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About Econometric Analysis of Factors Affecting the Change in the USD/AZN Rate

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Abstract. In the study, on the basis of real indicators covering the period from 01.01.2013 to 10.01.2017 [10], an econometric analysis of changes in the USD/AZN rate was conducted. As a result of study, the dependence of several factors provided a serious influence on the change in the USD/AZN rate and the relationship of interdependence with their endogenous variability were gained by carrying out empirical analysis. Verification of the optimality and adequacy of the model is tested using the tools of the software package Eviews. To build a regression equation for the model and test its coefficient of determination, F-Fisher statistics, t − Student criterion, etc., the execution of the Quick → Equation order of the Eviews software package is considered, to check the stationarity of factors, the execution of the test order Quick → Series statistics → Unit root and as a result, conclusions were drawn and recommendations were made for a predictive-analytical computing system.

Key Words and Phrases: Regression, correlation, determination, F-Fisher statistics, t-Student criterion, prediction, VAR, inpatient, Unit root test

JEL code: C10; C12; C13; C14; C15; C22; C32; C51; C53

The exchange rate in the system of international economic relations is a tool of dependence on the value indicators of world and national markets. The exchange rate, as an important component of the world monetary system, is one of the factors affecting the macroeconomic position of each country. The dynamics of the exchange rate, amplitude and frequency of its changes are clear evidence of the economic and political stability of the country. Formation of the exchange rate is a multifactorial process. These factors can be predictable and unpredictable internal and external factors, structural and opportunistic factors. The factors shaping exchange rates are fairly mobile, and their mutual influence can either strengthen or even neutralize the effect on the exchange rate. It should be noted that multifactor dependencies and other macroeconomic processes relevant to the case research were studied in relation to some fundamental economic indicators (for example, [7, 8, 9]). However, for the first time, an analysis of the correlation-regression dependence of the influence of factors with delay on the change in the USD / AZN exchange rate and the construction of the corresponding models are being studied.

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To build an econometric optimal model for changes in the USD / AZN exchange rate, at first each of the factors that can influence it was considered separately, and a general regression equation was established (Table 1).

Table 1

Dependent Variable: USD_AZN
Method: Least Squares
Date: 03/26/18 Time: 08:26
Sample: 2014M02 2017M10
Included observations: 45

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	1.096385	1.150521	0.952946	0.3478
GDP	1.88E-05	2.21E-05	0.847511	0.4030
TRADE_BALANCE	0.004072	0.007142	0.570128	0.5726
REPO_INTEREST	-0.012209	0.023162	-0.527117	0.6017
OIL	0.000560	0.001666	0.336206	0.7389
EXPORT	-0.004078	0.007136	-0.571378	0.5717
INFLATION	0.018750	0.008356	2.244049	0.0319
INPORT	0.003977	0.007132	0.557633	0.5810
GBP_EUR	0.452617	0.531777	0.851141	0.4010
FED	-0.044484	0.097009	-0.458554	0.6497
INTEREST	0.023441	0.028851	0.812476	0.4225
COUNTER_REPO_INTER	0.038649	0.007609	5.079250	0.0000
EUR_USD	-0.938357	0.461383	-2.033791	0.0503

R-squared	0.975141	Mean dependent var	1.254932
Adjusted R-squared	0.965819	S.D. dependent var	0.396210
S.E. of regression	0.073251	Akaike info criterion	-2.152987
Sum squared resid	0.171705	Schwarz criterion	-1.631063
Log likelihood	61.44222	Hannan-Quinn criter.	-1.958419
F-statistic	104.6065	Durbin-Watson stat	2.537828
Prob(F-statistic)	0.000000		

Table 1 summarizes both its own grades and the probable grades of several tests. Let's analyze some tests in the table separately. As you can see, the coefficient of determination (R-squared) and the adjusted coefficient of determination (Adjusted R-squared) are very large. This means that the factor signs of the coefficients of the established regression equation can explain 96–97% of the signs of the result. Let's take a look at the F-Fisher test. Since the probability value (F-statistic = 104.6, the probability value $p = 0$) is much less than $\alpha = 0.05$, we can consider the factors of the model as valid. Let's take a look at the Durbin-Watson test ($DW = 2.54$). If we compare the results obtained here with tabular prices, we must say that the existence of negative autocorrelation of residuals ($d_l = 0.79, d_u = 2.044, 4 - d_l = 2.21$ and $4 - d_u = 1.956$; $4 - d_l < 2.54 < 4$) accepted.

As a result of the study, let's analyze the question of whether the model in Table 2 was the optimal model that was established with the introduction of the Least Squares Method.

Table 2

Dependent Variable: USD_AZN_D
Method: Least Squares
Date: 10/29/18 Time: 12:38
Sample (adjusted): 2014M03 2017M10
Included observations: 44 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.000537	0.022135	0.024280	0.9808
EUR_USD_D(-1)	-1.281899	0.311269	-4.118302	0.0002
FED_D(-1)	0.233846	0.113924	2.052647	0.0470
INFLATION_D(-1)	0.050949	0.005287	9.636359	0.0000
OIL	0.003686	0.001505	2.449618	0.0190
OIL(-1)	-0.003686	0.001441	-2.557816	0.0146

R-squared	0.798188	Mean dependent var	0.020143
Adjusted R-squared	0.771634	S.D. dependent var	0.099869
S.E. of regression	0.047725	Akaike info criterion	-3.120601
Sum squared resid	0.086552	Schwarz criterion	-2.877302
Log likelihood	74.65321	Hannan-Quinn criter.	-3.030374
F-statistic	30.05882	Durbin-Watson stat	2.043172
Prob(F-statistic)	0.000000		

The analytical form of the model is as follows:

$$y_t = 0.0005 - 1.28x_{1,t-1} + 0.23x_{2,t-1} + 0.051x_{3,t-1} + 0.0037x_{4,t} - 0.0037x_{4,t-1}.$$

Here: x_1 is the first difference in the course of the EUR / USD exchange rate, x_2 is the 1st difference FED, x_3 is the first difference of inflation, and x_4 is the indicator of oil prices. In addition, t represents the value of the indicator itself, and $t-1$ represents the value of the delay from the 1st power.

Let us explain the results obtained in Table 2. If we look at the t-Student criteria for each of the factors of the model individually, we will see that the probability of all factors outside the constant c is less than 5%. This means that the model is individually significant for each factor. In general, let's look at the F-Fisher test statistics to check the importance of the model. As you can see, the probability is close to 0, which means that the model is usually considered important. In addition, since the Durbin Watson test model is close to 2, it can be said that there is no autocorrelation model (other tests were considered to check for the presence of autocorrelation). The coefficient of determination ($R^2 = 79.8188\%$) means the disclosure of about 80% of the model, which is not considered to be quite important. The main reason for this is that there is another factor that can affect fluctuations in the exchange rate of the US dollar / manat. Whether the constructed model is optimal is tested by the following tests.

The correlation coefficients of all factors were calculated in the **multicollinearity** test, and the following results were obtained as a result of the Quick → Group statistics → Correlations command of the Eviews software test (Table 3):

Table 3

	Trade_balance	GDP	REPO_INTEREST	OIL	EXPORT	IMPORT	INFLATION	GBP_EUR	FED	INTEREST	EUR_USD	COUNTER_REPO_INETEREST
Trade_balance	1,000	0,039	-0,361	0,858	0,931	-0,211	-0,594	-0,030	-0,391	-0,354	0,803	-0,471
GDP	0,039	1,000	0,555	0,004	0,068	0,078	0,386	-0,590	0,564	0,584	0,027	0,454
REPO_INTEREST	-0,361	0,555	1,000	-0,322	-0,398	-0,087	0,833	-0,782	0,796	0,990	-0,408	0,892
OIL	0,858	0,004	-0,322	1,000	0,881	0,037	-0,596	-0,120	-0,376	-0,305	0,902	-0,400
EXPORT	0,931	0,068	-0,398	0,881	1,000	0,160	-0,644	-0,065	-0,352	-0,365	0,872	-0,486
IMPORT	-0,211	0,078	-0,087	0,037	0,160	1,000	-0,116	-0,096	0,118	-0,020	0,162	-0,026
INFLATION	-0,594	0,386	0,833	-0,596	-0,644	-0,116	1,000	-0,597	0,843	0,846	-0,582	0,908
GBP_EUR	-0,030	-0,590	-0,782	-0,120	-0,065	-0,096	-0,597	1,000	-0,694	-0,825	-0,154	-0,763
FED	-0,391	0,564	0,796	-0,376	-0,352	0,118	0,843	-0,694	1,000	0,840	-0,316	0,821
INTEREST	-0,354	0,584	0,990	-0,305	-0,365	-0,020	0,846	-0,825	0,840	1,000	-0,366	0,919
EUR_USD	0,803	0,027	-0,408	0,902	0,872	0,162	-0,582	-0,154	-0,316	-0,366	1,000	-0,421
COUNTER_REPO_INETEREST	-0,471	0,454	0,892	-0,400	-0,486	-0,026	0,908	-0,763	0,821	0,919	-0,421	1,000

Let's explain the results. In (Table 3), the highest value is the correlation coefficient of interest rates with repo percentage. That is, these indicators explain 99% of each other. The high correlation coefficient is evidence of the multicollinearity problem in the embedded model, demonstrating a strong correlation between the indicators. To eliminate multicollinearity, at least one of these factors should be excluded. To do this, review the t-Student values for both indicators in (Table 1). Note that among these two factors, the value of the t-Student criterion is higher at the repo rate. Therefore, this factor should be excluded from the model. Once the factor was removed, the model was re-modeled, and the results were closer to the results in Table 1. Thus, this rule excludes several other factors from the model.

Stationarity. One of the most important tasks is to test the stationarity of an optimal econometric model. Thus, for each factor, the stationary test in the Eviews software package was checked by the Quick → Series statistics → Unit root tests command to determine that several factors (including FED, Inflation, EUR / USD, etc.), are considered to be non- stationary , oil (at the level of 10% significance) and the trade balance are considered stationary.

Granger test. The overall result, including all factors included in the regression equation, was first used to process this test for a computer package. The main goal here is to check, with the removal of multicollinearity, whether Granger is the cause of the USD / AZN indicators of all factors, including the excluded factors. The Eviews software package revealed Granger's causal relationship for 5 factors that directly or indirectly affect the change in the USD / AZN exchange rate, so the test results can be compiled in the following table (Table 4) compactly. The (+) sign is a causal link, and (-) indicates the absence of this link).

Table 4

Granger Causality Tests					
EUR/USD	$\underset{-}{\rightarrow}$	USD/AZN	EUR/USD	$\underset{-}{\rightarrow}$	USD/AZN
USD/AZN	$\underset{-}{\rightarrow}$	EUR/USD	USD/AZN	$\underset{-}{\rightarrow}$	EUR/USD
FED	$\underset{+}{\rightarrow}$	USD/AZN	FED	$\underset{+}{\rightarrow}$	USD/AZN
USD/AZN	$\underset{-}{\rightarrow}$	FED	USD/AZN	$\underset{-}{\rightarrow}$	FED
İnflation	$\underset{+}{\rightarrow}$	USD/AZN	İnflation	$\underset{+}{\rightarrow}$	USD/AZN
USD/AZN	$\underset{-}{\rightarrow}$	İnflation	USD/AZN	$\underset{-}{\rightarrow}$	İnflation
Oil	$\underset{+}{\rightarrow}$	USD/AZN	Oil	$\underset{+}{\rightarrow}$	USD/AZN
USD/AZN	$\underset{-}{\rightarrow}$	Oil	USD/AZN	$\underset{-}{\rightarrow}$	Oil
Trade balance	$\underset{+}{\rightarrow}$	USD/AZN	Trade balance	$\underset{+}{\rightarrow}$	USD/AZN
USD/AZN	$\underset{-}{\rightarrow}$	Trade balance	USD/AZN	$\underset{-}{\rightarrow}$	Trade balance
EUR/USD	$\underset{-}{\rightarrow}$	FED	EUR/USD	$\underset{-}{\rightarrow}$	FED
FED	$\underset{-}{\rightarrow}$	EUR/USD	FED	$\underset{-}{\rightarrow}$	EUR/USD
EUR/USD	$\underset{+}{\rightarrow}$	İnflation	EUR/USD	$\underset{+}{\rightarrow}$	İnflation
İnflation	$\underset{-}{\rightarrow}$	EUR/USD	İnflation	$\underset{-}{\rightarrow}$	EUR/USD
EUR/USD	$\underset{+}{\rightarrow}$	Oil	EUR/USD	$\underset{+}{\rightarrow}$	Oil

Note that the check of this test is carried out on the basis of the probable value of α (prob) and is estimated by the probability $\alpha = 5\%$. If we look at the values of the probabilities, we get that FED ($\alpha = 0.13\%$), Oil ($\alpha = 4.64\%$), Inflation ($\alpha = 1,256 \cdot 10^{-9}\%$) can be counted as a Granger-cause of USD / AZN. In addition, we note that the oil exchange rate ($\alpha = 0.69\%$) and the EUR / USD exchange rate are the Granger-cause of oil ($\alpha = 0.23\%$) and inflation ($\alpha = 4.51\%$).

Testing heteroscedasticity. Let's look at the implementation of the White test [3, pp. 386-387] to test heteroscedasticity (Table 5).

