mathbb{N},$$

which implies

$$||u_n||_3 \to \infty \text{ as } n \to \infty.$$

It is obvious that

$$u_n = \lambda_n f_\infty \mathcal{L} u_n + \lambda_n \mathcal{G}(u_n). \tag{22}$$

Let $v_n = \frac{u_n}{||u_n||_3}$. Then by (22) v_n satisfies the relations

$$v_n = \lambda_n f_\infty \mathcal{L} v_n + \lambda_n \frac{\mathcal{G}(u_n)}{||u_n||_3} \tag{23}$$

By virtue of completely continuity of operators $\mathcal{L}$ and $\mathcal{G}$, and the boundedness of $\{\lambda_n\}_{n=1}^\infty$ it follows from (23) that there exists a subsequence of the sequence $\{(\lambda_n, v_n)\}_{n=1}^\infty$ (which we will relabel as $\{(\lambda_n, v_n)\}_{n=1}^\infty$) which is convergent to $(\tilde{\lambda}, v)$ in $\mathbb{R}^{\mathrm{sgn}k} \times E$, with $||v||_3 = 1$, $v \in S_1^\nu$ and

$$v = \tilde{\lambda} f_\infty \mathcal{L} v. \tag{24}$$

Then by Theorem 2.1 it follows from (24) that

$$\tilde{\lambda} = \frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}.$$

Hence

$$(\lambda_n, u_n) \to \left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right) \text{ as } n \to \infty,$$

which by (21) implies that

$$\mathcal{D}_k^\nu \backslash Q \subset \mathfrak{L}_k^\nu. \tag{25}$$

Moreover, it follows from the proof of [25; Corollary of Theorem 2.4] that $\mathcal{D}_k^\nu$ contains a subcontinuum $\mathfrak{D}_k^\nu$ lying in $\mathbb{R} \times S_1^\nu$ such that either $\mathfrak{D}_k^\nu \backslash Q$ is unbounded or intersects the line $\mathcal{R} = \{(\lambda, 0) \in \mathbb{R} \times E\}$ of trivial solutions at $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)$. Consequently, by (25) we have $\mathfrak{L}_k^\nu = \mathfrak{D}_k^\nu$. The proof of this theorem is complete.

Corollary 1. *Let r be a real constant such that*

$$r \in \left(\frac{\lambda_1^{\mathrm{sgn}k}\mathrm{sgn}k}{f_\infty}, \frac{\lambda_1^{\mathrm{sgn}k}\mathrm{sgn}k}{f_0}\right)$$

or

$$r \in \left(\frac{\lambda_1^{\mathrm{sgn}k}\mathrm{sgn}k}{f_0}, \frac{\lambda_1^{\mathrm{sgn}k}\mathrm{sgn}k}{f_\infty}\right), \ k = -1 \text{ or } k = 1.$$

where $f_0 \neq f_\infty$. Then the problem

$$\begin{aligned} &(\ell u)(t) = rg(t)f(u(t)), \ t \in (0,1), \\ &u \in B.C. \end{aligned}$$

has at least one negative and one positive solutions.

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On Wiman-Valiron Type Estimations for Evolution Equations

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Abstract. In the paper we establish Wiman-Valiron-type estimates for evolution equations in Hilbert spaces containing a pseudo-differential operator of the Hormander class $L^m_{p,\delta}$. Using asymptotic formulas for the function $N(\lambda)$ for the given operator in the equation, we prove Wiman-Valiron-type theorems characterizing the behavior of the solution depending on the properties of Fourier coefficients of the solution.
Key Words and Phrases: evolution equation, discrete spectrum, pseudo-differential operator, Wiman-Valiron type estimates, distribution function, Hilbert space, parabolic equation, Fourier series.
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1. Introduction

Let $f(z) = \sum\limits_0^\infty a_n z^n$ be an integer

$$M(r) = \max_{|z|=r} |f(z)|, \quad \mu(r) = \max_n |a_n| r^u, \ \mu(r) \to \infty, \ M(r) \to \infty, \ r \to \infty.$$

