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Interpolation Theorems on the Nikolskii-Morrey type Spaces

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Abstract. In the paper was studied a differential and differential-difference properties of functions from intersection of Nikolski-Morrey type spaces $H^{l^{\mu}}_{p_{\mu},\varphi,\beta}(G_{\varphi}), (\mu = 1, 2, \ldots, N)$.

Key Words and Phrases: Nikolskii-Morrey type spaces, integral representation, generalized Hölder condition.

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1. Introduction

In the paper, we study some differential properties functions from spaces type $\bigcup\limits_{\mu=1}^{N} H^{l^{\mu}}_{p_{\mu},\varphi,\beta}(G_{\varphi})$, more precisely we prove inequality type Riesz-Torin for functions from spaces type $H^{l^{\mu}}_{p_{\mu},\varphi,\beta}(G_{\varphi}), (\mu = 1, 2, \ldots, N)$, and also we prove that for the functions from intersection this spaces, the generalized mixed derivatives $D^{\nu} f$ satisfy the Holder condition in the metric $L_q(G)$ and $C(G)$. The space $H^{l}_{p,\varphi,\beta}(G)$is defined in [1] as a linear normed space of functions f, on G with the finite norm $(m_i > l_i - k_i > 0, i = 1, 2, ..., n)$

$$\|f\|_{H^{l}_{p,\varphi,\beta}(G)} = \|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^{n} \sup_{0<h<h_0} \frac{\left\| \Delta_i^{m_i}\left(\varphi_i\left(h\right), G_{\varphi(h)}\right) D_i^{k_i} f \right\|_{p,\varphi,\beta}}{\varphi_i\left(h\right)^{l_i - k_i}}, \tag{1}$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{x\in G, t>0} \left(\left|\varphi\left([t]_1\right)\right|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right), \tag{2}$$

$\left|\varphi\left([t]_1\right)\right|^{-\beta} = \prod\limits_{j=1}^{n} \left(\varphi_j\left([t]_1\right)\right)^{-\beta_j}$, $\beta_j \in [0, 1]$, $j = 1, 2, ..., n$; $l \in (0, \infty)^n$, $m_i \in \mathbb{N}$, $k_i \in \mathbb{N}_0$, $p \in [1, \infty)$, $[t]_1 = \min\{1, t\}$, and vector-functions $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))$ with Lebesgue measurable functions $\varphi_j(t) > 0$, $t > 0$, $\lim\limits_{t\to+0} \varphi_j(t) = 0$, $\lim\limits_{t\to+\infty} \varphi_j(t) = L \leq \infty$, $j = 1, 2, ..., n$. Denote by $\mathbb{A}$ the set of vector functions φ.

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For any $x \in R^n$,

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), \quad j = 1, 2, ..., n \right\},$$

Let for any $t > 0$, $|\varphi([t]_1)| \leq C$, where C is some positive constant. Then the embeddings $L_{p,\varphi,\beta}(G) \to L_p(G)$ and $H^l_{p,\varphi,\beta}(G) \to H^l_p(G)$ hold, i.e.

$$\|f\|_{p,G} \leq c\|f\|_{p,\varphi,\beta;G}, \text{ and } \|f\|_{H^l_p(G)} \leq c\,\|f\|_{H^l_{p,\varphi,\beta}(G)}. \tag{3}$$

Note that the spaces $L_{p,\varphi,\beta}(G)$ and $H^l_{p,\varphi,\beta}(G)$ are Banach spaces. The space $H^l_{p,\varphi,\beta}(G)$, in the case $\beta_j = 0\,(j = 1, ..., n)$ it coincides with the Nikolski space $H^l_p(G)$. The spaces of such type with different norms were introduced and studied in [3]-[8].

2. Preliminaries

Assuming that $\varphi_j(t)$ $(j = 1, 2, ..., n)$ are also differentiable on $[0, T]$.

Let $\lambda_\mu \geq 0$ $(\mu = 1, 2, ..., N)$ and $\sum\limits_{\mu=1}^{N} \lambda_\mu = 1$, $\frac{1}{p} = \sum\limits_{\mu=1}^{N} \frac{\lambda_\mu}{p_\mu}$, $\frac{1}{q} = \sum\limits_{\mu=1}^{N} \frac{\lambda_\mu}{q_\mu}$, $l = \sum\limits_{\mu=1}^{N} l^\mu \lambda_\mu$, and $\Omega(\cdot, y)$, $M_i(\cdot, y) \in C_0^\infty(R^n)$ be such that

$$S(M_i) = \operatorname{supp} M_i \subset I_{\varphi(t)} = \left\{ y : |y_j| < \frac{1}{2}, \quad j = 1, 2, ..., n \right\}.$$

Assume that for any $0 < T \leq 1$ is a fixed number:

$$V = \bigcup_{0 < t \leq T} \left\{ y : \frac{y}{\varphi(t)} \in S(M_i) \right\}.$$

It is clear that $V \subset I_{\varphi(t)}$ and suppose that $U + V \subset G$. Assume $\varphi(t)(j = 1, 2, \ldots, n)$ are also differentiable on $[0, T]$.

Lemma 1. *Let $1 \leq p_\mu \leq q_\mu \leq r_\mu \leq \infty$; $0 < \eta$, $t < T \leq 1$, $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $\nu_j \geq 0$ are integers, $j = 1, 2, ..., n$; $\Delta_i^{m_i}(\varphi_i(t)) \in L_{p,\varphi,\beta}(G)$ and let*

$$Q^i_T = \int\limits_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right)} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1 - \sum\limits_{\mu=1}^{N} l_i^\mu \lambda_\mu}} dt < \infty,$$

$$A(x) = \prod_{j=1}^{n} \int\limits_{R^n} \int\limits_{R^n} f(x + y + z)\Omega^\nu\left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{2\varphi(T)}\right)$$

$$\times \Omega\left(\frac{z}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{2\varphi(T)}\right) f(x + y + z)\, dydz. \tag{4}$$

$$A^i_\eta(x) = \int\limits_0^\eta L_i(x,t) \prod_{j=1}^n (\varphi_j(t))^{\nu_j - 2} \frac{\varphi'_i(t)}{\varphi_i(t)} dt \tag{5}$$

$$A^i_{\eta T}(x) = \int\limits_\eta^T L_i(x,t) \prod_{j=1}^n (\varphi_j(t))^{\nu_j - 2} \frac{\varphi'_i(t)}{\varphi_i(t)} dt \tag{6}$$

where

$$L_i(x,t) = \int\limits_{R^n} \int\limits_{-\infty}^{+\infty} M_i\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}\right)$$
$$\times \zeta_i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t),x)}{2\varphi_i(t)}, \frac{1}{2}\rho'_i(\varphi_i(t),x)\right) \Delta_i^{m_i}(\varphi_i(\delta)u) f(x+y+ue_i)\, dudy \tag{7}$$

Then for any $\overline{x} \in U$ *the following inequalities*

$$\sup_{\overline{x}\in U} \|A\|_{qU_{\psi(\xi)}(\overline{x})} \le C_1 \prod_{\mu=1}^N \{\|f\|_{p_\mu,\varphi,\beta;G}\}^{\lambda_\mu} \times$$
$$\times \prod_{j=1}^n (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j=1}^n (\psi_j[\xi]_1)^{\beta_j \frac{p}{q}}, \tag{8}$$

$$\sup_{\overline{x}\in U} \left\|A^i_\eta\right\|_{qU_{\psi(\xi)}(\overline{x})} \le C_2 \prod_{\mu=1}^N \left\{\left\|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p_\mu,\varphi,\beta;G}\right\}^{\lambda_\mu}$$
$$\times \left|Q^i_\eta\right| \prod_{j=1}^n (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}}, \tag{9}$$

$$\sup_{\overline{x}\in U} \left\|A^i_{\eta T}\right\|_{qU_{\psi(\xi)}(\overline{x})} \le C_2 \prod_{\mu=1}^N \left\{\left\|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G}\right\}^{\lambda_\mu}$$
$$\times \left|Q^i_{\eta T}\right| \prod_{j=1}^n (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}}, \tag{10}$$

where $U_{\psi(\xi)}(\overline{x}) = \left\{x : |x_j - \overline{x}_j| < \frac{1}{2}\psi_j(\xi), j = 1,2,...,n\right\}$ *and* $\psi \in A$, C_1, C_2 *are the constants independent of* φ, ξ, η *and* T.

Proof. Using the Minkowsky inequality for any $\overline{x} \in U$

$$\left\|A^i_\eta\right\|_{qU_{\psi(\xi)}(\overline{x})} \le \int\limits_0^\eta \|L_i(\cdot,t)\|_{qU_{\psi(\xi)}(\overline{x})} \prod_{j=1}^n (\varphi_j(t))^{\nu_j - 2} \frac{\varphi'_i(t)}{\varphi_i(t)} dt, \tag{11}$$

and we get

$$\|L_i(\cdot,t)\|_{qU_{\psi(\xi)}(\overline{x})} \le C_1 \left(\int\limits_{U_{\psi(\xi)}(\overline{x})} \prod_{\mu=1}^{N} \{|L_i(\cdot,t)|\}^{\lambda_\mu q} dx \right)^{\frac{1}{q}}.$$

Once again using the Hölder's inequality with indication $\alpha_\mu = \frac{q_\mu}{q\lambda_\mu}$, $\mu = 1,2,\ldots,N$ $(\sum\limits_{\mu=1}^{N} \frac{1}{\alpha_\mu} = q \sum \frac{\lambda_\mu}{q_\mu} = 1)$. Then we have

$$\|L_i(\cdot,t)\|_{qU_{\psi(\xi)}(\overline{x})} \le C_2 \prod_{\mu=1}^{N} \{\|L_i(\cdot,t)\|_{q_\mu U_{\psi(\xi)}(\overline{x})}\}^{\lambda_\mu}. \tag{12}$$

Taking Hölder inequality $(q_\mu \le r_\mu)$ we get

$$\|L_i(\cdot,t)\|_{qU_{\psi(\xi)}(\overline{x})} \le \|L_i(\cdot,t)\|_{rU_{\psi(\xi)}(\overline{x})} \prod_{j=1}^{n} (\psi_j(\xi))^{\frac{1}{q_\mu}-\frac{1}{r_\mu}}. \tag{13}$$

Let X be a characteristic function of the set $S(M_i) = supp\, M_i$. Noting that $1 \le p_\mu \le r_\mu \le \infty$, $s_\mu \le r_\mu$ $\left(\frac{1}{s_\mu} = 1 - \frac{1}{p_\mu} + \frac{1}{r_\mu}\right)$, and apply for $|L_i|$ the Hölder inequality $\left(\frac{1}{p_\mu} + \left(\frac{1}{p_\mu} - \frac{1}{r_\mu}\right) + \left(\frac{1}{s_\mu} - \frac{1}{r_\mu}\right) = 1\right)$, and we obtain

$$\|L_i(\cdot,t)\|_{r_\mu,U_{\psi(\xi)}(\overline{x})} \le$$

$$\le \sup_{x\in U_{\psi(\xi)}(\overline{x})} \left(\int\limits_{R^n} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t),x) \right) \right.\right.$$

$$\left.\left. \times \Delta_i^{m_i}(\varphi_i(t))\, f(x+y+ue_i)\, du\right|^{p_\mu} \chi\left(\frac{y}{\varphi(t)}\right) dy \right)^{\frac{1}{p_\mu}-\frac{1}{r_\mu}}$$

$$\times \sup_{y\in V} \left(\int\limits_{U_{\psi(\xi)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t),x) \right) \right.\right.$$

$$\left.\left. \times \Delta_i^{m_i}(\varphi_i(t)u)\, f(x+y+ue_i)\, du\right|^{p_\mu} dx \right)^{\frac{1}{p_\mu}}$$

$$\times \left(\int\limits_{R^n} \left| M_i\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x) \right) \right|^{s_\mu} dy \right)^{\frac{1}{s_\mu}} \tag{14}$$

(suppose that $|M_i(x,y,z)| \le C|\widetilde{M}_i(x)|$).

For any $x \in U$ we have

$$\int\limits_{R^n} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \right.$$

$$\times \Delta_i^{m_i}(\varphi_i(\delta) u) f(x + y + ue_i) du|^{p_\mu} \chi \left(\frac{y}{\varphi(t)} \right) dy$$

$$\leq \int\limits_{(U+V)_{\varphi(t)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \right.$$

$$\times \Delta_i^{m_i}(\varphi_i(\delta) u) f(y + ue_i) du|^{p_\mu} dy \leq$$

$$\leq \varphi_i(t)^{p_\mu + p_\mu l_i^\mu} \left\| \varphi_i(t)^{-l_i^\mu} \Delta_i^{m_i} \left(\varphi_i(\delta) u, G_{\varphi(t)} \right) \right\|_{p_\mu, \varphi, \beta}^{p_\mu} \prod_{j=1}^{n} (\varphi_j(t))^{\beta_j p_\mu} . \tag{15}$$

For $y \in V$

$$\int\limits_{U_{\psi(\xi)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \Delta_i^{m_i}(\varphi_i(\delta) u) f(x + y + ue_i) du \right|^{p_\mu} dx$$

$$\leq \int\limits_{G_{\varphi(\xi)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \Delta_i^{m_i}(\varphi_i(\delta) u) f(x + ue_i) du \right|^{p_\mu} dx$$

$$\leq (\varphi_i(t))^{p_\mu l_i^\mu} \left\| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \right.$$

$$\left. \times \varphi_i(t)^{-l_i^\mu} \Delta_i^{m_i} \left(\varphi_i(\delta) u, G_{\varphi(t)} \right) f du \right\|_{p_\mu, G_{\varphi(t)}(\overline{x})}^{p_\mu}$$

$$\leq \varphi_i(t)^{p + p l_i^\mu} \left\| \varphi_i(t)^{-l_i^\mu} \Delta_i^{m_i} \left(\varphi_i(\delta), G_{\varphi(t)} \right) \right\|_{p_\mu, \varphi, \beta}^{p_\mu} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j p_\mu} \tag{16}$$

$$\left(\int\limits_{R^n} \left| \widetilde{M}_i \left(\frac{y}{\varphi(t)} \right) \right|^{s_\mu} dy \right)^{\frac{1}{s_\mu}} = \left\| \widetilde{M}_i \right\|_{s_\mu}^{s_\mu} \cdot \prod_{j=1} \varphi_j(t). \tag{17}$$

From inequalities (11)-(17) for $(r_\mu = q_\mu)$ and for any $\overline{x} \in U$ reduce to the estimation

$$\left\| A_\eta^i \right\|_{q U_{\psi(\xi)}(\overline{x})} \leq C_1 \prod_{\mu=1}^{N} \left\{ \left\| (\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(\delta) u) f \right\|_{p_\mu, \varphi, \beta; G} \right\}^{\lambda_\mu}$$

$$\times \left|Q_{\eta}^{i}\right| \prod_{j=1}^{n} \left(\psi_j([\xi]_1)\right)^{\beta_j \frac{p}{q}} \quad \left(Q_{\eta}^{i} < \infty\right). \tag{18}$$

In the case $Q_{\eta,T}^{i} < \infty$ inequality (10) and (8) is proved in the same way.

From last inequalities it follows that

$$\left\|A_{\eta}^{i}\right\|_{q,\psi,\beta^1;U} \le C^1 \prod_{\mu=1}^{N} \left\{ \left\| (\varphi_i(t))^{-l_i^{\mu}} \, \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu}, \tag{19}$$

$$\left\|A_{\eta T}^{i}\right\|_{q,\psi,\beta^1;U} \le C^2 \prod_{\mu=1}^{N} \left\{ \left\| (\varphi_i(t))^{-l_i^{\mu}} \, \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu}. \tag{20}$$

C_1' and C_2' are the constants independent of φ.

3. Main results

Theorem 1. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn[1], $1 \le p_\mu \le q_\mu \le \infty, \mu = 1, 2, \ldots, N$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \ge 0$ be entire $j = 1, 2, ..., n$, $Q_T^i < \infty$ $(i = 1, 2, ..., n)$ and let $f \in \bigcup_{\mu=1}^{N} H_{p_\mu,\varphi,\beta}^{l^\mu}(G_\varphi)$. Then the following embedding hold*

$$D^{\nu} : \bigcup_{\mu=1}^{N} H_{p_\mu,\varphi,\beta}^{l^\mu}(G_\varphi) \to L_{q,\psi,\beta^1}(G)$$

more precisely, for $f \in \bigcup_{\mu=1}^{N} H_{p_\mu,\varphi,\beta}^{l^\mu}(G_\varphi)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are valid:

$$\|D^\nu f\|_{q,G} \le C_1 H(t) \prod_{\mu=1} \left\{ \|f\|_{H_{pmu,\varphi,\beta}^{l^\mu}(G_\varphi)} \right\}^{\lambda_\mu}, \tag{21}$$

$$\|D^\nu f\|_{q,\psi,\beta^1;G} \le C_2 \prod_{\mu=1} \left\{ \|f\|_{H_{pmu,\varphi,\beta}^{l^\mu}(G_\varphi)} \right\}^{\lambda_\mu}, \; p \le q < \infty. \tag{22}$$

In particular, if

$$Q_{T,0}^{i} = \int_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\frac{1}{p}} \times$$

$$\times \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-\sum\limits_{\mu=1}^{N} l^\mu \lambda_\mu}} dt < \infty, (i = 1, 2, \ldots, n),$$

then the function $D^{\nu}f\left(x\right)$ *is continuous on* G, *and*

$$\sup_{x\in G}|D^{\nu}f(x)|\leq C_1H_0(t)\prod_{\mu=1}\left\{\|f\|_{H^{l^{\mu}}_{p^{\mu},\varphi,\beta}(G_{\varphi})}\right\}^{\lambda_{\mu}} \tag{23}$$

where $H(T)=\sum\limits_{i=0}^{n}|Q_T^i|$, $H_0(T)=\sum\limits_{i=0}^{n}|Q_{T,0}^i|$,

$$Q_T^0=\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-(1-\beta_jp)(\frac{1}{p}-\frac{1}{q})}$$

$0<T\leq\min\{1,T_0\}$, T_0 *is a fixed number;* C_1, C_2 *are the constants independent of* f, *also* C_1 *is independent from* T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^{\nu}f$ on G. Indeed, from the condition $Q_T^i<\infty$ for all $(i=1,2,...,n)$ it follows that for $f\in H^{l^{\mu}}_{p_{\mu},\varphi,\beta}(G)\rightarrow H^{l^{\mu}}_{p_{\mu}}(G)$, there exists $D^{\nu}f\in L_{p_{\mu}}(G)$ and for almost every point of $x\in G$ integral representation is valid.

$$D^{\nu}f(x)=f^{(\nu)}_{\varphi(t)}(x)+(-1)^{|\nu|}\sum_{i=1}^{n}\int\limits_{0}^{T}\int\limits_{-\infty}^{+\infty}\int\limits_{R^n}K_i^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}\right)$$

$$\times\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi_i(t),x\right)\right)$$

$$\times\Delta_i^{m_i}\left(\varphi_i\left(\delta\right)u\right)f\left(x+y+ue_i\right)\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dtdudy, \tag{24}$$

$$f^{(\nu)}_{\varphi(T)}(x)=\prod_{j=1}^{n}\left(\varphi_j(T)\right)^{-2-\nu_j}\times$$

$$\times\int\limits_{R^n}\int\limits_{R^n}\Omega^{(\nu)}\left(\frac{y}{\varphi(T)},\frac{\rho\left(\varphi(T),x\right)}{2\varphi(T)}\right)\Omega\left(\frac{z}{\varphi(T)},\frac{\rho\left(\varphi(T),x\right)}{2\varphi(T)}\right)f(x+y+z)dydz, \tag{25}$$

Applying the Minkowsky inequality we have

$$\|D^{\nu}f\|_{q,G}\leq\left\|f^{(\nu)}_{\varphi(T)}\right\|_{q,G}+\sum_{i=1}^{n}\left\|A_T^i\right\|_{q,G}. \tag{26}$$

By means of inequality (8) and (9) for $M_i=K_i^{(\nu)}$, $\eta=T$ we get inequality (21). By means of inequality (19) for $M_i=K_i^{(\nu)}$, $\eta=T$ we get inequality (22).

Now let conditions $Q_T^i<\infty(i=1,2,\ldots,n)$, then take into account (24),and (25), from inequality (26) we get

$$\left\|D^{\nu}f-f^{(\nu)}_{\varphi(T)}\right\|_{\infty,G}\leq$$

$$\leq C\sum_{i=1}^{n}\left|Q_T^i\right|\prod_{\mu=1}\left\{\sup_{0<t<t_0}\left\|\frac{\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f}{\left(\varphi_i(t)\right)^{l_i^{\mu}}}\right\|_{p,\varphi,\beta;G}\right\}^{\lambda_\mu}.$$

As $T\to 0$, the left side of this inequality tends to zero, since $f_{\varphi(T)}^{(\nu)}(x)$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the absolutely convergence. Consequently, the derivative function is continuous G.

Let γ be an n-dimensional vector.

Theorem 2. *Let all the conditions of Theorem 1 be fulfilled. Then for $Q_T^i<\infty$ $(i=1,2,...,n)$ the derivative $D^\nu f$ satisfies on G the Hölder generalized condition, i.e. the following inequality is valid:*

$$\left\|\Delta\left(\gamma,G\right)D^\nu f\right\|_{q,G}\leq$$

$$\leq C\prod_{\mu=1}\left\{\|f\|_{H^{l^\mu}_{p^\mu,\varphi,\beta}(G_\varphi)}\right\}^{\lambda_\mu}\cdot\left|R\left(|\gamma|,\varphi;T\right)\right|, \tag{27}$$

in particular, if $Q_{T,0}^i<\infty$, $(i=1,2,\ldots,n)$, then

$$\sup_{x\in G}\left|\Delta\left(\gamma,G\right)D^\nu f\left(x\right)\right|\leq C\prod_{\mu=1}\left\{\|f\|_{H^{l^\mu}_{p^\mu,\varphi,\beta}(G_\varphi)}\right\}^{\lambda_\mu}\cdot\left|R_0\left(|\gamma|,\varphi,T\right)\right|. \tag{28}$$

where $R\left(|\gamma|,\varphi,T\right)=\max\limits_i\left\{|\gamma|,Q^i_{|\gamma|},Q^i_{|\gamma|,T}\right\}$ $\left(h_0\left(|\gamma|,\varphi,T\right)\ =\max\limits_i\left\{|\gamma|,Q^i_{|\gamma|,0},Q^i_{|\gamma|,T,0}\right\}\right)$.

Proof. According to Lemma 8.6 from [2] there exists a domain

$$G_\omega\subset G\left(\omega=\zeta r\left(x\right),\ \zeta>0\ r\left(x\right)=\rho\left(x,\partial G\right),\ x\in G\right)$$

and assume that $|\gamma|<\omega$, then for any $x\in G_\omega$ the segment connecting the points $x,x+\gamma$ is contained in G. For all the points of this segment,from identies (24), (25) after same transformations we get

$$\left\|\Delta\left(\gamma,G\right)D^\nu f\left(x\right)\right\|\leq\left\|B\left(\cdot,\gamma\right)\right\|_{q,G}+$$

$$+\sum_{i=1}^{n}\left(\left\|B_1\left(\cdot,\gamma\right)\right\|_{q,G}+\left\|B_2\left(\cdot,\gamma\right)\right\|_{q,G}\right), \tag{29}$$

where

$$B\left(x,\gamma\right)=\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-2-\nu_j}$$

$$\times\int\limits_{R^n}\int\limits_{R^n}\left|f\left(x+y+z\right)\right|\left|\Omega^{(\nu)}\left(\frac{y-\gamma}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{2\varphi(t)}\right)\right.$$

$$-\Omega^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{2\varphi\left(T\right)}\right)\Bigg|\,dydz\leq\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-2-\nu_j}\times$$

$$\times\int_0^{|\gamma|}d\zeta\int_{R^n}\int_{R^n}\left|f\left(x+\zeta e_\zeta+z\right)\right|\left|D_j\Omega^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{2\varphi(t)}\right)-\right.$$

$$\left.-\Omega^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{2\varphi\left(T\right)}\right)\right|dydz,$$

$$B_1\left(x,\gamma\right)=\int_0^{|\gamma|}\int_{R^n}\int_{-\infty}^{+\infty}\left|\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i\left(t,x\right)\right)}{\varphi_i(t)},\frac{1}{2}\rho'\left(\varphi(t),x\right)\right)\right|\times$$

$$\times\left|K_i^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi\left(t,x\right)\right)}{\varphi(t)}\right)\right|\left|\Delta_i^{m_i}\left(\varphi_i\left(\delta\right)u\right)f\left(x+y+ue_i\right)\right|dydudt$$

$$B_2\left(x,\gamma\right)=\int_{|\gamma|}^{T}\int_{R^n}\int_{-\infty}^{+\infty}\left|K_i^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi\left(t,x\right)\right)}{\varphi(t)}\right)\right|\times$$

$$\times\left|\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i\left(t,x\right)\right)}{\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi(t),x\right)\right)\right|\times$$

$$\times\int_0^1\left|\Delta_i^{m_i}\left(\varphi_i\left(\delta\right)u\right)f\left(x+y+v\gamma\right)\right|dvdudydt.$$

Here $0<T\leq\min\{1,T_0\}$. Additionally, we assume that $|\gamma|<T$, then $|\gamma|<\min\left(\omega,T\right)$ and for $x\in G\setminus G_\omega$ then

$$\Delta\left(\gamma,G\right)D^\nu f\left(x\right)=0.$$

Taking into account $\xi e_\gamma+G_\omega\subset G$, based around the generalized Minkowsky inequality, from inequality (8) for $U=G$, we have

$$\left\|B\left(\cdot,\gamma\right)\right\|_{q,G_\omega}\leq C_1\left|\gamma\right|\prod_{\mu=1}\left\{\|f\|_{H^{l^\mu}_{p^\mu,\varphi,\beta}(G_\varphi)}\right\}^{\lambda_\mu}. \tag{30}$$

By means of inequality (9),(10) for $U=G$, $M_i=K_i^{(\nu)}$, $\eta=|\gamma|$ we get

$$\left\|B_1\left(\cdot,\gamma\right)\right\|_{q,G_\omega}\leq C_2\left|Q^i_{|\gamma|}\right|\times$$

$$\times\prod_{\mu=1}\left\{\left\|\left(\varphi_i(t)\right)^{-l_i^\mu}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p_\mu,\varphi,\beta;G}\right\}^{\lambda_\mu} \tag{31}$$

$$\left\|B_2\left(\cdot,\gamma\right)\right\|_{q,G_\omega}\leq C_3\left|Q^i_{|\gamma|,T}\right|\times$$

$$\times \prod_{\mu=1} \left\{ \left\| (\varphi_i(t))^{-l_i^{\mu}} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu}. \tag{32}$$

From inequalities (29) –(32) we get the required inequality (27).

Let $|\gamma| \geq \min(\omega, T)$, then

$$\|\Delta(\gamma, G) D^\nu f\|_{q,G} \leq 2 \|D^\nu f\|_{q,G} \leq C(\omega T) \|D^\nu f\|_{q,G} |R(|\gamma|, \varphi; T)|.$$

Estimating for $\|D^\nu f\|_{q,G}$ by means of inequality (21), in this case we get estimation (27).

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On an Employment Period of a Class of Service Systems

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Abstract. The condition of the property of the employment period a one class of service systems is found in work under enough general conditions on a stream of demands and character of service. It is established that for the considered service system the necessary and sufficient condition of the property of the employment period is equivalent to the ergodicity condition for the same system.

Key Words and Phrases: Markov process, service system, reliability, ergodicity condition.
2010 Mathematics Subject Classifications: 60A10, 60J25, 60G10

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1. Introduction

Investigation of any service system makes necessary to analyze a random process related with transmission of this system from one state to another one. A lot of these systems with nonreliable devices are described by homogeneous Markov process with two components.

Investigations on the reliability of the service was founded by B.V. Gnedenko [1]. Then these investigations were continued by various authors. Statement of the problems and investigated in these works process mainly have different characters.

In the work by N.N. Yejov, T. Annaev [2] the period of employment of the service system with non-reliable devices, when we have non-homogeneous and Puasson input flow. G.P. Basharin [1] considered the systems with bounded turn of non-reliable devices, three possible service subjects (direct, inverse, random choice of the demand from the turn) and when input flow consists of ˆ the sum of finite number of simple Hows, each which corresponds to its own parameter of the exponential service low. Enough general one dimensional system with non-reliable devices was studied by G.P. Klimov [4], using the method of included Markov chain. In the work [5] using the methods of the functional analysis the process is investigated that describe a wide class of service systems considered in [2,3]. Considering the result of [2] and [5] in [6] the ergodicity condition is found for the service system with $n\,(n \geq 1)$ number of non-reliable devices and non-homogeneous input of the demands.

As is known independently from the character of the service the employment period is the main characteristics of the service system. Namely by these characteristics each service system works.

In the present work the property condition of the period of employment of a class of service systems is investigated.

The advantage of the obtained results consists of the fact that for the considered service system the necessary and sufficient condition of property of the employment period, related with some functional equation is equivalent to the ergodicity condition for the same system. This condition is found in the sense of mathematical expectation in the form of inequality.

Let's consider homogeneous Markov process $\{\xi_t, \eta_t\}$, $t \geq 0$ with phase process $\{0^+, 1^\pm, 2^\pm, ...\}\, R^+$ where $R^+ = \{x : x \geq 0\}$ and satisfies to the following transmission probabilities by $\Delta \downarrow 0$

$$\left(0^+, x\right) \begin{cases} (0^+, x+\Delta) : 1 - \lambda\,(x)\,\Delta + o\,(\Delta) \\ (k^+, 0) : 1 - \lambda_k\,(x)\,\Delta + o\,(\Delta)\,, k \geq 1 \end{cases}$$

$$\left(k^+, \Delta\right) \begin{cases} (k^+, x+\Delta) : 1 - \left[\lambda^+(x) + \mu^-(x) + \nu\,(x)\right]\Delta + o\,(\Delta) \\ \left((k^+, r)^+, x+\Delta\right) : \lambda_r^+(x)\,\Delta + o\,(\Delta)\,, r \geq 1 \\ \left((k-r)^+, 0\right) : \nu\,(x)\,\Delta + o\,(\Delta) \\ (k^-, 0) : \mu^-(x)\,\Delta + o\,(\Delta)\,, \end{cases} \tag{1}$$

$$\left(k^-, \Delta\right) \begin{cases} (k^-, x+\Delta) : 1 - \left[\lambda^-(x) + \mu^+(x) + \nu\,(x)\right]\Delta + o\,(\Delta) \\ \left((k+r)^-, x+\Delta\right) : \lambda_r^-(x)\,\Delta + o\,(\Delta)\,, r \geq 1 \\ \left((k-1)^+, 0\right) : \nu\,(x)\,\Delta + o\,(\Delta) \\ (k^+, 0) : \mu^+(x)\,\Delta + o\,(\Delta)\,, \end{cases}$$

where $\nu\,(x)\,, \mu^\pm(x)\,, \lambda_2^\pm(x)\,, r \geq 1$- non negative functions and

$$\lambda^\pm(x) = \sum_{k=1}^{\infty} \lambda_k^\pm(x)\,, \qquad \lambda\,(x) = \sum_{k=1}^{\infty} \lambda_k\,(x)\,.$$

Such kind of process are met in the investigation of the service systems with non-reliable devices and intensities depending on some parameter $x \in R^+ = [x, +\infty\}$. In this work by means of mathematical expectation the property condition and Laplace transformation is found for the employment period of the considered service system.

Let to the one-line service system with expectation non-homogeneous Puasson flow input with intensity $\lambda^\pm(x)\,, \lambda^\pm(x) = \sum\limits_{i=1}^{\infty} \lambda_k^\pm(x)$. The service period has intensity $\nu\,(x)$. The device may be broken during the service then repaired. The intensity of breakage and repairing are $\mu^-(x)\,, \mu^+(x)$ correspondingly.

The state $(0^+, x)$ means that at the considered time moment the device is free, capable, and the stopping period is equal to x. The state $(k^\pm, x)\,, x \geq 1$ means that at the considered time moment there exist k number of demands to the system, the device is capable (non-capable) x units of time (the time expended to repairing is also equal to x).

It is clear that this system will be described by the process $\{\xi_t, \eta_t\}\,, t \geq 0$ given in (1).

If the initial state of the system is $(k^+, 0)\,, k \geq 1$ then ξ^+ defines the time till the end of the service, or breakage of the device. If the initial state is $(k^-, 0)\,, k \geq 1$ then ξ^- is the period expended for the repairing of the device.

2. Main result

Let's investigate employment period of transmissions probabilities (1) .

Let τ_k^+ be the period of transmissions of the process $\{\xi_t, \eta_t\}$ from the state $(k^+, 0), k \geq 1$ to the state $(0, 0)$.

The quantity τ_k^+ is called the employment period of the service system following to Takara [7].

Let's define by $\pi^{\pm}(x)$ the Puasson process with local characteristics $\lambda^{\pm}(u), k \geq 1$ by $\xi^{\pm} = x$. Then

$$\begin{array}{ll} \tau_k^+ = \xi^+_{k+S^+(\xi^+)-1} \quad \text{with probability} & \frac{\nu(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}, \\ \tau_k^+ = \xi^+_{k+S^+(\xi^+)} \quad \text{with probability} & \frac{\mu^-(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}, \end{array} \tag{2}$$

It is easy to check that

$$\tau_k^- = \xi^- + \tau^+_{k+\pi^-(\xi^-)}. \tag{3}$$

Introduce

$$Me^{-s\tau_k^{\pm}} = \varphi_k^{\pm}(s), \quad (k \geq 1)$$

Thus

$$\tau_k^+ = \tau_k^{+k-1} + \tau_{k-1}^{+k-2} + \ldots + \tau_1^{+0} \tag{4}$$

where all terms are independent and have such distribution that $\tau_1^+ \equiv \tau_1^{+0}$ and considering (4)

$$\varphi_k^+(s) = [\omega(s)]^k,$$

where $\omega(s) = \varphi_1^+(s)$. Then from (3) we obtain

$$\varphi_k^-(s) = Me^{-s\xi-}[\omega(s)]^{k+\pi^-(\xi^-)}.$$

Considering this from (2) we can get

$$\omega^k = Me^{-s\xi^+}\left[\frac{\nu(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}\omega^{k+\pi^+(\xi^+)-1} + \right.$$

$$\left. + \frac{\mu^-(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}Me^{-s\xi^-}\omega^{k+\pi^+(\xi^+)+\pi^-(\xi^-)}\right]$$

or

$$Me^{-s\xi^+}\frac{\nu(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}\omega^{\pi^+(\xi^+)-1} +$$

$$\left. + Me^{-s\xi^+}\frac{\mu^-(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}\omega^{\pi^+(\xi^+)}Me^{-s\xi^-}\omega^{\pi^-(\xi^-)} = 1\right] \tag{5}$$

Define

$$L(s,\omega) = Me^{-s\xi^+}\frac{\nu(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}\omega^{\pi^+(\xi^+)} +$$

$$+\omega Me^{-s\xi^{+}}\frac{\mu^{-}\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}\omega^{\pi^{+}\left(\xi^{+}\right)}Me^{-s\xi^{-}}\omega^{\pi^{-}\left(\xi^{-}\right)}.$$

Then (5) may be rewritten in the form of equation

$$\omega=L\left(s,\omega\right). \tag{6}$$

Now let's consider the equation (6) in unit interval $0\leq\omega\leq 1$.