Table 5

Heteroskedasticity Test: White

F-statistic	1.710014	Prob. F(18,25)	0.1061
Obs*R-squared	24.27976	Prob. Chi-Square(18)	0.1461
Scaled explained SS	42.40716	Prob. Chi-Square(18)	0.0010

Test Equation:
Dependent Variable: RESID^2
Method: Least Squares
Date: 10/29/18 Time: 13:08
Sample: 2014M03 2017M10
Included observations: 44
Collinear test regressors dropped from specification

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.011034	0.013201	0.835844	0.4112
EUR_USD_D(-1)^2	1.800454	1.572769	1.144767	0.2631
EUR_USD_D(-1)*FED_D(-1)	-6.915193	6.724561	-1.028349	0.3136
EUR_USD_D(-1)*INFLATION_D(-1)	0.159528	0.061246	2.604691	0.0153
EUR_USD_D(-1)*OIL	-0.025398	0.013356	-1.901649	0.0688
EUR_USD_D(-1)*OIL(-1)	0.024083	0.013136	1.833325	0.0787
EUR_USD_D(-1)	0.085939	0.131043	0.655809	0.5179
FED_D(-1)^2	7.341197	8.748055	0.839180	0.4093
FED_D(-1)*INFLATION_D(-1)	-0.135257	0.153497	-0.881172	0.3866
FED_D(-1)*OIL	-0.035316	0.042686	-0.827330	0.4159
INFLATION_D(-1)^2	0.000993	0.002324	0.427231	0.6729
INFLATION_D(-1)*OIL	-0.001207	0.000845	-1.427448	0.1658
INFLATION_D(-1)*OIL(-1)	0.001109	0.000696	1.594613	0.1234
INFLATION_D(-1)	0.013920	0.012300	1.131683	0.2685
OIL^2	6.10E-05	3.16E-05	1.927163	0.0654
OIL*OIL(-1)	-0.000121	6.48E-05	-1.858851	0.0749
OIL	2.60E-05	0.000701	0.037124	0.9707
OIL(-1)^2	6.20E-05	3.45E-05	1.796556	0.0845
OIL(-1)	-0.000378	0.000730	-0.518380	0.6088

R-squared	0.551813	Mean dependent var	0.001967
Adjusted R-squared	0.229118	S.D. dependent var	0.004306
S.E. of regression	0.003781	Akaike info criterion	-8.019412
Sum squared resid	0.000357	Schwarz criterion	-7.248967
Log likelihood	195.4271	Hannan-Quinn criter.	-7.733694
F-statistic	1.710014	Durbin-Watson stat	1.768087
Prob(F-statistic)	0.106141		

The model is considered to be homoscedastic, since the significance level of trial prices in the upper right-hand corner of the table exceeds 5% significance level.

To test the autocorrelation of the residual model, 2 tests are used for the Q-statistical (AR) and Serial L_m tests (MA). To verify the accuracy of the hypothesis of the absence of autocorrelation, consider the following tables (Tables 6 and 7):

Table 6

	AC	PAC	Q-Stat	Prob
1	-0.024	-0.024	0.0277	0.868
2	-0.078	-0.078	0.3173	0.853
3	-0.121	-0.126	1.0458	0.790
4	0.000	-0.014	1.0458	0.903
5	0.186	0.169	2.8325	0.726
6	-0.147	-0.159	3.9856	0.679
7	0.009	0.028	3.9899	0.781
8	-0.026	-0.003	4.0277	0.855
9	-0.005	-0.042	4.0294	0.909
10	-0.047	-0.084	4.1620	0.940
11	-0.026	0.022	4.2033	0.964
12	-0.105	-0.163	4.9027	0.961
13	-0.125	-0.149	5.9191	0.949
14	0.077	0.064	6.3210	0.958
15	0.027	-0.010	6.3714	0.973
16	0.011	-0.048	6.3799	0.983
17	-0.094	-0.034	7.0399	0.983
18	0.074	0.092	7.4618	0.986
19	0.026	-0.065	7.5173	0.991
20	0.006	0.013	7.5204	0.995

Table 7

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	0.250479	Prob. F(4,34)	0.9074
Obs*R-squared	1.259484	Prob. Chi-Square(4)	0.8682

Test Equation:
Dependent Variable: RESID
Method: Least Squares
Date: 10/29/18 Time: 13:16
Sample: 2014M03 2017M10
Included observations: 4
Presample missing value lagged residuals set to zero.

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.001815	0.023200	-0.078220	0.9381
EUR_USD_D(-1)	-0.136491	0.358360	-0.380876	0.7057
FED_D(-1)	-0.020787	0.126753	-0.163999	0.8707
INFLATION_D(-1)	0.000974	0.005747	0.169497	0.8664
OIL	0.000492	0.001665	0.295244	0.7696
OIL(-1)	-0.000458	0.001595	-0.287320	0.7756
RESID(-1)	-0.051468	0.177744	-0.289564	0.7739
RESID(-2)	-0.119908	0.192764	-0.622047	0.5381
RESID(-3)	-0.148864	0.179426	-0.829668	0.4125
RESID(-4)	-0.031735	0.186675	-0.170003	0.8660

R-squared	0.028625	Mean dependent var	5.78E-17
Adjusted R-squared	-0.228504	S.D. dependent var	0.044865
S.E. of regression	0.049727	Akaike info criterion	-2.967825
Sum squared resid	0.084074	Schwarz criterion	-2.562327
Log likelihood	75.29214	Hannan-Quinn criter.	-2.817447
F-statistic	0.111324	Durbin-Watson stat	1.977868
Prob(F-statistic)	0.999211		

Here, the null hypothesis is that there is no autocorrelation, and an alternative hypothesis is the existence of autocorrelation.

Table 6 shows that this model was tested for an autoregressive model with 20 lags and received more than 5% for each lag (the lowest probability was observed at the 6th delay $\alpha = 67.9\%$). This means that the model we establish indicates acceptance of the null hypothesis as a result of the Q-statistical test (i.e. there is no autocorrelation in the model we established).

Now let's explain the results of Table 7. Here the null hypothesis is the absence of autocorrelation of residuals, and the alternative hypothesis is the existence of autocorrelation of residues. Remind that the results of this test, as a rule, are checked with 5% probable accuracy. To verify the test, 4 lag cases were considered. When choosing the optimal variant, the condition is assumed that the probable value, like the Q-statistical test, will be more than 5%. As can be seen from the table, the probable values are rather large than the 5% probability values. If we specify the result with the hypothesis, the results will be the adoption of the null hypothesis and the failure of the alternative hypothesis. That is, there is no autocorrelation of residuals on the model.

To determine which lags are included in the model, the VAR is selected in the Eviews software package instead of the Equation tool, and by executing the Lag sturucture → Lag length criteria command in an open window, a new table is formed (Table 8).

Table 8

VAR Lag Order Selection Criteria
Endogenous variables: USD_AZN_D EUR_USD_D FED_D
INFLYASIYA_D NEFT TICARET_BALANSI
Exogenous
variables: C
Date: 10/21/18
Time: 20:16
Sample:
2013M01
2017M10
Included observations: 40

Lag	LogL	LR	FPE	AIC	SC	HQ
0	-337.8910	NA	1.182168	17.19455	17.44788	17.28615
1	-234.2502	171.0073*	0.041052*	13.81251	15.58584*	14.45369*
2	-203.5986	41.37970	0.060773	14.07993	17.37325	15.27069
3	-174.9210	30.11149	0.124541	14.44605	19.25936	16.18639
4	-114.4260	45.37126	0.081436	13.22130	19.55460	15.51122
5	-59.71112	24.62169	0.192247	12.28556*	20.13885	15.12506

* indicates lag order selected by the criterion
LR: sequential modified LR test statistic (each test
at 5% level) FPE: Final prediction error
AIC: Akaike
information criterion
SC: Schwarz
information criterion
HQ: Hannan-Quinn information criterion

4th of the star symbols indicate an inevitable delay to the 1st degree, and 1 to a delay to the 5th degree. Since the first lag is taken basic by the 4th criteria, the model was re-estimated using the least squares method, introducing the 1st lag (Table 9).

Table 9

Dependent Variable: USD_AZN_D
Method: Least Squares
Date: 10/21/18 Time: 20:22
Sample (adjusted): 2014M03 2017M10
Included observations: 44 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.038211	0.048931	0.780918	0.4406
~~USD_AZN_D(-1)~~	-0.074854	0.086008	-0.870312	0.3906
EUR_USD_D	0.054378	0.359104	0.151426	0.8806
EUR_USD_D(-1)	-1.435419	0.353779	-4.057385	0.0003
FED_D	-0.106016	0.122313	-0.866763	0.3925
FED_D(-1)	0.210132	0.127133	1.652850	0.1081
INFLATION_D	-0.004672	0.005978	-0.781536	0.4402
INFLATION_D(-1)	0.049029	0.005947	8.243755	0.0000
OIL	0.003844	0.001667	2.305641	0.0278
OIL(-1)	-0.004556	0.001916	-2.377944	0.0236
TRADE_BALANCE	8.98E-06	3.75E-05	0.239448	0.8123
TRADE_BALANCE(-1)	1.94E-05	3.73E-05	0.520867	0.6060

R-squared	0.814241	Mean dependent var	0.020143
Adjusted R-squared	0.750386	S.D. dependent var	0.099869
S.E. of regression	0.049896	Akaike info criterion	-2.930759
Sum squared resid	0.079667	Schwarz criterion	-2.444162
Log likelihood	76.47669	Hannan-Quinn criter.	-2.750305
F-statistic	12.75146	Durbin-Watson stat	1.996700
Prob(F-statistic)	0.000000		

Although the results are considered normal by many criteria, the results of the t-Student test are not considered acceptable. To eliminate this drawback, we need to remove some factors from the model. After subtracting the negative factors, we get the results of the optimal model, i.e. Table 2.

Forecasting. The following operations must be performed sequentially to make predictions through the built model:

First, the regression equation for the model is again set. The main difference between this regression equation and the original regression equation is that the equation is not executed for all observed moments, but from the time it starts to the moment when the observation prices at that moment are used for forecasting. The results for the newly created regression equation are shown below (Table 10):

Table 10

Dependent Variable: USD_AZN_D
Method: Least Squares
Date: 10/21/18 Time: 20:58
Sample (adjusted): 2014M03 2016M06
Included observations: 28 after adjustments

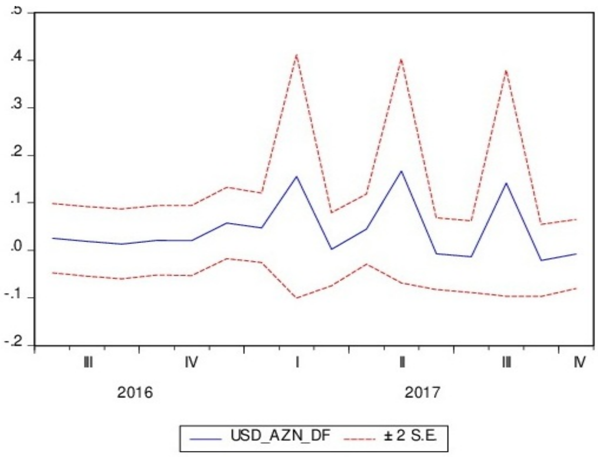
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.008925	0.019400	0.460057	0.6500
EUR_USD_D(-1)	-0.972143	0.283976	-3.423321	0.0024
FED_D(-1)	0.640711	0.476624	1.344271	0.1926
INFLATION_D(-1)	0.042808	0.011207	3.819926	0.0009
OIL	0.001839	0.001208	1.521712	0.1423
OIL(-1)	-0.001931	0.001166	-1.656244	0.1119

R-squared	0.925163	Mean dependent var	0.025246
Adjusted R-squared	0.908155	S.D. dependent var	0.115460
S.E. of regression	0.034991	Akaike info criterion	-3.680029
Sum squared resid	0.026937	Schwarz criterion	-3.394556
Log likelihood	57.52040	Hannan-Quinn criter.	-3.592757
F-statistic	54.39447	Durbin-Watson stat	2.136338
Prob(F-statistic)	0.000000		

Analysis of the results shows that there have been some changes in the values of the indicators. This change is a result of the difference in moments when the moments used in the model were not used in the prediction.

Now let's look at the prediction results for the remaining moments:

Table 11

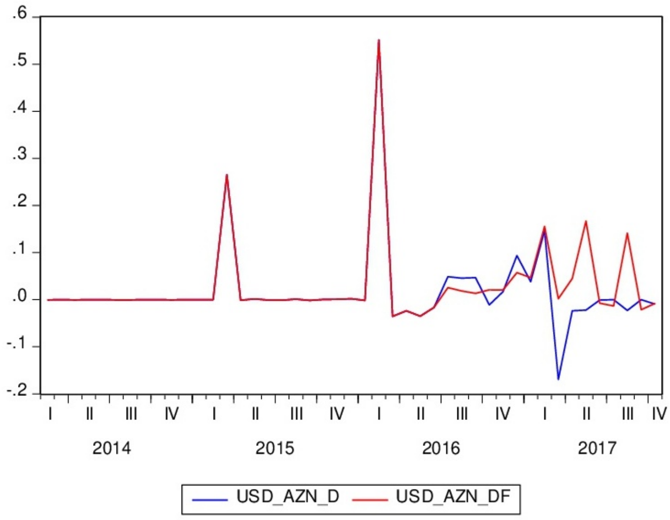


Forecast: USD_AZN_DF	
Actual: USD_AZN_D	
Forecast sample: 2016M07 2017M10	
Included observations: 16	
Root Mean Squared Error	0.079929
Mean Absolute Error	0.050603
Mean Abs. Percent Error	1655.816
Theil Inequality Coefficient	0.581606
Bias Proportion	0.144156
Variance Proportion	0.005912
Covariance Proportion	0.849933
Theil U2 Coefficient	7.103336
Symmetric MAPE	121.3647

Each test interval is two times longer than the standard error ($\sigma^2 \approx 0,08$). Note that the closer the standard error is to zero, the more accurate the model prediction can be.

Now let's look at the following chart to compare the forecast of the USD / AZN exchange rate curve (Chart 1):

Chart 1



Here, the USD / AZN exchange rate curve is shown in blue, and the projected exchange rate curve is shown in red.

As you can see, the curve model obtained using the forecast was located at some distance from the curve itself. This difference is due to the fact that the model is not fully explained by the factors mentioned.

Conclusion

Thus, as a result of comparative testing of many tests using the Eviews software package, the optimal regression model was tested, which shows that the model covering the time segment 01.01.2013-01.10.2017 changed significantly depending on four factors. A separate analysis of the results of each test shows that the model residues are homoscedastic, do not depend on autocorrelation, and can be considered to be generally significant. At the end of the model, the most optimistic version was predicted.

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Some Differential Properties of Generalized Nikolskii-Morrey Type Spaces

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Abstract. In the paper a generalized Nikolski-Morrey type spaces were introduced and studied. With help a integral representation are obtained Sobolev type inequalities for functions from this spaces.

Key Words and Phrases: Nikolskii-Morrey type spaces, integral representation, embedding theorems, generalized Holder condition.