The estimation $\mu(r) \leq M(r)$ is always true. But it is very important to get the estimation $M(r)$ from above by $\mu(r)$. In the papers of Wiman [1] and Valiron, the estimation of the following form

$$M(r) \leq \mu(r)(\log \mu(r))^{\frac{1}{2}+\varepsilon}$$

that is fulfilled out of some set $E \subset (0, \infty)$ of finite logarithmic measure, was established. In 1966, Rosenbloom [3] established more general result: for some class of functions $\varphi(y) > 0$, $y > 0$ the estimation of type

$$M(r) \leq \mu(r)\sqrt{\varphi(\log M(r))} \tag{1}$$

is fulfilled out of some set of weighted measure. In 1966, Kovari [4] established similar results for power series with finite radius of convergence. In the author's (see [5]) theory of Wiman-Valiron-Rosenbloom type estimations was constructed for evolution equations in Hilbert space. In the present paper we establish estimations of type (1) for evolution (parabolic) equations containing pseudo-differential operator of the Hormander class.

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2. Problem statement

Let us consider the equation

$$u'(t) + A(t)\,u(t), \tag{2}$$

where $A(t) \in L^m_{p,\delta}$ is a positive self-adjoint pseudo-differential (2) operator with a discrete spectrum. Let on $D(t)$ the strong derivative $A'(t)$ be determined and for $U \in D(A)$ the condition of the form

$$\left(A'(t)\,u, u\right) \le k(t)\,(A(t)\,u, u), \quad 0 < k(t) \in L_1(0, \infty) \tag{$*$}$$

be fulfilled.

Denote by $N(\lambda)$ the amount of all eigen values $\lambda_k(t)$ of the operator $A(t)$ not exceeding λ (with regard to multiplicity). The following lemma was proved in the paper (see [5], p. 84) of the author.

Lemma 1. *The following differential inequality*

$$e^{2g(t)} \le \mu(t)\,P\left(g', g''\right), \tag{3}$$

where $0 < t < T$, $g(t) = \frac{1}{2}\log(u(t), u(t))$ *the* $u(t)$ *is solution of equation (2),*

$$\mu(t) = \max_k |(u(t), \varphi_k(t))|,$$

$$P(a; b) = N\left(a + C\sqrt{b + k(t)\,a}\right) - N\left(a - C\sqrt{b + k(t)\,a}\right). \tag{4}$$

(Here and in the sequel denotes C absolute an constant, but not always identical). We briefly note the basic idea (fragments) of the method of proof based on probability. This method was constructed by us and is a very significant and strong modification of Rosembloom's problem constructed by him only for entire functions.

Associate to the function $u(t)$ some random variable ξ whose range of values is the set of eigen-values $\lambda_k(t)$ of the operator $A(t)$, and distribution of probabilities (dependent on parameter t) we define by the

$$P_k = P(\xi = \lambda_k(t)) = C_k(t)^2\,\|u(t)\|^2,$$

where $C_k(t) \equiv (u(t), \varphi_k(t))$ are the Fourier coefficients of the function $u(t)$ with respect to orthonormed system $\{\varphi_k(t)\}$ of eigen functions of the operator $A(t)$.

Having calculated the mathematical expectation $M\xi$, and variance $D\xi$, we find:

$$M\xi = -g'(t), \quad D\xi \le g''(t) \quad\quad k(t)\,g'(t).$$

Applying the Chebyshev known inequality from probability theory

$$P(|\xi - M\xi| > \varepsilon) \le D\xi|\,\varepsilon^2,$$

we get (take $\varepsilon = C\sqrt{D\xi}$)

$$\left|-P\left(\left|\xi+g'\right| \leq C\sqrt{D\xi}\right) \leq C\sqrt{D\xi} \leq 1\right| C^2.$$