Let $c>0$. Then $L\left(s,\omega\right)$ is down convex in $\omega\in[0,1]$ function and the curve, describing this function has unique joint point ω^{*} with straight line $y=\omega$. Consequently in this case the equation (6) has unique solution ω^{*}.

Let's find necessary and sufficient condition for the property of the random quantity τ_1^{+}. For this purpose it is enough to check that the inequality

$$\left.\frac{dL\left(0,\omega\right)}{d\omega}\right|_{\omega=1}<1 \tag{7}$$

is valid. From this that $\omega\left(0\right)=1$, i.e. τ_1^{+} is eigenrandom quantity. For $L\left(0,\omega\right)$ we have

$$L\left(0,\omega\right)=M\frac{\nu\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}\omega^{\pi^{+}\left(\xi^{+}\right)}+\omega M\frac{\mu^{-}\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}\omega^{\pi^{+}\left(\xi^{+}\right)}M\omega^{\pi^{-}\left(\xi^{-}\right)},$$

$$\left.\frac{dL\left(0,\omega\right)}{d\omega}\right|_{\omega=1}=M\pi^{+}\left(\xi^{+}\right)\frac{\nu\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}\omega^{\pi^{+}\left(\xi^{+}\right)}+$$

$$+\omega M\frac{\mu^{-}\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}\omega^{\pi^{+}\left(\xi^{+}\right)}M\omega^{\pi^{-}\left(\xi^{-}\right)},$$

$$\left.\frac{dL\left(0,\omega\right)}{d\omega}\right|_{\omega=1}=M\pi^{+}\left(\xi^{+}\right)\frac{\nu\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}+M\frac{\mu^{-}\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}+$$

$$+M\frac{\mu^{-}\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}M\pi^{-}\left(\xi^{-}\right)+M\pi^{+}\left(\xi^{+}\right)\frac{\mu^{-}\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}.$$

From the condition (7) one can get

$$M\pi^{+}\left(\xi^{+}\right)+M\frac{\mu^{-}\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)}M\pi^{-}\left(\xi^{-}\right)<M\frac{\nu\left(\xi^{+}\right)}{\nu\left(\xi^{+}\right)+\mu^{-}\left(\xi^{+}\right)} \tag{8}$$

As

$$M\theta^{\pi^{\pm}(x)}=\exp\left\{\int_0^x\lambda^{\pm}\left(u,\theta\right)du\right\},$$

where

$$\lambda^{\pm}\left(u,\theta\right)=\sum_{i=1}^{\infty}\lambda_k^{\pm}\left(u\right)\theta^{k}$$

then

$$M\pi^{\pm(x)} = \int_0^x \dot{\lambda}^{\pm}(u,1)\,du,$$

(here $z(x,1)$ is a derivative of the function $z(x,\theta)$ by $x=0$).

Considering last relations the inequality (8) takes a form

$$M\int_0^{\xi^+} \dot{\lambda}^{+}(u,1)\,du + M\frac{\mu^-(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)} M\int_0^{\xi^-} \dot{\lambda}^{-}(u,1)\,du < M\frac{\nu(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}. \quad (9)$$

If $\varphi_1^+(s)$ is known, then it is possible to define $\varphi_k^+(s)$ for any $k \geq 2$.

In [8] is proved that the inequality (9) is ergodicity condition for the service system described above. We establish that this inequality is necessary and sufficient condition of property of the employment period of the service system.

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Conducting of Monitoring and Experiments in Toxic Substances of Poisoning

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Abstract. According to statistical data, with the development of oil, chemical, gas industries cases of poisoning caused by toxic substances employed in these branches have become more frequent recently. A special place among them is occupied by carbon monoxide, poisoning with which has been growing steadily. Considering such consequences of similar-poisonings as myocardial infarction, Parkinson's disease u.a. it is expedient to perform monitoring of a patient after staying in a stationary hospital which determines optimum time of its performance, kind and the number of analysis required for developing an intelligent system. This paper proposes an elaboration of an intelligent information system for monitoring in cases of poisonings with toxic substances using carbon monoxide as an example.

Key Words and Phrases: Carbon monoxide, poisoning, monitoring, parametric criteria, non-parametric criteria, biostatistical methods.

2010 Mathematics Subject Classifications: 004.891.3

1. Introduction

The quote adopted by the World Health Organization in 1998, says: " .... prompt and adequate treatment of acute poisoning can save lives by minimizing the impact of poisoning ". İf the poisoning was discovered and treated in ti_me, so its results can show up after a long period of time. A few weeks later, parkinsonism, heart muscle damage inflicted deaths can occur. Clearly, these people are poisoned by toxic substances or other doses are in need of long-term monitoring [1]. Monitoring changing position of the object, and its performance is desirable observation or comparison with the previous ones.

Along with the diagnosis of poisoning by carbon monoxide poisoning in order to forecast has a great importance for the consequences of monitoring. It is to be observed during a certain time health status of persons poisoned. During the monitoring, in order to solve the problem of statistical data analysis methods can be applied.

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2. Statistics and Literature review

The number of people affected by carbon monoxide in the Central Europe and Southwestern Europe over the 2004 is shown in Table 1 [2, 3].

Table 1.

Consolidated Table of carbon monoxide poisonings

Country/Region	**Extrapolated Incidence**	**Population Estimated Used**
Carbon monoxide poisoning in Central Europe (Extrapolated Statistics)		
Austria	3,406	8,174,762
Czech Republic	519	10,246,178
Germany	34,343	82,424,609
Hungary	4,180	10,032,375
Liechtenstein	13	33,436
Poland	16,094	38,626,349
Slovakia	2,259	5,423,567
Slovenia	838	2,011,473
Switzerland	3,104	7,450,867
Azerbaijan	3,278	7,868,385
Portugal	4,385	10,524,145
Spain	16,783	40,280,780
Georgia	1,955	4,693,892

The different aspect of toxic substances, including carbon monoxide poisoning is diagnose making, management and treatment for her tactics based on antidote therapy complement each other and are carried out under the supervision of a doctor, but one of the main problems seen in a long time the importance of the patient's condition after treatment. Because a number of studies have shown that carbon monoxide poisoning are not only harmful effects on the human body, its results are still manifests itself after a long period of time. This is mainly disorders of the nervous system, cardiovascular system diseases. According to scientists poisoned by carbon monoxide, 37% of patients suffered from cardiac muscle damage.1/4 of the poisoned people had died after 7 years. Professor Timothy Henry, head of the research process in the U.S, says: "The main result of the study of carbon monoxide poisoning in the delivery of long-term negative impact on health." Professor Henry said that the number of patients a result of poisoning observed in disorders of cardiac activity was higher in all the possibilities of scientists [4].

The last years innovations in the construction industry; in the narrow streets surrounded on both sides with "skyscrapers", the speed of vehicles is reduced, the carbon dioxide emitted from machines gas is accumulated in the air close to the ground surface at the respiratory level of people.

At low air condition, carbon dioxide creates a hazardous situation for people's health. One of the reasons for the increase in the number of cardiovascular diseases is carbon dioxide poisoning. Table 5.1. was drawn by the information of Baku City Emergency and Urgent Medical Help Station (BCEUMHS).

Table 5.1

Information of BCEUMHS about myocardial infarction for 2006-2013 years.

Years	**Appeal an hospitalization**	**Interest rate indicator**
2006	**1573**	**1,7%**
	1205	**76,6%**
2007	**1627**	**1.1%**
	1321	**81,2%**
2008	**1797**	**1,7%**
	1472	**81,9%**
2009	**1601**	**1,5%**
	1362	**85%**
2010	**1311**	**0.3%**
	1153	**88%**
2011	**1289**	**0.2%**
	1216	**94%**
2012	**1211**	**0.2%**
	1154	**95%**
2013	**1054**	**0.2%**
	1006	**95,4%**

Comings of carbon monoxide poisoning the concentration of toxic substance in the course, the amount to be included in the body of the organism, the situation timely, adequate medical assistance provided. In general, the higher the percentage of fatal outcome. However, fixed in 2% of patients had severe poisoning neuro psychological found quitting the gap is observed. More than 10.8% of the patients after 3 years neuro physical disorders (memory disturbance, personality disorders) suffer.

In recent years, innovations in the chemical and construction industry for example, streets surrounded by tall and thin skyscrapers in both sides, traffic congestion is reduced with respect to the speed of vehicles, carbon monoxide which are removed from vehicles accumulates in the air near-surface where people breathe in a closed environment and carbon monoxide collected in the atmosphere, in less windy conditions creates a dangerous situation for the health of people. All of these lead to chronic intoxication. One of the reasons for the increase of cardiovascular diseases is chronic intoxication. For these reasons, the followings need to be considered:

- Differential diagnosis of patients in comatose;

- Health surveillance to poisoned person after a certain period of time.

3. Methods

Solution of the first problem is carried out using mathematical and artificial intelligence methods in intelligent information system. The second is monitring issue after receiving treatment outcome. Monitoring is advisable for both once poisoned also for persons affected by chronic intoxication.

Monitoring needs to be conducted after successful treatment in hospital. Therefore, starting time of monitoring should coincide with the end of the treatment. Functional parameters and biochemical analysis of carbon monoxide victim needs to be examined from time to time during the monitoring (fixed time interval). Particularly, type of poisoning, more affected poisoning of the body and more nervous and cardiovascular systems, mainly due to the majority of these indicators are checked by selecting from among the more specific ones. In most cases, the determination of patient treatment verification of indicators reflecting the health of people selects in the process of stationer treatment. Analysis in the selected interval of time should be checked for prevention and prognoses of consequences after poisoning.

Analysis should be checked in a certain time interval to carbon monoxide victim after antidote therapy and appropriate treatment in order to control the situation. Double autocorrelation and non-parametric methods [5] is used for comparison and detection of analyze results with most specificity. Any change or signs in the toxicated or treated people could be observed by using these methods. The application of appropriate methods allows the assessment of independent indications,symptoms assessment of before and after treatment, assessment of the differences between the dynamics of change and plays an important role in the detection of change differences.

4. Monitoring Tips

Time changes in carbon monoxide poisoning should be controlled after receiving treatment in order to avoid the consequences of toxication. Time series method is used in those situations. The basis of time series analysis is that former happenings have important indications for future happenings. Time series data is a sequence of successive moments of time, which reflects to the situation. In contrast to randomly selected analysis, time series based on observation data of equal times. Time series can be often found in medicine. Time series analysis has two goals: determination the nature of queue and prognosis. In both cases, the model must be specified before the turn to the interpretation of the data.

According to the analysis of time series, data consists of systematic component and a random voice complication detection components which arranged in a regular variable. Majority of research methods allows to observe the change in the index on a regular basis using a variety of methods for filtering noise. Routine variables of time series have two classes: either the trend or seasonal components.

Change dynamics reflects the trend. Trend consists of the variable components changed through the time organized in a systematic linear or non-linear. The seasonal component is repeated periodically [6].

Time series process used to identify prognostic factors of data in the past, today linked to a similar effect in the near future. Analysis of observations is a continuous process which estimated in a certain discrete moments of time (when you can evenly across the distribution). For that reason indications which can cause a dangerous development in near future should be selected (months, sometimes years).

There is not "automate" methods for detection of time series. If the trend (increasing or decreasing) is monotone, the queue is not difficult to analyze. If the time series has enough offense in that case smoothing process should be conducted primarily as a method of filtration. Smoothing process is a kind of moderation of data. In this case, the non-systematic errors repel each other. The most common method of smoothing method is moving average, when m the members of the neighboring row of each member shall be replaced by a simple average, m – is a price of intervals. Also, the trend is to be used for the detection of exponential smoothing. Many monotone time series described by linear to express analytically. If there is non-linear component, set of data needs to be carried out to remove it. For this reason, most of the time logarithmic, exponential, or polynomial transformations can be used. In some cases, the least squares method is carried out in the smoothing. All of these methods are given the relatively smooth line noise filtering, transforms to circle.

Moving average method determine the start of a new trend, also warns of the end or return. This method allows you to keep track of the development process, it can be viewed as trend lines. However, this method is not used for making predictions, because it follows a trend, but it can't predict only shows the start of a new trend. Smoothed curve and the trend observed during the performance of the simplified average, short-term floating-average rate reflect dynamics more accurately for long intervals calculation.

Moving average is defined as follows:

$$y_t = \frac{1}{m} \sum_{i=t-p}^{t+p} y_i \,, \tag{1}$$

wherey_i, - value of the i-th level; m - the number of levels from smooth intervals - $(m = 2p + 1)$; y_t dynamic row of the current level; i- number smooth level range; p-m single range value$p = (m - 1)/2$.

Smooth change interval depends on the determination of the indicators. Thus, indicators of irregular, small changes smooth interval assumed to be more. If you are required to take into account changes in smoothing, small gap becomes smaller.

Moving average method is used if time series is organized in straight lines. Because this time is not misrepresent the dynamics of the index. If the range is non-linear, usage of this method can cause distortion of indicators . It is used when smooth is exponential [7].

Analytical smoothing method is an identification of development trends as time series function.

$$\hat{y}_t = f(t) \,, \tag{2}$$

where$\hat{y}_t$- theoretical value of time series with analytical expression for the time t-time.

Theoretical value are derived from the mathematical model.

Indicating the trend of development, the following features are implemented:

1. The linear function with straight line graphs:

$$\hat{y}_t = a_0 + a_t t$$

2. Exponential function

$$\hat{y}_t = a_0 \, * a_1^t \,,$$

3. Exponential function second degree (parabola)

$$\hat{y}_t = a_0 + a_1 \, * t + a_2 t^2 \,;$$

4. Logarithmic function:

$$\hat{y}_t = a_0 \, + a_1 \ln \, t \,.$$

Estimation of functions parameters are carried out by least squares method. In this case, the solution is the minimum value of the sum of theoretical and empirical levels squares:

$$\sum (\hat{y}_t - y_i)^2 \to \min \,, \tag{3}$$

where $\hat{y}_t$, — calculated,y_t — real levels.

Smooth on a straight line is used in cases where the increments are fixed.

Smooth with exponential function is applied in geometric changes in the when there is a steady increase in the ratios.

Secondary exponential function smooth is used to changes dynamic range and stable chain increases.

The smooth on logarithmic function reflects growth of the number of decrease, the recent increase in the time series.

Counting accuracy of the analytical expressions is defined as follows: sum of empirical series of price must coincide with the sum of the smoothed series levels. In this case, small errors can occur due to the calculated values:

$$\sum y = \sum \hat{y}_t \,. \tag{4}$$

Autocorrelation is used to determine patterns of additional data change in time series smooth method. Autocorrelation function, determine indication whether it is increasing or decreasing based on seasonal fluctuations.

Determination model is used to assess the trend model accuracy:

$$R^2 = \frac{\sigma_{\hat{y}}^2}{\sigma_y^2} \,, \tag{5}$$

where $\sigma_{\hat{y}}^2$- theoretical model dispersion of the data variance,σ_y^2- empirical dispersion of the data.

Trend model shows development tendency of R^2 close to 1 indicators in values. According to the time series method, data processing is carried out in three stages:

In the first phase filtering is carried out not to take into consideration distortions resulting from seasonal or other changes. The main goal of filtration is to find out$y$-changes affected from x-changes, eliminate factors that will affect that relationship further. A few known methods for filtering floating above the average value is the most widely used.

According to the moving average price at the time of moving to and from in the price index is calculated by determining the average number. In this situation, the long-term periods doesn't show accurately value compared to the changes in the short-term periods. However, filtration should be conducted carefully. Important information may be lost as a result of the smoothing filter. Therefore, filtration should be carried out in several ways, the results should be verified with the help of correlation analysis.

The second stage is a conduction of the forecast index. For this reason regression model selection and installation is carried out.

Regression analysis is used for two reasons:

- Detection of relationship between the measured parameters;
- Prognoses of the value of a variable based on the value of regression equation for non-dependent.

Monitoring with carbon monoxide poisoning shows interesting facts according to the method of time series in the monitoring of indicators to determine whether certain moments of time, but also forecast of the change indicators. Time series method is using to show the changed indicators of regression equation by time to time. Single regression equation shows the variation of the moments and observation of a person poisoned by a factor. Changes of signs in time, creates time series of dynamic rows. The characteristics of that rows is time factor (x), and dependant variable (y) factor, the sign of the value change. The dependence between them can be shown as regression equation.

The changes indications by using the method of time series depends on single factor regression equation or multivariate factor of regression equation. In addition, the figure forecast in a single-factor regression equation is given by:

$$y = a + b\ * x \tag{6}$$

where a - the free member; b - determines the slope of the regression line rectangular axes. According to the least squares method to determine the parameters of the equations will be as follows:

$$a * n + b \sum x = \sum y \tag{7}$$

$$a \sum x\ + b \sum x^2 = \sum y * x \tag{8}$$

Formulas given for determination of parameters:

$$a = y - b * x \tag{9}$$

$$b = \frac{y * x - \bar{y} * \bar{x}}{x^2 - \bar{x}^2} \tag{10}$$

Multivariate regression equation is used to monitor and prediction of the dynamics of change of many traits at the same time:

$$\hat{y}_x = a + b_1 x_1 + b_2 x_2 + ... + b_m x_m \tag{11}$$

Based on the assumption of multivariate regression testing is not possible, dependants becomes more obvious on the basis of the probabilities. Because the regression coefficients for the various tendencies traits values cause a shift in the regression line, and can change direction. Even one trait value in the presence causes a change in the outcome. Despite it is necessary to monitor the observation of a large number of indicators in carbon monoxide poisoning, more realistic indication of each individual was considered more appropriate to the forecast by the factorial regression. This has been confirmed in numerous experiments. The prognosis by regression equation is given for a certain time after the end of the monitoring period.

In the third stage, the quality of the model should be estimated.
The regression is carried out by adequacy of the model determination:

$$R^2 = \sum_{i=1}^{N} (\widehat{y}_i - \bar{y})^2 / \sum_{i=1}^{N} (y_i - \bar{y})^2. \tag{12}$$

where - $\hat{y}_i$- relevant to $x_i$the theoretical or estimated valuey_i.

Determination coefficient shows variables depending on the degree of compared dispersion. The adequacy of the regression equation is increasing in respect to R^2 high value. Determination coefficient regression model useful for prediction. The regression equation for the determination of a criteria of Fisher are used:

$$F = \frac{R^2}{1' - R^2} * \frac{n - m - 1}{m} \tag{13}$$

where R - determination coefficient, n - number of observations, m - number of parameters in x variables (the number of factors in linear regression model).

This criterion assesses the significance factors included in the regression equation. Calculated F-value of the significance level αup, are compared with 1 and n-$m - 1$ in table value. If the calculated Fvalue exceeds the value of the table, i.e., $F \geq F_{table}$, then x factor included in the model is statistical significance. If the calculated F is less than table value, x variable doesn't affect to y variables changes and the inclusion in the model is inappropriate.

Determination coefficient with the help of correlation is defined as follows:

$$r = \sqrt{R^2} \tag{14}$$

Determination coefficient, -1, +1 varies in correlation coefficient. Determination coefficient is close +1 shows close relation of y variables with x factor to prove that indicator is the most significant factors for formalization of consequences. In this regard, the regression model can be used to forecast the indicator.

The indicators selected for monitoring medicine will be:

$$x_i \in \{X\} \ , \quad i = \overline{1, n}$$

where x_i– indicator.

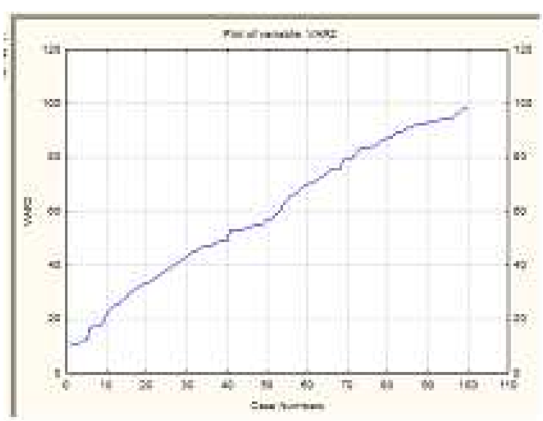
There are ending regulatory values for given parameters. Based on this, there is specific change interval for $\forall x_i$ (in some cases, the standards are different for men and women). Standards in accordance with the upper and lower boundaries is y_i and z_i. Then

$$y_i < x_i < z_i$$

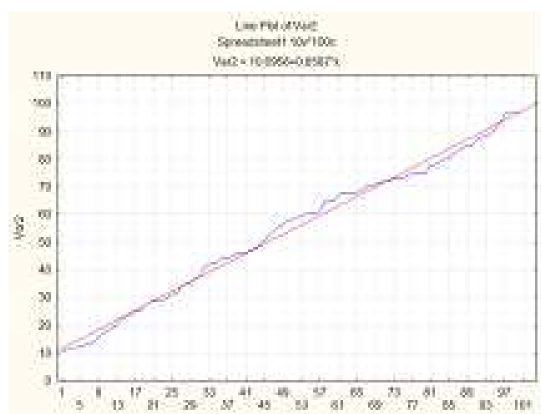
should be. Each x_i is observed in $T = \{t_1,\, t_2, ...,\, t_k\}$ time. k– is the number of measurements. Then x_i^j can be described as an arbitrary parameter, where $i = 1, 2, ..., n$, $j = 1, 2, ..., k$. Lower and upper variables can be considered as pathology:

$x_i^j < y_i$ or $x_i^j > z_i$

Autocorrelation functions are established for observation of any change of variable x_i ^{j} in $T = \{t_1,\, t_2, ...,\, t_k\}$time. It should be noted that the numbers do not reflect the cost of health indicators, the random number generator has been used. Kas the number of points used in the determination during the observation period, sometimes it means the number of years or months. For example, 100-point numbers with a given distribution (fig.2a), trend (fig.2b), smoothing curve (moving average) (fig.2c), forecasting (fig.2d), shown a certain time autocorrelation (fig.2e) and partial autocorrelation function (fig.2d). This series show ascending value of numbers.



a)



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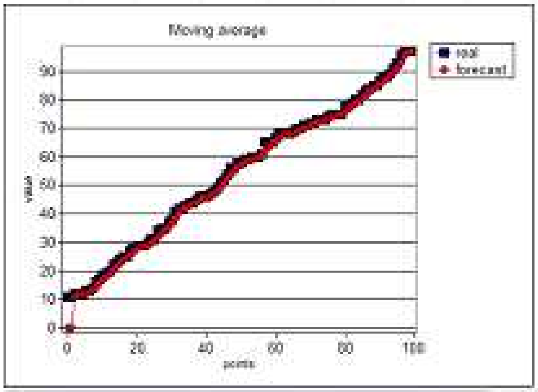
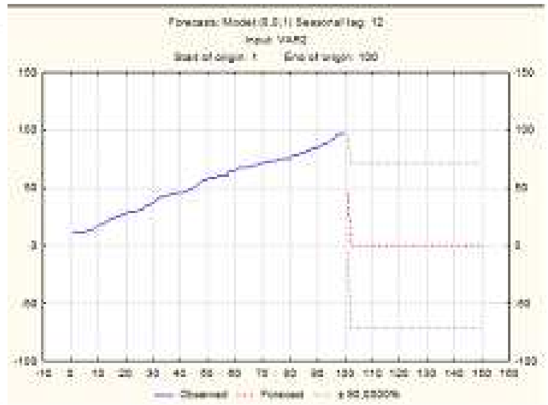
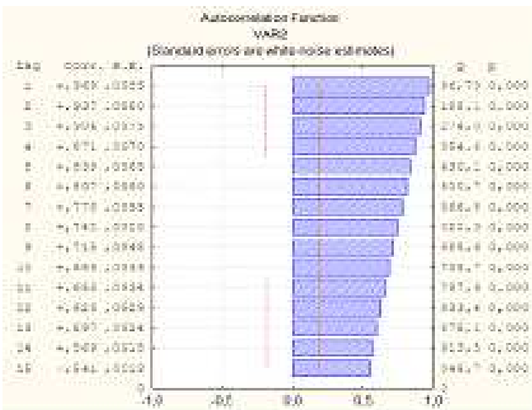
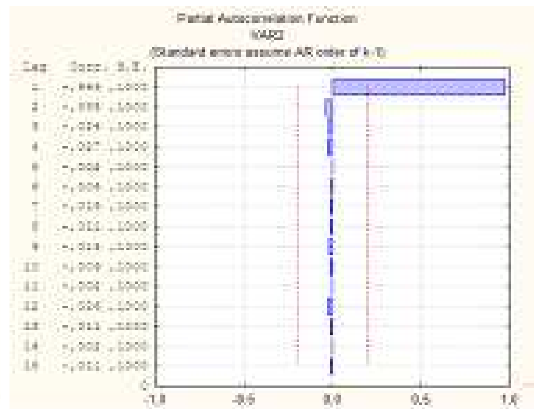




c) d)



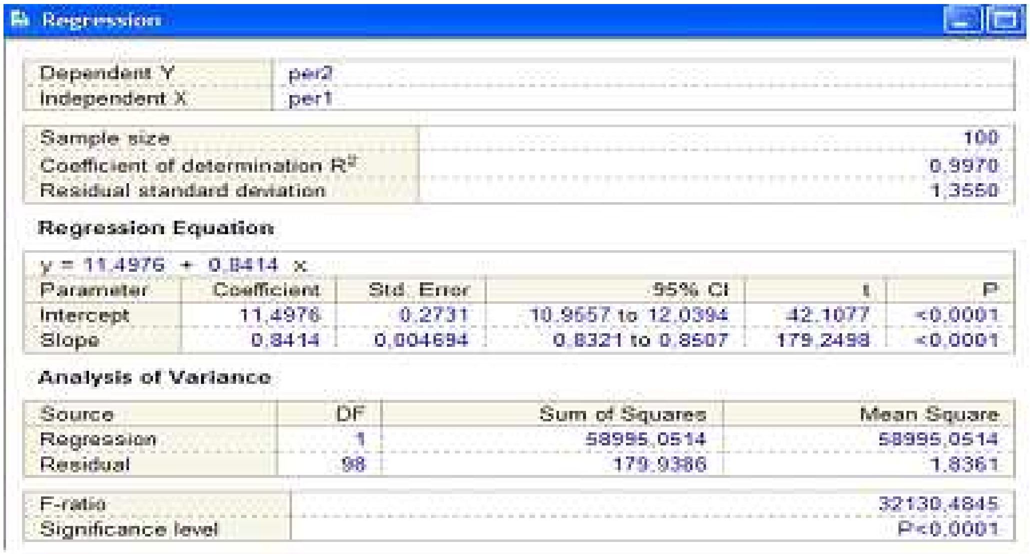


e) f)

Fig.2. Characteristic of time series with ascending numbers

Partial correlation shows variables between two random variables, when taken the effect of internal values of autocorrelation doesn't take into account. Partial autocorrelation is almost same with simple autocorrelation in small moving. In practice, the periodic dependence of the specific autocorrelation is showing as "clean". The appearance of autocorrelation and partial autocorrelation depends on the length of the time series. Autocorrelation function shows the model accurately when the series is long. When the range is short, correlation loses its accuracy and autocorrelation and autocorrelation estimation degree is decreasing. Meanwhile, the trend shows that there is not a periodic function in autocorrelation changes.

Regression equation for distribution, coefficient of determination (fig.3a), scatter regression of the order given as follows (fig.3b):



Regression

Dependent Y	per2
Independent X	per1

Sample size	100
Coefficient of determination R^2	0,9970
Residual standard deviation	1,3550

Regression Equation

y = 11,4976 + 0,8414 x

Parameter	Coefficient	Std. Error	95% CI	t	P
Intercept	11,4976	0,2731	10,9557 to 12,0394	42,1077	<0,0001
Slope	0,8414	0,004694	0,8321 to 0,8507	179,2498	<0,0001

Analysis of Variance

Source	DF	Sum of Squares	Mean Square
Regression	1	58995,0514	58995,0514
Residual	98	179,9386	1,8361

F-ratio	32130,4845
Significance level	P<0,0001

Fig.3a

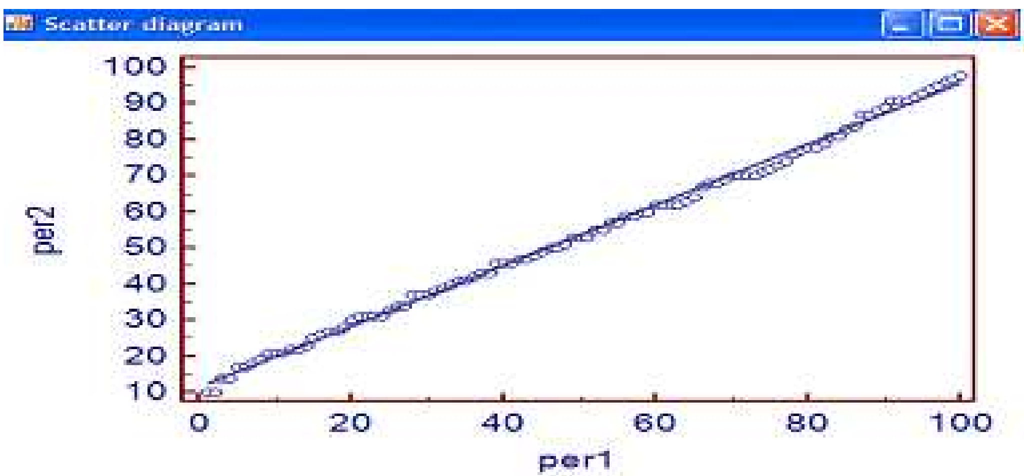


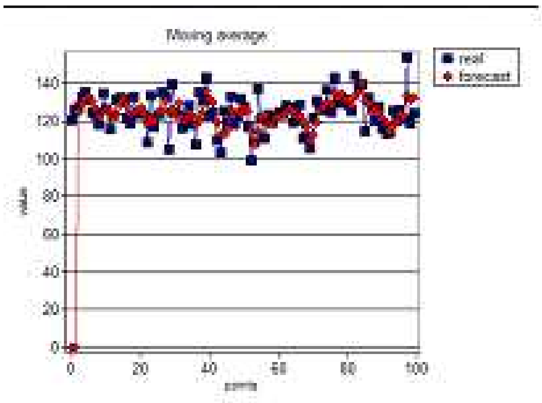
Fig.3b

According to Fisher criteria, this statistics is significance.

Another example for number of shows with normal distribution in fig.4a,b,c,d,e,f [8].

a)

b)



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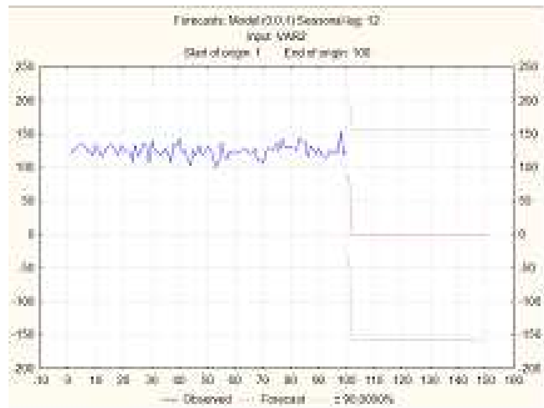
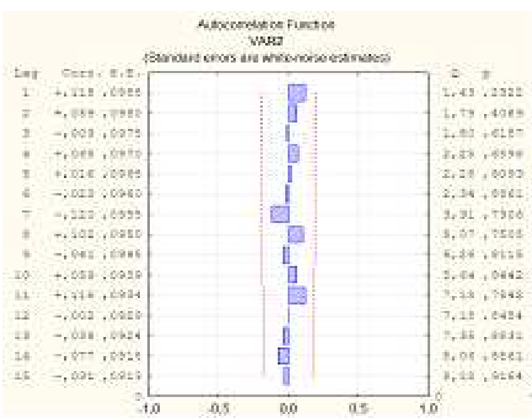
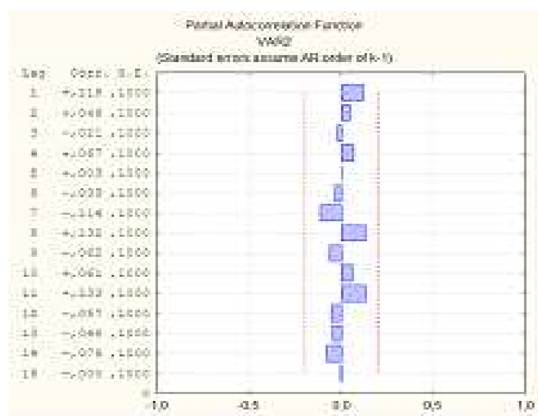


d)



e)



f)

Fig.4. Characteristic of time series with normal distribution numbers

Regression equation for distribution, coefficient of determination, scatter and regression of the order given as follows (fig.5a,b):

Regression

Dependent Y	per2
Independent X	per1

Sample size	100
Coefficient of determination R^2	0.004941
Residual standard deviation	10.4833

Regression Equation

$y = 123.6061 + 0.02533\ x$

Parameter	Coefficient	Std. Error	95% CI	t	P
Intercept	123.6061	2.1125	119.4140 to 127.7983	58.5120	<0.0001
Slope	0.02533	0.03632	-0.04674 to 0.09740	0.6976	0.4871

Analysis of Variance

Source	DF	Sum of Squares	Mean Square
Regression	1	53.4811	53.4811
Residual	98	10770.2386	109.9004

F-ratio	0.4866
Significance level	P=0.4871

Fig.5a

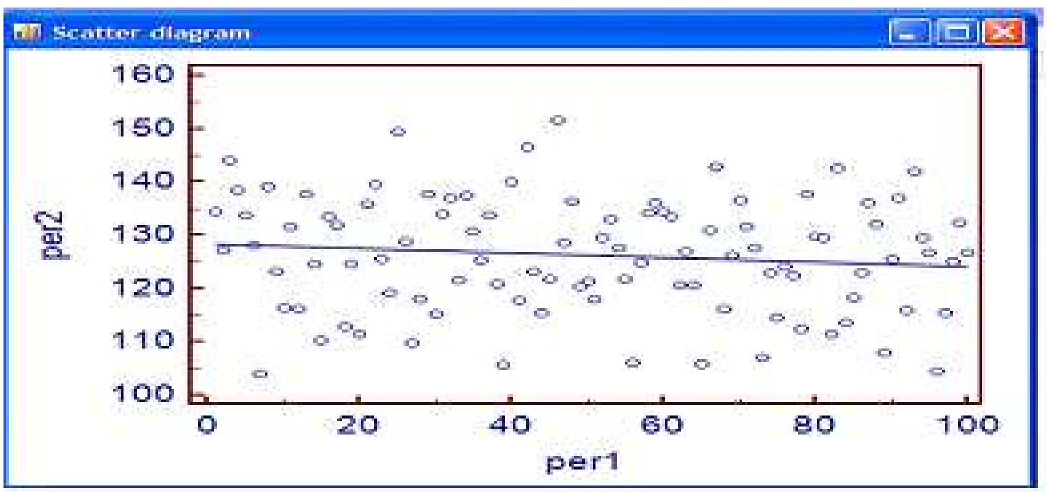


Fig.5b

Smoothing curve,autocorrelation and special autocorrelation functions shows that there is trend in that range. Determination coefficient value shows that forecast is impossible. According to Fisher theory, the value of indication is not significance.