2010 Mathematics Subject Classifications: 46E35, 46E30, 26D15

1. Introduction

In the paper, we introduce a generalized Nikolski-Morrey type spaces

$$\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi). \tag{1}$$

and help of method inetgral representation we study differential-difference properties of functions from this spaces. Let $G \subset R^n$;$1 \leq p^i < \infty$; $l^i \in (0,\infty)^n$,$i = 0,1,\ldots,n$; $l^0_j \geq 0, l^i_j \geq 0 (i \neq j = 1,2,\ldots,n), l^i_i \geq 0 (i = 1,2,\ldots,n)$; $\beta \in [0,1]^n$; $[t]_1 = \min\{1,t\}$, and let vector-functions $\varphi(t) = (\varphi_1(t),\ldots,\varphi_n(t))$, with Lebesgue measurable functions $\varphi_j(t) > 0, (t > 0)$, $\lim\limits_{t\to+0} \varphi_j(t) = 0$, $\lim\limits_{t\to+\infty} \varphi_j(t) = L \leq \infty$, $j = 1,2,\ldots,n$. Denote by $\mathbb{A}$ the set of vector functions φ. Let $m^0 = (m^0_1,\ldots,m^0_n), m^0_j \in N_0 (j = 1,\ldots,n)$, $m^i = (m^i_1,\ldots,m^i_n), m^i_j \in N_0 (i \neq j = 1,\ldots,n), m^i_i \in N (i = 1,\ldots,n)$ $k^0 = (k^0_1,\ldots,k^0_n), k^i_j \in N_0 (j = 1,\ldots,n, i = 1,\ldots,n)$.

Definition 1. *The space type* $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)$ *we denote the spaces of all functions* $f \in L^{loc}(G)$ $(m^i_j > l^i_j - k^i_j \geq 0, i \neq j = 1,\ldots,n; m^i_i > l^i_i - k^i_i \geq 0, i = 1,2,...,n)$ *with the finite norm*

$$\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)} = \sum_{i=0}^{n} \sup_{0<h<h_0} \frac{\left\|\Delta^{m_i}\left(\varphi(h), G_{\varphi(h)}\right) D^{k_i} f\right\|_{p^i,\varphi,\beta}}{\prod\limits_{j=1}^{n} \varphi_j(h)^{l^i_j - k^i_j}}, \tag{2}$$

where

$$\|f\|_{p^i,\varphi,\beta;G} = |f|_{L_{p^i,\varphi,\beta}(G)} = \sup_{x\in G,t>0}\left(|\varphi\left([t]_1\right)|^{-\beta}\|f\|_{p^i,G_{\varphi(t)}(x)}\right), \quad (3)$$

$\left|\varphi\left([t]_1\right)\right|^{-\beta} = \prod_{j=1}^{n}\left(\varphi_j\left([t]_1\right)\right)^{-\beta_j}$, $\Delta^{m_i}\left(\varphi\left(h\right), G_{\varphi(h)}\right) f$ =?, h_0 *it is positive fixed number, and let for any* $x \in R^n$

$$G_{\varphi(t)}\left(x\right) = G \cap I_{\varphi(t)}\left(x\right) = G \cap \left\{y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), \quad j = 1, 2, ..., n\right\},$$

For any $t > 0$,suppose $|\varphi\left([t]_1\right)| \leq C$, then the embeddings $\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi) \to \bigcap_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)$ and hold, i.e.

$$\|f\|_{\bigcap_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)} \leq c\|f\|_{\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)}, \quad (4)$$

Note that the spaces $\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)$ and is Banach space. The space (1) when $l^0 = (0,\ldots,0)$, $l^i = (0,\ldots,0,l_i,0,\ldots,0)$, $p^i = p(i = 0,1,\ldots,n)$ coincides with the space $H_{p,\varphi,\beta}^{l}\left(G_\varphi\right)$ introduced and studied in [1], in the case $\beta_j = 0\,(j = 1,...,n)$ it coincides with generalized Nikolski space $\bigcap_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)$.The spaces of such type with different norms introduced and studied [3]-[13].

Lemma 1. *Let* $G \subset R^n$, $1 \leq p^i \leq \infty$, *and* $f \in \bigcap_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)$. *Then we can construct the sequence* $h_s = h_s(x)$ $(s = 1,2,\ldots)$ *of infinitely differentiable finite in* R^n *functions for which*

$$\lim_{s\to\infty}\|f - h_s\|_{\bigcap_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)} = 0. \quad (5)$$

Proof. Let $G = \bigcup_{\lambda=1}^{M} G^\lambda$ and for obtaining equality (5) we estimate the norm

$$\|f - h_s\|_{\bigcap_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)} = \sum_{i=0}^{n}\omega_i^{l^i}(f - h_s). \quad (6)$$

$$\omega_i^{l^i}(f - h_s) = \sup_{0<h<h_0}\frac{\left\|\Delta^{m_i}\left(\varphi\left(h\right), G_{\varphi(h)}\right) D^{k_i} f\right\|_{p^i,\varphi,\beta}}{\prod_{j=1}^{n}\varphi_j\left(h\right)^{l_i-k_i}} \quad (7)$$

The sequence $h_s(x)$ $(s = 1,2,\ldots)$ is determined by the equality

$$h_s(x) = F(x,\varphi(t))|_{t=\frac{1}{s}} = \sum_{\lambda}^{M}\eta_\lambda(x)f_{\varphi^\lambda(t)}(x),$$

here the averaging functions are determined as follows:

$$f_{\varphi^\lambda(t)}(x)=\int\limits_{R^n} f(x+\varphi^\lambda(t)y)K_\lambda(y)dy,$$

where $K_\lambda(y)\in C_0^\infty(R^n)$ $(\lambda=1,2,\ldots,M)$ $\sup pK_\lambda(\cdot)\subset[-1;1]$

$$\int\limits_{R^n} K_\lambda(y)dy=1,$$

the functions $\eta_\lambda=\eta_\lambda(x)$ $(\lambda=1,2,\ldots,M)$ determine the expansion of a unit in the domain G, i.e.

1)$1\le\eta_\lambda(x)\le1$ in R^n;

2) $\eta_\lambda(x)=0$ in $G\setminus G_\lambda$ for all $\lambda=1,2,\ldots,M$;

3) $|D^\alpha\eta_\lambda(x)|\le C_\lambda$, $C_\lambda=const$ for all $\lambda=1,2,\ldots,M$ and $\alpha\ge0$.

Obviously,

$$\|f(\cdot)-h_s(\cdot)\|_{\bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)}\le\sum_{\lambda}^{M}\|\eta_\lambda(\cdot)(f(\cdot)-f_{\varphi^\lambda(t)}(\cdot))\|\le$$
$$\le C\sum_{\lambda}^{M}\|(f(\cdot)-f_{\varphi^\lambda(t)}(\cdot))\|_{\bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)},\tag{8}$$

As much as small for rather small, t, as a consequence of continuity of L_p- average functions, belonging to the space $L_p(G_\varphi^\lambda)$,from (6),(7) and (8) it follows

$$\|f(\cdot)-h_s(\cdot)\|_{\bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)}<\varepsilon,$$

in other words,

$$\lim_{s\to\infty}\|f-h_s\|_{\bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)}=0.$$

Assuming that $\varphi_j(t)$ $(j=1,2,...,n)$ are also differentiable on $[0,T]$, we can show that for $f\in\bigcap\limits_{i=0}^{n}L_{p^i}^{<l^i>}(G_\varphi)$ determined in n- dimensional domains, satisfying the condition of flexible φ-horn, it holds the following integral representation $(\forall x\in U\subset G)$

$$D^\nu f(x)=(-1)^{|\nu|+|l^0|}\prod_{j=1}^{n}(\varphi_j(T))^{-\nu_j-1}\int\limits_{R^n}\int\limits_{-\infty}^{+\infty}K_0^{(\nu)}\left(\frac{y}{\varphi(T)},\frac{\rho(\varphi(T,x))}{\varphi(t)}\right)$$
$$\times\zeta_i\left(\frac{u}{\varphi_i(T)},\frac{\rho_i(\varphi_i(T,x))}{\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi(T),x)\right)\Delta_i^{m^0}(\varphi_i(\delta)u)$$

$$\times f\left(x+y+u_1+\ldots+u_n\right) dydu+\sum_{i=1}^{n}(-1)^{|\nu|+|l^i|}\int_0^T\int_{R^n}\int_{-\infty}^{+\infty}K_i^{(\nu)}\times$$

$$\times\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi\left(t,x\right)\right)}{\varphi(t)}\right)\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i\left(t,x\right)\right)}{\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi(t),x\right)\right)\Delta_i^{m^i}\left(\varphi_i\left(\delta\right)u\right)$$

$$\times f\left(x+y+u_1+\ldots+u_n\right) dydu\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dtdudy, \quad (9)$$

Let $\Phi_i\left(\cdot,y\right)\in C_0^{\infty}\left(R^n\right)$ be such that

$$S\left(\psi_i\right)\subset I_{\varphi(t)}=\left\{y:|y_j|<\frac{1}{2}\varphi_j(t),\quad j=1,2,...,n\right\}.$$

for any $0<T\leq 1$ assume that

$$V=\bigcup_{0<t\leq T}\left\{y:\frac{y}{\varphi(t)}\in S\left(\psi_i\right)\right\}.$$

It is clear that $V\subset I_{\varphi(t)}$ and suppose that $U+V\subset G$.

Lemma 2. *Let $1\leq p^i\leq p\leq r\leq\infty$; $0<\eta,t<T\leq 1$, $\nu=(\nu_1,\nu_2,...,\nu_n)$, $\nu_j\geq 0$ are integers, $j=1,2,...,n$; $\Delta_i^{m^i}\left(h\right)\in L_{p^i,\varphi,\beta}(G)$ and let*

$$F\left(x\right)=\prod_{j=1}^{n}(-1)^{|\nu_j|-1}\int_{R^n}\int_{-\infty}^{+\infty}K_0^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}\right)$$

$$\times\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi_i(t),x\right)\right)$$

$$\times\Delta^{m^0}\left(\varphi_i\left(\delta\right)u\right)f\left(x+y+u\right)dxdudy \quad (10)$$

$$F_\eta^i\left(x\right)=\int_0^\eta L_i\left(x,t\right)\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-2}\prod_{j\in m^i}\frac{\varphi_j'(t)}{\varphi_j(t)}dt \quad (11)$$

$$F_{\eta T}^i\left(x\right)=\int_\eta^T L_i\left(x,t\right)\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{\nu_j-2}\prod_{j\in m^i}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \quad (12)$$

$$Q_T^i=\int_0^T\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-(1-\beta_jp)\left(\frac{1}{p^i}-\frac{1}{p}\right)}\prod_{j\in l^i}\frac{\varphi_i'(t)}{\left(\varphi_i(t)\right)^{1-l_i}}dt<\infty$$

where

$$L_i\left(x,t\right)=\int_{R^n}\int_{-\infty}^{+\infty}M_i^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}\right)$$

$$\times\,\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi_i(t),x\right)\right)\Delta_i^{m_i}\left(\varphi_i\left(\delta\right)u\right)f\left(x+y+ue_i\right)dudy \quad (13)$$

Then for any $\overline{x}\in U$ the following inequalities are true

$$\sup_{\overline{x}\in U}\|F\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_1\left\|\prod_{j=1}^{n}\left(\varphi_i(t)\right)^{-l_j^0}\Delta^{m^0}\left(\varphi_i(T),G_{\varphi(T)}\right)f\right\|_{p^0,\varphi,\beta;G}$$

$$\times\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p^i}-\frac{1}{p}\right)}\prod_{j=1}^{n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p}{q}}, \quad (14)$$

$$\sup_{\overline{x}\in U}\left\|F_\eta^i\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_2\left\|\prod_{j=1}^{n}\left(\varphi_i(t)\right)^{-l_j^i}\Delta^{m^i}\left(\varphi_i(T),G_{\varphi(T)}\right)f\right\|_{p^i,\varphi,\beta;G}$$

$$\times|Q_T^i|\prod_{j=1}^{n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p^i}{p}}, \quad (15)$$

$$\sup_{\overline{x}\in U}\left\|F_{\eta T}^i\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_3\left\|\left(\varphi_i(t)\right)^{-l_j^i}\Delta^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p^i,\varphi,\beta;G}$$

$$\times\left|Q_{\eta T}^i\right|\prod_{j=1}^{n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p^i}{p}}, \quad (16)$$

is hold, where $U_{\psi(\xi)}\left(\overline{x}\right)=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\psi_j\left(\xi\right),j=1,2,...,n\right\}$ and $\psi\in A$, C_1, C_2 are the constants independent of φ, ξ, η and T.

Corollary 1.

$$\|F\|_{p,\psi,\beta^1;U}\leq C_1'\left\|\prod_{j=1}^{n}\left(\varphi_i(t)\right)^{-l_j^0}\Delta^{m^0}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p^0,\varphi,\beta;G}, \quad (17)$$

$$\left\|F_\eta^i\right\|_{p,\psi,\beta^1;U}\leq C_2'\left\|\prod_{j=1}^{n}\left(\varphi_i(t)\right)^{-l_j^i}\Delta^{m^i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p^i,\varphi,\beta;G}. \quad (18)$$

$$\left\|F_{\eta,T}^i\right\|_{p,\psi,\beta^1;U}\leq C_3'\left\|\prod_{j=1}^{n}\left(\varphi_i(t)\right)^{-l_j^i}\Delta^{m^i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p^i,\varphi,\beta;G}. \quad (19)$$

The proof is similar to the proof of Lemma 2 in [1].

2. Main results

Prove two theorems on the properties of the functions from the space $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)$.

Theorem 1. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn, $1 \le p^i \le p \le \infty$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \ge 0$ be entire $j = 1, 2, ..., n$, $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ and let $f \in \bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)$. Then the following embeddings hold*

$$D^\nu : \bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi) \to L_{q,\psi,\beta^1}(G)$$

i.e. for $f \in \bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are true

$$\|D^\nu f\|_{p,G} \le$$

$$\le C_1 \sum_{i=1}^{n} \left|Q^i_T\right| \sup_{0<t<t_0} \left\| \prod_{j=1}^{n} (\varphi_i(t))^{l^i_j} \Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p,\varphi,\beta;G}, \tag{20}$$

$$\|D^\nu f\|_{q,\psi,\beta^1;G} \le C_2 \|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)}, \; p^i \le p < \infty. \tag{21}$$

In particular, if

$$Q^i_{T,0} = \int_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\frac{1}{p}} \prod_{j \in l^i} \frac{\varphi'_i(t)}{(\varphi_i(t))^{1-l_i}} dt < \infty,$$

then $D^\nu f(x)$ is continuous on G, i.e.

$$\sup_{x\in G} |D^\nu f(x)| \le \sum_{i=1}^{n} \left|Q^i_{T,0}\right| \sup_{0<t<t_0} \left\| \prod_{j=1}^{n} (\varphi_i(t))^{l^i_j} \Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i,\varphi,\beta;G} \tag{22}$$

$0 < T \le \min\{1, T_0\}$, T_0 is a fixed number; C_1, C_2 are the constants independent of f, C_1 are independent also on T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^\nu f$ on G. Indeed, from the condition $Q^i_T < \infty$ for all $(i = 1, 2, ..., n)$ it follows that for $f \in \bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi) \to \bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i}(G_\varphi)$, there exists $D^\nu f \in L_p(G)$ and for it integral representation (9) with the same kernels is valid.