Hence we have

$$1-\frac{1}{C^2} \leq P\left(\left|\xi+g'\right| \leq \varepsilon\right) = \sum_{k\in I} p_k = \frac{T}{\|u\|^2}\sum_{k\in I} C_k^2,$$

where $I = \{k : |\lambda_k + g'| \leq \varepsilon\}$. Consequently, we have:

$$\|u(t)\|^2 \leq C\mu(t)^2 \sum_{k\in I} 1 \tag{5}$$

It is clear that by (4),

$$\sum_{k\in I} 1 = N\left(\left|g'\right| + C\sqrt{g''+k(t)\,g'}\right) - N\left(\left|g'\right| - C\sqrt{g''+k(t)\,g'}\right) \equiv$$

$$\equiv P\left(\left|g'\right|, g''\right).$$

Then, (5) yields the estimation of the form

$$\|u(t)\|^2 \leq C\mu(t)^2 P\left(\left|g'\right|, g''\right). \tag{6}$$

If we find a function $\Psi(y)$, $y > 0$ such that in some sense the inequality of following form

$$P\left(\left|g'\right|, g''(t)\right) \leq \Psi(g(t)), \tag{7}$$

is valid, then, from (6) we get that it holds the estimation of type

$$\|u(t)\| \leq C\mu(t)\sqrt{\psi(\log\|u(t)\|)},$$

that is Riman-Valiron type estimation for evolution equation (2). The conditions under which inequalities of type (8) are fulfilled, were studied in the papers (see [5]).

In the following theorem there is an assumption on asymptotic behavior of the function $N(\lambda)$, and Wiman-Valiron-Rosenbloom type estimations for solving equation (2) are established on its bases.

Theorem 1. *Let the function $N(\lambda)$ for the operator $A \in L^m_{p,\delta}(\Omega)$ of the Hormander class satisfy the following conditions:*

$$N(\lambda) \leq C\lambda^{s+1}\ln\lambda, \quad s+1>0, \quad C>0 \tag{8}$$

and for $\lambda > \delta > 0$, $\lambda \to \infty$ the inequality of type $(0 \leq \nu \leq 1)$:

$$\Delta N(\lambda,\delta) \equiv N(\lambda+\delta) - N(\lambda-\delta) \leq C\delta\lambda^s\left(1+\lambda^\nu\right)(1+lg\lambda). \tag{9}$$

Let the function $\varphi(y) > 0$, $y > 0$ do not decrease and be such that for some $\alpha > 0$, following the condition be fulfilled

$$\int^{\infty}\left(\int^{y}\varphi(t)\,dt\right)^{-\alpha}dy < +\infty. \tag{10}$$

Then, out of possibly some certain set $EC(0,\infty)$ of finite measure, the following Wiman-Valirob type estimation is valid:

$$\|u(t)\| \le C\mu(t)\sqrt[4]{\varphi(\log\|u(t)\|)}. \tag{11}$$

Proof. We immediately note that conditions (9) and (10) satisfy the function, for example, of type:

$$N(\lambda) = \lambda^{p}\ln\lambda, \quad N(\lambda) = \lambda^{\frac{m}{n}} + O\left(\lambda^{\frac{n-1}{m}}\right), \quad N(\lambda) = \lambda^{p} l(\lambda),$$

where $l(\lambda)$ is a slowly growing function, i.e. $\lim\limits_{\lambda\to\infty}\lambda\frac{l'(\lambda)}{l(\lambda)} = 0$. For example, the functions $l(\lambda) = \ln\lambda$, $l(y) = \ln\ln\lambda$, $l(\lambda) = (\ln\lambda)^{2}$, $\alpha > 0$ and others are this type functions.

Introduce a change of variables:

$$\xi(t) = \int_{0}^{t}\Phi(p)\,d\rho, \tag{12}$$

where

$$\Phi(\rho) = \int_{\rho}^{t}k(\tau)\,d\tau.$$

For any function $h(t)$ denote $\widetilde{h}(\xi) = h(t(\xi))$, where $t(\xi)$ is determined from the relation (13). We get:

$$\sqrt{g''(t) + k(t)\,g'(t)} = \sqrt{\widetilde{g}''(t)}\Phi(t), \quad \widetilde{g}'' > 0.$$