During the course of the monitoring indicators of each time interval along with the observation of one or several indicators needs to be found observed. Mann-Whitney criteria is used for the evaluation of the difference between two independent indicators, Wilcoxon T-criterion is used for evaluation of monitoring from treatment period, any indication of a change in a certain time, Friedman method is used to measure the difference between double monitoring difference evaluation and Kruskal-Wallis criterion is applied for assessment of presence of indicators in several measurements.

Conclusions. This work proposes a time series method for monitoring the state of a patient after treatment of carbon monoxide poisoning. The said method allows to trace dynamics of indices in time intervals and detect a more important index for observation of treatment resistant symptoms and elimination of excessive checks. For comparison of the indices in time intervals parametric and non-parametric criteria of biostatistics are employed.

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A Variable Exponent Hardy's Inequality Approach for Some Nonlinear Eigenvalue Problem

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Abstract. Applying a new bounded ness and compactness result for Hardy's operator $(\int_0^x f(t)\,dt)$ and its conjugate $(\int_x^l f(t)\,dt)$ in variable exponent spaces $L^{p(\cdot)}(0,l)$ and applying the Mountain Pass Theorem approaches in this paper it has been proved an existence result for the eigenvalue problem

$$\begin{cases} -\left(|y'|^{p(x)-2}y'\right)' = \lambda\, y^{p(x)-1} + \left(\frac{y}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}\frac{a(x)}{x^\alpha(l-x)^\alpha}, \\ y(x) > 0, \qquad 0 < x < l, \\ y(0) = y(l) = 0. \end{cases}$$

where the exponent function $p:(0,l)\to(1,\infty)$ is monotone near the origin and l also satisfying a log-regularity conditions in this points.

Key Words and Phrases: variable exponent spaces, inequality, eigenvalue problem, mountain pass theorem, functional.

2010 Mathematics Subject Classifications: 26D10, 42B37, 35D05, 35J60, 35P30

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1. Introduction

In this paper, we shall study an existence result for the nonlinear eigenvalue problem

$$\begin{cases} -\left(\,|y'|^{p(x)-2}y'\right)' = \lambda\, y^{p(x)-1} + \left(\frac{y}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}\frac{a(x)}{x^\alpha(l-x)^\alpha}, \\ y(x) > 0, \quad 0 < x < l, \\ y(0) = y(l) = 0. \end{cases} \tag{1}$$

Let $Lip_0(0,l)$ be a class of Lipshitsz continuous functions $f:(0,l)\to R$ with $f(0) = f(l) = 0$. Close this class of functions in a norm

$$\|f\|_{\dot W^1_{p(.)}(0,l)} = \|f'\|_{L^{p(\cdot)}(0,l)}.$$

*Corresponding author.

The obtained variable exponent Sobolev type space denote as $\dot{W}^1_{p(\cdot)}(0,l)$. This is a reflexive Banach space if $1 < p^- := \inf\limits_{(0,l)} p(x), \quad p^+ := \sup\limits_{(0,l)} p(x) < \infty$ (see, e.g. [14, 15])

In space $\dot{W}^1_{p(\cdot)}(0,l)$, consider an eigenvalue problem (1) with Dirichlet conditions in the ends of a finite interval $(0,l)$.

Let λ_1 be the first eigenvalue number of the $p(x)$-Laplace's operator. In other words,

$$\lambda_1 = \inf_{\{y\in AC(0,l),\, y\neq 0,\, y(0)=y(l)=0\}} \frac{\int\limits_0^l |y'(x)|^{p(x)}\,dx}{\int\limits_0^l |y_(x)|^{p(x)}\,dx} \tag{2}$$

It is satisfied

$$\begin{cases} -\frac{d}{dx}\left(\left|\frac{dy_1}{dx}\right|^{p(x)-2}\frac{dy_1}{dx}\right) = \lambda_1 \left|y_1(x)\right|^{p(x)-2} y_1(x), \\ y(x) > 0, \quad 0 < x < l, \\ y(0) = y(l) = 0. \end{cases} \tag{3}$$

for the first eigenvalue λ_1 and the eigenfunction $y_1(x)$ of the problem (2). It has been shown in [3] that there are infinitely many discreet eigenvalues $0 \le \lambda_1 < \lambda_2 ... < \lambda_k ...$ of the problem (3) such that $\lambda_k \to \infty \quad \text{as} \quad k \to \infty$. At that, the first eigenvalue may be no strongly positive. In the cited work, it was stated that the first eigenvalue is strongly positive ($\lambda_1 > 0$) if one dimensional case and a monotony exponent function $p(x)$ be considered.

To prove the existence of solution of problem (1), we shall apply a Montain pass theorem due to Ambrosetti and Rabinowitz [2, 1]. In order to carry out this, we need some new variable exponent boundedness and compactness results for Hardy's operator and its conjugate [7, 8, 12, 9].

Theorem 1. *Let $q, p : (0,l) \to (1,\infty)$ be measurable functions such that $1 < p^- \le p(x) \le q(x) \le q^+ < \infty$. Assume that, $\alpha \in (1 - \frac{1}{p^+}, 1)$, and be satisfied the conditions:*

$$\limsup_{x\to 0} |f(x) - f(0)| \ln \frac{1}{x} < \infty, \quad \limsup_{x\to l} |f(x) - f(l)| \ln \frac{1}{l-x} < \infty, \tag{4}$$

moreover,

$$p^+ \le q^- < \frac{1}{\alpha - 1 + \frac{1}{p^+}}. \tag{5}$$

holds.

Then the set of functions $\{y(t) \in AC(0,l) : y(0) = y(l) = 0\}$ with bounded norm

$$\left\|y'(x)\right\|_{L^{p(\cdot)}(0,l)}$$

are compactly embedded into the class of functions with finite norm

$$\left\|\frac{y}{x^\alpha (l-x)^\alpha}\right\|_{L^{q(\cdot)}(0,l)}. \tag{6}$$

For an exact characterization of the Hardy's inequality in variable exponent spaces not using the regularity conditions (4) on the exponent functions see, the recent works [11, 13])

Theorem 2. *Let $p : (0, l) \to (1, \infty)$ be measurable function, such that, $1 < p^- \leq p(x) \leq p^+ < \infty$. Assume that, p satisfies (4) near the origin and l. Then it holds an inequality*

$$\left\| \frac{y(x)}{x(l-x)} \right\|_{p(x);(0,l)} \leq \frac{C}{l} \left\| y'(x) \right\|_{p(x);(0,l)} \tag{7}$$

for all absolutely continuous functions $u : (0, l) \to R$ with $u(0) = u(l) = 0$. Moreover, a positive constant C in (7) depends on p^-, p^+, C_1, C_2.

From Theorem 2 one gets easily the following Sobolev type inequality

$$\frac{1}{lC} \|y\|_{L^{p(\cdot)}(0,l)} \leq \|y'\|_{L^{p(\cdot)}(0,l)} \tag{8}$$

for any absolutely continuous function y in $(0, l)$ with limits $y(0) = y(l) = 0$.

Theorem 3. *Let $q, p : (0, l) \to (1, \infty)$ be measurable functions, such that, $1 < p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ < \infty$, and the conditions (4) be satisfied. Let the exponent function p be monotony near the origin and l. Assume a real positive number α satisfies (5). Then there exists a positive solution of the problem (1) from space $\dot{W}^1_{p(\cdot)}(0, l)$ for any $\lambda < \lambda_1$ and $a(x) \in L^\infty(0, l)$.*

The proof of the above result relies on the celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz [1] in the following variant.

Theorem 4. *Let X be a real Banach space and let $F : X \to \mathbb{R}$ be C^1-functional. Suppose that F satisfies the Palas-Smale condition and the following geometric assumptions:*

- *1) there exists positive constants ρ, c_0 such that $F(u) \geq c_0$ for all $u \in X$ with $\|u\| = \rho$;*
- *2) $F(0) < c_0$ and there exists $v \in X$ such that $\|v\| > \rho$ and $F(v) < c_0$.*

Then the functional F posseses at least a critical point.

For the multidimensional case $n \geq 3$ and constant exponents $p = 2,\ 2 < q < \frac{2n}{n-2}$, $\alpha = 0$, $a(x) = 1$ we refer to [4], where an enhanced description of nonlinearities and eigenvalue number ranges, enabling multiplicity of solutions for the problem (1) is given applying the Lusternik-Schnirelman category approache in manifold. For the variable exponent setting, we cite [6], where constant exponents q, $\alpha = 0$, $a(x) = 1$ has been considered in case $n \geq 2$.

For a solution of problem (1) we call a function $y \in \dot{W}^1_{p(\cdot)}(0,l)$ that satisfies the integral identity

$$\int_0^l |y'|^{p(x)-2} y'v'\,dx - \lambda \int_0^l y_+^{p(x)-1} v\,dx \\ - \int_0^l \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)-1} \frac{va(x)}{x^\alpha (l-x)^\alpha}\,dx = 0 \tag{9}$$

for any test function $v \in \dot{W}^1_{p(\cdot)}(0,l)$.

Consider in $\dot{W}^1_{p(\cdot)}(0,l)$ the functional $I : \dot{W}^1_{p(\cdot)}(0,l) \to R$ defined as

$$I(y) = \int_0^l \frac{1}{p(x)} |y'|^{p(x)}\,dx - \int_0^l \frac{\lambda}{p(x)} y_+^{p(x)}\,dx - \int_0^l \frac{a(x)}{q(x)} \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)}\,dx, \tag{10}$$

where $y_+ = \max(y(x), 0)$.

Correct setting a solution notion. Verify correctness of the solution notion and the functional $I(y)$ setled in $E := \dot{W}^1_{p(\cdot)}(0,l)$. The first integral in (9) is well defined by virtue of Holder's inequality and $y, v \in \dot{W}^1_{p(\cdot)}(0,l)$. By virtue of (7) and Holder's inequalities second and third integrals are well-defined:

$$\int_0^l |y_+|^{p(x)-2} |y_+ v|\,dx \le c_0 \big\| |y_+|^{p(x)-1} \big\|_{L^{p'(\cdot)}(0,l)} \cdot \big\|v\big\|_{L^{p(\cdot)}(0,l)}$$

$$\le c_0 \Big(1 + \|y_+\|^{p^+-1}_{L^{p(\cdot)}(0,l)}\Big) \big\|v\big\|_{L^{p(\cdot)}(0,l)}$$

$$c_0 l \Big(1 + C l^{p^+-1} \|y'\|^{p^+-1}_{L^{p(\cdot)}(0,l)}\Big) \|v'\|_{L^{p(\cdot)}(0,l)}.$$

For the third integral by use of Young's inequality, it follows

$$\int_0^l \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)-1} \Big|\frac{v}{x^\alpha (l-x)^\alpha}\Big|\,dx$$

$$\int_0^l \frac{a(x)}{q'(x)} \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)}\,dx + \int_0^l \frac{a(x)}{q(x)} \Big|\frac{v}{x^\alpha (l-x)^\alpha}\Big|^{q(x)}\,dx = i_1 + i_2$$

For every summand here we have the inequalities

$$i_1 \le \int_0^l \frac{K^{q(x)}}{q^-} \Big(\frac{y_+}{K x^\alpha (l-x)^\alpha}\Big)^{q(x)}\,dx \le \frac{1 + K^{q^+}}{q^-} \int_0^l \Big(\frac{y_+}{K x^\alpha (l-x)^\alpha}\Big)^{q(x)}\,dx$$

$$\le \frac{1+K^{q^+}}{q^-}\Big(1+\big(\varepsilon\|y'\|_{L^{p(\cdot)}(0,l)}+C_\varepsilon\|y\|_{L^{p(\cdot)}(0,l)}\big)^{q^+-1}\Big)$$
$$\le \frac{1}{q^-}+\frac{1}{q^-}\big(\varepsilon\|y'\|_{L^{p(\cdot)}(0,l)}+C_\varepsilon l^2\Big\|\frac{y}{x(l-x)}\Big\|_{L^{p(\cdot)}(0,l)}\big)^{q^+-1}$$
$$\le \frac{1}{q^-}+\frac{1}{q^-}\big(\varepsilon+C_2C_\varepsilon l\big)^{q^+-1}\|y'\|^{q^+-1}_{L^{p(\cdot)}(0,l)}\le C_3\big\|y\big\|_{\dot W^1_{p(\cdot)}(0,l)}$$

with
$$K=\Big\|\frac{y_+}{x^\alpha(l-x)^\alpha}\Big\|_{L^{q(\cdot)}(0,l)},\quad \varepsilon>0.$$

Notice, here it has been used the inequality
$$\big\|y\big\|_Y\le\varepsilon\big\|y\big\|_X+C_\varepsilon\big\|y\big\|_Z \tag{11}$$
for a triple Banach spaces $Y\subset X\subset Z$ with the imbedding $Y\subset\subset X$ to be compactly [10] and Theorem 1 and Theorem 2.

Same chain of inequalities hold for the i_2 too.

The Gatox derivative of $I(y)$ and its continuity.

Show that the functional $I(u)$ has a continuous Gatox derivative $I'(u)\in E^*$ and for every $v\in E$ it holds
$$<\ I'(u),v>=\int_0^l|y'|^{p(x)-2}y'v'\ dx-\lambda\int_0^l|y|^{p(x)-2}yv\,dx$$
$$-\int_0^l a(x)\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-2}\frac{y_+}{x^\alpha(l-x)^\alpha}\cdot\frac{v}{x^\alpha(l-x)^\alpha}\,dx. \tag{12}$$

Derivatives of $J(y)$. For a functional $J(u)=\int_0^l|y'|^{p(x)}\,dx$ number r, a function $v\in E$ using the mean value theorem, and Lebesgue's limit theorem tending $r\to0$, it follows
$$\frac{J\left(y+rv\right)-J(y)}{r}=\int_0^l\frac{1}{p(x)}\frac{1}{r}\Big(|y'(x)+rv'(x)|^{p(x)}-|y'(x)|^{p(x)}\Big)\,dx$$
$$=\int_0^l|y'(x)+\theta rv'(x)|^{p(x)-2}y'(x)v'(x)\,dx\to\int_0^l|y'(x)|^{p(x)-2}y'(x)v'(x)\,dx, \tag{13}$$
where $\theta\in(0,1)$ depends on $x,y(x)$.

We have used that $|y'(x)+\theta rv'(x)|^{p(x)-2}\to|y'(x)|^{p(x)-2}$ as $r\to0$ a.e. $x\in(0,l)$. We have also used that there exists an integrable majorant function for all $r\in(-1,1)$ in order to apply the Legesgue theorem:
$$\Big||y'(x)+\theta rv'(x)|^{p(x)-2}y'(x)v'(x)\Big|$$

$$\leq \Big(|y'(x)|+|r||v'(x)|\Big)^{p(x)-2}\Big(\frac{|y'|+|v'|}{2}\Big)^2$$

$$\leq \Big(|y'(x)|+|r||v'(x)|\Big)^{p(x)} \leq 2^{p(x)-1}\Big(|y'(x)|^{p(x)}+|r|^{p(x)}|v'(x)|^{p(x)}\Big).$$

Therefore, the upper passage to the limit in (13) is legitimately.

The continuity of derivatives $J(y)$. Show $J \in C^1(E, E^*)$. Let $y_n \to y$ in E. Then for a $v \in E$ we have

$$|< J'(y_n) - J'(y), v >| = \Big|\int\limits_0^1 \big(|y_n'|^{p(x)-2}y_n' - |y'|^{p(x)-2}y'\big)v'\,dx\Big|$$

Using Egorov's theorem, there is a set $A \subset (0,l)$ with $|A| < \delta$ such that $y_n' \to y'$ uniformly in $(0,l) \setminus A$. Let $N(\varepsilon) \in N$ be such that $|y_n'(x) - y'(x)| < \varepsilon, \quad x \in (0,l) \setminus A$ as $n > N(\varepsilon)$. Then

$$|< J'(y_n) - J'(y), v >| \leq \int\limits_{(0,1)\setminus A} \big||y_n'|^{p(x)-2}y_n' - |y'|^{p(x)-2}y'\big||v'|\,dx$$

$$+\int\limits_A \big||y_n'|^{p(x)-1} + |y'|^{p(x)-1}\big||v'|\,dx$$

$$\leq C\varepsilon\|v\|_{\dot W^1_{p(\cdot)}(0,l)} + c_0\|v\|_{\dot W^1_{p(\cdot)}(0,l)}\Big(\|y_n'\|^{p^+}_{L_{p(\cdot)}(A)} + \|y'\|^{p^-}_{L_{p(\cdot)}(A)}\Big)$$

Therefore and since $y_n' \to y'$ in $L_{p(\cdot)}(0,1)$,

$$\|J(y_n) - J(y)\|_{E^*} \leq C\varepsilon + c_0\|y_n'\|_{L_{p(\cdot)}(A)} + c_0\|y'\|_{L_{p(\cdot)}(A)}$$

$$\leq (C+1)\varepsilon + 2c_0\|y'\|_{L_{p(\cdot)}(A)} < \epsilon$$

choosing sufficiently small $\delta > 0$ and ε.

Derivatives of $F(y)$. For a functional

$$F(y) = \int\limits_0^l y_+^{p(x)}\,dx, \quad \text{where} \quad y_+(x) = \max\{y(x), 0\},$$

show that

$$< F'(y), v >= \int\limits_0^l y_+^{p(x)-2}y_+ v\,dx.$$

By the same way, as above,

$$\frac{F\,(y+rv) - F(y)}{r} = \int\limits_0^l \frac{1}{p(x)} \cdot \frac{(y+rv)_+^{p(x)} - y_+^{p(x)}}{r}\,dx$$

$$= \int_0^l \zeta_+^{p(x)-1} v\,dx \to \int_0^l y_+^{p(x)-1} v\,dx \quad \text{as} \quad r \to 0,$$

where ζ is a number between y_+ and $(y+rv)_+$.

Continuity of derivatives of $F(y)$. To show $F \in C^1(E, E^*)$ let $y_n \to y$ in E. From Theorem 2 it follows $y_n \to y$ in $L^{p(\cdot)}(0,l)$. For a fixed $v \in E$ we have

$$| < F'(y_n) - F'(y), v > | = \Big| \int_0^l \Big((y_n)_+^{p(x)-1} - y_+^{p(x)-1} \Big) v\,dx \Big|$$

Since $y_n \to y$ in $L^{p(\cdot)}(0,l)$ there exists a subsequence y_{n_k} converging y almost everywhere in $(0,l)$. Denote it again y_n. Using Egorov's theorem there exists a set $|A| < \delta$ with any small $\delta > 0$, such that, the convergence y_n to y is uniformly on $(0,l) \setminus A$.

Then since $|(y_n)_+ - y_+| \le |y_n - y|$, it follows

$$| < F'(y_n) - F'(y), v > | = \Big| \int_{(0,1)\setminus A} \Big((y_n)_+^{p(x)-1} - y_+^{p(x)-1} \Big) v\,dx \Big|$$

$$+ \Big| \int_A \Big((y_n)_+^{p(x)-1} - y_+^{p(x)-1} \Big) v\,dx \Big|$$

$$\le \varepsilon \int_{(0,1)\setminus A} |v|\,dx + \int_A (y_n)_+^{p(x)-1} |v|\,dx + \int_A y_+^{p(x)-1} |v|\,dx$$

Applying Holder's inequality here one gets

$$| < F'(y_n) - F'(y), v > |$$

$$\le \Big(C\varepsilon + \|(y_n)_+^{p(x)-1}\|_{L^{p'(\cdot)}(A)} + \|y_+^{p(x)-1}\|_{L^{p'(\cdot)}(A)} \Big) \|v\|_{L^{p(\cdot)}(0,l)} \tag{14}$$

Applying for any $g \in L^{p(\cdot)}$ the inequality

$$\|g^{p(\cdot)-1}\|_{L^{p'(\cdot)}} \le \|g\|_{L^{p(\cdot)}}^{p^+-1} + \|g\|_{L^{p(\cdot)}}^{p^--1}$$

in the right hand side (14) one gets

$$| < F'(y_n) - F'(y), v > |$$

$$\Big((C+1)\varepsilon + 3\|y\|_{L^{p(\cdot)}(A)}^{p^--1} \Big) \|v\|_{L^{p(\cdot)}(0,l)}$$

Choosing sufficiently small $\delta > 0$ and applying inequality (11) this is exceeded

$$(C+2)C_1\varepsilon \|v\|_E.$$

Hence
$$\|F(y_n) - F(y)\|_{E^*} \leq (C+2)C_1\varepsilon,$$
which proves the continuity of derivative of functional F.

Derivatives of $G(y)$. By the same way, find the Gatox derivative of the functional
$$G(u) = \int_0^l \frac{a(x)}{q(x)} \Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)} dx$$
in E and show its continuity. Show that
$$< G'(y), v >= \int_0^l a(x) \Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1} \cdot \frac{v}{x^\alpha(l-x)^\alpha}\, dx. \tag{15}$$
By the same way, as above,
$$\frac{G\,(y+rv) - G(y)}{r} = \int_0^l \frac{a(x)}{q(x)} \cdot \frac{1}{r}\Big(\Big(\frac{(y+rv)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)} - \Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)}\Big) v\, dx$$
$$= \int_0^l \frac{a(x)}{q(x)} \cdot \frac{1}{x^\alpha(l-x)^\alpha}\Big(\frac{(y+rv)_+^{q(x)} - y_+^{q(x)}}{r}\Big) v\, dx$$
Using the mean value formula this equals
$$\int_0^l a(x) \cdot \frac{1}{x^\alpha(l-x)^\alpha} \theta^{q(x)-1} v\, dx,$$
where θ is a quantity ranged between y_+ and $(y+rv)_+$. Tending $r \to 0$ and applying Lebesgue convergence theorem from this one gets (15). For this, it has been used that $a \in L^\infty$ and $v, \theta \in L^{q(\cdot)}(0,l)$. The last inclusion follows from Holder's inequality and Theorem 2:
$$\|\theta\|_{L^{q(\cdot)}(0,l)} \leq \|(y+rv)_+\|_{L^{q(\cdot)}(0,l)} + \|y_+\|_{L^{q(\cdot)}(0,l)}$$
$$\leq 2\|y\|_{L^{q(\cdot)}(0,l)} + r\|v\|_{L^{q(\cdot)}(0,l)}$$
$$\leq 2l^{2\alpha}\Big\|\frac{y}{x^\alpha(l-x)^\alpha}\Big\|_{L^{q(\cdot)}(0,l)} + rl^{2\alpha}\Big\|\frac{v}{x^\alpha(l-x)^\alpha}\Big\|_{L^{q(\cdot)}(0,l)}.$$
Applying the compact embedding result from Theorem 2 by using inequality (11) from here we get
$$\|\theta\|_{L^{q(\cdot)}(0,l)} \leq \varepsilon 2l^{2\alpha} C_1\Big(\|y'\|_{L^{p(\cdot)}(0,l)} + r\|v'\|_{L^{p(\cdot)}(0,l)}\Big) +$$
$$C_\varepsilon 2l^{2\alpha}\Big(\|y\|_{L^{p(\cdot)}(0,l)} + 2l^{2\alpha} r\|v\|_{L^{p(\cdot)}(0,l)}\Big).$$

This guaranties the limiting prosses using Lebesgue Theorem.

Continuity of derivatives of $G(y)$. Show the continuity of derivative of the functional G. Let $y_n \to y$ in E. Show that $G'(y_n) \to G'(y)$ in E^*. In this way, let $v \in E$ be any function.

We have

$$| < G'(y_n) - G'(y), v > |$$

$$= \Big| \int_0^l a(x) \Big(\Big(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \Big)^{q(x)-1} - \Big(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \Big)^{q(x)-1} \Big) \cdot \frac{v}{x^\alpha (l-x)^\alpha} \, dx \Big|$$

As the preceding estimates since $|(y_n)_+ - y_+| \le |y_n - y|$, we have

$$| < G'(y_n) - G'(y), v > | = \Big| \int_{(0,1)\setminus A} \frac{|a(x)|}{(x^\alpha (l-x)^\alpha)^{q(x)}} \Big((y_n)_+^{q(x)-1} - y_+^{q(x)-1} \Big) v \, dx \Big|$$

$$+ \Big| \int_A \frac{|a(x)|}{(x^\alpha (l-x)^\alpha)^{q(x)}} \Big((y_n)_+^{q(x)-1} - y_+^{q(x)-1} \Big) v \, dx \Big|$$

$$\le \varepsilon \int_{(0,1)\setminus A} |a(x)| \cdot \frac{|v|}{x^\alpha (l-x)^\alpha} \, dx + \int_A \frac{|a(x)| \, (y_n)_+^{q(x)-1} |v|}{(x^\alpha (l-x)^\alpha)^{q(x)}} \, dx$$

$$+ \int_A \frac{|a(x)| \, y_+^{q(x)-1} |v|}{(x^\alpha (l-x)^\alpha)^{q(x)}} \, dx$$

(we have included a little neighborhoods of origin and l to the set A).

Applying Holder's inequality in the preceding inequality, one gets

$$| < G'(y_n) - G'(y), v > |$$

$$\le \Big[C\varepsilon + \Big\| \Big(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \Big)^{q(x)-1} \Big\|_{L^{q'(\cdot)}(A)}$$

$$+ \Big\| \Big(\frac{y_+}{x^\alpha (l-x)^\alpha} \Big)^{q(x)-1} \Big\|_{L^{q'(\cdot)}(A)} \Big] \cdot \Big\| \frac{v}{x^\alpha (l-x)^\alpha} \Big\|_{L^{q(\cdot)}(0,l)} \tag{16}$$

Applying in the case $g(x) = \Big(\frac{y_+(x)}{x^\alpha (l-\alpha)} \Big)^{q(x)-1}$ and $p(x) = q(x)$ the inequality

$$\|g^{p(\cdot)-1}\|_{L^{p'(\cdot)}} \le \|g\|_{L^{p(\cdot)}}^{p^+-1} + \|g\|_{L^{p(\cdot)}}^{p^--1}$$

in the right hand side (16) one gets

$$| < G'(y_n) - G'(y), v > |$$

$$\le \Big((C+1)\varepsilon + 3 \Big\| \frac{y}{x^\alpha (l-x)^\alpha} \Big\|_{L^{q(\cdot)}(A)}^{q^- - 1} \Big) \Big\| \frac{v}{x^\alpha (l-x)^\alpha} \Big\|_{L^{q(\cdot)}(0,l)}$$

Choosing sufficiently small $\delta > 0$ and applying inequality (11) this is exceeded

$$(C+2)C_1\varepsilon\|v\|_E.$$

This entails

$$\|G'(y) - G'(y_n)\|_{E^*} \leq C_1\varepsilon,$$

which proves the continuity of functional G'.

Weak lower semi continuity of $I(y)$.

Lower semi continuity of $J(y)$. First show the weak lower semi continuity (w.l.s.c.) of $J(y)$. (In order to show this, some people use the fact from [5] asserting that a convex functional is w.l.s.c. if it is a strongly lower semi continuous).

Show that $J(y)$ is convex in E. For any $\theta \in (0,1)$ and $y, z \in E$ we have

$$J\big(\theta y + (1-\theta)z\big) = \int_0^l \big|\theta y'(x) + (1-\theta)z'(x)\big|^{p(x)}\,dx,$$

by convexity of the function x^p,

$$\leq \theta \int_0^l \big|y'(x)\big|^{p(x)} + (1-\theta)\int_0^l \big|z'(x)\big|^{p(x)}\,dx$$

To show the strong lower semi continuity of $J(y)$ in E set $y_n \to y$. We have

$$\int_0^l |y_n'|^{p(x)}\,dx - \int_0^l |y'|^{p(x)}\,dx = \int_0^l \frac{d}{dt}|y' + t(y_n' - y')|^{p(x)}\,dx$$

$$= \int_0^l p(x)|y' + t(y_n' - y')|^{p(x)-2}\big(y' + t(y_n' - y')\big)(y_n' - y')\,dx$$

$$= \int_0^l p(x)\Big(|y' + t(y_n' - y')|^{p(x)-2}\big(y' + t(y_n' - y')\big) - |y'|^{p(x)-2}y'\Big)\big(y' + t(y_n' - y') - y'\big)\,\frac{dx}{t}$$

$$+ \int_0^l |y'|^{p(x)-2}y'(y_n' - y')\,dx,$$

since the first integral is positive by the convexity it holds an inequality, $\big(|a|^{p-2}a - |b|^{p-2}\big)(a-b) \geq 0$, for any $a, b \in \mathbb{R}$ that entails $|b|^{p-2}b \geq |a|^{p-2}a + p|a|^{p-2}a(b-a)$, therefore,

$$\geq \int_0^l |y'|^{p(x)-2}y'(y_n' - y')\,dx.$$

Now, it remains to take a limit in the preseeding inequality, in order to show that $J(y)$ is weakly lower semi continues in E :

$$\liminf_{n\to\infty}\int_0^l |y_n'|^{p(x)}\,dx \geq \int_0^l |y'|^{p(x)}\,dx + \liminf_{n\to\infty}\int_0^l |y'|^{p(x)-2}y'(y_n'-y')\,dx$$

$$\geq \int_0^l |y'|^{p(x)}\,dx,$$

i.e.

$$\liminf_{n\to\infty} J(y_n) \geq J(y)$$

Lower semi continuity of $I(y)$. Let $\{y_n\}\subset E$ be a weakly convergent subsequence of E tending to $y\in E$, i.e. $y_n \rightharpoonup y$. Show that $\liminf\limits_{n\to\infty} I(y_n)\geq I(y)$. By Theorem 1 the space E compactly imbedded into the class (6). By this, there exists a subsequence y_{n_k} that converges strongly to y in the norm $\left\|\Big(x(l-x)\Big)^{-\alpha}\cdot\right\|_{L^{q(\cdot)}(0,l)}$. and $\|\|_{L^{p(\cdot)}(0,l)}$ This means

$$\liminf_{n\to\infty} I(y_{n_k}) = \liminf_{n\to\infty}\int_0^l \frac{1}{p(x)}\,|y_{n_k}'|^{p(x)}\,dx$$

$$-\lim_{n\to\infty}\int_0^l \frac{\lambda}{p(x)}\,|(y_{n_k})_+|^{p(x)}\,dx - \lim_{n\to\infty}\int_0^l \frac{a(x)}{q(x)}\left|\frac{(y_{n_k})_+}{x^\alpha(l-x)^\alpha}\right|^{q(x)}dx$$

$$\geq \int_0^l \frac{1}{p(x)}\,|y'|^{p(x)}\,dx - \int_0^l \frac{\lambda}{p(x)}\,|y_+|^{p(x)}\,dx$$

$$-\int_0^l \frac{a(x)}{q(x)}\left|\frac{y_+}{x^\alpha(l-x)^\alpha}\right|^{q(x)}dx = I(y)$$

Therefore,

$$\liminf_{n\to\infty} I(y_{n_k}) \geq I(y),$$

that proves lower semi continuity of $I(y)$.

Palas-Smale condition (PS). Recall the notion of PS -condition.

Let $\{y_n\}\subset E$ be a sequence such that

1) $I(y_n)$ is bounded ;

2) $I'(y_n)\to I'(y)$ in E^*.

Then there exists a subsequence y_{n_k} that converges to y strongly in E. Since $I(y_n)$ is bounded, we may assume that $I(y_{n_k}) \to c$ by some real number $c \in R$. To save simplicity, denote y_{n_k} as y_n.

Boundedness of y_n' in E . From condition 1) it follows that there exists an $M > 0$ not depending on n such that $|I(y_n)| \le M$, i.e.

$$\int_0^l \frac{1}{p(x)} \left(|y_n'|^{p(x)} - \lambda (y_n)_+^{p(x)} \right) dx - \int_0^l \frac{a(x)}{q(x)} \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \right)^{q(x)} dx \le M,$$

or

$$\int_0^l \frac{a(x)}{q(x)} \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \right)^{q(x)} dx \ge \int_0^l \frac{1}{p(x)} \left(|y_n'|^{p(x)} - \lambda (y_n)_+^{p(x)} \right) dx - M.$$

Then by assumption $\lambda_1 > 0$ it follows that

$$\int_0^l a(x) \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \right)^{q(x)} dx \ge \frac{q^-}{p^+} \int_0^l |y_n'|^{p(x)} dx - \int_0^l \frac{\lambda q^-}{p^+} (y_n)_+^{p(x)} dx - M q^-. \tag{17}$$

On other hand, from condition 2) it follows that

$$| < I'(y_n), v > | \le o(1) \|v\|_{W^1_{p(\cdot)}(0,l)},$$

i.e.

$$\int_0^l |y_n'|^{p(x)-2} y_n' v' \, dx - \lambda \int_0^l (y_n)_+^{p(x)-1} v \, dx$$

$$- \int_0^l a(x) \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \right)^{q(x)-1} \cdot \frac{v}{x^\alpha (l-x)^\alpha} \, dx$$

$$= o(1) \|v'\|_{L^{p(\cdot)}(0,l)}.$$

Inserting here $v = y_n$ this yields

$$\int_0^l |y_n'|^{p(x)} dx - \lambda \int_0^l (y_n)_+^{p(x)} dx$$

$$- \int_0^l a(x) \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \right)^{q(x)} dx = o(1) \|y_n'\|_{L^{p(\cdot)}(0,l)}$$

or

$$\int_0^l a(x) \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \right)^{q(x)} dx \le \int_0^l \left(|y_n'|^{p(x)} - \lambda (y_n)_+^{p(x)} \right) dx$$

$$+o(1)\|y_n'\|_{L^{p(\cdot)}(0,l)}. \tag{18}$$

From (18) and (17) and the assumption $q^- > p^+$ it follows that

$$\Big(\frac{q^-}{p^+}-1\Big)\int\limits_0^l |y_n'|^{p(x)}\,dx \le \lambda\Big(\frac{q^-}{p^+}-1\Big)\int\limits_0^l (y_n)_+^{p(x)}\,dx + Mq^- + o(1)\|y_n'\|_{L^{p(\cdot)}(0,l)}$$

or

$$\int\limits_0^l |y_n'|^{p(x)}\,dx \le \lambda\int\limits_0^l (y_n)_+^{p(x)}\,dx + \frac{Mq^-p^+}{q^--p^+} + o(1)\|y_n'\|_{L^{p(\cdot)}(0,l)}$$

Now assuming $\lambda < \lambda_1$ and a strong positivity of the first eigenvalue λ_1 in (2), (3) from this it follows

$$\int\limits_0^l |y_n'|^{p(x)}\,dx \le O(1).$$

The bounded ness of y_n in E has been proved.