Based around the Minkowsky inequality, from identities (9) we get

$$\|D^\nu f\|_{q,G} \le \|F\|_{q,G} + \sum_{i=1}^{n} \|F_i\|_{p,G} \,. \tag{23}$$

By means of inequality (14) for $U = G$, $M_i = K_i^i$, $t = T$ we get

$$\|F\|_{p,G} \le C_1 |Q_T^0| \left\| \prod_{j=1}^{n} (\varphi_i(t))^{-l_j^0} \Delta^{m^0} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^0,\varphi,\beta;G}, \tag{24}$$

and by means inequality (15) for $\eta = T$, $M_i = K_i^i$, $U = G$, we get

$$\|F_i\|_{q,G} \le C_2 |Q_T^i| \left\| \prod_{j=1}^{n} (\varphi_i(t))^{-l_j^i} \Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i,\varphi,\beta;G}, \tag{25}$$

Substituting (25) and (24) in (23), we get inequality (20). By means of inequalities (17), (18) and (19) for $\eta = T$ we get inequality (21).

Now let conditions $Q_T^i < \infty$ $(i = 1, 2, ..., n)$ be satisfied, then based around identities (9) from inequality (23) we get

$$\left\| D^\nu f - f_{\varphi(T)}^{(\nu)} \right\|_{\infty,G} \le C \sum_{i=1}^{n} \left|Q_{T,0}^i\right| \sup_{0<t<t_0} \left\| \frac{\Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f}{\prod\limits_{j=1}^{n} (\varphi_i(t))^{l_j^i}} \right\|_{p^i,\varphi,\beta;G}.$$

As $T \to 0$, the left side of this inequality tends to zero, since $f_{\varphi(T)}^{(\nu)}(x)$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f$ is continuous on G.

Theorem 1 is proved.

Let γ be an n-dimensional vector.

Theorem 2. Let all the conditions of theorem 1 be fulfilled. Then for $Q_T^i < \infty$ $(i = 1, 2, ..., n)$ the derivative $D^\nu f$ satisfies on G the Holder generalized condition, i.e. the following inequality is valid:

$$\|\Delta(\gamma, G) D^\nu f\|_{q,G} \le C \|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)} \cdot |H(|\gamma|, \varphi; T)|, \tag{26}$$

where C is a constant independent of f, $|\gamma|$ and T.

In particular, if $Q_{T,0}^i < \infty$, $(i = 1, 2, \ldots, n)$, then

$$\sup_{x \in G} |\Delta(\gamma, G) D^\nu f(x)| \le C \|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)} \cdot |H_0(|\gamma|, \varphi, T)|. \tag{27}$$

where $H(|\gamma|, \varphi, T) = \max\limits_i \left\{|\gamma|, Q_{|\gamma|}^i, Q_{|\gamma|,T}^i\right\}$ $\left(H_0(|\gamma|, \varphi, T) = \max\limits_i \left\{|\gamma|, Q_{|\gamma|,0}^i, Q_{|\gamma|,T,0}^i\right\}\right)$

Proof. According to lemma 8.6 from [2] there exists a domain

$$G_\omega \subset G \left(\omega = \zeta r(x),\ \zeta > 0\ r(x) = \rho(x, \partial G),\ x \in G\right)$$

and assume that $|\gamma| < \omega$, then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identities (9) with the same kernels are valid. After same transformations, from (9) and (4) we get

$$
\begin{aligned}
&|\Delta(\gamma, G) D^\nu f(x)| \le C_1 \prod_{j=1}^{n} (\varphi_j(t))^{-1-\nu_j} \times \\
&\times \int\limits_{R^n} \int\limits_{-\infty}^{+\infty} \left| K_0^{(\nu)}\left(\frac{y-\gamma}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)}\right) - K_0^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(T)}\right)\right| dydz \times \\
&\times |\Delta^{m^0}(\varphi(\delta)u)(x+y+u_1+\ldots+u_n)| \cdot |\zeta^0\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{2\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi_i(t), x)\right) |dudy+ \\
&+ C_2 \sum_{i=1}^{n} \left\{ \int\limits_{0}^{|\gamma|} \int\limits_{R^n} \int\limits_{-\infty}^{+\infty} \left| \zeta^i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t,x))}{\varphi_i(t)}, \frac{1}{2}\rho'(\varphi(t), x)\right)\right| \times - \right. \\
&\times \left|\Delta^{m^i}(\varphi_i(\delta) u) f(x+y+u_1+\ldots+u_n)\right| \prod_{j=1}^{n} (\varphi_j(t))^{\nu_j-2} \prod_{j\in m^i} \frac{\varphi_i'(t)}{\varphi_i(t)} dydudt \\
&+ \int\limits_{|\gamma|}^{T} \int\limits_{R^n} \int\limits_{-\infty}^{+\infty} \left| K_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t,x))}{\varphi(t)}\right)\right| \left|\zeta_i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t,x))}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x)\right)\right| \\
&\left. \times \int\limits_{0}^{1} \left|\Delta^{m^i}(\varphi_i(\delta) u) f(x+y+u_1+\ldots+u_n\gamma)\right| \prod_{j=1}^{n} (\varphi_j(t))^{\nu_j-2} \prod_{j\in m^i} \frac{\varphi_i'(t)}{\varphi_i(t)} dvdudydt \right\}. \\
&= C_1 F(x,\gamma) + C_2 \sum_{i=1}^{n} \left(F_1(x,\gamma) + F_2^i(x,\gamma)\right), \qquad (28)
\end{aligned}
$$

where $0 < T \le \{1, T_0\}$ we also assume that $|\gamma| < T$. Consequently, $|\gamma| < \min(\omega, T)$. If $x \in G \setminus G_\omega$ then by definition

$$\Delta(\gamma, G) D^\nu f(x) = 0.$$

Based around (28) we have

$$
\begin{aligned}
\|\Delta(\gamma, G) D^\nu f\|_{q,G} \le &\left\|F_1^i(\cdot,\gamma)\right\|_{q,G_\omega} \\
&+ \sum_{i=1}^{n} \left(\|E(\cdot,\gamma)\|_{q,G_\omega} + \left\|F_2^i(\cdot,\gamma)\right\|_{q,G_\omega}\right), \qquad (29)
\end{aligned}
$$

$$F(x,\gamma) \le \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j-2} \int\limits_{0}^{|\gamma|} d\zeta \int\limits_{R^n} \int\limits_{R^n} |f(x+y+u_1+\ldots+u_n)|$$

$$\times \left| D_j K^{(\nu)} \left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)} \right) \Omega^{(\nu)} \left(\frac{z}{\varphi(T)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)} \right) \right| dydz.$$

Taking into account $\xi e_\gamma + G_\omega \subset G$, based around the generalized Minkowsky inequality, from inequality (19) for $U = G$, we have

$$\|F(\cdot, \gamma)\|_{p,G_\vartheta} \leq C_1 |\gamma| \left\| \prod_{j=1}^{n} (\varphi_i(t))^{-l_j^0} \Delta^{m^0} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i,\varphi,\beta;G} \tag{30}$$

By means of inequality (16), for $U = G$, $\eta = |\gamma|$ we get

$$\left\|F_1^i(\cdot, \gamma)\right\|_{q,G_\omega} \leq C_2 \left| Q^i_{|\gamma|} \right| \left\| \prod_{j=1}^{n} (\varphi_i(t))^{-l_j^i} \Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i,\varphi,\beta;G} \tag{31}$$

and by means of inequality (10) for $U = G$, $\eta = |\gamma|$ we get

$$\left\|F_2^i(\cdot, \gamma)\right\|_{q,G_\omega} \leq C_3 \left| Q^i_{|\gamma|,T} \right| \left\| \prod_{j=1}^{n} (\varphi_i(t))^{-l_j^i} \Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i,\varphi,\beta;G}. \tag{32}$$

From inequalities (29) -(32) we get the required inequality.

Now suppose that $|\gamma| \geq \min(\omega, T)$. Then

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \leq 2 \|D^\nu f\|_{p,G} \leq C(\vartheta T) \|D^\nu f\|_{p,G} |H(|\gamma|, \varphi; T)|.$$

Estimating for $\|D^\nu f\|_{p,G}$ by means of inequality (20), in this case we get estimation (26).

Theorem 2 is proved.

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Boundedness of the Fractional Maximal Operator in Local and Global Morrey-type Spaces on the Heisenberg Group

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Abstract. We study the boundedness of the fractional maximal operator M_α on the Heisenberg group $\mathbb{H}^n$ in local and global Morrey-type spaces $LM_{p\theta,w}(\mathbb{H}^n)$ and $GM_{p\theta,w}(\mathbb{H}^n)$, respectively. We give a characterization of strong and weak type boundedness for the operator M_α in local Morrey-type spaces $LM_{p\theta,w}(\mathbb{H}^n)$.

Key Words and Phrases: fractional maximal operator, local Morrey-type space, Heisenberg group.

2010 Mathematics Subject Classifications: 42B25, 42B35, 43A15

1. Introduction

In this paper, we establish the norm inequalities for the fractional maximal operator in local Morrey-type spaces on Heisenberg group. The Heisenberg group [6, 7, 15, 17] appears in quantum physics and many fields of mathematics, including harmonic analysis, the theory of several complex variables and geometry. We begin with some basic notation. The Heisenberg group $\mathbb{H}_n$ a non-commutative nilpotent Lie group with the product spaces $\mathbb{R}^{2n+1}$ that have the multiplication

$$xy = \Big(x' + y', x_{2n+1} + y_{2n+1} + 2\sum_{k=1}^{n} x_k y_{n+k} - x_{n+k} y_k\Big),$$

where $x = (x', x_{2n+1})$, and $y = (y', y_{2n+1})$. By the definition, the identity element on $\mathbb{H}_n$ is $0 \in \mathbb{R}^{2n+1}$, while the inverse element of $x = (x', t)$ is $x^{-1} = (-x', -t)$.

The corresponding Lie algebra is generated by the left-invariant vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial x_{2n+1}}, X_{n+j} \quad = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial x_{2n+1}}, X_{2n+1} = \frac{\partial}{\partial x_{2n+1}},\ j = 1, \ldots, n.$$

The only non-trivial commutator relations are

$$\big[X_j, X_{n+j}\big] = -4X_{2n+1}, \quad j = 1, \dots, n.$$

The non-isotropic dilation on $\mathbb{H}_n$ is defined as $\delta_t(x', x_{2n+1}) = (tx', t^2 x_{2n+1})$ for $t > 0$. The Haar measure dx on this group coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. It is easy to check that $d\big(\delta_t x\big) = r^Q dx$. In the above, $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. The norm of $x = (x', x_{2n+1}) \in \mathbb{H}_n$ is given by $|x|_{\mathbb{H}} = (|x'|^4 + x_{2n+1}^2)^{1/4}$, where $|x'|^2 = \sum_{k=1}^{2n} x_k^2$. The norm satisfies the triangle inequality and leads to the left-invariant distance $d(x, y) = |xy^{-1}|_{\mathbb{H}}$. With this norm we define the Heisenberg ball, $B(x, r) = \{y \in \mathbb{H}_n : |xy^{-1}|_{\mathbb{H}} < r\}$, where x is the center and r is the radius. The volume of $B(x, r)$ is $d_n r^{2n+2}$, where dC_n is the volume of the unit ball $B_1 \equiv B(e, 1)$. Let $S_H = \{x \in \mathbb{H}_n : |x|_{\mathbb{H}} = 1\}$ be the unit sphere in $\mathbb{H}_n$ equipped with the normalized Haar surface measure $d\sigma$.

The fractional maximal function $M_\alpha f$, $0 < \alpha < Q$ on the Heisenberg groups of a function $f \in L_1^{\mathrm{loc}}(\mathbb{H}_n)$ is defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{Q}} \int_{B(x,t)} |f(y)| dy.$$

If $\alpha = 0$, then $M \equiv M_0$ is the maximal operator on the Heisenberg groups. It is well known that the fractional maximal operator on the Heisenberg groups play an important role in harmonic analysis (see [7, 16]).

The main purpose of [10] is to give some sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups $\mathbb{G}$ in local Morrey-type space $LM_{p\theta,w_1}(\mathbb{G})$. In a series of papers by Burenkov V., Guliyev H. and Guliyev V. etc. (see, for example [2, 3, 4]) be given some necessary and sufficient conditions for the boundedness of fractional maximal operators, fractional integral operators and singular integral operators in local Morrey-type spaces $LM_{p\theta,w_1}(\mathbb{R}^n)$.

In this paper, we study the boundedness of the fractional maximal operator M_α on the Heisenberg group $\mathbb{H}^n$ in local Morrey-type spaces $LM_{p\theta,w}(\mathbb{H}^n)$. Also we give a characterization of strong and weak type boundedness for the operator M_α in local Morrey-type spaces $LM_{p\theta,w}(\mathbb{H}^n)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. For a number p, $p' = p/(p-1)$ denotes the conjugate exponent of p.

2. Local and global Morrey-type spaces on the Heisenberg group

Let $0 < p, \theta \le \infty$. Denote by Ω_θ a set of all non-negative measurable functions $w(r)$ on $(0, \infty)$ such that $w(t) \ne 0$ on the set of positive measure and $\|w(r)\|_{L_\theta(t_1,\infty)} < \infty$ for some $t_1 > 0$. The set $\Omega_{p,\theta}$ consists of the functions $w(r) \in \Omega_\theta$ such that $\|w(r) r^{Q/p}\|_{L_\theta(0,t_2)} < \infty$

for some $t_2 > 0$ (see [2]). Let $w_1 \in \Omega_\theta$, $w_2 \in \Omega_{\theta,p}$. Recall that in 1994 the doctoral thesisis [10] (see also [11]) by Guliyev introduced the local Morrey-type space $LM_{p\theta,w_1}$ and in [1] (see also [2, 3, 4]) by Burenkov, Guliyev introduced the global Morrey-type space $GM_{p\theta,w_1}$.