From condition (10) we get an inequality of the form $\left(\lambda = \widetilde{g}', \quad \delta = \sqrt{\widetilde{g}''}\right)$:

$$\Delta N(\lambda,\delta) \le C\sqrt{\widetilde{g}''}\widetilde{g}'^{s}\left(1 + \widetilde{g}'^{\nu}\right). \tag{13}$$

Thus, the problem is reduced to the fact that it is necessary to find such a function $\varphi(y)$ that the inequality (in the sequel, instead of $\widetilde{g}$ we simply write g):

$$\sqrt{g''}g'\left(1 + g'\nu\right) \le \sqrt{\varphi(g)} \tag{14}$$

is fulfilled.

Thus, we get the system of differential inequalities

$$\left\{ \begin{array}{c} \sqrt{g''}g'^{s+\nu} \leq \alpha\sqrt{\varphi(g)} \\ \sqrt{g''}g'^{s} \leq \beta\sqrt{\varphi(g)} \end{array} \right. \qquad (\alpha+\beta\leq 1). \tag{15}$$

Let $E=\left\{\sqrt{g''}g'^{s+\nu} > \alpha\sqrt{\varphi(g)}\right\}$. Consider the first inequality of the system:

$$g''g'^{2(s+\nu)+1} \leq \alpha^2\varphi(g)\,g'$$

$$\left(g'^p\right)' \leq C\varphi(g)\,g', \qquad p=2(s+\nu)+2,$$

$$g' \leq \left(C\int^{g}\varphi(t)\,dt\right)^{1/p} \equiv \psi_1(g).$$

Then

$$E=\left\{g'(t) > \varphi_1(g)\right\}.$$

From condition (11) we have

$$mesE=\int_E dt < \int_E \frac{g'(t)\,dt}{\varphi_1(g)} \leq \int_{g(E)} \frac{dg}{\varphi_1(g)} \leq \int_0^\infty \frac{dg}{\varphi_1(g)} = \int^\infty \left(\int^\infty \varphi(t)\,dt\right)^{-\frac{1}{p}} dg < \infty.$$

Thus, subject to the condition (11), out of the set E, $mesE<\infty$ the first inequality of the system (16) is fulfilled. In the similar way, we obtain that the second inequality of this system is also fulfilled out of some set of finite measure. Consequently, the system (1) is true out of E, $mesE<\infty$. Then the inequality (15) is fulfilled out of E. Consequently, $\Delta N(g',g'') \leq \varphi(g)$.

Then, by Lemma 1, estimation (12) is valid.

Theorem 2. *Let for $\lambda>\delta>0$, $\lambda\to\infty$ the condition of the form*

$$\Delta N(\lambda,\delta) \leq C\lambda^{n/m}\left(\delta+\lambda^{-\frac{1}{m}}\right)(1+\ln\lambda) \tag{16}$$

be fulfilled. Then estimation of type (12) is valid. The proof is similar.

Remark 1. *The condition of type (12) appears when for differential operator $A\in L^m_{p,\delta}(R^n)$ of order m in R^n, the function $N(\lambda)$ grows as $\lambda\to\infty$ faster than power λ, for example as $\lambda^p\ln\lambda$.*

In Shubin's monograph [p. 130], there is an example of the operator for which the function $N(\lambda)$ grows faster than power λ

$$N(\lambda)=C\lambda^{k_0}(\ln\lambda)^l, \quad \lambda\to\infty,$$

where $k_0>0$ while l is a natural number that is equal to the order of the pole at the point $z=-k_0$ of zeta function

$$\zeta(z)=\int_0^\infty t^z dN(t).$$

For a self-adjoint positive elliptic operator of order m, Hormander [see [7], p. 134] obtained for $N(\lambda)$ [see also [8]-[10]] the exact formula of the form

$$N(\lambda) = C\lambda^{n/m} + O\left(\lambda^{\frac{n-1}{m}}\right).$$

In Shubin's papers this result was proved by the original method owing to which this formula was developed.