Now, after establishment of the bounded ness $\{y_n\}$ in E, we may apply the the weak convergence for some subsequence $\{y_{n_k}\}$. Moreover, show the strong convergence $y_n \to y$ in E. Remaining the notation y_n in place of y_{n_k}, the weak convergence $y_n \to y$ in E, we have the equality for PS-sequence:

$$\begin{aligned}&\int\limits_0^l |y_n'|^{p(x)-2}y_n'v'\,dx - \lambda\int\limits_0^l (y_n)_+^{p(x)-1}v\,dx\\ &-\int\limits_0^l a(x)\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\cdot\frac{v}{x^\alpha(l-x)^\alpha}\,dx\\ &= o(1)\|v'\|_{L^{p(\cdot)}(0,l)}.\end{aligned} \tag{19}$$

Inserting in (19) $v = y_n - y$, we get

$$\begin{aligned}&\int\limits_0^l |y_n'|^{p(x)-2}y_n'(y_n'-y')\,dx - \lambda\int\limits_0^l (y_n)_+^{p(x)-1}(y_n-y)\,dx\\ &-\int\limits_0^l a(x)\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\cdot\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx\\ &= o(1)\|y_n'-y'\|_{L^{p(\cdot)}(0,l)}.\end{aligned} \tag{20}$$

From (20), we easily get

$$\int_0^l \Big(|y_n'|^{p(x)-2}y_n' - |y'|^{p(x)-2}y'\Big)(y_n'-y')\,dx + \int_0^l |y'|^{p(x)-2}y'(y_n'-y')$$

$$= \lambda \int_0^l \big((y_n)_+^{p(x)-1} - y_+^{p(x)-1}\big)(y_n-y)\,dx + \lambda \int_0^l y_+^{p(x)-1}(y_n-y)$$

$$+ \int_0^l a(x)\,\Big[\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1} - \Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\Big]\cdot \frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx$$

$$+ \int_0^l a(x)\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx$$

$$+o(1)\|y_n'-y'\|_{L^{p(\cdot)}(0,l)}. \tag{21}$$

Now, since $y_n \to y$ weakly in E , we see that the additional terms in (21) tend to zero: those are

$$\lim_{n\to\infty}\int_0^l |y'|^{p(x)-2}y'(y_n'-y') = 0 \tag{22}$$

that is implied from the fact that for $y \in E$ it is $|y'|^{p(x)-2}y' \in E^*$ (that is $|y'|^{p(x)-2}y' \in L^{p'}$).

The convergence

$$\lim_{n\to\infty}\int_0^l y_+^{p(x)-1}(y_n-y) = 0 \tag{23}$$

follows from the fact that $y_+^{p(x)-1} \in E^*$, and $y_n \to y$ weakly in E since

$$\Big|\int_0^l y_+^{p(x)-1}(y_n-y)\,dx\Big| \le C(l)\int_0^l \big(\frac{y_+}{x(l-x)}\big)^{p(x)-1}\big|\frac{y_n-y}{x(l-x)}\big|\,dx, \tag{24}$$

where $C(l) = l^2 \max\left\{l^{p^+-1},\, l^{p^--1}\right\}$. Applying inequality (7) to the expression (24) we find that is exceeded

$$\le C(l)\Big\|\frac{y_n-y}{x(l-x)}\Big\|_{L^{p(\cdot)}}\Big\|\frac{y_+^{p(x-1)}}{x(l-x)}\Big\|_{L^{p'(\cdot)}}$$

$$\le C_2^2 C(l)\big\|y_n'-y'\big\|_{L^{p(\cdot)}}\Big(\|y'\|_{L^{p(\cdot)}}^{p^+-1} + \|y'\|_{L^{p(\cdot)}}^{p^--1}\Big) \le C_3\big\|y_n'-y'\big\|_E.$$

The convergence

$$\lim_{n\to\infty}\int\limits_0^l a(x)\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx=0 \tag{25}$$

follows from the fact that $\frac{a(x)}{x^\alpha(l-x)^\alpha}\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\in E^*$, since

$$\Big|\int\limits_0^l a(x)\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx\Big|$$

$$\leq C_1(l)\|a\|_{L^\infty}\cdot\Big\|\frac{y_n-y}{x(l-x)}\Big\|_{L^{q(\cdot)}}\cdot\Big\|\Big(\frac{y_+}{x(l-x)}\Big)^{q(x)-1}\Big\|_{L^{q'(\cdot)}}$$

$$\leq C_1(l)C_2^2\|a\|_{L^\infty}\cdot\Big\|y_n'-y'\Big\|_{L^{p(\cdot)}}\cdot\Big(\|y'\|^{q^+-1}_{L^{p(\cdot)}}+\|y'\|^{q^--1}_{L^{p(\cdot)}}\Big)\leq C_4\big\|y_n-y\big\|_E,$$

where $C_1(l)=\max\big\{l^{2(1-\alpha)q^+},\, l^{2(1-\alpha)q^-}\big\}$. Applying the limits (22), (23), (25) it follows from (21) that

$$\int\limits_0^l\Big(|y_n'|^{p(x)-2}y_n'-|y'|^{p(x)-2}y'\Big)(y_n'-y')\,dx$$

$$=\lambda\int\limits_0^l\big((y_n)_+^{p(x)-1}-y_+^{p(x)-1}\big)(y_n-y)\,dx$$

$$+\int\limits_0^l a(x)\,\Big[\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}-\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\Big]\cdot\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx$$

$$+o(1)\|y_n'-y'\|_{L^{p(\cdot)}(0,l)}+o(1). \tag{26}$$

We need the following two inequalities for $a,b\in R$ (see e.g., [**?**] in the case of n -dimensional vectors)

$$\Big(|a|^{p-1}a-|b|^{p-2}\Big)(a-b)\geq\gamma_1(p)|a-b|^p\quad\text{if}\qquad p\geq 2,$$
$$\Big(|a|^{p-1}a-|b|^{p-2}\Big)(a-b)\geq\gamma_2(p)\frac{|a-b|^2}{\Big(|a|+|b|\Big)^{2-p}}\quad\text{if}\quad p\leq 2. \tag{27}$$

In order to finish the proof of convergence $y_n\to y$ in E , we shall use Egorov's theorem in order to show a convergence to zero of the first summand in the right hand side (26), and compact imbedding theorem, to show the convergence of second summand.

For $\lambda \geq 0$, $p \geq 2$ using (27) it follows from (21) that

$$\gamma_1(p) \int\limits_0^l |y_n' - y'|^{p(x)}\,dx \leq \lambda \int\limits_0^l \big((y_n)_+^{p(x)-1} - y_+^{p(x)-1}\big)(y_n - y)\,dx$$

$$+ \int\limits_0^l a(x)\left[\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1} - \Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\right]\cdot \frac{y_n - y}{x^\alpha(l-x)^\alpha}\,dx$$

$$+o(1)\|y_n' - y'\|_{L^{p(\cdot)}(0,l)} + o(1). \tag{28}$$

Using mean value theorem, the last integral (28) is estimated as

$$\Big|\int\limits_0^l a(x)\left[\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1} - \Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\right]\cdot \frac{y_n - y}{x^\alpha(l-x)^\alpha}\,dx\Big|$$

$$\leq (q^+ - 1)\|a(x)\|_{L^\infty} \cdot \int\limits_0^l \Big(\frac{y_n - y}{x^\alpha(l-x)^\alpha}\Big)^2 \cdot \frac{|y_n|^{q(x)-2} + |y|^{q(x)-2}}{\big(x^\alpha(l-x)^\alpha\big)^{q(x)-2}}\,dx$$

Further, applying Holder's inequality in the right hand side it is exceeded

$$\leq (q^+ - 1)\|a(x)\|_{L^\infty}\Big\|\frac{y_n - y}{x^\alpha(l-x)^\alpha}\Big\|^2_{L^{q(\cdot)}} \cdot \Big(\Big\|\frac{|y_n| + |y|}{x^\alpha(l-x)^\alpha}\Big\|_{L^{q(\cdot)}}\Big)^{q^+-2} \to 0 \tag{29}$$

as $n \to \infty$ by using the compact embedding E into the weighted class (6) in Theorem 2.

Using Egorov's theorem there exists a set $|A| < \delta$ with any small $\delta > 0$, such that the convergence y_n to y is uniformly on $A^c = (0,l) \setminus A$. Applying that, and the Holder inequality, we see

$$\int\limits_0^l \big((y_n)_+^{p(x)-1} - y_+^{p(x)-1}\big)(y_n - y)\,dx$$

$$\leq \varepsilon \int\limits_{A^c} \big((y_n)_+^{p(x)-1} + y_+^{p(x)-1}\big)\,dx \tag{30}$$

$$+ \int\limits_A \big(|y_n|^{p(x)} + |y_n|^{p(x)-2}|y|^2 + |y_n|^2|y|^{p(x)-2} + |y|^{p(x)}\big)\,dx < (M+4)\varepsilon$$

choosing sufficiently small $\delta > 0$ and large n.

Inserting in (28) the estimates (30), (29) we get the strong convergence $y_n \to y$ in E for the case $p(x) \geq 2$.

It remains to consider the case $p < 2$. Inserting the second inequality (27) in (28) and applying the Holder inequality, we get

$$\gamma_2(p)\int_0^l \frac{|y_n'-y'|^2}{\left(|y_n'|+|y'|\right)^{2-p}}\,dx \le \lambda\int_0^l \left((y_n)_+^{p(x)-1}-y_+^{p(x)-1}\right)(y_n-y)\,dx$$

$$+\int_0^l a(x)\left[\left(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}-\left(\frac{y_+}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}\right]\cdot\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx \tag{31}$$

$$+o(1)\|y_n'-y'\|_{L^{p(\cdot)}(0,l)}+o(1)$$

The second integral in the right hand side (31) is estimated as

$$\left|\int_0^l a(x)\left[\left(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}-\left(\frac{y_+}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}\right]\cdot\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx\right|$$

$$\le \|a(x)\|_{L^\infty}\cdot\int_0^l\left(\frac{y_n-y}{x^\alpha(l-x)^\alpha}\right)\cdot\frac{|y_n|^{q(x)-1}+|y|^{q(x)-1}}{\left(x^\alpha(l-x)^\alpha\right)^{q(x)-1}}\;dx$$

On base of Holder's inequality

$$\le\left\|\frac{y_n-y}{x^\alpha(l-x)^\alpha}\right\|_{L^{q(\cdot)}}\cdot\left[\left\|\frac{y_n}{x^\alpha(l-x)^\alpha}\right\|_{L^{q(\cdot)}}^{q^+-1}+\left\|\frac{y}{x^\alpha(l-x)^\alpha}\right\|_{L^{q(\cdot)}}^{q^+-1}\right]\to 0$$

as $n\to\infty$ on base of compactness Theorem 2.

By the same way, it is not difficult to show that

$$\int_0^l\left((y_n)_+^{p(x)-1}-y_+^{p(x)-1}\right)(y_n-y)\;dx\to 0\quad\text{as}\quad n\to\infty$$

Therefore,

$$\int_0^l\frac{|y_n'-y'|^2}{\left(|y_n'|+|y'|\right)^{2-p}}\;dx=o(1)\quad\text{as}\quad n\to\infty.$$

Applying it Holder's inequality, we get

$$\|y_n'-y'\|_{L^{p(\cdot)}}^4\le c_0\left(\int_0^l\frac{|y_n-y|^2}{(|y_n|+|y|)^{2-p}}\;dx\right)\;\left(\|y_n\|_{L^{p(\cdot)}}+\|y_n\|_{L^{p(\cdot)}}\right)^{2-p^+}=o(1).$$

as $n\to\infty$.

This proves the PS-property of the functional $I(y)$. Now, we are ready to the application of Mountain pass theorem in order to get an existence result for the problem (1).

Mountain pass theorem. Let y be a fixed function in E. Inserting ty in place of y we see that

$$I(ty) = \int_0^l \frac{t^{p(x)}}{p(x)} |y'|\, dx - \lambda \int_0^l \frac{t^{p(x)}}{p(x)} y_+^{p(x)}\, dx - \int_0^l \frac{t^{q(x)}}{q(x)} \frac{y_+^{q(x)}}{\left(x^\alpha(l-x)^\alpha\right)^{q(x)}}\, dx$$

For sufficiently large $t > 0$ we have the estimation

$$I(ty_0) \leq \frac{t^{p^+}}{p^-} \int_0^l |y'|^{p(x)}\, dx - \lambda \frac{t^{p^-}}{p^+} \int_0^l y_+^{p(x)}\, dx - \frac{t^{q^-}}{q^+} \int_0^l \frac{y_+^{q(x)}}{\left(x^\alpha(l-x)^\alpha\right)^{q(x)}}\, dx$$

Using the condition $q^- > p^+$ from this it follows $I(y) < 0$ for sufficiently large $t > 0$.

On other hand, $I(y) > 0$ for sufficiently small norm $\|y'\|_{L^{p(\cdot)}}$. Indeed, for such $y \in E$ it holds the estimates

$$I(y) \geq C_1 \int_0^l \Big(\frac{|y'|}{\|y'\|_{L^{p(\cdot)}}}\Big)^{p(x)} \|y'\|_{L^{p(\cdot)}}^{p(x)}\, dx - \int_0^l \frac{1}{q^-} \Big(\frac{y_+^{q(x)}}{N x^\alpha (l-x)^\alpha}\Big)^{q(x)} N^{q(x)}\, dx$$

$$\geq C_1 \|y'\|_{L^{p(\cdot)}}^{p^+} \int_0^l \Big(\frac{|y'|}{\|y'\|_{L^{p(\cdot)}}}\Big)^{p(x)} dx - \frac{N^{q^+}}{q^-} \int_0^l \Big(\frac{y_+^{q(x)}}{x^\alpha (l-x)^\alpha}\Big)^{q(x)} dx$$

$$\geq C_1 \|y'\|_{L^{p(\cdot)}}^{p^+} - \frac{N^{q^+}}{q^-},$$

where $N = \|\frac{y_+}{x^\alpha(l-x)^\alpha}\|_{L^{q(\cdot)}}$, using the Theorem 2, $N \leq C\|y'\|_{L^{p(\cdot)}}$,

$$\geq C_1 \|y'\|_{L^{p(\cdot)}}^{p^+} - \frac{1}{q^-} \|y'\|_{L^{p(\cdot)}}^{q^-} \geq \frac{C_1}{2} \|y'\|_{L^{p(\cdot)}}^{p^+}$$

choosing $\|y'\|_{L^{p(\cdot)}} = \left(\frac{q^- C_1}{2}\right)^{\frac{1}{q^- - p^+}}$.

Therefore, all conditions of Mountain pass theorem is satisfied by the sphere $\|y\|_E = \rho$ with $\rho = \left(\frac{q^- C_1}{2}\right)^{\frac{1}{q^- - p^+}}$. Then there exists a point $y_0 \in E$ such that $I(\hat{y}) = c = \inf I(y)$ and $c = \inf \sup I(y)$ and such that $I'(\hat{y}) = 0$, i.e. for any $v \in E$ it holds

$$0 =< I'(\hat{y}), v >= \int_0^l |\hat{y}'|^{p(x)-2} \hat{y}' v'\, dx - \lambda \int_0^l \hat{y}_+^{p(x)-1} v\, dx$$

$$-\int_0^l a(x)\left(\frac{\hat{y}_+}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}\cdot\frac{v}{x^\alpha(l-x)^\alpha}\,dx.$$

that is a solution of the problem (1). It remains to show that y_0 is positive. Insert in the preceding equality $v = y_- := (-y)_+$. Then

$$0=\int_0^l |\hat{y}'|^{p(x)-2}\hat{y}'\hat{y}_-'\,dx-\lambda\int_0^l \hat{y}_+^{p(x)-1}\hat{y}_-\,dx$$

$$-\int_0^l a(x)\left(\frac{\hat{y}_+}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}\cdot\frac{\hat{y}_-}{x^\alpha(l-x)^\alpha}\,dx.$$

$$=\int_0^l |\hat{y}_-'|^{p(x)}\,dx$$

Therefore, $\hat{y}_- = 0$, i.e $\hat{y}$=0, then $\hat{y}$ is a positive solution of the problem (1).

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On the Boundedness of Commutators Dunkl-type Maximal Operator in the Dunkl-type Morrey Spaces

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Abstract. In this paper we consider the generalized shift operator, associated with the Dunkl operator and we investigate maximal commutators associated with the generalized shift operator. The boundedness of the Dunkl-type maximal commutator $M_{b,\alpha}$ from the Dunkl-type Morrey space $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{p,\lambda,\alpha}(\mathbb{R})$ for all $1 < p < \infty$ when $b \in BMO_\alpha(\mathbb{R})$ are proved.

Key Words and Phrases: commutator, generalized shift operator, Dunkl-type maximal function, Dunkl-type B–Morrey space, BMO_α space.

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1. Introduction

The Hardy–Littlewood maximal function, fractional maximal function and fractional integrals are important technical tools in harmonic analysis, theory of functions and partial differential equations. On the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In the works [2, 11, 21, 29] the maximal operator associated with the Dunkl operator on $\mathbb{R}$ were studied. Let b be a locally integrable function on $\mathbb{R}^n$ and T be a Calderon-Zygmund operator. The commutator is defined for smooth functions f as

$$[b, T]f = bT(f) - T(bf).$$

Coifman, Rochberg and Weiss [8] stated that $[b, T]$ is a bounded operator on $L_p(\mathbb{R}^n), 1 < p < \infty$, when b is a BMO function. Chanillo [7] proved that the commutators of the Riesz potentials characterize the function space BMO. In [10] proved that $b \in BMO(\mathbb{R}^n)$ if and only if the maximal commutator M_b is bounded from the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$. In this work, we study the maximal commutator (Dunkl-type Dunkl-type maximal commutator) associated with the Dunkl operator on $\mathbb{R}$. We obtain the necessary and sufficient conditions for the boundedness of the Dunkl-type maximal commutator.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball centered at x of radius r.

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Let $f \in L_1^{loc}(\mathbb{R}^n)$. The maximal operator M and the Riesz potential I^β are defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy.$$

The operator M play important role in real and harmonic analysis (see, for example [30]).

In the theory of partial differential equations Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ play an important role. They were introduced by C. Morrey in 1938 [26] and defined as follows: *For* $0 \le \lambda \le n$, $1 \le p < \infty$, $f \in \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ *if* $f \in L_p^{loc}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

If $\lambda = 0$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, if $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

Also by $W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak L_p-space.

F. Chiarenza and M. Frasca [9] studied the boundedness of the maximal operator M in Morrey spaces $\mathcal{M}_{p,\lambda}$. Their results can be summarized as follows:

Theorem A. *Let* $0 < \alpha < n$ *and* $0 \le \lambda < n$, $1 \le p < \infty$.

1) If $1 < p < \infty$, *then* M *is bounded from* $\mathcal{M}_{p,\lambda}$ *to* $\mathcal{M}_{p,\lambda}$.

2) If $p = 1$, *then* M *is bounded from* $\mathcal{M}_{1,\lambda}$ *to* $W\mathcal{M}_{1,\lambda}$.

2. Definitions, notation and preliminaries

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1}\, dx.$$

For every $1 \le p \le \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(\mathbb{R}, d\mu_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{L_{p,\alpha}} = \left(\int_{\mathbb{R}} |f(x)|^p\, d\mu_\alpha(x)\right)^{1/p} < \infty \quad \text{if} \quad p \in [1,\infty),$$

and

$$\|f\|_{\infty,\alpha} \equiv \|f\|_{L_\infty} = ess\sup_{x\in\mathbb{R}} |f(x)| \quad \text{if} \quad p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}(\mathbb{R})$, the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions f with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r \left(\mu_\alpha \{x \in \mathbb{R} \ : \ |f(x)| > r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \quad \text{and} \quad \|f\|_{WL_{p,\alpha}} \le \|f\|_{p,\alpha} \text{ for all } f \in L_{p,\alpha}(\mathbb{R}).$$

Let $B(x,t) = \{y \in \mathbb{R} : |y| \in]\max\{0, |x|-t\}, |x|+t[\ \}$ and $B_t \equiv B(0,t) =]-t, t[$, $t > 0$. Then

$$\mu_\alpha B_t = b_\alpha t^{2\alpha+2},$$

where $b_\alpha = \left[2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)\right]^{-1}$.

Let $M_\alpha^\sharp$ be the Dunkl-type sharp maximal function defined by

$$M_\gamma^\sharp f(x) = \sup_{r>0} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - f_{B_r}(x)| \, d\mu_\alpha(y),$$

where $f_{B_r}(x) = \frac{1}{\mu_\alpha B_r} \int_{B_r} \tau_x f(y) \, d\mu_\alpha(y)$.

We denote by $BMO_\alpha(\mathbb{R})$ (Dunkl-type BMO space) the set of locally integrable functions f with finite norm

$$\|f\|_{BMO_\alpha} = \sup_{r>0, \ x\in\mathbb{R}} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - f_{B_r}(x)| \, d\mu_\alpha(y) < \infty$$

or

$$\|f\|_{BMO_\alpha} = \inf_C \sup_{r>0, \ x\in\mathbb{R}} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - C| \, d\mu_\alpha(y).$$

$BMO\,(X,\nu)$ space is defined as the space of locally integrable functions f with the following finite norm

$$\|f\|_* = \sup_{t>0, \ x\in X} \nu(B(x,t))^{-1} \int\limits_{B(x,t)} |f(y) - f_{B(x,t)}| d\nu(y) < \infty,$$

where $f_{B(x,t)} = \nu(B(x,t))^{-1} \int_{B(x,t)} f(y) d\nu(y)$.

Theorem 1. *[24] Let $f \in BMO\,(X,\nu)$ and ν doubling measure. For any $r > 0$, then*

$$\nu\left|\left\{y \in B(x,t) \ : \ |f(x) - f_{B(x,t)}| > r\right\}\right| \le C\nu(B(x,t)) e^{\frac{-cr}{\|f\|_*}},$$

where the constants C and c are independent of f and r.

It is clear that $BMO(X,\nu) = BMO_p(X,\nu)$ if the John-Nirenberg inequality holds. The following theorem holds.

Theorem 2. *1) Let $f \in L^{loc}_{1,\alpha}(\mathbb{R})$. If*

$$\sup_{t>0,\ x\in\mathbb{R}} \left(\mu_\alpha(B_t)^{-1} \int\limits_{B_t} |\tau_y f(x) - f_{B_t}|^p d\mu_\alpha(y) \right)^{1/p} = \|f\|_{BMO_{p,\alpha}} < \infty$$

then for any $1 < p < \infty$,

$$\|f\|_{BMO_\alpha} \le \|f\|_{BMO_{p,\alpha}} \le A_p \|f\|_{BMO_\alpha},$$

where the constant A_p depends only on p.

2) Let $f \in BMO_\alpha(\mathbb{R})$. Then, there is a constant $C > 0$ such that

$$|f_{B_r} - f_{B_t}| \le C\|f\|_{BMO_\alpha} \ln\frac{t}{r},\ \ 0 < 2r < t,$$

where C is independent of f, x, r and t.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (X, ρ, ν). By this we mean a topological space $X = \mathbb{R}$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu B(x, 2r) \le C_0 \nu B(x, r) \tag{1}$$

with a constant C_0 being independent of x and $r > 0$. Here $B(x, r) = \{y \in X : \rho(x, y) < r\}$, $\rho(x, y) = |x - y|$. Let (X, ρ, ν) be a space of homogeneous type, where $X = \mathbb{R}$, $d\nu(x) = d\mu_\alpha(x)$. It is clear that this measure satisfies the doubling condition (1).

Since

$$\int\limits_{B_r} \tau_y |f(x)| d\mu_\alpha(y) \approx \int\limits_{B(x,r)} |f(y)|\ d\nu(y)$$

we get

$$\sup_{t>0,\ x\in\mathbb{R}} \mu(B_t)^{-1} \int\limits_{B_t} |\tau_y f(x) - C| d\mu_\alpha(y)$$

$$\begin{aligned} &\approx \sup_{t>0,\ x\in X} \nu(B(x,t))^{-1} \int\limits_{B(x,t)} |f(y) - C|\ d\nu(y) \\ &= \|f\|_{BMO(X,\nu)} \approx \|f\|_{BMO_p(X,\nu)} \approx \|f\|_{BMO_{p,\alpha}}. \end{aligned}$$

Similarly, we can prove

$$|f_{B_r} - f_{B_t}| \le C\|f\|_{BMO_\alpha} \ln\frac{t}{r},\ \ 0 < 2r < t.$$

For all x, y, $z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\Delta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x,\, y \in \mathbb{R}\setminus\{0\}, \\ 0 & \text{otherwise} \end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z) = \begin{cases} d_\alpha \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } |z| \in A_{x,y}, \\ 0, & \text{otherwise,} \end{cases}$$

where $d_\alpha = (\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2}))$ and $A_{x,y} = [||x|-|y||, |x|+|y|]$.

Properties 1. *(see Rösler [32]) The signed kernel W_α is even with respect to all variables and satisfies the following properties*

$$W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y),$$

$$W_\alpha(x,y,z) = W_\alpha(-z,y,-x) = W_\alpha(-x,-y,-z)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|\, d\mu_\alpha(z) \le 4.$$

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} = \begin{cases} W_\alpha(x,y,z)\, d\mu_\alpha(z) & \text{if } x,\, y \in \mathbb{R}\setminus\{0\}, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0. \end{cases}$$

Definition 1. *For x, $y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put*

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)\, d\nu_{x,y}(z).$$

The operators τ_x, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form (see [32])

$$\tau_x f(y) = c_\alpha \int_0^\pi f_e\left((x,y)_\theta\right)\, h_1(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta$$

$$+ c_\alpha \int_0^\pi f_o\left((x,y)_\theta\right)\, h_2(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta,$$

where $(x,y)_\theta = \sqrt{x^2+y^2-2|xy|\cos\theta}$, $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f, with

$$c_\alpha \equiv \left(\int_0^\pi (\sin\theta)^{2\alpha}\, d\theta\right)^{-1} = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)},$$

$$h_1(x,y,\theta) = 1 - sgn(xy)\cos\theta$$

and

$$h_2(x,y,\theta) = \begin{cases} \frac{(x+y)\,[1-sgn(xy)\cos\theta]}{(x,y)_\theta}, & \text{if } \ xy \neq 0, \\ 0, & \text{if } \ xy = 0. \end{cases}$$

Using the change of variable $z = (x,y)_\theta$, we have also (see [4])

$$\tau_x f(y) = c_\alpha \int_0^\pi \{f\left((x,y)_\theta\right) + f\left(-(x,y)_\theta\right) \\ + \frac{x+y}{(x,y)_\theta}\left[f\left((x,y)_\theta\right) - f\left(-(x,y)_\theta\right)\right]\}(1-\cos\theta)(\sin\theta)^{2\alpha}\, d\theta.$$

Now we define the Dunkl-type fractional maximal function by

$$M_{\beta,\alpha} f(x) = \sup_{r>0} \big(\mu_\alpha B_r\big)^{\frac{\beta}{2\alpha+2}-1} \int_{B_r} \tau_x |f|(y)\, d\mu_\alpha(y), \quad 0 \leq \beta < 2\alpha+2.$$

If $\beta = 0$, then $M_\alpha \equiv M_{0,\alpha}$ is the Hardy-Littlewood maximal operator associated with the Dunkl operator (see [2, 11, 21, 29]).

The following theorem is our main result in which we obtain the necessary and sufficient conditions for the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ to be bounded from the spaces $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$, $1 < p < q < \infty$ and from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\alpha}(\mathbb{R})$, $1 < q < \infty$.

Theorem 3. *([12]) Let $0 < \beta < 2\alpha + 2$ and $1 \leq p \leq \frac{2\alpha+2}{\beta}$.*

1) If $1 < p < \frac{2\alpha+2}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $M_{\beta,\alpha}$ from $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $M_{\beta,\alpha}$ from $L_{1,\alpha}(\mathbb{R})$ to $WL_{q,\alpha}(\mathbb{R})$.

3) If $p = \frac{2\alpha+2}{\beta}$, then $M_{\beta,\alpha}$ is bounded from $L_{p,\alpha}(\mathbb{R})$ to $L_\infty(\mathbb{R})$.

Theorem 4. *[11]*

1. If $f \in L_{1,\omega,\alpha}(\mathbb{R})$ and $\omega \in A_{1,\alpha}(\mathbb{R})$, then $M_\alpha f \in WL_{1,\omega,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{WL_{1,\omega,\alpha}} \leq C_{1,\alpha} \|f\|_{L_{1,\omega,\alpha}},$$

where $C_{1,\alpha}$ depends only on α.

2. If $f \in L_{p,\omega,\alpha}(\mathbb{R})$, $1 < p < \infty$ and $\omega \in A_{p,\alpha}(\mathbb{R})$, then $M_\alpha f \in L_{p,\omega,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{L_{p,\omega,\alpha}} \leq C_{p,\alpha} \|f\|_{L_{p,\omega,\alpha}},$$

where $C_{p,\alpha}$ depends only on p, α.

Definition 2. *Let $1 \le p < \infty$, $0 \le \lambda \le 2\alpha + 2$. We denote by $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R})$ Dunkl-type Morrey space ($\equiv$ D-Morrey space) as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with the finite norm*

$$\|f\|_{\mathcal{M}_{p,\lambda,\alpha}} = \sup_{t>0,\, x\in\mathbb{R}} \left(t^{-\lambda} \int_{B_t} [\tau_x |f(y)|\,]^p d\mu_\alpha(y) \right)^{1/p}.$$

Theorem 5. *[13]*

1. If $f \in \mathcal{M}_{1,\lambda,\alpha}(\mathbb{R})$, $0 \le \lambda < 2\alpha + 2$, then $M_\alpha f \in W\mathcal{M}_{1,\lambda,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{W\mathcal{M}_{1,\lambda,\alpha}} \le C_{1,\lambda,\alpha} \|f\|_{\mathcal{M}_{1,\lambda,\alpha}},$$

where $C_{1,\lambda,\alpha}$ depends only on λ,α and n.

2. If $f \in \mathcal{M}_{p,\lambda,\alpha}(\mathbb{R})$, $1 < p < \infty$,$0 \le \lambda < 2\alpha + 2$, then $M_\alpha f \in \mathcal{M}_{p,\lambda,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{\mathcal{M}_{p,\lambda,\alpha}} \le C_{p,\lambda,\alpha} \|f\|_{\mathcal{M}_{p,\lambda,\alpha}},$$

where $C_{p,\lambda,\alpha}$ depends only on p,λ,α and n.

Theorem 6. *[13] Let $0 < \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2 - \beta$ and $1 \le p < \frac{2\alpha+2-\lambda}{\beta}$.*

1) If $1 < p < \frac{2\alpha+2-\lambda}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\alpha+2-\lambda}$ is necessary and sufficient for the boundedness $M_{\beta,\alpha}$ from $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R})$ to $\mathcal{M}_{q,\lambda,\alpha}(\mathbb{R})$.

2) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{2\alpha+2-\lambda}$ is necessary and sufficient for the boundedness $M_{\beta,\alpha}$ from $\mathcal{M}_{1,\lambda,\alpha}(\mathbb{R})$ to $W\mathcal{M}_{q,\lambda,\alpha}(\mathbb{R})$.

For $1 \le p, \theta \le \infty$, $0 \le \lambda \le 2\alpha + 2$ and $0 < s < 1$, the Besov-Morrey space for the Dunkl operators on $\mathbb{R}$ (Besov-Morrey-Dunkl space) $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ consists of all functions f in $L_{p,\lambda,\alpha}(\mathbb{R})$ so that

$$\|f\|_{B^s_{p\theta,\lambda,\alpha}} = \|f\|_{L_{p,\lambda,\alpha}} + \left(\int_{\mathbb{R}} \frac{\|\tau_x f(\cdot) - f(\cdot)\|^\theta_{L_{p,\lambda,\alpha}}}{|x|^{2\alpha+2+s\theta}} d\mu_\alpha(x) \right)^{1/\theta} < \infty. \tag{2}$$

Besov spaces in the setting of the Dunkl operators studied by C. Abdelkefi and M. Sifi [3, 4], R. Bouguila, M.N. Lazhari and M. Assal [5], L. Kamoun [19], Y.Y. Mammadov [22] and V.S. Guliyev, Y.Y. Mammadov [12]. In the following theorem we prove the boundedness of the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ in the Dunkl-type Besov spaces.

Theorem 7. *([12]) For $1 < p < q < \infty$, $0 \le \lambda \le 2\alpha + 2$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2-\lambda}$, $1 \le \theta \le \infty$ and $0 < s < 1$ the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ is bounded from $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ to $B^s_{q\theta,\lambda,\alpha}(\mathbb{R})$. More precisely, there is a constant $C > 0$ such that*

$$\|M_{\beta,\alpha} f\|_{B^s_{q\theta,\lambda,\alpha}} \le C \|f\|_{B^s_{p\theta,\lambda,\alpha}}$$

hold for all $f \in B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$.

For a real parameter $\alpha \geq -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right) \tag{3}$$

Note that $\Lambda_{-1/2} = d/dx$.

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial value problem :

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R}$$

has a unique solution $E_\alpha(\lambda x)$ called Dunkl kernel [6, 27, 33] and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},$$

where j_α is the normalized Bessel function of the first kind and order α [34], defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n (z/2)^{2n}}{n!\,\alpha(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

We can write for $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ (see Rösler [32], p. 295)

$$E_\alpha(-i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_{-1}^{1}(1-t^2)^{\alpha-1/2}\,(1-t)\,e^{i\lambda x t}\,dt.$$

Note that $E_{-1/2}(\lambda x) = e^{\lambda x}$.

The Dunkl transform $\mathcal{F}_\alpha$ of a function $f \in L_{1,\alpha}(\mathbb{R})$, is given by

$$\mathcal{F}_\alpha f(\lambda) := \int_{\mathbb{R}} E_\alpha(-i\lambda x)\, f(x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}.$$

Here the integral makes sense since $|E_\alpha(ix| \leq 1$ for every $x \in \mathbb{R}$ [32], p. 295.

Note that $\mathcal{F}_{-1/2}$ agrees with the classical Fourier transform $\mathcal{F}$, given by:

$$\mathcal{F} f(\lambda) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\lambda x}\, f(x) dx, \quad \lambda \in \mathbb{R}.$$

Proposition 1. *(see Soltani [28])*

(i) If f is an even positive continuous function, then $\tau_x f$ is positive.