Definition 1. *Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w}(\mathbb{H}^n)$, $GM_{p\theta,w}(\mathbb{H}^n)$, the local Morrey-type spaces, the global Morrey-type spaces on the Heisenberg group respectively, the spaces of all functions $f \in L_p^{\rm loc}(\mathbb{H}^n)$ with finite quasinorms*

$$\|f\|_{LM_{p\theta,w}(\mathbb{H}^n)} = \big\|w(r)\|f\|_{L_p(B(0,t))}\big\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p\theta,w}(\mathbb{H}^n)} = \sup_{x\in\mathbb{H}^n} \big\|w(r)\|f\|_{L_p(B(x,t))}\big\|_{L_\theta(0,\infty)}$$

respectively.

Note that

$$\|f\|_{LM_{p\infty,1}(\mathbb{H}^n)} = \|f\|_{GM_{p\infty,1}(\mathbb{H}^n)} = \|f\|_{L_p(\mathbb{H}^n)}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p}(\mathbb{H}^n)} \equiv M_{p,\lambda}(\mathbb{H}^n)$, $0 \le \lambda \le Q$.

For a measurable set $\mathbb{H}^n$ and a function v non-negative and measurable on $\mathbb{H}^n$, let $L_{p,v}(\mathbb{H}^n)$ be the weighted L_p-space of all functions f measurable on $\mathbb{H}^n$ for which $\|f\|_{L_{p,v}(\mathbb{H}^n)} = \|vf\|_{L_p(\mathbb{H}^n)} < \infty$.

If $0 < p \le \theta \le \infty$, then $\|f\|_{LM_{p\theta,w}(\mathbb{H}^n)} \le \|f\|_{L_{p,W}(\mathbb{H}^n)}$, and if $0 < \theta \le p \le \infty$, then $\|f\|_{L_{p,W}(\mathbb{H}^n)} \le \|f\|_{LM_{p\theta,w}(\mathbb{H}^n)}$, where for all $x \in \mathbb{H}^n$ $W(x) = \|w\|_{L_\theta(|x|_{\mathbb{H}},\infty)}$.

In particular, for $0 < p \le \infty$ $\|f\|_{LM_{pp,w}(\mathbb{H}^n)} = \|f\|_{L_{p,V}(\mathbb{H}^n)}$, where for all $x \in \mathbb{H}^n$ $V(x) = \|w\|_{L_p(|x|_{\mathbb{H}},\infty)(\mathbb{H}^n)}$.

We shall use the following theorem stating necessary and sufficient conditions for the validity of the following inequality

$$\|M_\alpha f\|_{L_{p_2,v_2}(\mathbb{H}^n)} \le c\|f\|_{L_{p_1,v_1}(\mathbb{H}^n)} \tag{1}$$

where v_1 and v_2 are functions non-negative and measurable on $\mathbb{H}^n$ and $c > 0$ is independent of f (see [5, 14]).

Given a set $\Omega \subset \mathbb{H}^n$, χ_Ω will denote the characteristic function of Ω.

Theorem 1. *Let $0 \le \alpha < Q$, $1 < p_1 \le p_2 < \infty$. Moreover, let v_1, v_2 be non-negative and measurable on $\mathbb{H}^n$. Then inequality (1) holds if, and only if, the following equivalent conditions are satisfied*

$$\mathcal{J} = \sup_{B\subset\mathbb{H}^n} |B|^{\frac{\alpha}{n}-1} \left\|v_1^{-1}\right\|_{L_{p_1'}(B)} \|v_2\|_{L_{p_2}(B)} < \infty \tag{2}$$

and

$$\sup_{B\subset\mathbb{H}^n} \left\|M_\alpha\left(\chi_B v_1^{p_1/(1-p_1)}\right)\right\|_{L_{p_2,v_2}(B)} \left\|v_1^{1/(1-p_1)}\right\|_{L_{p_1}(B)}^{-1} < \infty. \tag{3}$$

Moreover, the sharp (minimal possible) constant c^* in (1), satisfies the inequality $c\mathcal{J} \le c^* \le c\mathcal{J}$, where $c, c^* > 0$ are independent of v_1 and v_2.

3. Boundedness of the fractional maximal operator in local Morrey-type spaces on Heisenberg group

Let $0 < p, \theta \le \infty$. Denote by Ω_θ a set of all non-negative measurable functions $w(r)$ on $(0,\infty)$ such that $w(t) \neq 0$ on the set of positive measure and $\|w(r)\|_{L_\theta(t_1,\infty)} < \infty$ for some $t_1 > 0$. Let $w_1 \in \Omega_\theta$, $w_2 \in \Omega_{\theta,p}$. Recall that in 1994 the doctoral thesisis [10] (see also [11]) by Guliyev V.S. introduced the local Morrey-type space $LM_{p\theta,w_1}(\mathbb{H}^n)$ is given by

$$\|f\|_{LM_{p\theta,w_1}(\mathbb{H}^n)} = \|w_1(r)\|f\|_{B(0,r)}\|_{L_\theta(0,\infty)}.$$

To obtain necessary and sufficient conditions on w_1 and w_2 under which M_α is bounded for other parameter values and to obtain simpler conditions for the case $p = \theta_1 = \theta_2$ we reduce the problem of the boundedness of M_α in the local Morrey-type spaces to the problem of the boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions.

Lemma 1. *Let $0 \le \alpha < Q$, $1 < p_1 \le p_2 < \infty$ and $-\infty < \gamma < \infty$. Then the inequality*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \le c(r)\|f\|_{L_{p_1,(|x|_{\mathbb{H}}+r)^\gamma}(\mathbb{H}^n)}, \tag{4}$$

where $c(r) > 0$ is independent of f holds for all $f \in L^{\mathrm{loc}}_{p_1}(\mathbb{H}^n)$ if and only if

$$\gamma \ge -\frac{Q}{p_2} \quad \text{and} \quad Q\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \le \alpha \le \frac{Q}{p_1} + \gamma. \tag{5}$$

If (5) holds, then the minimal constant $c(r)$ in (4) satisfies

$$c(r) \asymp r^{\alpha - Q(1/p_1 - 1/p_2) - \gamma}.$$

Proof. We apply Theorem 1 to the pair of functions $v_2(x) = \chi_{B(0,r)}(x), v_1(x) = (|x|_{\mathbb{H}} + r)^\gamma$. Then

$$\begin{aligned}
\mathcal{I}(v_1,v_2) &= \sup_{R>0} R^{\alpha-Q}\left(\int_0^R t^{Q-1}\chi_{(0,r)}(t)dt\right)^{1/p_2}\left(\int_0^R t^{Q-1}(t+r)^{-\gamma p_1'}dt\right)^{1/p_1'}\\
&= r^{Q/p_2+Q/p_1'-\gamma}\sup_{R>0} R^{\alpha-Q}\left(\int_0^{\frac{R}{r}} \tau^{Q-1}\chi_{(0,1)}(\tau)d\tau\right)^{1/p_2}\left(\int_0^{\frac{R}{r}} \tau^{Q-1}(\tau+1)^{-\gamma p_1'}d\tau\right)^{1/p_1'}\\
&= r^{\alpha+Q/p_2-Q/p_1-\gamma}\sup_{\rho>0} \rho^{\alpha-Q}\left(\int_0^{\rho} \tau^{Q-1}\chi_{(0,1)}(\tau)d\tau\right)^{1/p_2}\left(\int_0^{\rho} \tau^{Q-1}(\tau+1)^{-\gamma p_1'}d\tau\right)^{1/p_1'}\\
&\equiv r^{\alpha+Q/p_2-Q/p_1-\gamma}K,
\end{aligned}$$

where $K = \max\{K_1, K_2\}$,

$$K_1 = \sup_{0<\rho\le 1} \rho^{\alpha-Q}\left(\int_0^{\rho} \tau^{Q-1}\chi_{(0,1)}(\tau)d\tau\right)^{1/p_2}\left(\int_0^{\rho} \tau^{Q-1}(\tau+1)^{-\gamma p_1'}d\tau\right)^{1/p_1'}$$

and

$$K_2 = \sup_{1<\rho\leq\infty} \rho^{\alpha-Q} \left(\int_0^\rho \tau^{Q-1} \chi_{(0,1)}(\tau) d\tau \right)^{1/p_2} \left(\int_0^\rho \tau^{Q-1} (\tau+1)^{-\gamma p_1'} d\tau \right)^{1/p_1'} .$$

Next,

$$K_1 < \infty \Leftrightarrow \sup_{0<\rho\leq 1} \rho^{\alpha+Q/p_2-Q/p_1} < \infty \Leftrightarrow \alpha + Q/p_2 - Q/p_1 \geq 0.$$

Moreover,

$$K_2 < \infty \Longleftrightarrow \sup_{1<\rho<\infty} \rho^{\alpha-Q} \left(\int_1^\rho \tau^{Q-1-\gamma p_1'} d\tau \right)^{1/p_1'} < \infty.$$

If $\gamma > Q/p_1'$, then $\int_1^\infty \tau^{Q-1-\gamma p_1'} d\tau < \infty$ and $K_2 < \infty$ since $\alpha < Q$.

If $\gamma = Q/p_1'$, then $K_2 < \infty \Leftrightarrow \sup\limits_{1\leq\rho<\infty} \rho^{\alpha-Q} \ln \rho < \infty$. Therefore again $K_2 < \infty$ since $\alpha < Q$.

If $\gamma < Q/p_1'$, then

$$K_2 < \infty \Longleftrightarrow \sup_{1\leq\rho<\infty} \rho^{\alpha-Q+Q/p_1'-\gamma} < \infty \Longleftrightarrow$$
$$\alpha - Q + \frac{Q}{p_1'} - \gamma \leq 0 \Longleftrightarrow \gamma \geq \alpha - \frac{Q}{p_1}.$$

Inequality $\alpha < Q$, implies that $\alpha p_1 - Q < Q(p_1 - 1)$. Hence $K_2 < \infty \Leftrightarrow \gamma \geq \alpha - Q/p_1$.

Corollary 1. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $Q\,(1/p_1 - 1/p_2)_+ \leq \alpha < Q$. Then there exists $c > 0$ such that*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \leq c r^{Q/p_2} \left(\int_{\mathbb{H}^n} \frac{|f(x)|^{p_1}}{(|x|_{\mathbb{H}} + r)^{Q-\alpha p_1}} dx \right)^{\frac{1}{p_1}}, \tag{6}$$

for all $r > 0$ and for all $f \in L_{p_1}^{loc}(\mathbb{H}^n)$.

Proof. In the case $1 < p_1 \leq p_2 < \infty$ (6) follows by Lemma 1 with $\gamma = \alpha - Q/p_1$.

If $0 < p_2 < p_1 < \infty$, by Hölder's inequality and (6) for $p_2 = p_1$ we have

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \leq (d_n r^Q)^{1/p_2-1/p_1} \|M_\alpha f\|_{L_{p_1}(B(0,r))} \leq c r^{Q/p_2} \|M_\alpha f\|_{L_{p_1}(B(0,r))},$$

where d_n is the volume of the unit ball in $\mathbb{H}^n$ and $c > 0$ depends only on Q, p_1 and p_2.

The following lemma was proved in [2].

Lemma 2. *Let $\beta > 0$ and φ be a function non-negative and measurable on $\mathbb{H}^n$. Then for all $r > 0$*

$$\beta\, 2^{-\beta} \int_r^\infty \left(\int_{B(0,t)} \varphi(x) dx \right) \frac{dt}{t^{1+\beta}} \leq \int_{\mathbb{H}^n} \frac{\varphi(x) dx}{(|x|_{\mathbb{H}} + r)^\beta} \leq \beta \int_r^\infty \left(\int_{B(0,t)} \varphi(x) dx \right) \frac{dt}{t^{1+\beta}}.$$

Corollary 2. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$ *and* $Q\,(1/p_1 - 1/p_2)_+ \le \alpha < Q/p_1$. *Then there exists* $c > 0$ *such that*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \le c r^{Q/p_2} \left(\int_r^\infty \left(\int_{B(0,r)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{Q-\alpha p_1+1}} \right)^{1/p_1} \tag{7}$$

for all $r > 0$ *and for all* $f \in L_{p_1}^{loc}(\mathbb{H}^n)$.

Proof. Inequality (7) follows from inequality (6) and Lemma 2.

Corollary 3. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$ *and* $Q\,(1/p_1 - 1/p_2)_+ \le \alpha \le Q/p_1$, *then there exists* $c > 0$ *such that*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \le c r^{\alpha - Q(1/p_1 - 1/p_2)} \|f\|_{L_{p_1}(\mathbb{H}^n)} \tag{8}$$

for all $r > 0$ *and for all* $f \in L_{p_1}(\mathbb{H}^n)$.