In the book [[7], p. 134] for a self-adjoint elliptic operator $A \in L^m_{p,\delta}(\Omega)$ on n-dimensional closed manifold $\Omega \subset R^n$ such that its main symbol $a_m(x,\xi)$ is positive, for the function $N(\lambda)$ it was established the following formula with the unimproved residue

$$N(\lambda) = V(\lambda)\left(1 + O\left(\lambda^{-\frac{1}{m}}\right)\right), \lambda \to \infty \tag{17}$$

where the function $V(\lambda)$ is determined by the main symbol $a_m(x,\xi)$ of the equality

$$V(\lambda) = \frac{1}{(2\pi)^n} \int\limits_{a_m(x,\xi)<\lambda} dx d\xi, \quad \lambda \to \infty.$$

In this case, as $\lambda \to \infty$ the asymptotics

$$N(\lambda) = V(\lambda) = C\lambda^{n/m},$$

where

$$C = \frac{1}{(2\pi)^n} \int\limits_{a_m(x,\xi)<\lambda} dx d\xi.$$

But if the operator $A \in L^m_{p,\delta}$ is a general pseudo-differential operator of order m with main symbol $a_m(x,\xi) > 0$, then as was shown in [7], the asymptotics of the function $N(\lambda)$, determined by formula (18), may also have a not power growth, for example as $\lambda^{n/m}\ln\lambda$. Just in such cases a condition of type (17) appears on $N(\lambda)$.

In formula (18) assume $V(\lambda) = \lambda^p \ln\lambda$, $\nu = -1/m$ and consider the difference

$$\Delta N(\lambda,\delta) = (\lambda+\delta)^p \ln(\lambda+\delta) - (\lambda-\delta)^p \ln(\lambda-\delta) +$$

$$+C(\lambda+\delta)^{p-\nu}\ln(\lambda+\delta) - (\lambda-\delta)^{p-\nu}\ln(\lambda-\delta) = A + B;$$

$$A = \lambda^p\left\{\left(1+\frac{\delta}{\lambda}\right)^p\left[\ln\lambda + \ln\left(1+\frac{\delta}{\lambda}\right)\right] - \left(1-\frac{\delta}{\lambda}\right)^p\left[\ln\lambda + \ln\left(1-\frac{\delta}{\lambda}\right)\right]\right\} =$$

$$= \lambda^p\left\{\left(1+p\frac{\delta}{\lambda}\right)\left(\ln\lambda + \frac{\delta}{\lambda}\right) - \left(1-p\frac{\delta}{\lambda}\right)\left(\ln\lambda - \frac{\delta}{\lambda}\right)\right\} =$$

$$2\delta\lambda^{p-1}(1+\ln\lambda).$$

Similarly, $B = 2\delta\lambda^{p-1-\nu}(1+\ln\lambda)$.

Consequently: for $\Delta N(\lambda,\delta)$ we get an inequality of type

$$\Delta N(\lambda,\delta)\leq\varepsilon\delta\lambda^{\frac{n}{m}-\nu-1}(1+\lambda^{\nu})(1+\ln\lambda). \tag{18}$$

If we assume $\lambda=g'$, $\delta=\sqrt{g''}$, then obtain $g''\leq Cg'^2$ (out of the set of finite measure). Thus, for the function ΔN we get an inequality of the form

$$\Delta N\left(g',g''\right)\leq C\sqrt{g''}g'^s\left(1+g'^{\nu}\right)\left(1+\ln g'\right),\quad 0<\nu<1,\quad s=\frac{n}{m}-1-\nu.$$

Let $A\in L^m_{p,\delta}$ be an elliptic operator with the main symbol $a_m(x,\xi)>0$. Consider the function

$$V(t)=\frac{1}{(2\pi)^n}\int\limits_{a_m(x,\xi)<\lambda}dxd\xi. \tag{19}$$

The following statement was established in the paper [7, p. 206].