(ii) For all $x \in \mathbb{R}$ the operator τ_x extends to $L_{p,\alpha}(\mathbb{R})$, $p \geq 1$ and we have for $f \in L_{p,\alpha}(\mathbb{R})$,

$$\|\tau_x f\|_{p,\alpha} \leq 4\|f\|_{p,\alpha}. \tag{4}$$

(iii) For all x, $\lambda \in \mathbb{R}$ and $f \in L_{1,\alpha}(\mathbb{R})$, we have

$$\mathcal{F}_\alpha\left(\tau_x f\right)(\lambda) = E_\alpha(i\lambda x)\,\mathcal{F}_\alpha f(\lambda).$$

Let f and g be two continuous functions on $\mathbb{R}$ with compact support. We define the generalized convolution $*_\alpha$ of f and g by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y)\, g(y)\, d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The generalized convolution $*_\alpha$ is associative and commutative [32]. Note that $*_{-1/2}$ agrees with the standard convolution $*$.

Proposition 2. *(see Soltani [28])*

(i) If f is an even positive function and g a positive function with compact support, then $f *_\alpha g$ is positive.

(ii) Assume that p, q, $r \in [1, +\infty[$ satisfying $1/p + 1/q = 1 + 1/r$ (the Young condition). Then the map $(f, g) \mapsto f *_\alpha g$, defined on $\mathcal{E}_c \times \mathcal{E}_c$, extends to a continuous map from $L_{p,\alpha}(\mathbb{R}) \times L_{q,\alpha}(\mathbb{R})$ to $L_{r,\alpha}(\mathbb{R})$, and we have

$$\|f *_\alpha g\|_{r,\alpha} \le 4\|f\|_{p,\alpha}\, \|g\|_{q,\alpha}.$$

(ii) For all $f \in L_{1,\alpha}(\mathbb{R})$ and $g \in L_{2,\alpha}(\mathbb{R})$, we have

$$\mathcal{F}_\alpha\left(f *_\alpha g\right) = \left(\mathcal{F}_\alpha f\right)\, \left(\mathcal{F}_\alpha g\right).$$

3. Maximal commutators in $L_{p,\lambda,\alpha}(\mathbb{R})$

The commutator generated by the Dunkl-type maximal operator M_α, for given a measurable function b is formally defined by

$$[M_\alpha, b]f = M_\alpha(bf) - bM_\alpha(f)$$

and for given a measurable function b, the Dunkl-type maximal commutator is defined by

$$M_{b,\alpha}(f)(x) := \sup_{r>0} \mu_\alpha(B_r)^{-1} \int\limits_{B_r} \tau_y |(b(x) - b(y)) f(x)| d\mu_\alpha(y),$$

for all $x \in \mathbb{R}$.

Lemma 1. *Let $1 < s < \infty$, $b \in BMO(\mathbb{R})$. Then, there exists $C > 0$ such that for all $x \in \mathbb{R}$*

$$M_\alpha^\sharp(M_{b,\alpha} f)(x) \le C\|b\|_{BMO_\alpha} \left(\left(M_\alpha (M_\alpha f)^s\right)^{\frac{1}{s}}(x) + M_\alpha\left(M_\alpha |f|^s\right)^{\frac{1}{s}}(x) \right)$$

holds.

Proof. From the boundedness of the Dunkl-type maximal operator M_α and the pointwise inequality we have

$$M_\alpha^\sharp(M_{b,\alpha} f)(x) \le 2M_\alpha(M_{b,\alpha} f)(x), \ x \in \mathbb{R}.$$

Since $M_{b,\alpha}(f)(y)=\sup\limits_{t>0} M_{b,t,\alpha}(f)(y)$ then, we get

$$\begin{aligned}
&M_{b,t,\alpha}(f)(y)=(\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y(|b(y)-b(z)||f(z)|)d\mu_\alpha(z)\\
\leq\ & (\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y(|b(z)-b_{B_t}||f(z)|)d\mu_\alpha(z)\\
+\ & |b(y)-b_{B_t}|(\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y|f(z)|d\mu_\alpha(z)\\
\leq\ & (\mu_\alpha B_t)^{-1}\left(\int\limits_{B_t}[\tau_y|b(z)-b_{B_t}|]^{s'}d\mu_\alpha(z)\right)^{\frac{1}{s'}}\left(\int\limits_{B_t}[\tau_y|f(z)|]^{s}d\mu_\alpha(z)\right)^{\frac{1}{s}}\\
+\ & |b(y)-b_{B_t}|(\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y|f(z)|d\mu_\alpha(z)\\
\leq\ & C\|b\|_{BMO_\alpha}\left(M_\alpha|f|^s\right)^{\frac{1}{s}}(y)+|b(y)-b_{B_t}|(\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y|f(z)|d\mu_\alpha(z).
\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}
&(\mu_\alpha B_r)^{-1}\int\limits_{B_r}\tau_x\left[|b(y)-b_{B_t}|(\mu_\alpha B_t)^{-1}\left(\int\limits_{B_t}\tau_y|f(z)|d\mu_\alpha(z)\right)\right]d\mu_\alpha(y)\\
\leq\ & (\mu_\alpha B_r)^{-1}\int\limits_{B_r}\tau_x\left[|b(y)-b_{B_t}|M_\alpha f(y)\right]d\mu_\alpha(y)\\
\leq\ & (\mu_\alpha B_r)^{-1}\left(\int\limits_{B_r}[\tau_x|b(y)-b_{B_r}|]^{s'}d\mu_\alpha(y)\right)^{\frac{1}{s'}}\left(\int\limits_{B_r}\tau_x(M_\alpha f)^s(y)d\mu_\alpha(y)\right)^{\frac{1}{s}}
\end{aligned}$$

$$\begin{aligned}
+\ & (\mu_\alpha B_r)^{-1}\int\limits_{B_r}\tau_x\left[|b_{B_t}-b_{B_r}|M_\alpha f(y)\right]d\mu_\alpha(y)\\
\leq\ & C\|b\|_{BMO_\alpha}\left(M_\alpha(M_\alpha f)^s\right)^{\frac{1}{s}}(x).
\end{aligned}$$

Therefore

$$M_\alpha(M_{b,\alpha}f)(x)=\sup_{r>0}(\mu_\alpha B_r)^{-1}\int_{B_r}\tau_x(M_{b,\alpha}f)(y)d\mu_\alpha(y)$$

$$\leq C\|b\|_{BMO_\alpha}\left((M_\alpha(M_\alpha f)^s)^{\frac{1}{s}}(x)+\sup_{r>0}(\mu_\alpha B_r)^{-1}\int_{B_r}\tau_x\left(M_\alpha|f|^s\right)^{\frac{1}{s}}(y)d\mu_\alpha(y)\right)$$

$$\leq C\|b\|_{BMO_\alpha}\left((M_\alpha(M_\alpha f)^s)^{\frac{1}{s}}(x)+M_\alpha\left(M_\alpha|f|^s\right)^{\frac{1}{s}}(x)\right). \tag{5}$$

Proposition 3. *[18] For all weights ω and all nonnegative function f satisfying $\nu(\{x\in X:f(x)>\beta\})<\infty$ for all $\beta>0$, there exists a positive constant C such that*

1. If $\nu(X)=\infty$, then

$$\int_X f(y)g(y)d\nu(y)\leq C\int_X M^\sharp f(y)Mg(y)d\nu(y).$$

2. If $\nu(X)<\infty$, then

$$\int_X f(y)g(y)d\nu(y)\leq C\int_X M^\sharp f(y)Mg(y)d\nu(y)+Cg(X)\nu_X(f),$$

where g is nonnegative function, $g(X)=\int_X g(x)d\nu(x)$, $\nu_X(f)=\frac{1}{\nu(X)}\int_X f(y)d\nu(y)$.

Lemma 2. *Let $1<p<\infty$, $\omega\in A_{p,\alpha}(\mathbb{R})$. Then*

$$\|f\omega^{\frac{1}{p}}\|_{L_{p,\alpha}}\leq C\|\omega^{\frac{1}{p}}M_\alpha^\sharp f\|_{L_{p,\alpha}}$$

where a constant $C>0$ is independent of f.

Proof. Let (X,ν) be a space of homogeneous type. According to Proposition 1, we have

$$\|f\omega^{\frac{1}{p}}\|_{L_{p,\alpha}}\leq C\sup_{\|g\|_{L_{p',\gamma}}\leq 1}\left|\int_{\mathbb{R}} f(y)g(y)\omega^{\frac{1}{p}}(y)d\mu_\alpha(y)\right|$$

$$=C\sup_{\|g\|_{L_{p'}}\leq 1}\left|\int_X f(y)g(y)\omega^{\frac{1}{p}}(y)d\nu(y)\right|\leq C\sup_{\|g\|_{L_{p'}}\leq 1}\left|\int_X M^\sharp f(y)M(g\omega^{\frac{1}{p}})(y)d\nu(y)\right|.$$

Hence

$$\|f\omega^{\frac{1}{p}}\|_{L_{p,\alpha}}\leq C\sup_{\|g\|_{L_{p',\gamma}}\leq 1}\left|\int_{\mathbb{R}} M_\alpha^\sharp f(y)M_\alpha(g\omega^{\frac{1}{p}})(y)d\mu_\alpha(y)\right|.$$

Finally by using the Hölder inequality and Theorem 4, we get

$$\|f\omega^{\frac{1}{p}}\|_{L_{p,\alpha}}\leq C\sup_{\|g\|_{L_{p',\gamma}}\leq 1}\|\omega^{\frac{1}{p}}M_\alpha^\sharp f\|_{L_{p,\alpha}}\|\omega^{-\frac{1}{p}}M_\alpha(g\omega^{\frac{1}{p}})\|_{L_{p',\gamma}}$$

$$\leq C\sup_{\|g\|_{L_{p',\gamma}}\leq 1}\|\omega^{\frac{1}{p}}M_\alpha^\sharp f\|_{L_{p,\alpha}}\|g\|_{L_{p',\gamma}}\leq C\|\omega^{\frac{1}{p}}M_\alpha^\sharp f\|_{L_{p,\alpha}}.$$

Theorem 8. *Let $b \in BMO_\alpha(\mathbb{R})$, $1 < p < \infty$, $\omega \in A_{p,\alpha}(\mathbb{R})$. Then $M_{b,\alpha}$ is bounded on the space $L_{p,\omega,\alpha}(\mathbb{R})$.*

Proof. By using Lemma 1, Lemma 2 and Theorem 4, we have $M_{b,\alpha}$ is bounded on the space $L_{p,\omega,\gamma}(\mathbb{R})$.

Operators $M_{b,\alpha}$ and $[M_\alpha, b]$ are essentially different from each other. For example, $M_{b,\alpha}$ is a positive and sublinear operator, but $[M_\alpha, b]$ is neither positive nor sublinear. However, if b satisfy some additional conditions, then operator $M_{b,\alpha}$ is controled by $[M_\alpha, b]$.

Theorem 9. *Let $1 < p < \infty$, $0 \le \lambda < 2\alpha + 2$.Then the commutator $M_{b,\alpha}$ is bounded on $L_{p,\lambda,\alpha}(\mathbb{R})$ if and only if $b \in BMO_\alpha(\mathbb{R})$.*

Proof. Sufficiency: Let $1 < p < \infty$, $0 \le \lambda < 2\alpha + 2$, $f \in L_{p,\lambda,\alpha}(\mathbb{R})$. We have

$$\int_{B_t} \tau_y \left[M_{b,\alpha} f\right]^p (x) d\mu_\alpha(y) \le \int_{\mathbb{R}} \tau_y \left[M_{b,\alpha} f\right]^p (x) (M_\alpha \chi_{B_t}(y))^\delta d\mu_\alpha(y), \ \ x \in \mathbb{R}.$$

Taking into account the properties of $A_{p,\alpha}(\mathbb{R})$, we can easily see that $(M_\alpha \chi_{B_t})^\delta \in A_{p,\alpha}(\mathbb{R})$, for any $0 < \delta < 1$. Then by using Lemma 2 and Theorem 8 we obtain

$$\begin{aligned}
\int_{B_t} \tau_y \left[M_{b,\alpha} f\right]^p (x) d\mu_\alpha(y) &\le C \ \|b\|^p_{BMO_\alpha} \int_{\mathbb{R}} \tau_y |f(x)|^p (M_\alpha \chi_{B_r}(y))^\theta d\mu_\alpha(y) \\
&\le C \ \|b\|^p_{BMO_\alpha} \int_{B_r} \tau_y |f(x)|^p d\mu_\alpha(y) \\
&+ C \, \|b\|^p_{BMO_\alpha} \sum_{j=1}^{\infty} \int_{B_{2^{j+1}r} \setminus B_{2^j r}} \tau_y |f(x)|^p (M_\alpha \chi_{B_r}(y))^\theta d\mu_\alpha(y) \\
&\le C \ \|b\|^p_{BMO_\alpha} \int_{B_r} \tau_y |f(x)|^p d\mu_\alpha(y) \\
&+ C \, \|b\|^p_{BMO_\alpha} \sum_{j=1}^{\infty} \int_{B_{2^{j+1}r} \setminus B_{2^j r}} \tau_y |f(x)|^p \frac{r^{(2\alpha+2)\theta}}{(|y| + r)^{(2\alpha+2)\theta}} d\mu_\alpha(y) \\
&\le C \ \|b\|^p_{BMO_\alpha} \|f\|^p_{L_{p,\lambda,\alpha}} \left(r^\lambda + \sum_{j=1}^{\infty} \frac{1}{(2^j + 1)^{(2\alpha+2)\theta}} (2^{j+1} r)^\lambda \right) \\
&\le C \ r^\lambda \, \|b\|^p_{BMO_\alpha} \|f\|^p_{L_{p,\lambda,\alpha}}.
\end{aligned}$$

Necessity: Let $M_{b,\alpha}$ be bounded from $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{p,\lambda,\alpha}(\mathbb{R})$, $1 < p < \infty$.

Obviously,

$$\|f\|_{L_{p,\lambda,\alpha}} = \sup_{t>0,\, x \in \mathbb{R}} \left(t^{-\lambda} \int_{B_t} \tau_y |f(x)|^p d\mu_\alpha(y) \right)^{1/p}.$$

Now we consider $f = \chi_{B_r}$. It is easy to compute that

$$\begin{aligned}
\|\chi_{B_r}\|_{L_{p,\lambda,\alpha}} &\approx \sup_{t>0,\, x\in\mathbb{R}} \left(t^{-\lambda} \int\limits_{B_t} \tau_y \chi_{B_r}(x) d\mu_\alpha(y) \right)^{1/p} \\
&\approx \sup_{t>0,\, x\in\mathbb{R}} \left(t^{-\lambda} \int\limits_{B_t} \chi_{B_r}(y) d\mu_\alpha(y) \right)^{1/p} \\
&\approx \sup_{B_t\subset B_r} \left(t^{-\lambda} \mu_\alpha(B_t \cap B_r) \right)^{1/p} \leq r^{\frac{2\alpha+2-\lambda}{p}}.
\end{aligned}$$

Then

$$\frac{1}{(\mu_\alpha B_t)} \int_{B_t} |\tau_z b(x) - f_{B_t}|\, d\mu_\alpha(z)$$

$$\begin{aligned}
&= \frac{1}{(\mu_\alpha B_t)} \int_{B_t} \left| \tau_z b(x) - \frac{1}{(\mu_\alpha B_t)} \int_{B_t} \tau_z b(y) d\mu_\alpha(y) \right| d\mu_\alpha(z) \\
&\leq \frac{1}{(\mu_\alpha B_t)} \int_{B_t} \frac{1}{(\mu_\alpha B_t)} \int_{B_t} |\tau_z b(x) - \tau_z b(y)|\, d\mu_\alpha(y) d\mu_\alpha(z) \\
&\leq \frac{1}{(\mu_\alpha B_t)} \int_{B_t} \frac{1}{(\mu_\alpha B_t)} \int_{B_t} |\tau_z (b(x) - b(y))|\, d\mu_\alpha(y) d\mu_\alpha(z) \\
&\leq \frac{1}{(\mu_\alpha B_t)} \int_{B_t} M_{b,\alpha} \chi_{B_t}(z) d\mu_\alpha(z) \\
&\leq Ct^{-2\alpha-2+\lambda} \|M_{b,\alpha}\chi_{B_t}\|_{L_{p,\lambda,\alpha}} \|\chi_{B_t}\|_{L_{p',\lambda,\alpha}} \leq Ct^{\frac{2\alpha+2-\lambda}{p'} + \frac{2\alpha+2-\lambda}{p} - 2\alpha - 2 + \lambda} \leq C.
\end{aligned}$$

Theorem 10. *Let $0 \leq \lambda < 2\alpha+2$, $b \in BMO_\alpha(\mathbb{R})$. Then the commutator $M_{b,\alpha}$ is bounded from $L_{1,\lambda,\alpha}(\mathbb{R})$ to $WL_{1,\lambda,\alpha}(\mathbb{R})$.*

Proof. Let $0 \leq \lambda < 2\alpha + 2$, $f \in L_{1,\lambda,\alpha}(\mathbb{R})$. This assertion is easily obtained from $f(x) \leq M_\alpha f(x)$. Finally, by using (5) and Theorem 5, we get

$$\begin{aligned}
\|M_{b,\alpha} f\|_{WL_{1,\lambda,\alpha}} &\leq \|M_\alpha (M_{b,\alpha} f)\|_{WL_{1,\lambda,\alpha}} \\
&\leq \|b\|_{BMO_\alpha} \|\, (M_\alpha (M_\alpha f)^s)^{\frac{1}{s}} + M_\alpha \left(M_\alpha |f|^s\right)^{\frac{1}{s}} \|_{WL_{1,\lambda,\alpha}} \\
&\leq C \|b\|_{BMO_\alpha} \|f\|_{L_{1,\lambda,\alpha}}.
\end{aligned}$$

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Oscillatory Integral Operators and Their Commutators on Vanishing Generalized Morrey Spaces with Variable Exponent

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Abstract. We consider the generalized Morrey spaces $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ with variable exponent $p(x)$ and a general function $\varphi(x,r)$ defining the Morrey-type norm. In case of unbounded sets $\Omega \subset \mathbb{R}^n$ we prove the boundedness of the conditions in terms of Calderón-Zygmund-type integral inequalities for oscillatory integral operators and its commutators in the vanishing generalized weighted Morrey spaces with variable exponent.

Key Words and Phrases: maximal operator, singular integral operators, Calderón-Zygmund-type integral inequalities for oscillatory integral operators, generalized weighted Morrey space with variable exponent, BMO space.

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1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [51] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [20, 22, 24, 51]. Mizuhara [52] and Nakai [55] introduced generalized Morrey spaces. Later, Guliyev [24] defined the generalized Morrey spaces $M^{p,\varphi}$ with normalized norm.

As it is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [18, 40, 43, 59], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein. For mapping properties of maximal functions and singular integrals on Lebesgue spaces with variable exponent we refer to [11, 12, 13, 15, 16, 17, 42, 45].

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [2] and [53] in the Euclidean setting and in [41] in the setting of metric measure spaces, in case of bounded sets. The boundedness of the maximal operator in variable exponent Morrey

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spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$, $\lambda(\cdot)$ was proved in [2]. In [54] the maximal operator was considered in a somewhat more general space, but under more restrictive conditions on $p(x)$. P. Hästö in [35] used his new "local-to-global" approach to extend the result of [2] on the maximal operator to the case of the whole space $\mathbb{R}^n$. The boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [41].

Generalized Morrey spaces of such a kind in the case of constant p were studied in [4], [46], [52], [55]. In the case of bounded sets the boundedness of the maximal operator, singular integral operators and the potential operator in generalized variable exponent Morrey type spaces was proved in [29], [30], [31] and in the case of unbounded sets in [32], see also [36, 37, 56].

In the case of constant p and λ, the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [58], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [14]; for further results in the case of constant p and λ (see, for instance, [3]– [5]).

We consider the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y)|dy.$$

A distribution kernel $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \le C|x-y|^{-n} \quad \text{for all} \quad x \neq y,$$

$$|K(x,y) - K(x,z)| \le C \frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \le C \frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|x-\xi|$$

Calderón-Zygmund type singular operator and the oscillatory integral operator are defined by

$$Tf(x) = \int_{\Omega} K(x,y) f(y) dy, \tag{1}$$

$$Sf(x) = \int_{\Omega} e^{P(x,y)} K(x,y) f(y) dy, \tag{2}$$

where $P(x,y)$ is a real valued polynomial defined on $\Omega \times \Omega$. Lu and Zhang [50] used L^2-boundedness of T to get L^p- boundedness of S with $1 < p < \infty$.

Let

$$T^* f(x) = \sup_{\varepsilon > 0} |T_{\varepsilon} f(x)|$$

be the maximal singular operator, where $T_{\varepsilon} f(x)$ is the usual truncation

$$T_{\varepsilon} f(x) = \int_{\{y \in \Omega : |x-y| \ge \varepsilon\}} K(x,y) f(y) dy.$$

We find the condition on the Morrey function $\varphi(x,r)$ for the boundedness of the oscillatory integral operator in generalized weighted Morrey space $\mathcal{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$ with variable $p(x)$ under the log-condition on $p(\cdot)$.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces. In Section 3 we treat oscillatory integral operators and its commutators in $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$.

The main results are given in Theorems 7, 8, 9, 11, 12, 13. We emphasize that the results we obtain for generalized weighted Morrey spaces are new even in the case when $p(x)$ is constant, because we do not impose any monotonicity type condition on $\varphi(x,r)$.

We use the following notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\})$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, by c,C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces

We refer to the book [16] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $(1,\infty)$. An open set Ω which may be unbounded throughout the whole paper. We mainly suppose that

$$1 < p_- \leq p(x) \leq p_+ < \infty, \tag{3}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x)$, $p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : \; I_{p(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent.

The space $L^{p(\cdot)}(\Omega)$ coincides with the space

$$\left\{f(x) : \left|\int_{\Omega} f(y)g(y)dy\right| < \infty \quad for\ all \quad g \in L^{p'(\cdot)}(\Omega)\right\} \tag{4}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}(\Omega)} \approx \sup_{\|g\|_{L^{p'(\cdot)}}\leq 1} \left| \int_\Omega f(y)g(y)dy \right| \tag{5}$$

see [47, Proposition 2.2], see also [44, Theorem 2.3], or [60, Theorem 3.5].

For the basics on variable exponent Lebesgue spaces we refer to [61], [44].
$\mathcal{P}(\Omega)$ is the set of bounded measurable functions $p : \Omega \to [1, \infty)$;
$\mathcal{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2} \quad x, y \in \Omega, \tag{6}$$

where $A = A(p) > 0$ does not depend on x, y;
$\mathcal{A}^{log}(\Omega)$ is the set of bounded exponents $p : \Omega \to \mathbb{R}^n$ satisfying the condition (6);
$\mathbb{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{log}(\Omega)$ with $1 < p_- \leq p_+ < \infty$;
for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}_\infty^{log}(\Omega)$, $\mathbb{P}_\infty^{log}(\Omega)$, $\mathcal{A}_\infty^{log}(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \leq \frac{A_\infty}{\ln(2+|x|)}, \quad x \in \mathbb{R}^n, \tag{7}$$

where $p_\infty = \lim\limits_{x\to\infty} p(x) > 1$.

We will also make use of the estimate provided by the following lemma (see [16], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leq C r^{\theta_p(x,r)}, \qquad x \in \Omega, \ p \in \mathbb{P}_\infty^{log}(\Omega), \tag{8}$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \leq 1, \\ \frac{n}{p(\infty)}, & r \geq 1. \end{cases}$

A locally integrable function $\omega : \Omega \to (0, \infty)$ is called a weight. We say that $\omega \in A_p(\Omega)$, $1 < p < \infty$, if there is a constant $C > 0$ such that

$$\left(\frac{1}{|\widetilde{B}(x,t)|} \int_{\widetilde{B}(x,t)} \omega(x)dx \right) \left(\frac{1}{|\widetilde{B}(x,t)|} \int_{\widetilde{B}(x,t)} \omega^{1-p'}(x)dx \right)^{p-1} \leq C,$$

where $1/p + 1/p' = 1$. We say that $\omega \in A_1(\Omega)$ if there is a constant $C > 0$ such that $M\omega(x) \leq C\omega(x)$ almost everywhere.

The extrapolation theorems (Lemma 1 and Lemma 2 below) are originally due to Cruz-Uribe, Fiorenza, Martell and Pérez [12]. Here we use the form in [16], see Theorem 7.2.1 and Theorem 7.2.3 in [16].

Lemma 1. *([16]). Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < \infty$, every $(f, g) \in \mathcal{F}$ and every $\omega \in A_1$,*

$$\int_\Omega |f(x)|^{p_0}\omega(x)dx \leq C_0 \int \Omega |g(x)|^{p_0}\omega(x)dx.$$

Let $p(\cdot) \in P(\Omega)$ with $p_0 \le p_-$. If maximal operator is bounded on $L^{\left(\frac{p(\cdot)}{p_0}\right)'}(\Omega)$, then there exists a constant $C > 0$ such that for all $(f, g) \in \mathcal{F}$,

$$\|f\|_{L^{p(\cdot)}(\Omega)} \le C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Lemma 2. *([16]). Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < q_0 < \infty$, every $(f, g) \in \mathcal{F}$ and every $\omega \in A_1$*

$$\left(\int_\Omega |f(x)|^{q_0}\omega(x)dx\right)^{\frac{1}{q_0}} \le C_0 \left(\int_\Omega |g(x)|^{p_0}\omega^{\frac{p_0}{q_0}}(x)dx\right)^{\frac{1}{p_0}}.$$

Let $p(\cdot) \in P(\Omega)$ with $p_0 \le p_-$ and $\frac{1}{p_0} - \frac{1}{q_0} < \frac{1}{p_+}$, and define $q(x)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}.$$

If maximal operator is bounded on $L^{\left(\frac{q(\cdot)}{q_0}\right)'}(\Omega)$, then there exists a constant $C > 0$ such that for all $(f, g) \in \mathcal{F}$,

$$\|f\|_{L^{q(\cdot)}(\Omega)} \le C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Singular operators within the framework of the spaces with variable exponents were studied in [17]. From Theorem 4.8 and Remark 4.6 of [17] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

Theorem 1. *([17, Theorem 4.8]) Let $\Omega \subset \mathbb{R}^n$ be a unbounded open set and $p \in \mathbb{P}^{log}(\Omega)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\ t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

respectively.

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}|dy,$$

where $f_{\widetilde{B}(x,t)}(x) = |\widetilde{B}(x,t)|^{-1} \int_{\widetilde{B}(x,t)} f(y)dy$.

Definition 1. *We define the $BMO(\Omega)$ space as the set of all locally integrable functions f with finite norm*

$$\|f\|_{BMO} = \sup_{x\in\Omega} M^\sharp f(x) = \sup_{x\in\Omega,\ r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}|dy.$$

Definition 2. *We define the $BMO_{p(\cdot)}(\Omega)$ space as the set of all locally integrable functions f with finite norm*

$$\|f\|_{BMO_{p(\cdot)}} = \sup_{x\in\Omega,\, r>0} \frac{\|(f(\cdot) - f_{\widetilde{B}(x,r)})\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}.$$

Theorem 2. *[47] Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$, then the norms $\|\cdot\|_{BMO_{p(\cdot)}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

Before proving the main theorems, we need the following lemma.

Lemma 3. *[34] Let $b \in BMO(\Omega)$. Then there is a constant $C > 0$ such that*

$$\left|b_{\widetilde{B}(x,r)} - b_{\widetilde{B}(x,t)}\right| \le C\|b\|_* \ln\frac{t}{r} \quad \textit{for} \quad 0 < 2r < t,$$

where C is independent of b, x, r, and t.

Everywhere in the sequel the functions $\varphi(x,r)$, $\varphi_1(x,r)$ and $\varphi_2(x,r)$ used in the body of the paper, are non-negative measurable functions on $\Omega\times(0,\infty)$. We find it convenient to define the generalized weighted Morrey spaces in the form as follows.

Definition 3. *Let $1 \le p(x) < \infty$, $x \in \Omega$. The variable exponent generalized Morrey space $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norms*

$$\|f\|_{\mathcal{M}^{p(\cdot),\varphi}} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r)t^{\theta_p(x,t)}}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},$$

respectively.

According to this definition, we recover the space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\varphi(x,r) = r^{\theta_p(x,r) - \frac{\lambda(x)}{p(x)}}$:

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega) = \mathcal{M}^{p(\cdot),\varphi(\cdot)}(\Omega)\Bigg|_{\varphi(x,r) = r^{\theta_p(x,r) - \frac{\lambda(x)}{p(x)}}}.$$

Definition 4. *(Vanishing generalized weighted Morrey space) The vanishing generalized weighted Morrey space $V\mathcal{M}^{p(\cdot),\varphi}_{\omega}(\Omega)$ is defined as the space of functions $f \in \mathcal{M}^{p(\cdot),\varphi}_{\omega}(\Omega)$ such that*

$$\lim_{r\to 0}\sup_{x\in\Omega} \frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}_{\omega}(\Omega)} = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\inf\limits_{x\in\Omega}\varphi(x,t)} = 0. \tag{9}$$

and

$$\sup_{0<r<\infty} \frac{1}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\inf\limits_{x\in\Omega}\varphi(x,t)} = 0. \tag{10}$$

which makes the spaces $V\mathcal{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$ non-trivial, because bounded functions with compact support belong then to this space.

Let $L_v^\infty(\mathbb{R}_+)$ be the weighted L^∞-space with the norm

$$\|g\|_{L_v^\infty(\mathbb{R}_+)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t).$$

In the sequel $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively. We also denote

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_1(0,t)}, \quad t \in (0,\infty).$$

The following theorem was proved in [3].

Theorem 3. *Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,t)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L_{v_1}^\infty(\mathbb{R}_+)$ to $L_{v_2}^\infty(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|_{L_\infty(0,\cdot)}^{-1} \right) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following results on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds, \;\; H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [26, 27].

Theorem 4. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 5. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{11}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (11).

3. Oscillatory integral operators and its commutators in $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderón [6, 7] studied a kind of commutators, appearing in Cauchy integral problems of Lipschitz curve. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [8] states that the commutator operator $[b,K]f = K(bf) - b\,Kf$ is bounded on $L_p(\mathbb{R}^n)$ for $1<p<\infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [9], [10], [19], [20], [22]).

Lemma 4. *(see [49]). If K is a standard Calderón-Zygmund kernel and the Calderón-Zygmund singular integral operator T is of type $(L^2(\Omega), L^2(\Omega))$, then for any real polynomial $P(x,y)$ and $\omega \in A_p$ $(1<p<\infty)$, there exists constants $C>0$ independent of the coefficients of P such that*

$$\|Sf\|_{L^p_\omega(\Omega)} \le C \|f\|_{L^p_\omega(\Omega)}.$$

Theorem 6. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$. Then the operator S is bounded in the space $L^{p(\cdot)}(\Omega)$.*

Proof. By the Lemma 1 and Lemma 4, we derive the operator S is bounded in the space $L^{p(\cdot)}(\Omega)$.

The following local estimates are valid.

Theorem 7. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$ and $f \in L^{p(\cdot)}(\Omega)$. Then*

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}, \tag{12}$$

where C does not depend on f, $x \in \Omega$ and t.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\Omega\setminus\widetilde{B}(x,2t)}(y), \quad t>0, \tag{13}$$

and have

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} + \|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By the Theorem 6 we obtain

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)},$$

so that

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}.$$

Taking into account the inequality

$$\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

we get

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{14}$$

To estimate $\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|Sf_2(z)| \le C \int\limits_{\Omega\setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$|Sf_2(z)| \le C \int_{\Omega\setminus \widetilde{B}(x,2t)} |x-y|^{-n}|f(y)|dy,$$

To estimate Sf_2, we first prove the following auxiliary inequality

$$\begin{aligned}&\int_{\Omega\setminus \widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy\\ &\le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}.\end{aligned} \tag{15}$$

To this end, we choose $\delta > 0$ and proceed as follows

$$\begin{aligned}&\int_{\Omega\setminus \widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \le \delta \int_{\Omega\setminus \widetilde{B}(x,t)} |x-y|^{-n+\delta}|f(y)|dy \int_{|x-y|}^\infty s^{-\delta-1}ds\\ &\le C\int_t^\infty s^{-n}\frac{ds}{s} \int_{\{y\in\Omega: 2t\le|x-y|\le s\}} |f(y)|dy \le C\int_t^\infty s^{-n}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \|\chi_{\widetilde{B}(x,s)}\|_{L^{p'(\cdot)}(\Omega)}\frac{ds}{s}\\ &\le C\int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}.\end{aligned} \tag{16}$$

Hence by inequality (16), we get

$$\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)} \int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}$$

$$= Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \tag{17}$$

From (14) and (17) we arrive at (12).

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $\varphi_1(x,t)$ and $\varphi_2(x,r)$ fulfill condition*

$$\int_t^\infty \frac{\operatorname{ess\,inf}\limits_{s<r<\infty} \varphi_1(x,r) r^{\theta_p(x,r)}}{s^{\theta_p(x,s)}}\frac{ds}{s} \le C\varphi_2(x,t), \tag{18}$$

where C does not depend on $x \in \Omega$ and t. Then the singular integral operators T and T^* are bounded from the space $\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$. As usual, when estimating the norm

$$\|Sf\|_{\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)} = \sup_{x\in\Omega,\, t>0} \varphi_2(x,t)^{-1} t^{-\theta_p(x,t)} \|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}. \tag{19}$$

We estimate $\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}$ in (19) by means of Theorem 7 and obtain

$$\begin{aligned}
&\|Sf\|_{\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)} \\
&\le C \sup_{x\in\Omega,\, t>0} \frac{t^{\theta_p(x,t)}}{\varphi_2(x,t)t^{\theta_p(x,t)}} \int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s} \\
&\le C \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi_1(x,t)t^{\theta_p(x,t)}}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} = C\|f\|_{\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)}.
\end{aligned}$$

It remains to make use of condition (18).

Theorem 9. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$ and $\varphi_1(x,t)$ and $\varphi_2(x,r)$ fulfill satisfy the conditions* (18) *and*

$$C_\gamma := \int_t^\infty \frac{\operatorname{ess\,inf}\limits_{s<r<\infty} \varphi_1(x,r) r^{\theta_p(x,r)}}{s^{\theta_p(x,s)}}\frac{ds}{s} < \infty \tag{20}$$

for every γ.

Then the singular integral operators S is bounded from the space $V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

Proof. The norm inequalities follow from Theorem 7, so we only have to prove that if

$$\begin{aligned}&\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_1(x,t)t^{\theta_p(x,t)}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}=0\Rightarrow\\&\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)t^{\theta_p(x,t)}}\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}=0\end{aligned}\tag{21}$$

otherwise.