Proof. If $0 < p_2 < \infty$, inequality (8) follows by inequality (6). For $0 < p_2 \le \infty$ and $\alpha = Q/p_1$ it also follows directly from the definition of $M_\alpha f$. Indeed, Hölder's inequality implies that

$$\|M_{Q/p_1} f\|_{L_\infty} \le \|f\|_{L_{p_1}(\mathbb{H}^n)}.$$

Hence

$$\|M_{Q/p_1} f\|_{L_{p_2}(B(0,r)} \le d_n^{1/p_2} r^{Q/p_2} \|f\|_{L_{p_1}(\mathbb{H}^n)}.$$

Let H be the Hardy operator

$$Hg = \int_0^r g(t)dt, \quad 0 < r < \infty.$$

Lemma 3. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\,(1/p_1 - 1/p_2)_+ \le \alpha < Q/p_1$, $0 < \theta \le \infty$ *and* $w \in \Omega_\theta$. *Then there exists* $c > 0$ *such that*

$$\|M_\alpha f\|_{LM_{p_2\theta,w}} \le c \|Hg\|_{L_{\theta/p_1,v}(0,\infty)}^{1/p_1}$$

for all $f \in L_{p_1}^{\mathrm{loc}}(\mathbb{H}^n)$, *where*

$$g(t) = \int_{B\left(0,t^{1/(\alpha p_1 - Q)}\right)} |f(y)|^{p_1} dy \tag{9}$$

and

$$v(r) = \left[w\left(r^{1/(\alpha p_1 - Q)}\right) r^{(Q/p_2 + 1/\theta)/(\alpha p_1 - Q) - 1/\theta} \right]^{p_1}. \tag{10}$$

Proof. By Corollary 2

$$\|M_\alpha f\|_{LM_{p_2\theta,w}} = \Big\| w(r) \|M_\alpha f\|_{L_{p_2}(B(0,r))} \Big\|_{L_\theta(0,\infty)}$$

$$\leq c \left\| w(r) r^{Q/p_2} \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{Q-\alpha p_1+1}} \right)^{1/p_1} \right\|_{L_\theta(0,\infty)}$$

$$= c(Q-\alpha p_1)^{-1/p_1} \left\| w(r) r^{Q/p_2} \left(\int_0^{r^{\alpha p_1 - Q}} \left(\int_{B(0,\tau^{1/(\alpha p_1 - Q)})} |f(x)|^{p_1} dx \right) d\tau \right)^{1/p_1} \right\|_{L_\theta(0,\infty)}$$

$$= c(Q-\alpha p_1)^{-1/p_1} \left(\int_0^\infty \left(w(r) r^{Q/p_2} \right)^\theta \left(\int_0^{r^{\alpha p_1 - Q}} g(\tau) d\tau \right)^{\theta/p_1} dr \right)^{\frac{1}{\theta}}$$

$$= c \left(\int_0^\infty \left(w\left(\rho^{1/(\alpha p_1 - Q)}\right) \rho^{Q/(p_2(\alpha p_1 - Q))} \right)^\theta \rho^{1/(\alpha p_1 - Q)-1} \left(\int_0^\rho g(\tau) d\tau \right)^{\theta/p_1} d\rho \right)^{\frac{1}{\theta}}$$

$$= c \|Hg\|^{1/p_1}_{L_{\theta/p_1,v}(0,\infty)},$$

where $c > 0$ depends only on Q, p_1, p_2 and α.

Corollary 4. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\,(1/p_1 - 1/p_2)_+ \leq \alpha < Q/p_1$, $0 < \theta \leq \infty$ and $w \in \Omega_{p_1,\theta}$. Then there exists $c > 0$ such that*

$$\|M_\alpha f\|_{GM_{p_2\theta,w}} \leq c \sup_{x\in\mathbb{H}^n} \|H\,(g(x,\cdot))\,\|^{1/p_1}_{L_{\theta/p_1,v}(0,\infty)}$$

for all $f \in L^{\rm loc}_{p_1}(\mathbb{H}^n)$, where v is defined by (10) and

$$g(x,t) = \int_{B\left(x,t^{1/(\alpha p_1 - Q)}\right)} |f(y)|^{p_1} dy = \int_{B(0,t^{1/(\alpha p_1 - Q)})} |f(y^{-1}\cdot x)|^{p_1} dy. \tag{11}$$

Theorem 2. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\,(1/p_1 - 1/p_2)_+ \leq \alpha < Q/p_1$, $0 < \theta_1, \theta_2 \leq \infty$, $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$. Assume that H is bounded from $L_{\theta_1/p_1,v_1}(0,\infty)$ to $L_{\theta_2/p_1,v_2}(0,\infty)$ on the cone of all non-negative functions φ non-increasing on $(0,\infty)$ and satisfying $\lim\limits_{t\to\infty} \varphi(t) = 0$, where*

$$v_1(r) = \left[w_1\left(r^{1/(\alpha p_1 - Q)}\right) r^{1/((\alpha p_1 - Q)\theta_1) - 1/\theta_1} \right]^{p_1}, \tag{12}$$

$$v_2(r) = \left[w_2\left(r^{1/(\alpha p_1 - Q)}\right) r^{(Q/p_2 + 1/\theta_2)/(\alpha p_1 - Q) - 1/\theta_2} \right]^{p_1}. \tag{13}$$

Then M_α is bounded from $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$ and from $GM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $GM_{p_2\theta_2,w_2}(\mathbb{H}^n)$. (In the latter case we assume that $w_1 \in \Omega_{p_1,\theta_1}$, $w_2 \in \Omega_{p_2,\theta_2}$.)

Proof. By Lemma 3 applied to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$

$$\|M_\alpha f\|_{LM_{p_2\theta_2,w_2}(\mathbb{H}^n)} \le c\|Hg\|^{1/p_1}_{L_{\theta_2/p_1,v_2}(0,\infty)},$$

where $c>0$ is independent of f.

Since g is non-negative, non-increasing on $(0,\infty)$ and $\lim_{t\to+\infty} g(t) = 0$ and H is bounded from $L_{\theta_1/p_1,v_1}(0,\infty)$ to $L_{\theta_2/p_1,v_2}(0,\infty)$ on the cone of functions containing g, we have

$$\|M_\alpha f\|_{LM_{p_2\theta_2,w_2}(\mathbb{H}^n)} \le c\|g\|^{1/p_1}_{L_{\theta_1/p_1,v_1}(0,\infty)},$$

where $c>0$ is independent of f.

Hence

$$\begin{aligned}
\|M_\alpha f\|_{LM_{p_2\theta_2,w_2}(\mathbb{H}^n)} &\le c\left(\int_0^\infty v_1(t)^{\theta_1/p_1}\|f\|^{\theta_1}_{L_{p_1}\left(B\left(0,t^{1/(\alpha p_1-Q)}\right)\right)}dt\right)^{1/\theta_1}\\
&= c\,Q^{\frac{1}{\theta_1}}\left(\int_0^\infty v_1(r^{\alpha p_1-Q})^{\theta_1/p_1} r^{\alpha p_1-Q-1}\|f\|^{\theta_1}_{L_{p_1}(B(0,r))}dr\right)^{1/\theta_1}\\
&= c\,Q^{\frac{1}{\theta_1}}\left(\int_0^\infty \left(w_1(r)\|f\|_{L_{p_1}(B(0,r))}\right)^{\theta_1}dr\right)^{1/\theta_1}\\
&= c\,Q^{\frac{1}{\theta_1}}\|f\|_{LM_{p_1\theta_1,w_1}(\mathbb{H}^n)},
\end{aligned}$$

where $c>0$ is independent of f.

In order to obtain explicit sufficient conditions on weight functions ensuring the boundedness of M_α, first we shall apply the following statement.

Lemma 4. *[2] Let $0<\theta_1\le\infty$, $0<\theta_2\le\infty$, v_1 and v_2 be functions positive and measurable on $(0,\infty)$. Then the condition*

$$\left\|v_2(r)\left\|t^{-(1-\theta_1)_+/\theta_1}v_1^{-1}(t)\right\|_{L_{\theta_1/(\theta_1-1)_+}(0,r)}\right\|_{L_{\theta_2}(0,\infty)}<\infty \tag{14}$$

is a sufficient conditions for the boundedness of H from $L_{\theta_1,v_1}(0,\infty)$ to $L_{\theta_2,v_2}(0,\infty)$ in the case $1\le\theta_1\le\infty$ and the boundedness H from $L_{\theta_1,v_1}(0,\infty)$ to $L_{\theta_2,v_2}(0,\infty)$ on the cone of all non-negative functions φ non-increasing on $(0,\infty)$ in the case $0<\theta_1<\infty$.

If $\theta_1=\infty$, then condition (14) is also necessary for the boundedness of H from $L_{\infty,v_1}(0,\infty)$ to $L_{\theta_2,v_2}(0,\infty)$.

Theorem 2 and Lemma 4 imply a sufficient condition for the boundedness of M_α from $LM_{p_1\infty,w_1}(\mathbb{H}^n)$ to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

Theorem 3. *Let $1<p_1<\infty$, $0<p_2<\infty$, $Q\,(1/p_1-1/p_2)_+\le\alpha<Q$, $0<\theta_2\le\infty$, $w_2\in\Omega_{\theta_2}$.*

1. For $\alpha<Q/p_1$, let $w_1\in\Omega_{\theta_1}$ and

$$\left\|w_2(r)r^{Q/p_2}\left\|w_1^{-1}(t)t^{\alpha-Q/p_1-1/\min\{p_1,\theta_1\}}\right\|_{L_s(r,\infty)}\right\|_{L_{\theta_2}(0,\infty)}<\infty. \tag{15}$$

where $s = p_1\theta_1/(\theta_1 - p_1)_+$. *(If* $\theta_1 \le p_1$, *then* $s = \infty$.*) Then* M_α *is bounded from* $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

2. For $\alpha = Q/p_1$, *let*

$$w_2(r) r^{\alpha - Q(1/p_1 - 1/p_2)} \in L_{\theta_2}(0,\infty). \tag{16}$$

Then M_α *is bounded from* $L_{p_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

Corollary 5. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\left(1/p_1 - 1/p_2\right)_+ \le \alpha < Q/p_1$, $0 < \theta_2 \le \infty$, $w_1 \in \Omega_\infty$, $w_2 \in \Omega_{\theta_2}$ *and let*

$$\left\| w_2(r) r^{Q/p_2} \left(\int_r^\infty \frac{dt}{w_1^{p_1}(t) t^{Q+1-\alpha p_1}} \right)^{1/p_1} \right\|_{L_{\theta_2}(0,\infty)} < \infty. \tag{17}$$

Then M_α *is bounded from* $LM_{p_1\infty,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$ *and from* $GM_{p_1\infty,w_1}(\mathbb{H}^n)$ *to* $GM_{p\theta_2,w_2}(\mathbb{H}^n)$. *(In the latter case we assume that* $w_1 \in \Omega_{p_1,\infty}$, $w_2 \in \Omega_{p_2,\theta_2}$.*)*

Corollary 6. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\left(1/p_1 - 1/p_2\right)_+ \le \alpha < Q/p_1$, $w_1 \in \Omega_\infty$, $w_2 \in \Omega_\infty$ *and let for some* $c > 0$ *for all* $r > 0$

$$\int_r^\infty \frac{dt}{w_1^{p_1}(t) t^{Q+1-\alpha p_1}} \le \frac{c}{w_2^{p_1}(r) r^{\frac{Qp_1}{p_2}}}. \tag{18}$$

Then M_α *is bounded from* $LM_{p_1\infty,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\infty,w_2}(\mathbb{H}^n)$ *and from* $GM_{p_1\infty,w_1}(\mathbb{H}^n)$ *to* $GM_{p_2\infty,w_2}(\mathbb{H}^n)$. *(In the latter case we assume that* $w_1 \in \Omega_{p_1,\infty}$, $w_2 \in \Omega_{p_2,\infty}$.*)*

Remark 1. *Note that, the Corollary 6 was proved in [8], see also [9, 12, 13].*

For the majority of cases the necessary and sufficient conditions for the validity of

$$\|H\varphi\|_{L_{\frac{\theta_2}{p_1},v_2}(0,\infty)} \le c\|\varphi\|_{L_{\frac{\theta_1}{p_1},v_1}(0,\infty)}, \tag{19}$$

where $c > 0$ is independent of φ, for all non-negative decreasing functions φ are known, for detailed information see [18], [19]. Application of any of those conditions gives sufficient conditions for the boundedness of the fractional maximal operator from $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$ and from $GM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $GM_{p_2\theta_2,w_1}(\mathbb{H}^n)$.

However, there is no guarantee that the application of the necessary and sufficient conditions on v_1 and v_2 ensuring the validity of (19) implies the necessary and sufficient conditions for the boundedness of M_α from $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

Fortunately for certain values of the parameters this is the case, namely for $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\left(1/p_1 - 1/p_2\right)_+ \le \alpha < Q/p_1$, $0 < \theta_1 \le \theta_2 < \infty$, $\theta_1 \le p_1$.

Note that in this case the necessary conditions (coinciding with the sufficient ones) for the validity of inequality (19) for decreasing functions are obtained by taking $\varphi = \chi_{(0,t)}$ with an arbitrary $t > 0$.

Since in the proof of Theorem 2 inequality (19) is applied to the function $\varphi = g$, where g is given by (9), it is natural to choose, as test functions, functions f_t, $t > 0$, for which $\int_{B\left(0,u^{1/(\alpha p_1-Q)}\right)} |h_t(y)|^{p_1} dy$ is equal or close to $B(t)\chi_{(0,t)}(u)$, $u>0$, where $B(t)$ is independent of u. The simplest choice of f satisfying this requirement is

$$f_t(x)=\chi_{B(0,2t)\setminus B(0,t)}(x), \quad x\in\mathbb{H}^n,\ t>0. \tag{20}$$

Note that,

$$\|f_t\|_{L_{p_1}(B(0,r))}=0,\ \ 0<r\le t, \quad \|f_t\|_{L_{p_1}(B(0,r))}\le ct^{n/p_1}, \quad t<r<\infty, \tag{21}$$

where $c>0$ depends only on Q and p_1.

For functions F,G defined on $(0,\infty)\times(0,\infty)$ we shall write $F\asymp G$ if there exist $c,c'>0$ such that $cF(r,t)\le G(r,t)\le c'F(r,t)$ for all $r,t\in(0,\infty)$.