Proposition 1. *Let at some $\varepsilon>0$, $\delta>0$, $c>0$ for $V(t)$ the condition of type*

$$\frac{V\left(t+Ct^{1-\varepsilon}\right)-V(t)}{V(t)}=O\left(t^{-\delta}\right),\quad t\to+\infty$$

be fulfilled. Then for the function $N(\lambda)$ the asymptotic formula

$$N(\lambda)=V(\lambda)\left(t+O\left(\lambda^{-\delta}\right)\right),\quad t\to+\infty \tag{20}$$

is valid.

Using the method of the paper [7, p. 206] we can formulate a proposition more convenient for application.

Proposition 2. *Let $V(t)>0$ grow for $t>t_0$ and for some $0\leq\alpha,\ \nu\leq1$ the condition of type*

$$V'(t)\,|V(t)=O\left(Et^{\alpha+\nu}\right),\quad t\to+\infty \tag{21}$$

be fulfilled.

Then for the function $N(\lambda)$ the asymptotic formula

$$N(\lambda)=V(\lambda)\left(1+O\left(\lambda^{-\nu}\right)\right) \tag{22}$$

is valid.

Indeed, we denote $\varphi(t)=V'(t)|\,V(t)$. Integrating, we get

$$\frac{V(t+at^{\alpha})-V(t)}{V(t)}=\exp\int\limits_t^{t+Ct^2}\varphi(\tau)\,d\tau-1. \tag{23}$$

As $|\varphi(t)| \leq C_1 t^{-(\alpha+\nu)}$, then for $a+\nu \neq 1$ we have $(\gamma = 1-(a+\nu))$

$$\int\limits_t^{t+Ct^2} \varphi(\tau)\,d\tau \leq C_1 t^{\gamma}\left[\left(1+Ct^{\alpha-1}\right)^{\gamma}-1\right] = C_1 t^{\gamma}\left[Ct^{\alpha-1}\right] = C_2 t^{-\nu}.$$

The same estimation is obtained for $\lambda+\nu=1$ as well.

Since $e^{\lambda}-1\tilde{\ }X$ as $X\to 0$ $nC_2t^{-\nu}\to 0$ then from (24) we get

$$\frac{V(t+Ct^{\alpha})-V(t)}{V(t)} = O\left(t^{-\nu}\right), \quad t\to+\infty.$$

Hence, formula (23) follows from the above mentioned result of the paper [7].

Note that the asymptotic function $N(\lambda)$ determined from formula (23) may have also a not power series. For example, for the function $V(t)=t^p\ln t$ we have

$$\frac{V'(t)}{V(t)} = \frac{p}{t}+\frac{1}{t\ln t} = O\left(t^{-1}\right), \quad t\to+\infty.$$

Consequently, in proposition 2, $a+\nu=1$. Then by virture of this proposition, for $N(\lambda)$ we get a formula of the form

$$N(\lambda) = \lambda^p \ln\lambda\left(1+O\left(\lambda^{-\nu}\right)\right). \tag{24}$$

Let us consider a simpler example. Then $V(t)=t^p l(t)$, where $l(t)>0$ is a slowly growing function, i.e.

$$\lim_{t\to\infty} t\frac{l'(t)}{l(t)} = 0. \tag{25}$$

For this function we get

$$\frac{V'(t)}{V(t)} = \frac{p}{t}+\frac{l'(t)}{l(t)}.$$

Taking (26) into account, hence we find $\frac{V'(t)}{V(t)} = O\left(t^{-1}\right)$, i.e. in proposition 2 we have $\alpha+\nu=1$. Consequently, for $N(\lambda)$ the following formula is valid

$$N(\lambda) = \lambda^p l(\lambda)\left(1+O\left(\lambda^{-\nu}\right)\right).$$

Remark 2. *Let the symbol $a(x,\xi)$ of the operator $A\in L^m_{p,\delta}$ satisfy the conditions of the form*

1) $a(y)\to+\infty$ as $|y|\to+\infty$, where $y=(x,\xi)$, $x,\xi\in R^n$

2) $a(y)^{1-\alpha}\leq C\,|(y,\nabla a(y))|$as $|y|\geq N$, $C>0$, $0\leq\alpha\leq 1$, where ∇ is the gradient of the function $a(y)$.