To show that $\sup\limits_{x\in\Omega}\frac{1}{\varphi_2(x,t)t^{\theta_p(x,t)}}\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}<\varepsilon$ for small r, we split the right-hand side of (12):

$$\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}_\omega(\Omega)}\le C_0\left(I_{1,\gamma}(x,r)+I_{2,\gamma}(x,r)\right),\tag{22}$$

where $\gamma>0$ will be chosen as shown below (we may take $\gamma<1$),

$$I_{1,\gamma}(x,r):=\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_t^{\gamma_0}\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

$$I_{2,\gamma}(x,r):=\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_{\gamma_0}^{\infty}\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{q(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

and it is supposed that $r<\gamma$. Now we choose any fixed $\gamma>0$ such that

$$\sup_{x\in\Omega}\frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}_\omega(\Omega)}<\frac{\varepsilon}{2CC_0},\ \text{for all } 0<t<\gamma,$$

where C and C_0 are constants from (18) and (22), which is possible since $f\in V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)$. Then

$$\sup_{x\in\Omega}CI_{1,\gamma}(x,r)<\frac{\varepsilon}{2},\ \ 0<r<\gamma,$$

by (21).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (10). We have

$$I_{2,\gamma}(x,r)\le C_\gamma\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)},$$

where C_γ is the constant from (20). Then, by (10) it suffices to choose r small enough such that

$$\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}<\frac{\varepsilon}{2CC_\gamma\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)}}$$

which completes the proof of (21).

Lemma 5. *(see [62]). If K is a standard Calderón-Zygmund kernel and the Calderón-Zygmund singular integral operator T is of type $(L^2(\Omega), L^2(\Omega))$, then for any real polynomial $P(x,y)$ and $\omega \in A_p$ $(1 < p < \infty)$, there exists constants $C > 0$ independent of the coefficients of P such that*

$$\|[b,S]f\|_{L^p_\omega(\Omega)} \le C\|b\|_*\|f\|_{L^p_\omega(\Omega)}.$$

Theorem 10. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $b \in BMO(\Omega)$, $p \in \mathbb{P}^{log}_\infty(\Omega)$. Then the commutator operator $[b,S]$ is bounded on the space $L^{p(\cdot)}(\Omega)$.*

Proof. By Lemma 1 and Lemma 5, we derive the operator $[b,S]$ is bounded in the space $L^{p(\cdot)}(\Omega)$.

The following weighted local estimates are valid.

Theorem 11. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$ and $b \in BMO(\Omega)$. Then*

$$\|[b,S]f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} C\|b\|_*\|t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \tag{23}$$

for every $f \in L^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t.

Proof. We represent function f as in (13) and have

$$\|[b,S]f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|[b,S]f_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} + \|[b,S]f_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By Theorem 10 we obtain

$$\begin{aligned} \|[b,S]f_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} &\le \|[b,S]f_1\|_{L^{p(\cdot)}(\Omega)} \\ &\le C\|b\|_*\|f_1\|_{L^{p(\cdot)}(\Omega)} = C\|b\|_*\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}, \end{aligned} \tag{24}$$

where C does not depend on f. From (24) we obtain

$$\|[b,S]f_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|b\|_* t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \tag{25}$$

easily obtained from the fact that $\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}$ is non-decreasing in t, so that $\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}$ on the right-hand side of (24) is dominated by the right-hand side of (25). To estimate $\|[b,S]f_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|[b,S]f_2(z)| \le C \int\limits_{\Omega\setminus B(x,2t)} |b(z)-b(y)| \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$|[b,S]f_2(z)| \le C \int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n}|b(z)-b(y)|\,|f(y)|dy.$$

To estimate $[b,S]f_2$, we first prove the following auxiliary inequality

$$\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(z)-b(y)||f(y)|dy$$
$$\le C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \tag{26}$$

To estimate $[b,S]f_2(z)$, we observe that for $z\in\widetilde{B}(x,t)$ we have

$$\begin{aligned}
&\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(z)-b(y)||f(y)|dy\\
&\le\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy\\
&+\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(z)-b_{\widetilde{B}(x,t)}||f(y)|dy=J_1+J_2.
\end{aligned}$$

To this end, we choose $\delta>0$, by Theorem 2 and Lemma 3 we obtain

$$\begin{aligned}
J_1=&\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy\\
&\le\delta\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n+\delta}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy\int_{|x-y|}^\infty s^{-\delta-1}ds\\
&\le C\int_t^\infty s^{-n-1}\int_{\{y\in\Omega:2t\le|x-y|\le s\}}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dyds\\
&\le C\int_t^\infty s^{-n-1}\|b(\cdot)-b_{\widetilde{B}(x,s)}\|_{L^{p'(\cdot)}(\widetilde{B}(x,s))}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds\\
&+C\int_t^\infty s^{-n-1}|b_{\widetilde{B}(x,t)}-b_{\widetilde{B}(x,s)}|\int_{\widetilde{B}(x,s)}|f(y)|dyds\\
&\le C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)-n-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds\\
&+C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)-n-1}\ln\frac{s}{t}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds\\
&\le C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}.
\end{aligned}$$

To estimate J_2, by (15), we have

$$\begin{aligned}
J_2=&|b(z)-b_{\widetilde{B}(x,t)}|\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|f(y)|dy\\
&\le C|B(x,t)|^{-1}\int_{\widetilde{B}(x,t)}|b(z)-b(y)|dy\int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}\\
&\le CM_b\chi_{B(x,t)}(z)\int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},
\end{aligned}$$

where C does not depend on x, t.

Hence by inequality (26), we get

$$\begin{aligned}
&\|[b,S]f_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \lesssim \|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}\|b\|_* \\
&\times \int_t^\infty \left(1+\ln\frac{s}{t}\right) s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s} \\
&= \|b\|_* t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \qquad (27)
\end{aligned}$$

From (25) and (27) we arrive at (23).

Theorem 12. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $b \in BMO(\Omega)$ and the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$\int_t^\infty \left(1+\ln\frac{s}{t}\right)\frac{\operatorname*{ess\,inf}\limits_{s<r<\infty}\varphi_1(x,r)r^{\theta_p(x,r)}}{s^{\theta_p(x,s)}}\frac{ds}{s} \le C\varphi_2(x,t). \qquad (28)$$

Then the operator $[b,S]$ is bounded from the space $\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$. We have

$$\|[b,S]f\|_{\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)} = \sup_{x\in\Omega,\, t>0}\frac{1}{\varphi_2(x,t)t^{\theta_p(x,t)}}\|[b,S]f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By (28), Theorems 4 and 11 we obtain

$$\begin{aligned}
&\|[b,S]f\|_{\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)} \\
&\le C\|b\|_* \sup_{x\in\Omega,\, t>0}\frac{t^{\theta_p(x,t)}}{\varphi_2(x,t)t^{\theta_p(x,t)}}\int_t^\infty s^{-\theta_p(x,s)}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s} \\
&\le C\|b\|_* \sup_{x\in\Omega,\, t>0}\frac{1}{\varphi_1(x,t)t^{\theta_p(x,t)}}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} = C\|b\|_*\|f\|_{\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)}
\end{aligned}$$

which completes the proof.

Theorem 13. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $b \in BMO(\Omega)$ and the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the conditions (28) and*

$$C_{\delta_0} := \int_t^\infty \left(1+\ln\frac{t}{s}\right)\frac{\operatorname*{ess\,inf}\limits_{s<r<\infty}\varphi_1(x,r)r^{\theta_p(x,r)}}{s^{\theta_p(x,s)}}\frac{ds}{s} < \infty \qquad (29)$$

for every δ_0.

Then the operator $[b,S]$ is bounded from the space $V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

Proof. The norm inequalities follow from Theorem 11, so we only have to prove that if

$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}=0\Rightarrow$$
$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|[b,S]f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}=0 \tag{30}$$

otherwise.

To show that $\sup\limits_{x\in\Omega}\frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|[b,S]f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}<\varepsilon$ for small r, we split the right-hand side of (23):

$$\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|[b,S]f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}\le C_0\left(I_{1,\delta_0}(x,r)+I_{2,\delta_0}(x,r)\right), \tag{31}$$

where $\delta_0>0$ will be chosen as shown below (we may take $\delta_0<1$),

$$I_{1,\delta_0}(x,r):=\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_t^{\delta_0}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

$$I_{2,\delta_0}(x,r):=\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_{\delta_0}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{q(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

and it is supposed that $r<\delta_0$. Now we choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\Omega}\frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}<\frac{\varepsilon}{2CC_0\|b\|_*},\ \text{for all } 0<t<\delta_0,$$

where C and C_0 are constants from (28) and (31), which is possible since $f\in V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$. Then

$$\sup_{x\in\Omega}CI_{1,\delta_0}(x,r)<\frac{\varepsilon}{2},\ \ 0<r<\delta_0,$$

by (30).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (10). We have

$$I_{2,\delta_0}(x,r)\le C_{\delta_0}\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|b\|_*\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)},$$

where C_{δ_0} is the constant from (29). Then, by (10) it suffices to choose r small enough such that

$$\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}<\frac{\varepsilon}{2CC_\delta\|b\|_*\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)}}$$

which completes the proof of (30).

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One 3D in the Geometrical Middle Problem in the Non-classical Treatment for one 3D Bianchi Integro-differential Equation with Non-smooth Coefficients

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Abstract. In this paper substantiated for a 3D Bianchi integro-differential equation with non-smooth coefficients a three dimensional in the geometrical middle problem- 3D in the geometrical middle problem with non-classical boundary conditions is considered, which requires no matching conditions. Equivalence of these conditions three dimensional boundary condition is substantiated classical, in the case if the solution of the problem in the isotropic S. L. Sobolev's space is found. The considered equation as a hyperbolic equation generalizes not only classic equations of mathematical physics (Laplace equation, telegraph equation, string vibration equation) and also many models differential equations (2D and 3D telegraph equation, 2D Bianchi equation, 2D and 3D wave equations and etc.). It is grounded that the in the middle boundary conditions in the classic and non-classic treatment are equivalent to each other. Thus, namely in this paper, the non-classic problem with 3D in the geometrical middle conditions is grounded for a hyperbolic equation of third-order. For simplicity, this was demonstrated for one model case in one of S.L. Sobolev isotropic space $W_p^{(1,1,1)}(G)$.

Key Words and Phrases: 3D in the geometrical middle problem, 3D Bianchi integro-differential equation, 3D mathematical modeling, hyperbolic equations, equation with non-smooth coefficients, equations with dominating mixed derivative.
2010 Mathematics Subject Classifications: 35L25, 35L35

1. Introduction

Hyperbolic equations are attracted for sufficiently adequate description of a great deal of real processes occurring in the nature, engineering and etc. In particular, many processes arising in the theory of fluid filtration in cracked media are described by non-smooth coefficient hyperbolic equations.

Urgency of investigations conducted in this field is explained by appearance of local and non-local problems for non-smooth coefficients equations connected with different applied problems. Such type problems arise for example, while studying the problems of moisture, transfer in soils, heat transfer in heterogeneous media, diffusion of thermal neutrons in inhibitors, simulation of different biological processes, phenomena and etc. [1-3].

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In the present paper, here consider three dimensional in the geometrical middle problem for 3D Bianchi integro-differential equation with non-smooth coefficients. The coefficients in this hyperbolic equation are not necessarily differentiable; therefore, there does not exist a formally adjoint differential equation making a certain sense. For this reason, this question cannot be investigated by the well-known methods using classical integration by parts and Riemann functions or classical-type fundamental solutions. The theme of the present paper, devoted to the investigation in the geometrical middle problem for 3D integro-differential Bianchi equations of hyperbolic type, according to the above-stated is very actual for the solution of theoretical and practical problems. From this point of view, the paper is devoted to the actual problems of applied mathematics and physics.

2. Problem statement

Consider 3D Bianchi integro-differential equation

$$\begin{gathered}(V_{1,1,1}u)(x,y,z) \equiv u_{xyz}(x,y,z) + A_{0,0,0}u(x,y,z) + A_{1,0,0}u_x(x,y,z)+ \\ A_{0,1,0}u_y(x,y,z) + A_{0,0,1}u_z(x,y,z) + A_{1,1,0}u_{xy}(x,y,z) + A_{0,1,1}u_{yz}(x,y,z)+ \\ +A_{1,0,1}u_{xz}(x,y,z) + \int\limits_{\sqrt{x_0x_1}}^{x}\int\limits_{\sqrt{y_0y_1}}^{y}\int\limits_{\sqrt{z_0z_1}}^{z}[K_{0,0,0}(\tau,\xi,\eta;x,y,z)u(\tau,\xi,\eta)+ \\ +K_{1,0,0}(\tau,\xi,\eta;x,y,z)u_x(\tau,\xi,\eta) + K_{0,1,0}(\tau,\xi,\eta;x,y,z)u_y(\tau,\xi,\eta)+ \\ +K_{0,0,1}(\tau,\xi,\eta;x,y,z)u_z(\tau,\xi,\eta) + K_{1,1,0}(\tau,\xi,\eta;x,y,z)u_{xy}(\tau,\xi,\eta)+ \\ +K_{0,1,1}(\tau,\xi,\eta;x,y,z)u_{yz}(\tau,\xi,\eta)+ \\ +K_{1,0,1}(\tau,\xi,\eta;x,y,z)u_{xz}(\tau,\xi,\eta)]\,d\tau d\xi d\eta = \varphi_{1,1,1}(x,y,z),\end{gathered} \tag{1}$$

$(x,y,z) \in G$.

Here $u(x,y,z)$ is a desired function determined on G; $A_{i,j,k} = A_{i,j,k}(x,y,z)$ are the given measurable functions on $G = G_1 \times G_2 \times G_3$, where $G_1 = (x_0,x_1)$, $x_0 \geq 0$, $G_2 = (y_0,y_1)$, $y_0 \geq 0$, $G_3 = (z_0,z_1)$, $z_0 \geq 0$,; $\varphi_{1,1,1}(x,y,z)$ is a given measurable function on G; $K_{i,j,k}(\tau,\xi,\eta;x,y,z)$ are the given measurable functions on $G \times G$.

Equation (1) is a three dimensional Bianchi integro-differential equation with three simple real characteristics $x = const,\ y = const,\ z = const$. Therefore, in some sense we can consider equation (1) as a hyperbolic equation. Equations of the form (1) are used in the modeling of vibration processes [4].

In the present paper 3D Bianchi integro-differential equation (1) is considered in the general case when the coefficients $A_{i,j,k}(x,y,z)$ are non-smooth functions satisfying only the following conditions:

$$A_{0,0,0}(x,y,z) \in L_p(G),$$

$$A_{1,0,0}(x,y,z) \in L^{x,y,z}_{\infty,p,p}(G),$$

$$A_{0,1,0}(x,y,z) \in L^{x,y,z}_{p,\infty,p}(G),$$

$$A_{0,0,1}(x,y,z) \in L^{x,y,z}_{p,p,\infty}(G),$$

$$A_{1,1,0}(x,y,z) \in L^{x,y,z}_{\infty,\infty,p}(G),$$

$$A_{0,1,1}(x,y,z) \in L^{x,y,z}_{p,\infty,\infty}(G),$$
$$A_{1,0,1}(x,y,z) \in L^{x,y,z}_{\infty,p,\infty}(G).$$

In addition, the kernels of integral operators are assumed to satisfy the following conditions: $K_{i,j,k}(\tau,\xi,\eta;x,y,z) \in L_\infty(G \times G)$.

Under these conditions, we'll look for the solution $u(x,y,z)$of equation (1) in S.L.Sobolev isotropic space

$$W_p^{(1,1,1)}(G) \equiv \{u(x,y,z) : D_x^i D_y^j D_z^m u(x,y,z) \in L_p(G),\ i,j,m = \overline{0,1}\},$$

where $1 \le p \le \infty$. $D_v^i = \partial^{'}/\partial v^{'}$ is a generalized differentiation operator in S.L.Sobolev sense, D_v^0 is an identity transformation operator. We'll define the norm in the space $W_p^{(1,1,1)}(G)$ by the equality

$$\|u\|_{W_p^{(1,1,1)}(G)} = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{m=0}^{1} \|D_x^i D_y^j D_z^m u\|_{L_p(G)}.$$

For 3D Bianchi integro-differential equation (1) we can give the classic form in the geometrical middle boundary conditions in the form :

$$\begin{cases} u/_{x=\sqrt{x_0x_1}} = \Phi(y,z), \\ u/_{y=\sqrt{y_0y_1}} = \Psi(x,z), \\ u/_{z=\sqrt{z_0z_1}} = g(x,y), \end{cases} \tag{2}$$

where $\Phi(y,z)$, $\Psi(x,z)$, and $g(x,y)$ are the given measurable functions on G . It is obvious that in the case of conditions (2), in addition to the conditions

$$\Phi \in W_p^{(1,1)}(G_2 \times G_3) \equiv \left\{ \approx{\Phi}(y,z) : D_y^j D_z^m \approx{\Phi}(y,z) \in L_p(G_2 \times G_3), \quad j,m = \overline{0,1} \right\}$$

$$\Psi \in W_p^{(1,1)}(G_1 \times G_3) \equiv \left\{ \approx{\Psi}(x,z) : D_x^i D_z^m \approx{\Psi}(x,z) \in L_p(G_1 \times G_3), \quad i,m = \overline{0,1} \right\}$$

and

$$g(x,y) \in W_p^{(1,1)}(G_1 \times G_2) \equiv \left\{ \approx{g}(x,y) : D_x^i D_y^j \approx{g}(x,y) \in L_p(G_1 \times G_2),\ i,\ j = \overline{0,1} \right\}$$

the given functions should also satisfy the following agreement conditions:

$$\begin{cases} \Phi\left(\sqrt{y_0y_1}\,, z\right) = \Psi\left(\sqrt{x_0x_1}, z\right), \\ \Phi\left(y, \sqrt{z_0z_1}\right) = g\left(\sqrt{x_0x_1}, y\right), \\ \Psi\left(x, \sqrt{z_0z_1}\right) = g\left(x, \sqrt{y_0y_1}\right), \end{cases} \tag{3}$$

Consider the following non-classical in the geometrical middle boundary conditions :

$$\begin{cases} V_{0,0,0}u \equiv u\left(\sqrt{x_0 x_1}, \sqrt{y_0 y_1}, \sqrt{z_0 z_1}\right) = \varphi_{0,0,0}, \\ (V_{1,0,0}u)(x) \equiv u_x\left(x, \sqrt{y_0 y_1}, \sqrt{z_0 z_1}\right) = \varphi_{1,0,0}(x), \\ (V_{0,1,0}u)(y) \equiv u_y\left(\sqrt{x_0 x_1}, y, \sqrt{z_0 z_1}\right) = \varphi_{0,1,0}(y), \\ (V_{0,0,1}u)(z) \equiv u_z\left(\sqrt{x_0 x_1}, \sqrt{y_0 y_1}, z\right) = \varphi_{0,0,1}(z), \\ (V_{1,1,0}u)(x,y) \equiv u_{xy}\left(x, y, \sqrt{z_0 z_1}\right) = \varphi_{1,1,0}(x,y), \\ (V_{0,1,1}u)(y,z) \equiv u_{yz}\left(\sqrt{x_0 x_1}, y, z\right) = \varphi_{0,1,1}(y,z), \\ (V_{1,0,1}u)(x,z) \equiv u_{xz}\left(x, \sqrt{y_0 y_1}, z\right) = \varphi_{1,0,1}(x,z), \end{cases} \tag{4}$$

where $\varphi_{0,0,0}$ is a given number, and $\varphi_{i,j,k}$ the rest are given functions that satisfy the following conditions:

$$\begin{gathered} \varphi_{1,0,0}(x) \in L_p(G_1), \\ \varphi_{0,1,0}(y) \in L_p(G_2), \\ \varphi_{0,0,1}(z) \in L_p(G_3), \\ \varphi_{1,1,0}(x,y) \in L_p(G_1 \times G_2), \\ \varphi_{0,1,1}(y,z) \in L_p(G_2 \times G_3), \\ \varphi_{1,0,1}(x,z) \in L_p(G_1 \times G_3). \end{gathered}$$

3. Methodology

Therewith, the important principal moment is that the considered equation possesses nonsmooth coefficients satisfying only some p -integrability and boundedness conditions i.e. the considered integro-differential operator $V_{1,1,1}$ has no traditional conjugated operator. In other words, the Riemann function for this equation can't be investigated by the classical method of characteristics. In the papers [5-7] the Riemann function is determined as the solution of an integral equation. This is more natural than the classical way for deriving the Riemann function. The matter is that in the classic variant, for determining the Riemann function, the rigid smooth conditions on the coefficients of the equation are required.

The Riemanns method does not work for hyperbolic equations with non-smooth coefficients. Especially it should be noted that a variety of boundary-value problems for the equations of Bianchi studied in [8-13] and etc.

In the present paper, a method that essentially uses modern methods of the theory of functions and functional analysis is worked out for investigations of such problems. In the main, this method it requested in conformity to integro-differential equations of third-order with simple real characteristics. Notice that, in this paper the considered equation is a generation of many model equations of some processes (for example, 2D and 3D telegraph equation, 2D Bianchi equation, 2D and 3D wave equations and etc).

If the function $u \in W_p^{(1,1,1)}(G)$ is a solution of the classical form 3D in the geometrical middle problem (1), (2), then it is also a solution of problem (1), (4) for $\varphi_{i,j,k}$ defined by the following equalities:

$$\begin{gathered}\varphi_{0,0,0}=\Phi\left(\sqrt{y_0y_1},\sqrt{z_0z_1}\right)=\Psi\left(\sqrt{x_0x_1},\sqrt{z_0z_1}\right)=g\left(\sqrt{x_0x_1},\sqrt{y_0y_1}\right),\\ \varphi_{1,0,0}(x)=\Psi_x\left(x,\sqrt{z_0z_1}\right)=g_x\left(x,\sqrt{y_0y_1}\right),\\ \varphi_{0,1,0}(y)=g_y\left(\sqrt{x_0x_1},y\right)=\Phi_y\left(y,\sqrt{z_0z_1}\right),\\ \varphi_{0,0,1}(z)=\Phi_z\left(\sqrt{y_0y_1},z\right)=\Psi_z\left(\sqrt{x_0x_1},z\right),\\ \varphi_{1,1,0}(x,y)=g_{xy}\left(x,y\right),\\ \varphi_{0,1,1}(y,z)=\Phi_{yz}\left(y,z\right),\\ \varphi_{1,0,1}(x,z)=\Psi_{xz}\left(x,z\right).\end{gathered}$$

The inverse one is easily proved. In other words, if the function $u\in W_p^{(1,1,1)}(G)$ is a solution of problem (1), (4), then it is also a solution of problem (1), (2) for the following functions:

$$\Phi(y,z)=\varphi_{0,0,0}+\int\limits_{\sqrt{y_0y_1}}^{y}\varphi_{0,1,0}(\beta)d\beta+\int\limits_{\sqrt{z_0z_1}}^{z}\varphi_{0,0,1}(\gamma)d\gamma+\int\limits_{\sqrt{y_0y_1}}^{y}\int\limits_{\sqrt{z_0z_1}}^{z}\varphi_{0,1,1}(\beta,\gamma)d\beta d\gamma,\tag{5}$$

$$\Psi(x,z)=\varphi_{0,0,0}+\int\limits_{\sqrt{x_0x_1}}^{x}\varphi_{1,0,0}(\alpha)d\alpha+\int\limits_{\sqrt{z_0z_1}}^{z}\varphi_{0,0,1}(\gamma)d\gamma+\int\limits_{\sqrt{x_0x_1}}^{x}\int\limits_{\sqrt{z_0z_1}}^{z}\varphi_{1,0,1}(\alpha,\gamma)d\alpha d\gamma,\tag{6}$$

$$g(x,y)=\varphi_{0,0,0}+\int\limits_{\sqrt{x_0x_1}}^{x}\varphi_{1,0,0}(\alpha)d\alpha+\int\limits_{\sqrt{y_0y_1}}^{y}\varphi_{0,1,0}(\beta)d\beta+\int\limits_{\sqrt{x_0x_1}}^{x}\int\limits_{\sqrt{y_0y_1}}^{y}\varphi_{1,1,0}(\alpha,\beta)d\alpha d\beta.\tag{7}$$

Note that the functions (5)-(7) possess one important property, more exactly, for all $\varphi_{i,j,k}$, the agreement conditions (3) possessing the above-mentioned properties are fulfilled for them automatically. Therefore, equalities (5)-(7) may be considered as a general kind of all the functions $\Phi(y,z),\Psi(x,z)$ and $g(x,y)$ satisfying the agreement conditions (3).

We have thereby proved the following assertion.

Theorem 1. *The 3D in the geometrical middle problem of the form (1), (2) and the non-classical form (1), (4) are equivalent.*

Note that the 3D in the geometrical middle problem in the non-classical treatment (1), (4) can be studied with the use of integral representations of special form for the functions $u\in W_p^{(1,1,1)}(G)$ [14-21]

$$u(x,y,z)=u\left(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1}\right)+\int\limits_{\sqrt{x_0x_1}}^{x}u_\xi\left(\xi,\sqrt{y_0y_1},\sqrt{z_0z_1}\right)d\xi+$$

$$+\int\limits_{\sqrt{y_0y_1}}^{y} u_\eta\left(\sqrt{x_0x_1},\eta,\sqrt{z_0z_1}\right)d\eta+\int\limits_{\sqrt{z_0z_1}}^{z} u_\gamma\left(\sqrt{x_0x_1},\sqrt{y_0y_1},\gamma\right)d\gamma+$$

$$+\int\limits_{\sqrt{x_0x_1}}^{x}\int\limits_{\sqrt{y_0y_1}}^{y} u_{\xi\eta}\left(\xi,\eta,\sqrt{z_0z_1}\right)d\xi d\eta+\int\limits_{\sqrt{y_0y_1}}^{y}\int\limits_{\sqrt{z_0z_1}}^{z} u_{\eta\gamma}\left(\sqrt{x_0x_1},\eta,\gamma\right)d\eta d\gamma+$$

$$+\int\limits_{\sqrt{x_0x_1}}^{x}\int\limits_{\sqrt{z_0z_1}}^{z} u_{\xi\gamma}\left(\xi,\sqrt{y_0y_1},\gamma\right)d\xi d\gamma+\int\limits_{\sqrt{x_0x_1}}^{x}\int\limits_{\sqrt{y_0y_1}}^{y}\int\limits_{\sqrt{z_0z_1}}^{z} u_{\xi\eta\gamma}\left(\xi,\eta,\gamma\right)d\xi d\eta d\gamma.$$

4. Result

So, the classical form 3D in the geometrical middle problem (1), (2) and in non-classical treatment (1), (4) are equivalent in the general case. However, the 3D in the geometrical middle problem (1), (4) is more natural by statement than problem (1), (2). This is connected with the fact that in statement of problem (1), (4) the right sides of boundary conditions don't require additional conditions of agreement type. Note that some boundary-value problems in non-classical treatments for hyperbolic and also pseudoparabolic equations were investigated in the authors papers [22-31].

5. Discussion and conclusions

In this paper a non-classical type 3D in the geometrical middle problem is substantiated for a 3D Bianchi integro-differential equation with non-smooth coefficients and with a third-order dominating derivative. Classic 3D in the middle conditions are reduced to non-classic 3D in the geometrical middle problem by means of integral representations. Such statement of the problem has several advantages: 1) No additional agreement conditions are required in this statement; 2) One can consider this statement as a 3D in the geometrical middle problem formulated in terms of traces in the S.L. Sobolev isotropic space $W_p^{(1,1,1)}(G)$; 3) In this statement the considered 3D Bianchi integro-differential equation is a generalization of many model differential equations of some processes (e.g. 2D and 3D telegraph equation, 2D Bianchi equation, 2D and 3D wave equations and etc.).

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Influence of Thickness of Reinforced Cylindrical Shell Filled by Liquid on Free Vibrations

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Abstract. Free oscillations of a reinforced cylindrical shell filled with a liquid are investigated. Based on the technical theory of cylindrical shells, the equations of motion are written down using classical equations in displacements. The fluid motion is potentially described by the wave equation. The liquid moves without detachment from the walls of the cylinders. The fluid pressure is taken into account in the equations of shell motion, and the fluid and shell velocities are equated at the boundaries. Representing the solution in a harmonic form, it converted into a system of transcendental equations. Comparison of the solution of the problem without a liquid with a solution in the presence of a liquid, we find the dependence of the frequency of the system without liquid with the frequency of the system with the liquid. In some values of the system parameters the natural frequencies of the cylinder oscillations are determined.

Key Words and Phrases: cylinder, density of cord filaments, the horizontal movement, the fluid density, volume fraction of cord.

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1. Introduction

Circular cylindrical covers emerge with the elements in designs of flying machines and engines, underwater and surface means of transportation, tanks and pipelines, vaulted systems of underwater and underground tunnels and storehouses. Cylindrical covers were widely adopted in the technique. One of the basic spheres of their application are hydraulic systems where such covers are applied in the quality of flexible inserts. Mathematical description of fluctuations of the reinforced covers with fluid is devoted to the set of works [1-7].

One of the most important points in the investigation of fluctuations of covers emerges to be determination of frequencies of free fluctuations that allows to avoid a resonance from external sources of fluctuations or on the contrary to use in need of heat exchange at hashing of liquid products. It is necessary to cancel that the majority considered works are devoted to the elementary special case or to the approached methods.

In work [8], free fluctuations of the thin-walled cylindrical cover containing the compressed liquid are investigated. At some values of parameters of system its own frequencies of fluctuations are defined and influence of geometrical and physical parameters of system a cylindrical cover-liquid on free fluctuation of the cylinder is investigated.

In work [9] frequencies and forms of free fluctuations of the spherical and cylindrical covers contacting to elastic and liquid environments are investigated. Asymptotical methods receive the approached simple formulas for calculation of frequency and definition of the form of fluctuations of the considered systems that limits use of the received results as possibility of carrying out of the qualitative analysis of investigated processes excludes in a number of important cases.

In work [10] the problem of movement of the firm cylinder keeping vertical position under the influence of superficial waves in a liquid is considered. Change of a surface of a liquid is separated to two parts: result of a falling harmonious wave and the indignation caused by presence of the cylinder which thus moves. The problem is accomplished with an operational method. For a finding of the original solution, considering that the image represents a denominator of tabular function, Voltaire's integrated equation of the first sort is used.

2. Problem statement

In the given work free fluctuations of the reinforced cylindrical cover filled with a liquid are investigated. The case of orthotropic covers when cord threads keeps within symmetrically concerning a cover meridian is being considered. The reinforced cover, represents multilayered composite consisting of layers filler and a cord. As the finding of own frequencies of system a cylindrical cover-liquid is connected with the decision of the transcendental equations, frequency of fluctuations of the cover which are not containing a liquid, is expressed through frequency of fluctuations of system in an explicit form that allows both analytically, and graphically to investigate spectra of frequencies of system.

For the description of movement of a cover will use the classical equations in movements [11]. Fluctuations of the liquid filling a cover, are described by the wave equation in cylindrical co-ordinates [12]. On border of contact of a cover with a liquid equality of radial speeds [13] is set.

Thus, fluctuations of considered system is described by the equations:

$$
\begin{aligned}
A_{11}u + A_{12}v + A_{13}w &= \rho_s h \frac{\partial^2 u}{\partial t^2};\\
A_{21}u + A_{22}v + A_{23}w &= \rho_s h \frac{\partial^2 v}{\partial t^2};\\
A_{31}u + A_{32}v + A_{33}w &= -\left(\rho_s h \frac{\partial^2 w}{\partial t^2} + \rho_f \frac{\partial \Phi}{\partial t}\right).
\end{aligned}
\tag{1}
$$

Here

$$
A_{11} = C_{11}\frac{\partial^2}{\partial x^2} + \frac{C_{66}}{R^2}\frac{\partial^2}{\partial \varphi^2};
$$

$$
A_{12} = A_{21} = \frac{C_{12} + C_{66}}{R}\frac{\partial^2}{\partial x \partial \varphi};
$$

$$A_{13} = A_{31} = \frac{1}{R}\left(C_{12}\frac{\partial}{\partial x}\right);$$

$$A_{22} = \left(C_{66} + \frac{4}{R^2}D_{66}\right)\frac{\partial^2}{\partial x^2} + \frac{1}{R^2}\left(C_{22} + \frac{1}{R^2}D_{22}\right)\frac{\partial^2}{\partial \varphi^2};$$

$$A_{23} = A_{32} = \frac{1}{R}\left(\frac{C_{22}}{R}\frac{\partial}{\partial \varphi} - \frac{1}{R}\left(D_{12} + 4D_{66}\right)\frac{\partial^3}{\partial x^2 \partial \varphi} - \frac{D_{22}}{R^3}\frac{\partial^3}{\partial \varphi^3}\right);$$

$$A_{33} = \frac{1}{R^2}C_{22} + D_{11}\frac{\partial^4}{\partial x^4} + \frac{2}{R^2}\left(D_{12} + 2D_{66}\right)\frac{\partial^4}{\partial x^2 \partial \varphi^2} + \frac{\partial^4}{\partial \varphi^4}. \tag{2}$$

Here $C_{ik} = hB_{ik}$ $D_{ik} = \frac{h^3}{12}B_{ik}$.

$$B_{11} = B'_{11}\cos^4\theta + 2\left(B'_{12} + 2B'_{66}\right)\sin^2\theta\cos^2\theta + B'_{22}\sin^4\theta B_{22} =$$
$$= B'_{11}\sin^4\theta + 2\left(B'_{12} + 2B'_{66}\right)\sin^2\theta\cos^2\theta + B'_{22}\cos^4\theta B_{12} =$$
$$= B'_{12} + \left(B'_{11} + B'_{22} - 2\left(B'_{12} + 2B'_{66}\right)\right)\sin^2\theta\cos^2\theta(3)B_{66} =$$
$$= B'_{66} + \left(B'_{11} + B'_{22} - 2\left(B'_{12} + 2B'_{66}\right)\right)\sin^2\theta\cos^2\theta, \tag{3}$$

where

$$B'_{11} = \frac{E_1}{1 - \nu_1\nu_2}; \quad B'_{22} = \frac{E_2}{1 - \nu_1\nu_2}; \;\; B'_{66} = G; \quad B'_{12} = \frac{\nu_2 E_1}{1 - \nu_1\nu_2} = \frac{\nu_1 E_2}{1 - \nu_1\nu_2},$$

$E_1,\ E_2,\ \nu_1,\ \nu_2$- composite parameters on the elasticity mainstreams, calculated under formulas [14]:

$$E_1 = E_bV_b + E_m(1 - V_b), \frac{1}{E_2} = \frac{V_b}{E_b} + \frac{(1 - V_b)}{E_{m,}}, \nu_1 = v_bV_b + v_m\left(1 - V_b\right), v_2 = v_2\frac{E_2}{E_1},$$

$$\frac{1}{G} = \frac{V_b}{G_b} + \frac{(1 - V_b)}{G_m},$$

where E_b, G_b, v_b- the module the Ship's boy, the module of shift and factor of Puassona, and E_m, $G_{m,}\, v_m$- corresponding parameters filler; V_b- a cord volume fraction.