Lemma 5. *If* $0\le\alpha<Q$, $0<p<\infty$, *then*

$$\|M_\alpha f_t\|_{L_p(B(0,r))}\asymp t^\alpha r^{Q/p}\begin{cases}\left(\frac{t}{r+t}\right)^{\min\{Q-\alpha,Q/q\}}, & p\ne\frac{Q}{Q-\alpha},\\ \left(\frac{t}{r+t}\right)^{Q/p}\ln\left(e+\frac{r}{t}\right), & p=\frac{Q}{Q-\alpha}.\end{cases}$$

Theorem 1. (1) *Let* $1<p_1\le\infty$, $0<p_2\le\infty$, $0\le\alpha<Q$, $0<\theta_1,\theta_2\le\infty$, $w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$. *If* M_α *is bounded from* $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$, *then there exists a constant* $C_1>0$ *such that for all* $t>0$,

$$t^{\alpha-\frac{Q}{p_1}+\min(Q-\alpha,Q/p_2)}\left\|\frac{w_2(r)r^{Q/p_2}}{(t+r)^{\min(Q-\alpha,Q/p_2)}}\right\|_{L_{\theta_2}(0,\infty)}\le C_1\|w_1\|_{L_{\theta_1(t,\infty)}}.$$

(2) *Let* $1<p_1<\infty$, $0<p_2<\infty$, $0<\theta_1\le\theta_2\le\infty$, $\theta_1\le p_1$, $Q\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+\le\alpha<\frac{Q}{p_1}$, $w_1\in\Omega_{\theta_1}$, $w_2\in\Omega_{\theta_2}$ *and the equality* $\left\|\frac{w_2(r)r^{Q/p_2}}{(t+r)^{Q/p_1-\alpha}}\right\|_{L_{\theta_2(0,\infty)}}\le C_2\|w_1\|_{L_{\theta_1(t,\infty)}}$ $(C_2>0)$ *be true for all* $t>0$; *then* M_α *is bounded from* $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$. If also $w_1\in\Omega_{p_1,\theta_1}$, $w_2\in\Omega_{p_2,\theta_2}$, then M_α is bounded from $GM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $GM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

(3) *In particular, for* $1<p_1<\infty$, $0<p_2<\infty$, $0<\theta_1\le\theta_2\le\infty$, $\theta_1\le p_1$, $Q\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\le\alpha<\frac{Q}{p_1}$, $w_1\in\Omega_{\theta_1}$, $w_2\in\Omega_{\theta_2}$ *the operator* M_α *is bounded from* $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$ *if and only if for all* $t>0$,

$$\|w_2(r)r^{Q/p_2}(t+r)^{-Q/p_2}\|_{L_{\theta_2(0,\infty)}}\le C_3\|w_1\|_{L_{\theta_1}(t,\infty)}.$$

Here the constant $C_3>0$ *is independent of* t.

Note that, in the Euclidean setting Theorem 1 was proved in [2].

Proof. Sufficiency. It is known [19] that for $\theta_1\le\theta_2\le\infty$ the necessary and sufficient condition for the validity of (19) for all non-negative decreasing on $(0,\infty)$ functions φ has the form: for some $c>0$

$$\|v_2(r)\min\{t,r\}\|_{L_{\theta_2/p_1}(0,\infty)}\le c\|v_1(r)\|_{L_{\theta_1/p_1}(0,t)}$$

for all $t > 0$. Applying this condition to the functions v_1 and v_2 given by (12) and (13) we obtain

$$\left\| w_2(r) \frac{r^{Q/p_2}}{(t+r)^{Q/p_1-\alpha}} \right\|_{L_{\theta_2}(0,\infty)} \leq c \|w_1\|_{L_{\theta_1}(t,\infty)}. \tag{22}$$

Indeed, taking into account equalities (12) and (13) and replacing $r^{-\frac{p_2}{Q}}$ by ρ and $t^{-\frac{p_2}{Q}}$ by τ, we get that for some $c > 1$

$$\left\| w_2(\rho) \rho^{Q/p_2} \min\{\tau^{\alpha - Q/p_1}, \rho^{\alpha - Q/p_1}\} \right\|_{L_{\theta_2}(0,\infty)} \leq c \left\| w_1 \right\|_{L_{\theta_1}(\tau,\infty)}$$

for all $\tau > 0$.

Hence (22) follows since

$$\rho^{Q/p_2} \min\{\tau^{\alpha - Q/p_1}, \rho^{\alpha - Q/p_1}\} \asymp \frac{\rho^{Q/p_2}}{(\rho+\tau)^{Q/p_1-\alpha}}.$$

Necessity. Assume that, for some $c > 0$ and for all $f \in LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$

$$\|M_\alpha f\|_{LM_{p_2\theta_2,w_2}(\mathbb{H}^n)} \leq c \|f\|_{LM_{p_1\theta_1,w_1}(\mathbb{H}^n)}. \tag{23}$$

In (23) take $f = f_t$, where f_t is defined by (20). Then by (21) the right-hand side of (23) does not exceed a constant multiplied by $t^{Q/p_1} \|w_1\|_{L_{\theta_1}(t,\infty)}$. Furthermore by Lemma 5 the left-hand side of inequality (23) is greater than or equal to a constant multiplied by

$$t^{\alpha + \min\{Q-\alpha, Q/p_2\}} \left\| w_2(r) \frac{r^{Q/p_2}}{(t+r)^{\min\{Q-\alpha, Q/p_2\}}} \right\|_{L_{\theta_2}(0,\infty)}.$$

This works foe the case $\alpha = \frac{n}{p_2'}$ too, since $\ln(e + \frac{r}{t}) \geq 1$.

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Open Currency Position Risk and Value at Risk Analysis in Commercial Banks

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Abstract. In this article, Risk of Open Currency Position which the banks are most commited and Value at Risk method which is used to measure market risk are investigated. Variance - Covariance and Historical Simulation Methods used in calculating risk expense were thoroughly investigated and analyzed its effects on banks.

Key Words and Phrases: Open Currency Position, Currency Risk, Value at Risk, Normal Distribution, Variance - Covariance, Historical Simulation.

1. Open Currency Position Risk

If the bank has a disparity between the total balance sheet assets and the total of the balance sheet liabilities in one currency, then the bank's balance position in that currency is open. If the balance sheet assets in the currency exceed the total amount of the total liabilities, then the bank position will be long in that currency, in the contrary, if the balance sheet liabilities in the currency exceed the total amount of the total assets, then the bank position becomes short.

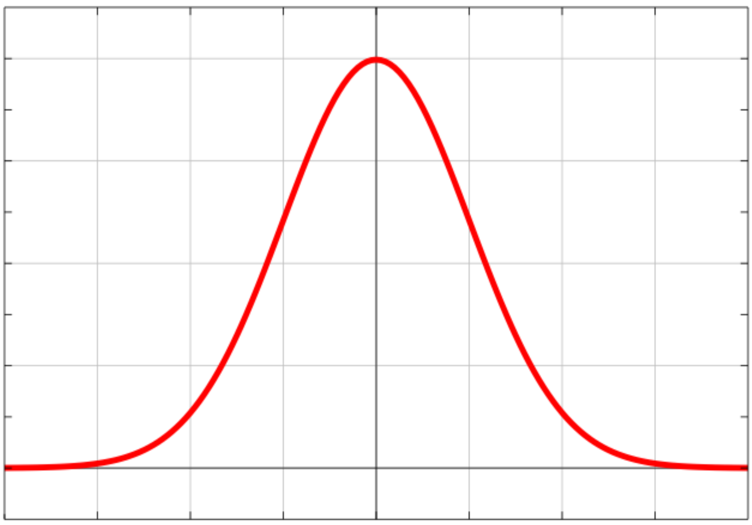
If the position of the bank in one currency is long, then the bank will suffer losses as a result of a decrese of exchange rate. On the contrary, if the position in one currency is short, then exchange rate of that currency will lead to the bank loss. If the position of the bank in any currency is open to the likelihood of an event that could lead to loss, then there would be a currency risk. This currency risk is called the Open Currency Position risk and this risk arises from an open position on the balance of the currency. Open Currency Position risk is one of the most exposed risks of banks.

2. Value at Risk Method

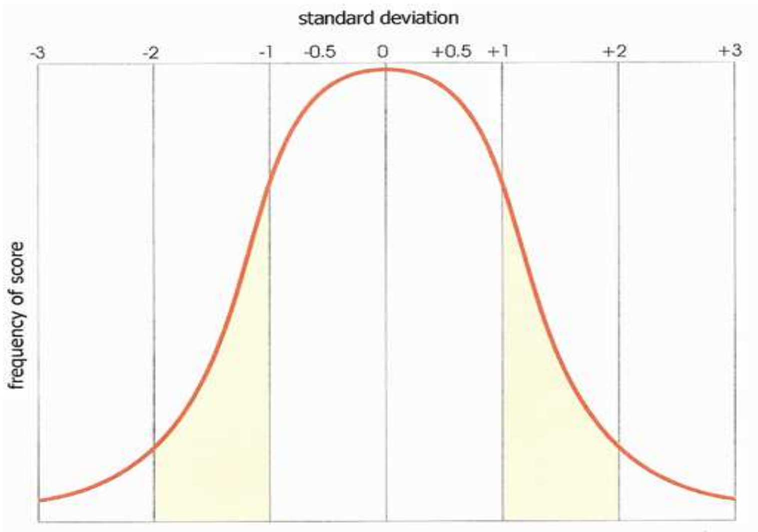
Banks open currency position either cumpulsorily or voluntarily in different currencies. Hence, banks are exposed to risks due to their open position. It is very important for banks to predict the effects of changes in exchange rates on the currency position within a certain period of time. The Bank should calculate to what extent it will expose its open position, and hence to what extent the exposure to risk. The method used to measure the open currency position risk is called the Value at Risk Method. Value at Risk is the maximum exposure to probable loss of a certain amount during the valuation period. Exposure to Risk is a method that measures the maximum loss of volatility in the financial markets, i.e. fluctuations in volatility.

As we mentioned, a value at risk is an expected maximum loss with a certain confidence level in a certain period of time As it is seen from the definition, Value at Risk includes two factors, such as the Time Span and the Level of Confidence. The time interval is a period it takes to close the position. Degree of precision of curency posiotn risk depends on confidence level. Degree to which extend a real risk exposure is less than calculated Value at risk is determined by confidence level. Here, the normal distribution and the features of this distribution are important.

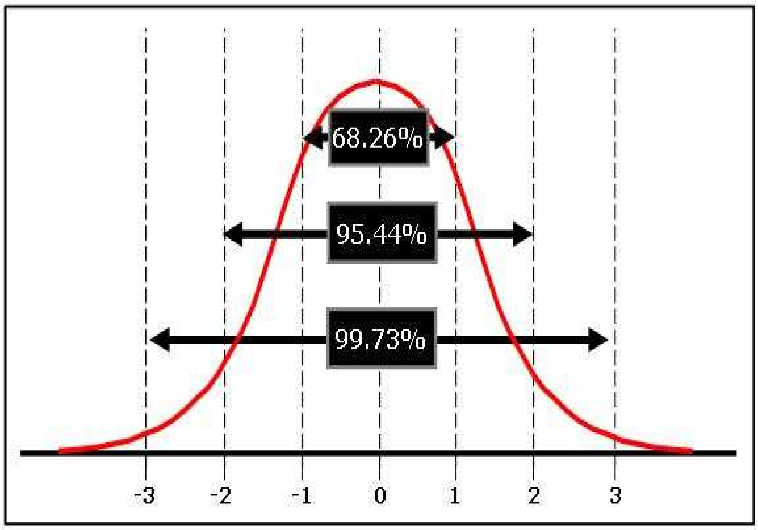
Normal distribution is such a distribution that is symmetrically average, and the average, median and mod are equal. These values are crossed at the same point as indicated by a curve. Mode is the most frequently number in a series. The median is the number in the middle of a sequence of numbers when it is put in order from smaller or larger in series. As noted above, the right and left sides of the intersection of the curve of the normal distribution are symmetric to each other. The normal distribution curve infinitly stretches to the left and right but does not cut the bottom line. Below is a schedule of normal distribution:



The standard deviation is taken at "0" at the hinge point of the normal distribution curve.



The standard distribution is such a distribution that the average value for that distribution is "0" and the variance is "1". 68%, 95% and 99% of values in normal distribution fall accordingly within (-1,+1), (-3,+3) and, (-2,+2) standard deviation:



Different levels of confidence are selected for estimating Value at Risk. The volatility of the market should be taken into consideration when selecting the level of confidence. In emerging economies, financial markets are highly volatile, thus the level of confidence should be taken higher in these countries, as the financial markets are less volatile in developed countries, so the level of confidence should be lower. However, the 99% confidence level is selected as a standard. The Basel Committee recommends that the level of confidence should be chosen as high as possible. The higher the risk level, the higher the value at risk.

The methods for calculating the value at risk are divided into two classes, including parametric and nonparametric methods. The Variance-Covariance method is used in the parametric method, however the Historical Simulation method is in the non-parametric method. parametric methods depend on the degree of confidence under the hypothesis that revenue is normally distributed. In non-parametric method, income is not dependent on any parameter. In other words, revenue is not based on any hypothesis. We will stay on these two methods which are commonly used to measure value at risk.

3. Parametric method: Variance - Covariance method

This method determines the parameters affecting the currency position and calculates the maximum loss value from the fluctuations that occur at a certain level of confidence.

Let's assume that the bank's total assets are 3,750,000 EUR and total liabilities are 5,730,000 EUR. Euro currency position is open and short. The open position is -1,980,000 EUR. Therefore, there will be loss as a result of the increase in exchange rate of the Euro.