Then from the results of the paper [7] (theorem 28.3) we have the estimation of the form

$$\frac{V'(t)}{V(t)} = O\left(t^{\alpha-1}\right), \quad t+\infty.$$

Herewith, if $a(y)$ is an elliptic polynomial with respect to y and of power m, then we can take $\alpha = 0$.

From [7, p. 206] for $\alpha < \nu < 1$ we have:

$$\frac{V\left(t + ct^{1-\nu}\right) - V(t)}{V(t)} = O\left(t^{\alpha-\nu}\right), \quad \to +\infty.$$

Then the estimations of the form

$$N(\lambda) = V(\lambda)\left(1 + O\left(\lambda^{\alpha-1}\right)\right), \quad t \to \infty \tag{26}$$

hold.

In particular, for $\alpha = 0$ we get

$$V'(t)\big| V(t) = O\left(\frac{1}{t}\right), \quad N(\lambda) = V(\lambda)\left(1 + O\left(\lambda^{-\nu}\right)\right), \quad 0 < \nu < 1.$$

Thus, for pseudo-differential operator $A \in L^m_{p,\delta}(R^n)$ with properties 1 and 2, the estimations (21) and (22) are valid, where the function $N(\lambda)$ possibly grows in not power way and theorems 1,2 are applicable in such situations.

In conclusion, let us consider as an application the results obtained in the paper, for example, the solutions of head conductivity equation in the domain $(0,T) \times \Omega$, $\Omega \subset R^n$ is a bounded domain with smooth boundary with homogeneous Dirichlet boundary condition on the plane $(0,T) \times \partial\Omega$ in the space $L_2(\Omega)$:

$$\frac{\partial u}{\partial t} = \Delta_x u, \quad u|_{(0,T)\times\partial\Omega} = 0, \quad u|_{t=0} = u_0(x). \tag{27}$$

Let $(u_0, \varphi_n) = \sqrt{n}$, $\lambda_n = \frac{n}{2}$. Then we get

$$\|u(t,\cdot)\|^2 = \sum (u_0, \varphi_n)^2 e^{-2t\lambda_k} = \sum n e^{-nt} = -\frac{d}{dt}\sum e^{nt} = -\frac{d}{dt}\frac{1}{1-e^{-t}} = \frac{e^{-t}}{(1-e^{-t})^2}$$

It is easy to see that $\frac{e^{-t}}{(1-e^{-t})^2} \sim \frac{1}{t^2,}$ $t \to 0$, we have $\|u(t)\|^2 \sim \frac{1}{t^2}$, $t \to 0$, indeed,

$$\lim_{t\to 0}\frac{e^{-t}}{(l-e^{-t})^2} = \lim_{t\to 0}\frac{t^2 e^{-t}}{1 - 2e^{-t} + e^{-2t}} = \lim_{t\to 0}\frac{t^2}{e^t - 2 + e^{-t}} = \lim_{t\to 0}\frac{2t}{e^t - e^{-t}} =$$

$$= \lim_{t\to 0}\frac{2}{e^t + e^{-t}} = 1.$$

Consequently, $\frac{e^{-t}}{(1-e^{-t})^2} \sim \frac{1}{t^2}$.

Calculate $\mu(t)$. Since

$$\mu^2(x) = \max_n n e^{-nt} = \max_n \psi(x),$$

where $\psi(x) = xe^{-xt}$. Then $\psi^l = e^{-xt} - t \times e^{-xt} = 0$, $1 - xt = 0$, $x = \frac{1}{t}$, $\psi\left(\frac{1}{t}\right) = \frac{1}{t}e^{-1}$. Consequently $\mu(t)^2 = \frac{1}{et}$, $\mu(t) = \frac{1}{\sqrt{et}}$. Then we get

$$\|u(t,\cdot)\|^2 = \frac{1}{t} = \frac{\sqrt{e}}{\sqrt{t}}\mu(t) = \mu(t)^2$$

i.e.

$$\|u(t,\cdot)\| = \mu(t) = \frac{1}{\sqrt{t}}, \quad t \to 0.$$

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