The density is defined from expression

$$\rho_s = \rho_bV_b + \rho_m\left(1 - V_b\right),$$

where ρ_b and ρ_m- density of threads of a cord and filler respectively.

Here following designations are accepted: ρ_s- cover density, ρ_f- liquid density, h- a thickness of a cover, R- radius of a median plane of a cover, B_{ik}- elastic parameters of the generalized law of Guka in cylindrical system of co-ordinates of a cover.

The potential Φ satisfies to the wave equation:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial \varphi^2} + \frac{\omega^2}{a^2}\Phi = 0. \tag{4}$$

On border between a liquid and a cover compatibility of movement answers a condition.

$$\frac{\partial w}{\partial t} = -\frac{\partial \Phi}{\partial r}\bigg|_{r=R}. \tag{5}$$

The decision of system (1) is represented in a kind:

$$\begin{aligned} u &= u_n \cos n\varphi \sin \omega t \cos \frac{\pi x}{l}, \\ \vartheta &= v_n \sin n\varphi \sin \omega t \sin \frac{\pi x}{l}, \\ w &= w_n \cos n\varphi \sin \omega t \sin \frac{\pi x}{l}, \end{aligned} \tag{6}$$

$$\Phi = \Phi_n(r) \cos n\varphi \cos \omega t \sin \frac{\pi x}{l}. \tag{7}$$

Here $\frac{\pi}{l} = k$. Let's substitute (7) in (4), we will receive

$$\Phi_n'' + \frac{1}{r}\Phi_n' + \left(\frac{\omega^2}{a^2} - k^2 - \frac{n^2}{r^2}\right)\Phi_n = 0. \tag{8}$$

In the cylinder the decision of the equation (8) looks as follows [15]:

$$\Phi_n(r) = CJ_n\left(\sqrt{\frac{\omega^2}{a^2} - k^2}\; r\right). \tag{9}$$

Considering (9) in (7)

$$\Phi = CJ_n\left(\sqrt{\frac{\omega^2}{a^2} - k^2}\; r\right) \cos n\varphi \cos \omega t \sin kx. \tag{10}$$

Here C- it is constant, $J_n\left(\sqrt{\frac{\omega^2}{a^2} - k^2}\; r\right)$- function of Bessel of an order n. And applying (6) and (10) in (5), we receive

$$C = -\frac{w_n \omega}{J_n'\left(\sqrt{\frac{\omega^2}{a^2} - k^2} R\right)}. \tag{11}$$

Having substituted (11) in (10), we will receive

$$\Phi = -\frac{w_n \omega J_n\left(\sqrt{\frac{\omega^2}{a^2} - k^2} R\right)}{J_n'\left(\sqrt{\frac{\omega^2}{a^2} - k^2} R\right)} \cos n\varphi \cos \omega t \sin kx. \tag{12}$$

Here a- speed of a sound in a liquid, ω- circular frequency, J_n, J_n'- functions of Bessel of an order n. Having substituted (6) and (12) in (1), we have

$$-\left(k^2C_{11}+\frac{n^2}{R^2}C_{66}\right)u_n+\frac{kn}{R}\left(C_{12}+C_{66}\right)\upsilon_n-\frac{k}{R}C_{12}w_n+\rho_s h\omega^2 u_n=0,$$

$$\frac{kn}{R}\left(C_{12}+C_{66}\right)u_n-\left(k^2\left(C_{66}+\frac{4}{R^2}D_{66}\right)-\frac{n^2}{R^2}\left(C_{22}+\frac{1}{R^2}D_{22}\right)+\rho_s h\omega^2\right)\vartheta_n+$$

$$+\frac{1}{R^2}\left(-nC_{22}+k^2n\left(D_{12}+4D_{66}\right)-\frac{n^3}{R}D_{22}\right)w_n=0, \tag{13}$$

$$-\frac{k}{R}C_{12}u_n+\frac{1}{R^2}\left(nC_{22}+k^2n\left(D_{12}+4D_{66}\right)+\frac{n^3}{R^2}D_{22}\right)\upsilon_n+$$

$$+\left(\frac{1}{R^2}C_{22}+k^4D_{11}+\frac{2k^2n^2}{R^2}\left(D_{12}+2D_{66}\right)+\frac{n^4}{R^4}D_{22}-\right.$$

$$\left.-\rho_s h\omega^2+\omega^2\,\rho_f\frac{J_n\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)}{J_n'\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)}\right)w_n=0.$$

For simplification it is entered (13) following designations:

$$\begin{gathered}\left(\alpha_{11}+\rho_s h\omega^2\right)u_n+\alpha_{12}\vartheta_n+\alpha_{13}w_n=0,\\ \alpha_{21}u_n+\left(\alpha_{22}+\rho_s h\omega^2\right)\upsilon_n+\alpha_{23}w_n=0,\\ \alpha_{31}u_n+\alpha_{32}\vartheta_n+\left(\alpha_{33}-\rho_s h\omega^2+\omega^2\,\rho_f\frac{J_n\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)}{J_n'\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)}\right)w_n=0.\end{gathered} \tag{14}$$

Here

$$\alpha_{11}=-\left(k^2C_{11}+\frac{n^2}{R^2}C_{66}\right),\alpha_{23}=\frac{kn}{R}\left(C_{12}+C_{66}\right),$$

$$\alpha_{12}=\alpha_{21}=\frac{kn}{R}\left(C_{12}+C_{66}\right),\alpha_{22}=-\left(k^2\left(C_{66}+\frac{4}{R^2}D_{66}\right)+\frac{n^2}{R^2}\left(C_{22}+\frac{1}{R^2}D_{22}\right)\right),$$

$$\alpha_{13}=\alpha_{31}=-\frac{k}{R}C_{12},\alpha_{23}=-\frac{n}{R^2}C_{22}+\frac{k^2n}{R^2}\left(D_{12}+4D_{66}\right)-\frac{n^3}{R^4}D_{22},$$

$$\alpha_{32}=\frac{n}{R^2}C_{22}+\frac{k^2n}{R^2}\left(D_{12}+4D_{66}\right)+\frac{n^3}{R^4}D_{22},$$

$$\alpha_{33}=\frac{1}{R^2}C_{22}+k^4D_{11}+\frac{2k^2n^2}{R^2}\left(D_{12}+2D_{66}\right)+\frac{n^4}{R^4}D_{22}.$$

Let's write out a condition non-triviality decision of system (14) rather $u_n,\ v_n,\ w_n$:

$$\begin{vmatrix} \alpha_{11}+\rho_s h\omega^2 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22}+\rho_s h\omega^2 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}-\rho_s h\omega^2+\omega^2\rho_f \frac{J_n\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)}{J_n'\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)} \end{vmatrix} = 0.$$

From here we will receive:

$$\begin{gathered} \left(\alpha_{11}+\rho_s h\omega^2\right)\left(\alpha_{22}+\rho_s h\omega^2\right)\left(\alpha_{33}-\rho_s h\omega^2+\omega^2\rho_f \frac{J_n\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)}{J_n'\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)}\right)+\alpha_{12}\alpha_{23}\alpha_{31}+ \\ +\alpha_{13}\alpha_{21}\alpha_{32}-\left(\alpha_{22}+\rho_s h\omega^2\right)\alpha_{13}\alpha_{31}- \\ -\alpha_{12}\alpha_{21}\left(\alpha_{33}-\rho_s h\omega^2+\omega^2\rho_f \frac{J_n\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)}{J_n'\left(\sqrt{\frac{\omega^2}{a^2}-k^2}R\right)}\right)- \\ -\alpha_{32}\left(\alpha_{11}+\rho_s h\omega^2\right)\alpha_{23}=0, \end{gathered} \tag{15}$$

$$\begin{gathered} -\rho_s^3 h^3\omega^6+\left(-\alpha_{11}-\alpha_{22}+\alpha_{33}\right)\rho_s^2 h^2\omega^4+ \\ +\left(\alpha_{11}\alpha_{33}+\alpha_{22}\alpha_{33}-\alpha_{11}\alpha_{22}-\alpha_{13}\alpha_{31}-\alpha_{32}\alpha_{23}+\alpha_{12}\alpha_{21}\right)\rho h\omega^{\ 2}+ \\ +\alpha_{11}\alpha_{22}\alpha_{33}+\rho_s^2 h^2\omega^6 a\rho_f\frac{J_n}{J_n'}+\alpha_{22}\rho_s h\omega^4\rho_f\frac{J_n}{J_n'}+\alpha_{11}\rho_s h\omega^4\rho_f\frac{J_n}{J_n'}+ \\ +\alpha_{11}\alpha_{22}\omega^2\rho_f\frac{J_n}{J_n'}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}- \\ -\alpha_{13}\alpha_{31}\alpha_{22}-\alpha_{11}\alpha_{32}\alpha_{23}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{12}\alpha_{21}\omega^2\rho_f\frac{J_n}{J_n'}=0. \end{gathered} \tag{16}$$

The equation (16) represents cubic the equation:

$$\Omega_1^3+A_1\Omega_1^2+A_2\Omega_1+A_3=0. \tag{17}$$

Here

$$\Omega_1=\rho_s h\omega^2, \tag{18}$$

$$A_1=\alpha_{11}+\alpha_{22}-\alpha_{33},$$

$$A_2 = -\alpha_{11}\alpha_{33} - \alpha_{22}\alpha_{33} + \alpha_{11}\alpha_{22} + \alpha_{13}\alpha_{31} + \alpha_{32}\alpha_{23} - \alpha_{12}\alpha_{21},$$

$$A_3 = -\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32} + \alpha_{13}\alpha_{31}\alpha_{22} + \alpha_{11}\alpha_{32}\alpha_{23} + \alpha_{12}\alpha_{21}\alpha_{33} -$$
$$-\rho_s^2 h^2 \omega^6 \rho_f \frac{J_n}{J_n'} - \alpha_{22}\rho_s h\omega^4 \rho_f \frac{J_n}{J_n'} - \alpha_{11}\rho_s h\omega^4 \rho_f \frac{J_n}{J_n'} -$$
$$-\alpha_{11}\alpha_{22}\omega^2 \rho_f \frac{J_n}{J_n'} + \alpha_{12}\alpha_{21}\omega^2 \rho_f \frac{J_n}{J_n'}.$$

Let's define [14] Ω_1 of (16)

$$\Omega_1 = y - \frac{A_1}{3}, \tag{19}$$

$$y^3 + py + q = 0,$$

$$y_1 = A + B; \quad y_{2,3} = -\frac{A+B}{2} \pm i\frac{A-B}{2}\sqrt{3}, \tag{20}$$

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}}; \quad B = \sqrt[3]{-\frac{q}{2} - \sqrt{Q}}; \quad Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{3}\right)^2,$$

$$p = -\frac{A_1^2}{3} + A_2; \quad q = 2\left(\frac{A_1}{3}\right)^3 - \frac{A_1 A_2}{3} + A_3.$$

In case of absence of a liquid ($\rho = 0$) the equation (17) will become

$$\left(\Omega_1^0\right)^3 + A_1^0\left(\Omega_1^0\right)^2 + A_2^0\Omega_1^0 + A_0^3 = 0. \tag{21}$$

Here

$$\Omega_1^0 = \rho_s h \left(\omega_0\right)^2, \tag{22}$$

where ω_0 - frequency of free fluctuations of a cover without a liquid.

$$A_1^0 = A_1; \quad A_2^0 = A_2;$$

$$A_3^0 = \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32} - \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{12}\alpha_{21}\alpha_{32}.$$

The decision of the equation (21) becomes [15]:

$$\Omega_1^0 = y_1^0 - \frac{A_1^0}{3}, \tag{23}$$

$$y_1^0 = A_0 + B_0; \quad y_{2,3}^0 = -\frac{A_0+B_0}{2} \pm i\frac{A_0-B_0}{2}\sqrt{3}, \tag{24}$$

$$A_0 = \sqrt[3]{-\frac{q_0}{2} + \sqrt{Q_0}}; \quad B_0 = \sqrt[3]{-\frac{q_0}{2} - \sqrt{Q_0}}; \quad Q_0 = \left(\frac{p_0}{3}\right)^3 + \left(\frac{q_0}{3}\right)^2,$$

$$p_0 = p,\ q_0 = 2\left(\frac{A_1^0}{3}\right)^3 - \frac{A_1^0 A_2^0}{3} + A_3^0.$$

Considering (20) in (19):

$$\Omega_1 = A + B - \frac{A_1}{3}, \tag{25}$$

and considering (24) in (23), we receive

$$\Omega_1^0 = A_0 + B_0 - \frac{A_1^0}{3},$$

from here:

$$\frac{\Omega_1^0}{\Omega_1} = \frac{A_0 + B_0 - \frac{A_1^0}{3}}{A + B - \frac{A_1}{3}}.$$

From here

$$\Omega_1^0 = \frac{A_0 + B_0 - \frac{A_1^0}{3}}{A + B - \frac{A_1}{3}}\Omega_1.$$

In other parties from (19) and (22):

$$\omega_0 = \sqrt{\frac{A_0 + B_0 - \frac{A_1^0}{3}}{A + B - \frac{A_1}{3}}\omega}. \tag{26}$$

The formula (26) expresses dependence ω_0 from ω.

The equation (26) connects free frequency of system with free frequency of a cover when there is lack of a liquid. The finding of frequencies of free fluctuations of system is associated with the decision of the transcendental equation (17) at which decision authors often resort to the approached methods, in particular to asymptotic. However, the decision of a return problem allows to build schedules of dependence of frequencies of fluctuations for various fashions of system from frequency of an empty cover that simplifies research, including definition of frequency of free fluctuations of system.

For some values of the system parameters, the dependence of the vibration frequencies for different modes of the system on the frequency of the empty shell is plotted. The influence of the geometric and physical parameters of the system on the free oscillation of the cylinder is studied.

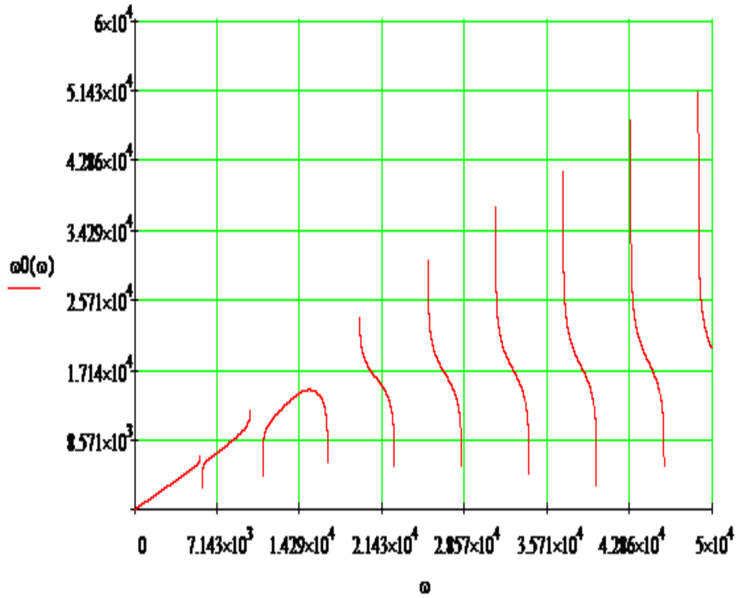

Fig.1
The effect of the thickness of a reinforced cylindrical shell filled with a liquid on free oscillations ($h = 0.08$; h- shell thickness)

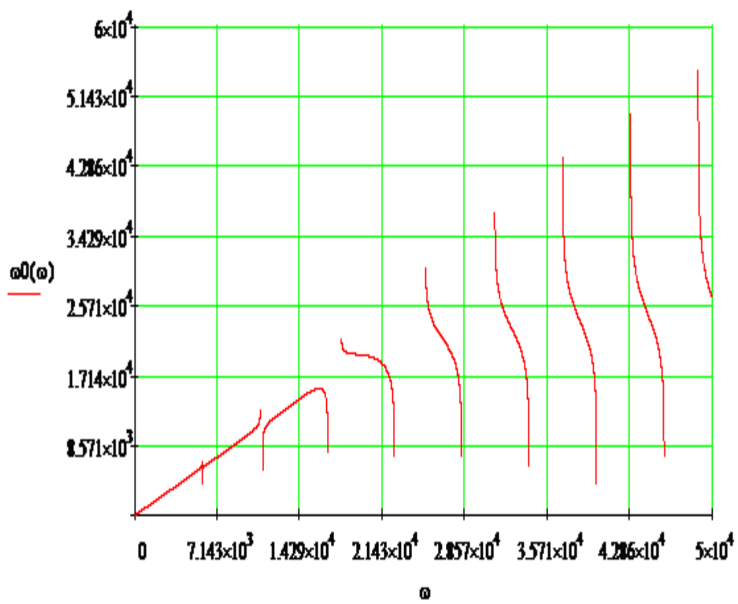

Fig.2
The effect of the thickness of a reinforced cylindrical shell filled with a liquid on free oscillations ($h = 0.5$; h- shell thickness)

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Integral Representations of Functions From the Spaces $S_p^l W(G)$, $S_{p,\theta}^l B(G)$ and $S_{p,\theta}^l F(G)$

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Abstract. In the paper we construct an integral representation of functions from $S_p^l W(G)$, $S_{p,\theta}^l B(G)$ and $S_{p,\theta}^l F(G)$, defined in n-dimensional domains and satisfying the flexible φ-horn condition.

Key Words and Phrases: integral representations, flexible φ-horn, the spaces type functions with dominant mixed derivatives.
2010 Mathematics Subject Classifications: 26E30, 26E35

1. Introduction

Integral representation of functions from the spaces with dominant mixed derivative Sobolev - $S_p^l W(G)$ Besov -$S_{p,\theta}^l B(G)$ Lizorkin-Triebel- $S_{p,\theta}^l F(G)$ in the case when the domain $G \subset R^n$ satisfies the conditions of rectangles, was first studied in the paper of A.J. Jabrailov [3], and then in the papers of R.A. Mashiev [5], M.K. Aliyev [1] and others, in the case when the domain $G \subset R^n$ satisfies the "flexible horn condition", in the papers of A.M. Najafov [5], [6], [7].

In this paper we construct an integral representation of functions from these spaces, defined in n-dimensional domains and satisfying the flexible φ-horn condition. Let vector functions $\varphi(t) = (\varphi_1(t_1), ..., \varphi_n(t_n))$ be differentiable continuous on $[0, T_j]$ $(0 < T_j < \infty)$, $\varphi_j(t_j) > 0\,(t_j > 0)$, $\lim\limits_{t_j \to +0} \varphi_j(t_j) = 0$, $\lim\limits_{t_j \to +\infty} \varphi_j(t_j) = A_j \leq \infty\,(j = 1, 2, ..., n)$. Suppose that $e_n = \{1, 2, ..., n\}$, $e \subseteq e_n$ and for each $x \in G$ consider the vector-function

$$\rho(\varphi(t), x) = (\rho_1(\varphi_1(t_1), x), ..., \rho_n(\varphi_n(t_n), x)), \quad 0 \leq t_j \leq T_j, \quad j \in e_n,$$

where $\rho_j(0, x) = 0$ for all $j \in e_n$, the functions $\rho_j(\varphi_j(t_j), x)$ are absolutely continuous on $[0, T_j]$ and $|\rho_j(\varphi_j(t_j), x)| \leq 1$ for almost all $t_j \in [0, T_j]$, $\rho'_j(u_j, x) = \frac{\partial}{\partial u_j} e_j(u_j, x)$, $j \in e_n$. Given $\theta[0,1]^n$, each of the sets $V(x, \theta) = \bigcup\limits_{0 < t_j \leq T_j} [\rho(\varphi(t), x) + \varphi(t)\theta I]$ and $x + V(x, \theta) \subset G$, where $I = [-1, 1]^n$, $\varphi(t)\theta I = \{(\varphi_1(t_1)\theta_1 y_1, ..., \varphi_n(t_n)\theta_n y_n) : y \in I\}$, is called a flexible φ-horn and the point x is called the vertex of the flexible φ-horn $x + V(x, \theta)$. In the case $\varphi_j(t_j) = t_j$ the set $x + V(x, \theta)$ is called the flexible horn introduced in [6], [7].

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Let $1^e = (\delta_1^e, ..., \delta_n^e)$, where $\delta_j^e = 1$ for $j \in e$ and $\delta_j^e = 0$ for j$\in e_n \backslash e = e'$. We suppose that $f \in L^{loc}(G)$ has all needed generalized derivatives on G. Introduce the average of f as follows:

$$f_{\varphi(t)}(x) = \prod_{j \in e_n} (\varphi_j(t_j))^{-1} \int_{R^n} f(x+y)\,\Omega\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}\right) dy, \tag{1}$$

where $\Omega(y,z) = \prod\limits_{j \in e_n} \omega_1(y_j, z_j)$, $\frac{y}{\varphi(t)} = \left(\frac{y_1}{\varphi_1(t_1)}, ..., \frac{y_n}{\varphi_n(t_n)}\right)$, in case $\varphi_j(t_j) = t^{\lambda_j}$ is the kernel introduced by O.V. Besov [2]. The average of (1) is constructed from the values of f at the points $x + y \in x + V(x, \theta) \subset G$. Let $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, $0 < \varepsilon_j < T_j$ $(j \in e_n)$. Then the following equality is valid:

$$f_{\varphi(\varepsilon)}(x) = \sum_{e \subseteq e_n} (-1)^{|1^e|} \int_{\varepsilon^{en}}^{T^e} D_t^{1^e} f_{\varphi\left(t^e + T^{e'}\right)}(x)\, dt^e, \tag{2}$$

where $t^e + T^{e'} = t_j$ $j = e$; $t^e + T^{e'} = T_j, j \in e^i$ and $\int\limits_{a^e}^{b^e} f(x)\, dx^e = \left(\prod\limits_{j \in e_n} \int\limits_{a_j}^{b_j} dx_j\right) f(x)$ i.e., integration is carried out only with respect to the variables x_j whose indices belong to e.

Differentiating with respect to t_j $(j \in e)$, and using [2], we obtain

$$D^{1^e} f_{\varphi\left(t^e + T^{e'}\right)}(x) = \prod_{j \in e} \frac{\partial}{\partial t_j} f_{\varphi\left(t^e + T^{e'}\right)} = (-1)^{|1^e|} \prod_{j \in e'} (\varphi_j(T_j))^{-1} \times$$

$$\times \int_{R^n} K_e^{(k^e + 1^e)} \left(\frac{y}{\varphi(t^e + T^{\varepsilon'})}, \frac{\rho\left(\varphi\left(t^e + T^{e'}\right), x\right)}{\varphi(t^e + T^{e'})}, \rho'\left(\varphi\left(t^e + T^{e'}\right), x\right)\right) \times$$

$$\times = (-1)^{|1^e|} \prod_{j \in e'} (\varphi_j(T_j))^{-1} \times (\varphi_j(t_j))^{-1} \prod_{j \in e} \frac{1}{\varphi_j(t_j)} \prod_{j \in e} \frac{\partial}{\partial t_j} \varphi_j(t_j)\, dy, \tag{3}$$

where $k = (k_1, ..., k_n)$, k_j -number in the kernel Ω can be chosen arbitrarily large,

$$K_e(x,y,z) = \prod_{j \in e'} \omega_j(x_j, y_j) \prod_{j \in e} \rho_j(x_j, y, z_j) \in C_0^\infty(R^n \times R^n \times R^n),$$

ρ_j -is defined in [7] and

$$K_e^{(\alpha)}(x,y,z) = D_x^{(\alpha)} K_e(x,y,z), \int_{R^n} K_e^{(\alpha)}(x,y,z)\, dx = 0 \;\text{ for all } y, z,$$

and α such that $|\alpha| > 0$.

By (2) from (3) we derive

$$f_{\varphi(\varepsilon)}(x)=\sum_{e\subseteq e_n}\prod_{j\in e'}(\varphi_j(T_j))^{-1}\int\limits_{\varepsilon^e}^{T^e}\int\limits_{R^n}K_e^{(k^e+1^e)}\times$$

$$\times\left(\frac{y}{\varphi\left(t^e+T^{\varepsilon'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)},\rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\times$$

$$\times f(x+y)\prod_{j\in e}(\varphi_j(t_j))^{-2}\prod_{j\in e}\varphi_j'(t_j)\,dt^e dy. \tag{4}$$

Then, in view of the Remark on Lemma 5.2 of [2], we have the following: if $f\in L^{loc}(G)$ and $1\le p<\infty$, then $f_{\varphi(\varepsilon)}\to f(x)$ as $\varepsilon_j\to 0\,(j\in e_n)$, moreover, for $p>1$ we have $f_{\varphi(\varepsilon)}(x)\to f(x)$ for almost all $x\in G$ by the Remark on Theorem 1.7 of [2]. Then it follows from (4) that

$$f(x)=\sum_{e\subseteq e_n}\prod_{j\in e'}(\varphi_j(T_j))^{-1}\times$$

$$\times\int\limits_{\varepsilon^e}^{T^e}\int\limits_{R^n}K_e^{(k^e+1^e)}\left(\frac{y}{\varphi\left(t^e+T^{\varepsilon'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)},\rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\times$$

$$\times f(x+y)\prod_{j\in e}(\varphi_j(t_j))^{-2}\prod_{j\in e}\varphi_j'(t_j)\,dt^e dy. \tag{5}$$

Let $l=(l_1,...,l_n)$, $l_j\in N$, $l^e=(l_1^e,...,l_n^e)$, $l_j^e=l_j$ for $j\in e$, $l_j^e=0$ for $j\in e'$, and let functions f having on G the generalized mixed derivatives $D^{l^e}f\in L^{loc}(G)$ and suppose that $l_j\le k_j$ for $j\in e$

$$f_{\varphi(\varepsilon)}(x)=\sum_{e\subseteq e_n}(-1)^{|l^e|}\prod_{j\in e'}(\varphi_j(T_j))^{-1}\times$$

$$\times\int\limits_{\varepsilon^e}^{T^e}\int\limits_{R^n}M_e\left(\frac{y}{\varphi\left(t^e+T^{\varepsilon'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)},\rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\times$$

$$\times D^{l^e}f(x+y)\prod_{j\in e}(\varphi_j(t_j))^{-2+l_j}\varphi_e'(t)\,dt^e dy. \tag{6}$$

where $\varphi_e'(t)=\prod_{j\in e}\varepsilon\varphi_j'(t_j)$, $M_e(x,y,z)=D_x^{k^e}+1^e-l^eK_e(x,y,z)$. Suppose that we obtain

$$f_{\varphi(\varepsilon)}^{(\nu)}(x)=\sum_{e\subseteq e_n}(-1)^{|\nu|+||l^e||}\prod_{j\in e'}(\varphi_j(T_j))^{-1-\nu_j}\times$$

$$\times\int\limits_{\varepsilon^e}^{T^e}\int\limits_{R^n} M_e^{(\nu)}\left(\frac{y}{\varphi\left(t^e+T^{\varepsilon'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)},\rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)D^{l^e}f\left(x+y\right)\times$$

$$\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{-2+l_j-\nu_j}\varphi_e'\left(t\right)dt^e dy. \tag{7}$$

Suppose that $\varphi_j\left(\varphi_j\left(t_j\right),x\right)$ and $\rho_j'\left(\varphi_j\left(t_j\right),x\right)$ as functions of $\left(\varphi_j\left(t_j\right),x\right)$ are locally summable on $(0,T_j]\times U$ $(j=1,...,n)$, where $U\subset G$ is an open set. Let $\nu=(\nu_1,...,\nu_n)\in N_0^n$; moreover, $l_j\le\nu_j+k_j$ for $j\in e$, and $l_j\le k_j$ for $j\in e'$. Applying the differentiation D_x^ν to both sides of (4) (and moving the differentiation onto the kernel in the summands on the right-hand side), we obtain.

Show now that if

$$\mu_j=l_j-\nu_j>0,\quad j\in e_n \tag{8}$$

then the generalized mixed derivative $D^\nu f\in L_p\left(G\right)$ exists on G. First, establish that

$$f_{\varphi(\varepsilon)}^{(\nu)}-f_{\varphi(\eta)}^{(\nu)}\to 0\quad\text{ as }\quad 0<\varepsilon_j<\eta_j\to 0,\ \ j\in e_n, \tag{9}$$

in $L^{loc}\left(U\right)$. Let $F\subset U$ be a compact set. Then $F+hI\subset U$ for some $h>0$. Put

$$M^{(\nu)}\left(x\right)=\max_{e\subseteq e_n}\max_{y_1 z\in I}\left|M_e^{(\nu)}\left(x,y,z\right)\right|.$$

By Minkowski's inequality, for sufficiently small ε and $T\equiv\eta$ we have

$$\left\|f_{\varphi(\varepsilon)}^{(\nu)}-f_{\varphi(\eta)}^{(\nu)}\right\|_{1,F+hI}\le C\sum_{e\subseteq e_n}\prod_{j\in e}\left(\varphi_j\left(\eta_j\right)\right)^{\mu_j-\nu_j}\left\|D^{\lambda^e}f\right\|_{1,F+hI}.$$

From here and (8) we obtain (9). Suppose that $D^\nu f$ exists on G, i.e.,

$$f_{\varphi(t)}^{(\nu)}\left(x\right)=D^\nu f\left(x\right)$$

for $x+C\varphi\left(t\right)I\subset G$ with some $C=(C_1,...,C_n)>0$. Pass to the limit in (7) as $\varepsilon_j\to 0\,(j\in e_n)$, observing that the limit exists in the sense of $L^{loc}\left(U\right)$ by (8) and almost everywhere on U by the relation $f_{\varphi(t)}^{(\nu)}\to f\left(x\right)$ as $\varphi_j\left(t_j\right)\to 0\,(j\in e_n)$ applied to $D^\nu f$. Then the equality

$$D^\nu f=\sum_{e\subseteq e_n}(-1)^{|\nu|+|l^e|}\prod_{j\in e}\left(\varphi_j\left(T_j\right)\right)^{-1-\nu_j}\int\limits_{o^e}^{T^e}\int\limits_{R^n}M_e^{(\nu)}\left(,,\right)D^{l^e}f\left(x+y\right)\times$$

$$\times\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{-2+l_j-\nu_j}\varphi_e'\left(t\right)dt^e dy \tag{10}$$

holds for almost all $x \in U$. Recall that the flexible φ- horn $x + V(x, \theta)$ is the support of the representation (10) for $x \in U$. We can assume that the kernels M_e and $M_e^{(\nu)}$ satisfy the following relations for all α and β

$$\int D_x^{\alpha} M_e(x, y, z)\, dx = 0, \quad \int D_x^{\beta} M_e(x, y, z)\, dx = 0.$$

Now we construct an integral representation for studying the properties of functions from $S^l_{p,\theta} B(G\varphi)$ defined in n-dimensional domains and satisfying the flexible φ-horn condition. Introduce the average of f as follows:

$$\overline{f}_{\varphi(t)}(x) = \left(f_{\varphi(t)}\right)_{\varphi(t)}(x) = \prod_{j \in e_n} (\varphi_j(t_j))^{-2} \int \int \Omega\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)}\right) \times$$

$$\times \Omega\left(\frac{z}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)}\right) f(x + y + z)\, dydz. \tag{11}$$

Obviously, $\Omega\left(\frac{y}{\varphi\left(t^e + T^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e + T^{e'}\right), x\right)}{2\varphi\left(t^e + T^{e'}\right)}\right) \neq 0$ is possible only for $\left|y_j - \frac{1}{2}\rho_j(\varphi_j(t_j), x)\right| < \sigma_j\left[1 + m_j + \frac{1}{2}m_j\right]\varphi_j(t_j)$, here σ_j, m_j are integers in formula of ω_j determined in [2]. Hence, it follows that double averaging was constructed by contraction of f on $x + \rho\left(\varphi\left(t^e + T^{e'}\right), x\right) + m\varphi\left(\sigma\left(t^e + T^{e'}\right)\right) I$ and was defined for $0 < \sigma < \frac{\eta}{m_0}$, $m_0 = \max(2 + 3m_j)$. Let

$$\overline{f}^{(\nu)}_{\varphi(t)}(x) = (-1)^{|\nu|} \prod_{j \in e_n} (\varphi_j(t_j))^{-2-\nu_j} \int_{R^n} \int_{R^n} \Omega\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)}\right) \times$$

$$\times \Omega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)}\right) f(x + y + z)\, dydz. \tag{12}$$

Note that if there exists $D^{\nu} f \in L^{loc}(G)$, then by $G(2)$ [2] $\overline{f}^{(\nu)}_{\varphi(t)}(x) = (D^{\nu} f)^{(x)}_{\varphi(t)}$ for $x \in U$, $0 < t_j \leq T_j$ $(j \in e_n)$. Applying the equality

$$g(z_j) = \int_{-\infty}^{\infty} \omega_j\left(\frac{y}{\varphi_j(t_j)}, \frac{\rho_j(\varphi_j(t_j), x)}{2\varphi_j(t_j)}\right) f(x + y + z)\, dy_j$$

we have

$$D_t^{1^e} \overline{f}_{\varphi\left(t^e + T^{e'}\right)}(x) = (-1)^{|1^e|} \prod_{j \in e_n} (\varphi_j(T_j))^{-1} \prod_{j \in e} A_j^{-1} \prod_{j \in e} (\varphi_j(t_j))^{-3} \prod_{j \in e} \varphi_j'(t_j) \times$$

$$\times \int_{R^n} \int_{-\infty}^{\infty} \psi_e\left(\frac{y}{\varphi\left(t^e + T^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e + T^{e'}\right), x\right)}{\varphi\left(t^e + T^{e'}\right)}\right) \times$$

$$\times\prod_{j\in e}S_j\left(\frac{u_j}{\varphi_j(t_j)}-\frac{\rho_j(\varphi_j(t_j),x)}{2\varphi_j(t_j)},\frac{1}{2}\rho_j'(\varphi_j(t_j),x)\right)\Delta^{m^e}(\varphi(\delta u))f(x+y+u^e)du^edy=$$

$$=(-1)^{|1^e|}\prod_{j\in e'}(\varphi_j(T_j))^{-1}\prod_{j\in e}(\varphi_j(t_j))^{-3}\prod_{j\in e}\varphi_j'(t_j)\times$$

$$\times\int_{R^n}\int_{-\infty}^{\infty}\psi_e\left(\frac{y}{\varphi(t^e+T^{e'})},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi(t^e+T^{e'})}\right)\times$$

$$\times S_e\left(\frac{u}{\varphi(t)}-\frac{\rho(\varphi(t),x)}{2\varphi(t)}\right)\Delta^{m^e}(\varphi(\delta u))f(x+y+u^e)dydu^e, \tag{13}$$

where $\Delta^{m^e}(t)f=\prod_{j\in e}\Delta_j^{m_j}(t_j)f$,

$$\psi_e\left(\frac{y}{\varphi(t^e+T^{e'})},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi(t^e+T^{e'})}\right)=$$

$$=2^{|1^e|}\prod_{j\in e'}A_j^{-1}\prod_{j\in e'}(\varphi_j(t_j))^{-1}\left\{\int\prod_{j\in e'}\omega_j\left(\frac{y_j}{\varphi_j(T_j)},\frac{\rho_j(\varphi_j(T_j),x)}{2\varphi_j(T_j)}\right)\times\right.$$

$$\left.\times\omega_j\left(\frac{z}{\varphi_j(T_j)},\frac{\rho_j(\varphi_j(T_j),x)}{2\varphi_j(T_j)}\right)dz^{e'}\right\}\times$$

$$\times\prod_{j\in e}\omega_j\left(\frac{y_j}{\varphi_j(t_j)},\frac{\rho_j(\varphi_j(t_j),x)}{2\varphi_j(t_j)}\right)\frac{\partial}{\partial y_j}\omega_j\left(\frac{z_j}{\varphi_j(t_j)},\frac{\rho_j(\varphi_j(t_j),x)}{2\varphi_j(t_j)}\right),$$

S_j, A_j are defined in [7, p. 88].

The equality (13) is valid in some vicinity of $x^{(0)}\in U$ also for the vector function $\rho\left(\varphi(t),x^{(0)}\right)$ instead of $\rho(\varphi(t),x)$. In this case, differentiating it with respect to x in the vicinity of the point $x^{(0)}$, taking into account possibility of carrying over the differentiation operation on the kernel, we have

$$D_t^{1^e}\overline{f}^{(\nu)}_{\varphi(t^e+T)}(x)=(-1)^{|\nu|+|1^e|}\prod_{j\in e'}(\varphi_j(T_j))^{-1-\nu_j}\prod_{j\in e}(\varphi_j(t_j))^{-3-\nu_j}\prod_{j\in e}\varphi_j'(t_j)\times$$

$$\times\int_{R^n}\int_{-\infty}^{\infty}\psi_e^{(\nu)}\left(\frac{y}{\varphi(t^e+T^{e'})},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi(t^e+T^{e'})}\right)\times$$

$$\times S_e\left(\frac{u}{\varphi(t)}-\frac{\rho(\varphi(t),x)}{2\varphi(t)},\frac{1}{2}\rho'(\varphi(t),x)\right)\Delta^{m^e}(\varphi(\delta u))f(x+y+u^e)dydu^e. \tag{14}$$

Hence we get

$$D^{\nu}f(x)=\sum_{e\subseteq e_n}(-1)^{|\nu|}\int\limits_{0^e}^{T^e}D_t^{1e}\overline{f}^{(\nu)}_{\varphi\left(t^e+T^{e'}\right)}(x)\,dt^e=$$

$$=\sum_{e\subseteq e_n}(-1)^{|\nu|}\prod_{j\in e'}(\varphi_j(T_j))^{-1-\nu_j}\times$$

$$\times\int\limits_{0^e}^{T^e}\int\limits_{R^n}\int\limits_{-\infty^e}^{\infty^e}\psi_e^{(\nu)}\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)}\right)\times$$

$$\times S_e\left(\frac{u}{\varphi(t)}-\frac{\rho(\varphi(t),x)}{2\varphi(t)},\frac{1}{2}\rho'(\varphi(t),x)\right)\Delta^{m^e}(\varphi(\delta u))f(x+y+u^e)\times$$

$$\times\prod_{j\in e}(\varphi_j(T_j))^{-3-\nu_j}\prod_{j\in e}\varphi_j'(t_j)\,dt^e dy du^e. \tag{15}$$

Note that $\psi_e(y,z)\in C^{\infty}(R^n\times R^n)$, i.e. is infinitely differentiable with respect to all variables, and $\psi_e(\cdot,z)$ is uniformly finite with respect to z from the arbitrary compact. The equality (15) is valid almost everywhere on V, the set $x+V(x,\theta)$ is a support of this representation.

Show that if the function f satisfies the conditions

$$\left(\int\limits_{0^e}^{\infty^e}\prod_{j\in e_n}(\varphi_j(t_j))^{-1-\theta_e l_j}\left\|\Delta^{m^e}(\varphi(t),E)f\right\|_p^{\theta_\varepsilon}dt^e\right)^{\frac{1}{\theta_e}}\leq A_e(E),\quad e\subseteq e_n,$$

where $A_e(E)$ are the constants independent of E and the vector $\nu=(\nu_1,...,\nu_n)$,$\nu_j\geq 0$ are entire $(j\in e_n)$ satisfy the conditions $\varepsilon_j=l_j-\nu_j>0\,(j\in e_n)$, then there exists the derivative $D^{\nu}f\in L_p^{loc}(G)$ and identity (15) is valid.

Let $\rho_j(\varphi_j(t_j),x)=0$, $0\leq t_j\leq T_j\,(j\in e_n)$ and the compact $F\subset G$. Then for all rather small $h=(h_1,...,h_n)$, $h_j>0\,(j\in e_n)\,F+hI$ is contained in some compact $E\subset G$. Based on (15), Minskovsky-Young and Holder generalized inequalities, we successively get

$$\left\|\overline{f}^{(\nu)}_{\varphi(\varepsilon)}-\overline{f}^{(\nu)}_{\varphi(T)}\right\|_{p,F}\leq$$

$$\leq\sum_{e\subseteq e_n}C_e\left\|\psi_e^{(\nu)}\right\|_1\|S_e\|_{\theta_e}A_e(E)\prod_{j\in e}(\varphi_j(T_j))^{l_j-\nu_j}$$

hence it follows that $\left\|\overline{f}^{(\nu)}_{\varphi(\varepsilon)}-\overline{f}^{(\nu)}_{\varphi(T)}\right\|_{p,F}\to 0$ for $0<\varepsilon_j<T_j\to 0$, $j\in e_n$. Then

$$\overline{f}_{\varphi(\varepsilon)}(x)\to f(x)$$

as $\varepsilon_j \to 0\,(j \in e_n)$ in the sense of convergence $L^{loc}(G)$. Based on lemma 6.2 [2] we deduce that there exists $D^{\nu} f \in L_p(G)$.

Now, for studying the space $S^l_{p,\theta}F(G)$ we construct integral representation of functions being some modification of the representation (15). Introduce the following averaging that differ from previous ones:

$$\widetilde{f}_{\varphi(t)}(x) = \left(\left(f^{(x)}_{\varphi(t)}\right)_{\varphi(t)}\right)_{\varphi(t)}(x) =$$

$$= \prod_{j \in e_n} (\varphi_j(t_j))^{-1} \int_{R^n} \Omega\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{3\varphi(t)}\right) f(x+y)\,dy, \tag{16}$$

and assume that $m\varphi(\sigma t) I \subset (m\varphi(\eta t) I)_{\varphi(\eta t)} \subset (\varphi(\eta t) I)_{\varphi(\sigma t)}$ σ_j, m_j are the numbers contained in the formula of ω_j that were determined in [2]. In other words, $\widetilde{f}_{\varphi(t)}(x)$ was constructed by contraction of f on $G_{\varphi(\sigma t)}$. Differentiating the equality (16) with respect to x in the neighborhood of the point $x^{(0)}$ taking into account possibility to transfer the differential operation on the kernel, we get

$$\widetilde{f}^{(\nu)}_{\varphi(t)}(x) = (-1)^{|\nu|} \prod_{j \in e_n} (\varphi_j(t_j))^{-1-\nu_j} \int_{R^n} \Omega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{3\varphi(t)}\right) f(x+y)\,dy, \tag{17}$$

Differentiating with respect to t_j and under the condition $0 < \varepsilon_j < T_j$, $j \in e_n$ from (17) we get

$$\widetilde{f}^{(\nu)}_{\varphi(\varepsilon)}(x) = \sum_{e \subseteq e_n} (-1)^{|1^{\varepsilon}|} \int_{\varepsilon^e}^{T^{\varepsilon}} D^{1^e}_t f_{\varphi\left(t^e + T^{e'}\right)}(x)\,dt^e =$$

$$= \sum_{e \subseteq e_n} (-1)^{|\nu| + |1^{\varepsilon}|} \prod_{j \in e'} (\varphi_j(T_j))^{-1-\nu_j} \times$$

$$\times \int_{\varepsilon^e}^{T^{\varepsilon}} \int_{R^n} M^{(\nu)}_e\left(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho\left(\varphi\left(t^e + T^{e'}\right), x\right)}{3\varphi(t^e + T^{e'})}\right) f_e(x+y, t) \times$$

$$\times \prod_{j \in e} (\varphi_j(t_j))^{-2-\nu_j} \prod_{j \in e} \varphi'_j(t_j)\,dt^e dy, \tag{18}$$

where $M^{(\nu)}_e(y,a) = D^{(\nu)}_y M_e(y,a)$; $y, a \in R^n$,

$$M^{(\nu)}_e(y,a) = 3 \prod_{j \in e} A_j^{-1} \left\{ \prod_{j \in e'} \omega_j\left(y_j - z_j - u_j, \frac{a_j}{3}\right) \omega_j\left(z_j, \frac{a_j}{3}\right) \omega_j, \frac{a_j}{3} \right\} \times$$

$$\times \prod_{j \in e} \frac{\partial}{\partial y_j} \omega_j\left(y_j, \frac{a_j}{3}\right),$$

$$f_e(x,t)=\prod_{j\in e}(\varphi_j(t_j))^{-2}\int\int\prod_{j\in e}\omega_j\left(\frac{z_j}{\varphi_j(t_j)},\frac{\rho(\varphi(t^e),x)}{3\varphi_j(t_j)}\right)\times$$

$$\times\rho_j\left(\frac{u_j}{\varphi_j(t_j)}-\frac{\rho_j(\varphi_j(t_j),x)}{3\varphi_j(t_j)},\frac{1}{3}\rho_j(\varphi_j(t_j),x)\right)\times$$

$$\times\Delta^{m^e}\left(\varphi(\delta u),G_{\varphi(\eta t)}\right)f(x+z^e+u^e)\,du^e dz^e,$$

furthermore, we can show that

$$|f_e(x,t)|\le C\int\limits_{-1^e}^{1^e}\delta^{m^e}(\varphi(\delta t))\,f(x+v\varphi(t))\,dv^e.$$

Note that under the condition $l_j-\nu_j>0\,(j\in n)$ there exists a generalized derivative $D^\nu f\in L^{loc}(G)$. Then $\widetilde{f}^{(\nu)}_{\varphi(\varepsilon)}(x)=\left(D^\nu\widetilde{f}\right)_{\varphi(\varepsilon)}(x)$ and as $\varepsilon_j\to 0\,(j\in n)\,\widetilde{f}_{\varphi(\varepsilon)}\to D^\nu f$ almost everywhere on G and b in the sense of $L^{loc}(G)$. Passing to the limit $\varepsilon_j\to 0$, $j\in e_n$ for $f\in S^l_{p,\theta}F(G)$

$$D^\nu f(x)=\sum_{e\subseteq e_n}(-1)^{|\nu|}\prod_{j\in e'}(\varphi_j(T_j))^{-1-\nu_j}\times$$

$$\times\int\limits_{0^e}^{T^e}\int\limits_{R^n}M_e^{(\nu)}\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{3\varphi\left(t^e+T^{e'}\right)}\right)f_e(x+y,t)\times$$

$$\times\prod_{j\in e'}(\varphi_j(t_j))^{-2-\nu_j}\prod_{j\in e}\varphi'_j(t_j)\,dt^e dy, \tag{19}$$

where the equality is fulfilled almost everywhere on G in the sense $L^{loc}(G)$ and the set $X+\bigcup\limits_{0<t_j\le T_j}[\rho(\varphi(t),x)+m\varphi(\sigma t)I]$ is the support of the representation (19). It should be noted that $M_e\in C_0^\infty(R^n xR^n)$ and $\int M_e^{(\nu)}(y,a)=0$; $ty,a\in R^n$, $\nu\in N_0^n$.

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On Global Bifurcation from Zero and Infinity in Fourth Order Nonlinear Eigenvalue Problems

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Abstract. In this paper we consider nonlinear eigenvalue problems for fourth order ordinary differential equations. We study bifurcation problems from zero and infinity simultaneously for these problems. We prove the existence of two pairs of unbounded continua of solutions corresponding to the usual nodal properties and bifurcating from intervals of the line of trivial solutions and infinity.

Key Words and Phrases: fourth order nonlinear eigenvalue problems, bifurcation point, bifurcation from infinity, bifurcation interval, connected component.
2010 Mathematics Subject Classifications: 34B24, 34C23, 34L15, 34L30, 47J10, 47J15

1. Introduction

We consider the following nonlinear eigenvalue problem

$$\ell y \equiv (py'')'' - (qy')' + r(x)y = \lambda \tau y + h(x, y, y', y'', y''', \lambda),\ x \in (0, l), \tag{1}$$

$$\begin{gathered} y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \\ y(0)\cos\beta + Ty(0)\sin\beta = 0, \\ y'(l)\ \cos\gamma + (py'')(l)\ \sin\gamma = 0, \\ y(l)\cos\delta - Ty(l)\sin\delta = 0, \end{gathered} \tag{2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $Ty \equiv (py'')' - qy'$, p is positive, twice continuously differentiable function on $[0, l]$, q is nonnegative, continuously differentiable function on $[0, l]$, r is real-valued continuous function on $[0, l]$, τ is positive continuous function on $[0, l]$ and α, β, γ, $\delta \in [0, \frac{\pi}{2}]$. The nonlinear term h has the form $h = f + g$, where f and g are real-valued continuous functions on $[0, l] \times \mathbb{R}^5$ and there exit $M > 0$ and sufficiently large $c_0 > 0$ such that

$$\left|\frac{f(x,y,s,v,w,\lambda)}{y}\right| \le M,\ x \in [0, l],\ y, s, v, w \in \mathbb{R},\ |y| + |s| + |v| + |w| \le \frac{1}{c_0},\ \lambda \in \mathbb{R}, \tag{3}$$

or

$$\left|\frac{f(x,y,s,v,w,\lambda)}{y}\right| \le M,\ x \in [0, l],\ y, s, v, w \in \mathbb{R},\ |y| + |s| + |v| + |w| \ge c_0,\ \lambda \in \mathbb{R}. \tag{4}$$

Moreover, for any bounded interval $\Lambda \subset \mathbb{R}$

$$g(x, y, s, v, w, \lambda) = o(|y| + |s| + |v| + |w|) \text{ as } |y| + |s| + |v| + |w| \to 0, \tag{5}$$

or

$$g(x, y, s, v, w, \lambda) = o(|y| + |s| + |v| + |w|) \text{ as } |y| + |s| + |v| + |w| \to \infty, \tag{6}$$

uniformly for $x \in [0, l]$ and $\lambda \in \Lambda$.

An important role in nonlinear analysis is played bifurcation theory of nonlinear eigenvalue problems. The bifurcation problem in nonlinear eigenvalue problems occurs in all fields of natural science (see, for example, [4, 5, 9, 10]). Note that, recently have been obtained fundamental results on local and global bifurcation in nonlinear eigenvalue problems for ordinary differential equations (see for example, [1-5, 7-20] and their references).

Similar problems for Sturm-Liouville equation has been considered before by Stuart [19], Toland [20], Rabinowitz [15, 16], Berestycki [7], Schmitt and Smith [18], Rynne [17], Ma and Dai [13], Przybycin [14]. For bifurcation problem from zero in [7, 13-15, 17, 18] the authors prove the existence of two families of global continua of solutions in $\mathbb{R} \times C^1$, corresponding to the usual nodal properties and bifurcating from the eigenvalues and intervals (in $\mathbb{R} \times \{0\}$, which we identify with $\mathbb{R}$) surrounding the eigenvalues of the corresponding linear problem. For bifurcation problem from infinity in [16, 17] show the existence of two families of unbounded continua of solutions bifurcating from the points and intervals in $\mathbb{R} \times \{\infty\}$ and having the usual nodal properties in the neighborhood of these points and intervals.

The nonlinear eigenvalue problem (1)-(2) under the conditions (3) and (5) has been considered by Aliyev [2] (see also [1]), under conditions (4) and (6) has been considered in our recent paper [3]. In these papers for bifurcation problems from zero and infinity we are able to obtain similar results as in the case of nonlinear Sturm-Liouville problems from above.

The purpose of this paper is to study the global bifurcation of nontrivial solutions of problem (1)-(2) in case when conditions (3), (5) and (4), (6) are satisfied simultaneously for f and g, respectively.

2. Preliminary

Let E be the Banach space of all continuously three times differentiable functions on $[0, l]$ which satisfy the conditions (2) and is equipped with its usual norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$, where $||u||_\infty = \max\limits_{x \in [0,l]} |u(x)|$.

Let $S = S_1 \cup S_2$, where $S_1 = \{u \in E : u^{(i)}(x) \neq 0, Tu(x) \neq 0, \, x \in [0, l], \, i = 0, \, 1, \, 2\}$ and $S_2 = \{u \in E :$ there exists $i_0 \in \{0, \, 1, \, 2\}$ and $x_0 \in (0, 1)$ such that $u^{(i_0)}(x_0) = 0$, or $Tu(x_0) = 0$ and if $u(x_0)u''(x_0) = 0$, then $u'(x)Tu(x) < 0$ in a neighborhood of x_0, and if $u'(x_0)Tu(x_0) = 0$, then $u(x)u''(x) < 0$ in a neighborhood of $x_0\}$.

Note that if $u \in S$ then the Jacobian $J = \rho^3 \cos\psi \sin\psi$ of the Prüfer-type transformation

$$\begin{cases} y(x) = \rho(x)\sin\psi(x)\cos\theta(x), \\ y'(x) = \rho(x)\cos\psi(x)\sin\varphi(x), \\ (py'')(x) = \rho(x)\cos\psi(x)\cos\varphi(x), \\ Ty(x) = \rho(x)\sin\psi(x)\sin\theta(x), \end{cases} \tag{7}$$

does not vanish in $(0, l)$ (see [1, 2, 5]).

For each $u \in S$ we define $\rho(u,x)$, $\theta(u,x)$, $\varphi(y,x)$ and $w(u,x)$ to be the continuous functions on $[0,l]$ satisfying

$$\rho(u,x) = u^2(x) + u'^2(x) + (p(x)u''(x))^2 + (Tu(x))^2,$$

$$\theta(u,x) = \operatorname{arctg}\frac{Tu(x)}{u(x)},\ \theta(u,0) = \beta - \pi/2\,,$$

$$\varphi(u,x) = \operatorname{arctg}\frac{u'(x)}{(pu'')(x)},\ \varphi(u,0) = \alpha\,,$$

$$w(u,x) = \operatorname{ctg}\psi(u,x) = \frac{u'(x)\cos\theta(u,x)}{u(x)\sin\varphi(u,x)},\ w(u,0) = \frac{u'(0)\sin\beta}{u(0)\sin\alpha}\,,$$

and $\psi(u,x) \in (0,\pi/2)$, $x \in (0,l)$, in the cases $u(0)u'(0) > 0$; $u(0) = 0$; $u'(0) = 0$ and $u(0)u''(0) > 0$, $\psi(u,x) \in (\pi/2,\pi)$, $x \in (0,l)$, in the cases $u(0)u'(0) < 0$; $u'(0) = 0$ and $u(0)u''(0) < 0$; $u'(0) = u''(0) = 0$, $\beta = \pi/2$ in the case $\psi(u,0) = 0$ and $\alpha = 0$ in the case $\psi(u,0) = \pi/2$.

It is apparent that ρ, θ, $\varphi, w : S \times [0,1] \to \mathbb{R}$ are continuous.

Remark 1. By (7) for each $u \in S$ the function $w(u,x)$ can de determined by one of the following relations

$$\text{a) } w(y,x) = \operatorname{ctg}\psi(y,x) = \frac{(py'')(x)\cos\theta(y,x)}{y(x)\cos\varphi(y,x)},\ w(y,0) = \frac{(py'')(0)\sin\beta}{y(0)\cos\alpha}\,,$$

$$\text{b) } w(y,x) = \operatorname{ctg}\psi(y,x) = \frac{(py'')(x)\sin\theta(y,x)}{Ty(x)\cos\varphi(y,x)},\ w(y,0) = -\frac{(py'')(0)\cos\beta}{Ty(0)\cos\alpha}\,,$$

$$\text{c) } w(y,x) = \operatorname{ctg}\psi(y,x) = \frac{y'(x)\sin\theta(y,x)}{Ty(x)\sin\varphi(y,x)},\ w(y,0) = -\frac{y'(0)\cos\beta}{Ty(0)\sin\alpha}\,.$$

For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ let by S_k^ν denote the subset of $y \in S$ such that

1) $\theta(y,l) = (2k-1)\pi/2 - \delta$, where $\delta = \pi/2$ in the case $\psi(y,l) = 0$;

2) $\varphi(y,l) = (k+1)\pi - \gamma$ or $\varphi(u,l) = k\pi - \gamma$ in the case $\psi(y,0) \in [0,\pi/2)$; $\varphi(y,l) = \pi - \gamma$ for $k = 1$, $\varphi(y,l) = k\pi - \gamma$ or $\varphi(y,l) = (k-1)\pi - \gamma$ for $k \geq 2$ in the case $\psi(y,0) \in [\pi/2,\pi)$, where $\gamma = 0$ in the case $\psi(y,l) = \pi/2$;

3) for fixed y, as x increases from 0 to l, the function $\theta(y,x)$ ($\varphi(y,x)$) strictly increasing takes values of $m\pi/2$, $m \in \mathbb{Z}$ ($s\pi$, $s \in \mathbb{Z}$); as x decreases, the function $\theta(y,x)$ ($\varphi(y,x)$), strictly decreasing takes values of $m\pi/2$, $m \in \mathbb{Z}$ ($s\pi$, $s \in \mathbb{Z}$);

4) the function $\nu y(x)$ is positive in a deleted neighborhood of $x = 0$.

It follows immediately from the definition of the sets S_k^+, S_k^- and $S_k = S_k^+ \cup S_k^-$, $k \in \mathbb{N}$, that they are disjoint and open in E.

By [2, Theorem 1.2] the eigenvalues of the linear problem

$$\begin{aligned} &\ell(y)(x) = \lambda\tau(x)y(x),\ x \in (0, l),\\ &y \in B.C., \end{aligned} \tag{8}$$

are real and simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$, where by $B.C.$ we denote the set of boundary conditions (2). Moreover, for each $k \in \mathbb{N}$ the eigenfunction $y_k(x)$ corresponding to the eigenvalue λ_k is lies in S_k (therefore $y_k(x)$ has $k-1$ simple nodal zeros in the interval $(0, l)$).

Lemma 1. [2, Lemma 2.2] *If* $(\lambda, y) \in \mathbb{R} \times E$ *is a solution of* (1)-(2) *and* $y \in \partial S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, *then* $y \equiv 0$.

Let $\mathcal{C} \subset \mathbb{R} \times E$ denote the set of solutions of problem (1)-(2). We say (λ, ∞) is a bifurcation point (or asymptotic bifurcation point) for problem (1)-(2) if every neighborhood of (λ, ∞) contains solutions of this problem, i.e. there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset \mathcal{C}$ such that $\lambda_n \to \lambda$ and $||u_n||_3 \to +\infty$ as $n \to \infty$ (we add the points $\{(\lambda, \infty) : \lambda \in \mathbb{R}\}$ to space $\mathbb{R} \times E$). Next for any $\lambda \in \mathbb{R}$, we say that a subset $D \subset \mathcal{C}$ meets (λ, ∞) (respectively, $(\lambda, 0)$) if there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset D$ such that $\lambda_n \to \lambda$ and $||u_n||_3 \to +\infty$ (respectively, $||u_n||_3 \to 0$) as $n \to \infty$. Furthermore, we will say that $D \subset \mathcal{C}$ meets (λ, ∞) (respectively, $(\lambda, 0)$) through $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, if the sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset D$ can be chosen so that $u_n \in S_k^\nu$ for all $n \in \mathbb{N}$ (in this case we also say that (λ, ∞) (respectively, $(\lambda, 0)$) is a bifurcation point of (1)-(2) with respect to the set $\mathbb{R} \times S_k^\nu$). If $I \in \mathbb{R}$ is a bounded interval we say that $D \subset \mathcal{C}$ meets $I \times \{\infty\}$ (respectively, $I \times \{0\}$) if D meets (λ, ∞) (respectively, $(\lambda, 0)$) for some $\lambda \in I$; we define $D \subset \mathcal{C}$ meets $I \times \{\infty\}$ (respectively, $I \times \{0\}$) through $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, similarly (see [16]).

When the functions f and g satisfies conditions (3) and (5) in [2] show that problem (1)-(2) has a nonempty set of bifurcation points, and if $(\lambda, 0)$ is a bifurcation point of this problem with respect to the set $\mathbb{R} \times S_k^\nu$, then $\lambda \in I_k$, where $I_k = [\lambda_k - \frac{M}{\tau_0}, \lambda_k + \frac{M}{\tau_0}]$, $\tau_0 = \min\limits_{x \in [0, l]} \tau(x)$.

For $k \in \mathbb{N}$ and $\nu \in \{+, -\}$ let $\tilde{\mathcal{C}}_k^\nu$ denote the union of the connected components $\mathcal{C}_{k,\lambda}^\nu$ of the solutions set of (1)-(2) under conditions (3) and (5) emanating from bifurcation points $(\lambda, 0) \in I_k \times \{0\}$ with respect to $\mathbb{R} \times S_k^\nu$. Let $\mathcal{C}_k^\nu = \tilde{\mathcal{C}}_k^\nu \cup I_k \times \{0\}$.

Theorem 1. *For each* $k \in \mathbb{N}$ *and each* $\nu \in \{+, -\}$ *the connected component* $\mathcal{C}_k^\nu$ *of* $\mathcal{C}$ *lies in* $(\mathbb{R} \times S_k^\nu) \cup (I_k \times \{0\})$ *and is unbounded in* $\mathbb{R} \times E$.

The proof of this theorem is similar to that of [2, Theorem 1.3] by using [2, Theorem 1.2].

In [3] it is prove that the set of asymptotic bifurcation points of problem (1)-(2) under conditions (4) and (6) with respect to the set $\mathbb{R} \times S_k^\nu$ is nonempty. Moreover, if (λ, ∞) is an asymptotic bifurcation point for (1)-(2) with respect to the set $\mathbb{R} \times S_k^\nu$, then $\lambda \in I_k$.

For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ we define the set $\mathcal{D}_k^\nu \subset \mathcal{C}$ to be the union of all the components of $\mathcal{C}$ which meet $I_k \times \{\infty\}$ through $\mathbb{R} \times S_k^\nu$. The set $\mathcal{D}_k^\nu$ may not be connected in $\mathbb{R} \times E$, but the set $\mathcal{D}_k^\nu \cup (I_k \times \{\infty\})$ is connected in $\mathbb{R} \times E$.

For any set $A \subset \mathbb{R} \times E$ we let $P_R(A)$ denote the natural projection of A onto $\mathbb{R} \times \{0\}$.

Theorem 2. *For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ for the set $\mathcal{D}_k^\nu$ at least one of the followings holds:*

(i) *$\mathcal{D}_k^\nu$ meets $I_{k'} \times \{\infty\}$ through $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$;*

(ii) *$\mathcal{D}_k^\nu$ meets $\mathcal{R}$ for some $\lambda \in \mathbb{R}$;*

(iii) *$P_R(\mathcal{D}_k^\nu)$ is unbounded.*

In addition, if the union $\mathcal{D}_k = \mathcal{D}_k^+ \cup \mathcal{D}_k^-$ does not satisfy (ii) or (iii) then it must satisfy (i) with $k' \neq k$.

3. Global bifurcation from zero and infinity of solutions of problem (1)-(2)

If conditions (3), (5) and (4), (6) are satisfied simultaneously for f and g, respectively, then we can improve Theorems 1 and 2 as follows.

Theorem 3. *Let the conditions (3)-(6) both hold. Then for each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ we have $\mathcal{D}_k^\nu \subset \mathbb{R} \times S_k^\nu$ and alternative (i) of Theorem 2 cannot hold. Furthermore, if $\mathcal{D}_k^\nu$ meets (λ, ∞) for some $\tilde{\lambda} \in \mathbb{R}$, then $\tilde{\lambda} \in I_k$. Similarly, if $\mathcal{C}_k^\nu$ meets $(\tilde{\lambda}, 0)$ for some $\tilde{\lambda} \in \mathbb{R}$, then $\tilde{\lambda} \in I_k$.*

Proof. It follows from Lemma 1 that if conditions (3)-(4) hold, then $\mathcal{C} \cap (\mathbb{R} \times \partial S_k^\nu) = \emptyset$. Hence the sets $\mathcal{C} \cap (\mathbb{R} \times S_k^\nu)$ and $\mathcal{C} \backslash (\mathbb{R} \times S_k^\nu)$ are mutually separated in $\mathbb{R} \times E$ (see [21, Definition 26.4]). Thus by [21, Corollary 26.6] it follows that any connected component of the set $\mathcal{C}$ must be a subset of one or another of the sets $\mathcal{C} \cap (\mathbb{R} \times S_k^\nu)$ and $\mathcal{C} \backslash (\mathbb{R} \times S_k^\nu)$. Since $\mathcal{D}_k^\nu$ is a connected component of $\mathcal{C}$ which intersect $\mathbb{R} \times S_k^\nu$, then $\mathcal{D}_k^\nu$ must be a subset of $\mathbb{R} \times S_k^\nu$, i.e. $\mathcal{D}_k^\nu \subset \mathbb{R} \times S_k^\nu$. But this shows that the alternative (i) of Theorem 2 cannot hold.

Now let $\mathcal{C}_k^\nu$ meets $(\tilde{\lambda}, \infty)$ for some $\tilde{\lambda} \in \mathbb{R}$. Then there exists a sequence $\{(\lambda_{k,n}, y_{k,n})\}_{n=1}^\infty \subset \mathcal{C}_k^\nu$ such that $\lambda_{k,n} \to \tilde{\lambda}$ and $||y_{k,n}||_3 \to \infty$ as $n \to \infty$ and

$$\ell y_{k,n} = \lambda_{k,n} \tau(x) y_{k,n} + f(x, y_{k,n}, y'_{k,n}, \lambda_{k,n}) + g(x, y_{k,n}, y'_{k,n}, \lambda_{k,n}).$$

Let $\lambda \notin I_k$ and

$$\tilde{\delta} = \frac{dist\{\tilde{\lambda}, I_k\}}{2}.$$

Then there exists $n_0 \in \mathbb{N}$ such that

$$\mathrm{dist}\,\{\lambda_{k,n}, I_k\} > \tilde{\delta}.$$

Obviously, $(\lambda_{k,n}, y_{k,n}) \in \mathcal{C}_k^\nu$ solves the nonlinear problem

$$\begin{cases} \ell y + \varphi_{k,n}(x) y = \lambda \tau(x) y + g(x, y, y', y'', y''', \lambda), \\ y \in B.C., \end{cases} \tag{9}$$

where

$$\varphi_{k,n}(x) = \begin{cases} -\dfrac{f\left(x, y_{k,n}(x), y_n'(x), y_{k,n}''(x), y_{k,n}'''(x), \lambda_{k,n}\right)}{y_{k,n}(x)} & \text{if } \ y_{k,n}(x) \neq 0, \\ 0 & \text{if } \ y_{k,n}(x) = 0. \end{cases}$$

By virtue of (5) we have $|\varphi_{k,n}(x)| \leq M$, $n \in \mathbb{N}$, $x \in [0, l]$. Since $y_n(x)$, $n \in \mathbb{N}$, has $k-1$ simple zeros on $(0, l)$ and is bounded on the closed interval $[0, l]$, it follows from [3, Lemma 5.2 and Remark 5.1] that the k-th eigenvalue $\lambda_{k,n}^*$ of the linear problem

$$\begin{cases} \ell y + \varphi_{k,n}(x) y = \lambda \tau(x) y, \ x \in (0, l), \\ y \in B.C. \end{cases}$$

lies in I_k. By [11, Ch. 4, § 3, Theorem 3.1] for each $n \in \mathbb{N}$ the point $(\lambda_{k,n}^*, \infty)$ is a unique asymptotic bifurcation point of (9) which corresponds to a continuous branch of solutions that meets this point through $\mathbb{R} \times S_k^\nu$. Hence for each sufficiently large $n > n_0$ we can assign a small $\delta_n > 0$ such that $\delta_n < \tilde{\delta}$ and $|\lambda_{k,n} - \lambda_{k,n}^*| < \delta_n$. Then it follows that $\operatorname{dist}\{\lambda_{k,n}, I_k\} < \tilde{\delta}$, contradicting $\operatorname{dist}\{\lambda_{k,n}, I_k\} > \tilde{\delta}$. Thus $\mathcal{C}_k^\nu$ can only meet $(\tilde{\lambda}, \infty)$ if $\tilde{\lambda} = \lambda_k$. Similarly is proved that $\mathcal{D}_k^\nu$ can only meet $(\tilde{\lambda}, 0)$ if $\tilde{\lambda} = \lambda_k$. The proof of this theorem is complete.

The naturally question arises whether or not $\mathcal{C}_k^\nu$ intersects $\mathcal{D}_k^\nu$. The following examples show that, both cases are possible.

Example 1. *Now we consider the boundary problem*

$$\begin{cases} y^{(4)}(x) = \lambda y(x) + 2y(x) + \lambda \tilde{g}(x, y(x), y'(x), y''(x), y'''(x))\, y(x), \ 0 < x < l, \\ y(0) = y''(0) = y(l) = y''(l) = 0, \end{cases} \tag{10}$$

It is obvious that in this case $f(x, y, s, v, w, \lambda) = 2y$ and $g(x, y, s, v, w, \lambda) = \lambda \tilde{g}(x, y, s, v, w)\, y$.

We assume that the function $\tilde{g}$ is satisfied the following conditions:

(i) there exist positive constants K, d and θ such that

$$|\tilde{g}(x, u, s, v, w)| \leq K(|u| + |s| + |v| + |w|)^{-\theta}$$

for all $(x, u, s, v, w) \in [0, l] \times \mathbb{R}^4$ with $|u| + |s| + |v| + |w| \geq d$;

(ii) $\tilde{g}$ is continuous in $[0, l] \times \mathbb{R}^4$ and $f(x, 0, 0, 0, 0) = 0$ for $x \in [0, l]$.

These two conditions ensures that for the function $g(x, u, s, v, w, \lambda) = \lambda \tilde{g}(x, u, s, v, w)$ conditions (4) and (6) both hold.

Then it follows from [3, Example 4.1] that if $\tilde{g}(x, u, s, v, w) \geq 0$ for $(x, u, s, v, w) \in [0, l] \times \mathbb{R}^4$, then $\mathcal{C}_1^\nu \cap \mathcal{D}_1^\nu \neq \emptyset$, and if $\tilde{g}(x, u, s, v, w) \leq 0$ for $(x, u, s, v, w) \in [0, l] \times \mathbb{R}^4$, then $\mathcal{C}_1^\nu \cap \mathcal{D}_1^\nu = \emptyset$.

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