We need exchange rates of at least two months to calculate value at risk with parametric method. We obtain the latest two-month euro exchange rate on the Central Bank's website. In order to apply the Variance-Covariance method it is important that the distribution of the exchange rates ought to be normal. Otherwise, the value at risk will yield an erronous result. We'll use skewness and kurtosis to detect whether the distribution is a normal distribution. Skewness is a number that shows the symmetry of distribution. If skewness is equal to zero the distribution will be symmetric, if it is smaller than zero, ie negative numbers, the distribution will be right-handed and ultimately, if it is more than zero, that is, positive numbers, the distribution will be left-handed. The number of kurtosis is a value associated with the sharpness or the cavity of the distribution.If the kurtosis is 3, then the distribution will be a normal distribution, if the distribution is less than 3 then distribution will be spike and if it is greater than 3 then it will be concave. In order to calculate skewness, we can use SKEW in the Excel program and KURT to calculate kurtosis. If we calculate the skewness and kurtosis values, we get the following:

Date	**Rate**	**Skewness**	**Kurtosis**	**Mode**	**Median**	**Average**
03.09.2018	1.9729	-0.080875	-0.685801	1.972900	1.967050	1.965113
04.09.2018	1.9727					
05.09.2018	1.9712					
06.09.2018	1.9778					
07.09.2018	1.9770					
10.09.2018	1.9630					
11.09.2018	1.9722					
12.09.2018	1.9699					
13.09.2018	1.9762					

Date	Rate	Skewness	Kurtosis	Mode	Median	Average
14.09.2018	1.9883					
17.09.2018	1.9781					
18.09.2018	1.9876					
19.09.2018	1.9848					
20.09.2018	1.9852					
21.09.2018	2.0024					
24.09.2018	1.9967					
25.09.2018	1.9960					
26.09.2018	1.9996					
27.09.2018	1.9976					
28.09.2018	1.9799					
01.10.2018	1.9712					
02.10.2018	1.9668					
03.10.2018	1.9681					
04.10.2018	1.9494					
05.10.2018	1.9567					
08.10.2018	1.9571					
09.10.2018	1.9546					
10.10.2018	1.9561					
11.10.2018	1.9655					
12.10.2018	1.9729					
15.10.2018	1.9635					
16.10.2018	1.9673					
17.10.2018	1.9649					
18.10.2018	1.9550					
19.10.2018	1.9479					
22.10.2018	1.9570					
23.10.2018	1.9474					
24.10.2018	1.9496					
25.10.2018	1.9398					
26.10.2018	1.9325					
29.10.2018	1.9366					
30.10.2018	1.9348					
31.10.2018	1.9281					
01.11.2018	1.9284					
02.11.2018	1.9396					
05.11.2018	1.9353					

As you can see, skewness is -0.080875 and kurtosis -0.685801. Skewness is different from "0" and kurtois is "3". At the same time, we have noted that in normal distribution the average, median and mod are equal. Here, all of the values are different from each other. Given all this, we can say that distribution is not a normal distribution. Nevertheless, we can normalize this distribution. We use the NORMDIST function in the Excel program. This function converts distribution to a normal distribution with a specified average value (standard average) and standard deviation. However, each distribution can not be converted to a normal distribution. In order to use the NORMDIST function, we need to calculate the average value and standard deviation values. To calculate the average, we sum up these amounts and divide them into their number. The average number is calculated by the following formula:

$$\overline{r} = \frac{\sum_{i=1}^{n} r_i}{n}$$

Where, n – number of values, r_i - values and $\overline{r}$- average of the values. After obtaining the mean, we need to calculate the variance to calculate the standard deviation. Variance is a statistical value that shows how far each of these exchange rates is from the average. To calculate the variance, we calculate the sum of the squares of the differences from average of all the rates and then divide their number to one minus. The following formula is used to calculate variance:

$$\sigma^2 = \frac{\sum_{i=1}^{n} (r_i - \overline{r})^2}{n-1}$$

Where, σ^2- the variance of these rates. Then we need to calculate the standard deviation. Standard deviation indicates a deviation of the estimated figure from from the mean of the set of values Standard deviation is the root of the variance and is calculated by the following formula:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (r_i - \overline{r})^2}{n-1}}$$

Where, σ- the standard deviation of given rates. The detailed description of the calculation is as follows:

Date	**Rate**	**Average**	**Square of deviation**	**Variance**	**Standard deviation**
03.09.2018	1.9729	1.965113	0.000061	0.000388	0.019710
04.09.2018	1.9727		0.000058		
05.09.2018	1.9712		0.000037		
06.09.2018	1.9778		0.000161		
07.09.2018	1.977		0.000141		
10.09.2018	1.963		0.000004		
11.09.2018	1.9722		0.000050		
12.09.2018	1.9699		0.000023		
13.09.2018	1.9762		0.000123		
14.09.2018	1.9883		0.000538		
17.09.2018	1.9781		0.000169		
18.09.2018	1.9876		0.000506		
19.09.2018	1.9848		0.000388		
20.09.2018	1.9852		0.000403		
21.09.2018	2.0024		0.001390		
24.09.2018	1.9967		0.000998		
25.09.2018	1.996		0.000954		
26.09.2018	1.9996		0.001189		
27.09.2018	1.9976		0.001055		
28.09.2018	1.9799		0.000219		
01.10.2018	1.9712		0.000037		
02.10.2018	1.9668		0.000003		
03.10.2018	1.9681		0.000009		
04.10.2018	1.9494		0.000247		
05.10.2018	1.9567		0.000071		
08.10.2018	1.9571		0.000064		
09.10.2018	1.9546		0.000111		
10.10.2018	1.9561		0.000081		
11.10.2018	1.9655		0.000000		
12.10.2018	1.9729		0.000061		
15.10.2018	1.9635		0.000003		
16.10.2018	1.9673		0.000005		
17.10.2018	1.9649		0.000000		

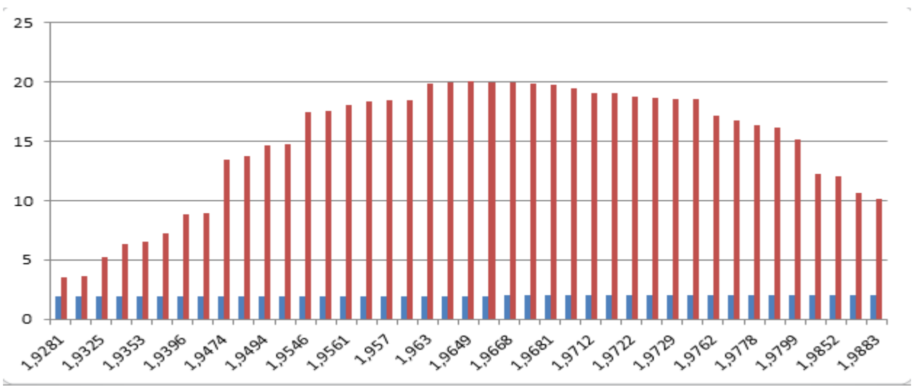
Date	**Rate**	**Average**	**Square of deviation**	**Variance**	**Standard deviation**
18.10.2018	1.955		0.000102		
19.10.2018	1.9479		0.000296		
22.10.2018	1.957		0.000066		
23.10.2018	1.9474		0.000314		
24.10.2018	1.9496		0.000241		
25.10.2018	1.9398		0.000641		
26.10.2018	1.9325		0.001064		
29.10.2018	1.9366		0.000813		
30.10.2018	1.9348		0.000919		
31.10.2018	1.9281		0.001370		
01.11.2018	1.9284		0.001348		
02.11.2018	1.9396		0.000651		
05.11.2018	1.9353		0.000889		

We can calculate variance and standard deviation by using Excel functions, VAR and STDEV. Let us Normalize distribution with NORMDIST function:

Rate	**Normalization**
1.9281	3.567192
1.9284	3.667930
1.9325	5.246246
1.9348	6.294925
1.9353	6.537772
1.9366	7.192703
1.9396	8.821054
1.9398	8.934681
1.9474	13.486201
1.9479	13.786019
1.9494	14.670538
1.9496	14.786353
1.9546	17.418853
1.955	17.600747
1.9561	18.073221
1.9567	18.312725
1.957	18.427399
1.9571	18.464853
1.963	19.907433

Rate	Normalization
1.9635	19.954187
1.9649	20.018521
1.9655	20.015891
1.9668	19.948059
1.9673	19.899468
1.9681	19.796030
1.9699	19.450303
1.9712	19.107172
1.9712	19.107172
1.9722	18.792848
1.9727	18.620038
1.9729	18.548090
1.9729	18.548090
1.9762	17.149081
1.977	16.756785
1.9778	16.347095
1.9781	16.189329
1.9799	15.201675
1.9848	12.289117
1.9852	12.045393
1.9876	10.591266
1.9883	10.173373
1.996	6.022602
1.9967	5.699944
1.9976	5.300748
1.9996	4.478074
2.0024	3.476943

The distribution table will be as follows:



With The Variance-Covariance Method (VAR) The Value at Risk is calculated as follows:

$$VAR = Open\ Position * \sigma * \alpha$$

Where, σ – standard deviation, α – the level of assurance. Here, α is equal to 1,645, at 95% confidence level in normal distribution, as noted. Similarly, the value that corresponds to 99% confidence level in normal distribution is 2,326. In our case, the open position is -1,980,000 EUR and standard deviation is 0.019710. Therefore,

$$VAR95\% = 64196.55 \quad \text{and} \quad VAR99\% = 90772.75$$

The Value at Risk we obtained means that the maximum amount of our loss at 99% confidence level within 1 day will be AZN 90 772.75. The probability exceeding of our loss from this amount is 1%. If we multiply the calcaulated Value at Risk for 1 day by the square root of holding period we will get a value at risk for that period The formula will be:

$$VAR = Open\ position * \sigma * \alpha * \sqrt{t}$$

Where, t – time interval, or, in other words, a retention period.

4. Non-parametric method: Historical Simulation method

This method is based on the assumption that the history repeats itself. This method uses Historical Values to calculate Value at Risk. That is, we can calculate the amount of loss for tomorrow by using the prior days' rates and losses, assuming that tomorrow will be like the days we left behind. There will also be a level of confidence.

Suppose that the total assets of the Bank is 9,750,000 GBP and the total liabilities is 7,350,000 GBP. Pound currency position is open and long. The open position is 2,400,000 GBP. Therefore, there will be loss as a result of the decrease of the exchange rate of the Pound.

With a Historical Simulation Method, we need exchange rates for at least the last two months to calculate the value at risk. We obtain the latest two-month exchange rate currencies from the Central Bank's website. Then we calculate the daily fluctuations of these exchange rates. The following formula is used to calculate the change:

$$\left(rate_{current\ day} - rate_{the\ day\ before}\right) / rate_{the\ day\ before}$$

Then the open position is re-evaluated by multiplying to the change of these exchange rates. These re-evaluations are then sorted from large to small. After the rankings the total rows are multiplied by 95% or 99%. There are 45 rows in our example. If we multiply this figure to 95% or 99%, we'll get approximately 43 and 44 respectively. The amount in row 43 with probability 95%, and the amount in row 44 with probability 99% are The Value at Risk. The detailed illustration of the calculation is as follows:

Date	**Rate**	**Change**	**Impact**	**Ranking**	**Open position**	**Value at Risk (95%)**
						-20201.66
03.09.2018	2.1980				2400000	
04.09.2018	2.1865	-0.523%	-12556.87	28795.16		**Value at Risk (99%)**
05.09.2018	2.1861	-0.018%	-439.06	25884.10		-22608.31
06.09.2018	2.1953	0.421%	10100.18	23178.90		
07.09.2018	2.1984	0.141%	3389.06	22331.57		
10.09.2018	2.1951	-0.150%	-3602.62	18663.53		
11.09.2018	2.2163	0.966%	23178.90	15288.02		
12.09.2018	2.2116	-0.212%	-5089.56	14071.80		
13.09.2018	2.2172	0.253%	6077.05	13381.90		
14.09.2018	2.2302	0.586%	14071.80	12606.06		
17.09.2018	2.2239	-0.282%	-6779.66	11263.58		
18.09.2018	2.2363	0.558%	13381.90	10234.31		
19.09.2018	2.2354	-0.040%	-965.88	10100.18		
20.09.2018	2.2354	0.000%	0.00	8782.47		
21.09.2018	2.2562	0.930%	22331.57	6077.05		
24.09.2018	2.2236	-1.445%	-34677.78	5705.83		
25.09.2018	2.2275	0.175%	4209.39	5229.47		
26.09.2018	2.2392	0.525%	12606.06	5047.88		
27.09.2018	2.2364	-0.125%	-3001.07	4209.39		
28.09.2018	2.2243	-0.541%	-12985.15	3389.06		
01.10.2018	2.2141	-0.459%	-11005.71	2882.82		
02.10.2018	2.2157	0.072%	1734.34	2643.05		
03.10.2018	2.2094	-0.284%	-6824.03	1734.34		

Date	**Rate**	**Change**	**Impact**	**Ranking**	**Open position**	**Value at Risk (95%)**
04.10.2018	2.1978	-0.525%	-12600.71	0.00		
05.10.2018	2.2118	0.637%	15288.02	-439.06		
08.10.2018	2.2290	0.778%	18663.53	-965.88		
09.10.2018	2.2278	-0.054%	-1292.06	-1292.06		
10.10.2018	2.2373	0.426%	10234.31	-2715.05		
11.10.2018	2.2478	0.469%	11263.58	-3001.07		
12.10.2018	2.2505	0.120%	2882.82	-3602.62		
15.10.2018	2.2293	-0.942%	-22608.31	-5060.27		
16.10.2018	2.2346	0.238%	5705.83	-5089.56		
17.10.2018	2.2393	0.210%	5047.88	-6779.66		
18.10.2018	2.2267	-0.563%	-13504.22	-6824.03		
19.10.2018	2.2135	-0.593%	-14227.33	-11005.71		
22.10.2018	2.2216	0.366%	8782.47	-12489.16		
23.10.2018	2.2029	-0.842%	-20201.66	-12556.87		
24.10.2018	2.2077	0.218%	5229.47	-12600.71		
25.10.2018	2.1907	-0.770%	-18480.77	-12985.15		
26.10.2018	2.1793	-0.520%	-12489.16	-13504.22		
29.10.2018	2.1817	0.110%	2643.05	-14227.33		
30.10.2018	2.1771	-0.211%	-5060.27	-18409.81		
31.10.2018	2.1604	-0.767%	-18409.81	-18480.77		
01.11.2018	2.1837	1.079%	25884.10	-20201.66		
02.11.2018	2.2099	1.200%	28795.16	-22608.31		
05.11.2018	2.2074	-0.113%	-2715.05	-34677.78		

As mentioned above, the Value at Risk that we have obtained likewise means that the maximum amount of loss we will have is 22 608.31 AZN at 99% confidence level within 1 day. The probability of the loss exceeding this amount is 1%.

Acknowledgement. *Dedicated to the bright memory of a great scientist and mathematician Prof. Dr. Gusein Sh. Guseinov.*

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Boundary Value Problems for Cauchy-Riemann Inhomogeneous Equation with Nonlocal Boundary Conditions in a Rectangle

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Abstract. A boundary value problem for a first order elliptic type equation with nonlocal boundary conditions in a rectangular domain is considered. The problem statement is such that four points of the boundary simultaneously move along the boundaries(every point is situated in one of sides of the rectangle). These points move so that the Carleman conditions are fulfilled, i.e. the neighboring points either move away from one boundary point or they approach to one of the boundary points. Carleman has called such problems the well-psed problems.

Key Words and Phrases: Cauchy-Riemann equation, nonlocal boundary condition, necessary condition, singularity, regularization, Fredholm property.

2010 Mathematics Subject Classifications: 35F15, 35C60

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1. Introduction

As is known from the course of mathematical functions equations and partial equations, boundary value problems with local conditions are mainly considered for elliptic type equations [7], [8], [10],[11] .

Further, a boundary value problem with local boundary conditions Dirichlet condition was considered for a first order elliptic type equation (Cauchy-Riemann equation) though such problems are ill-posed [2],[4].

Note that for an ordinary linear differential equation, the number of both initial and boundary conditions coincide with the order of the equation under consideration [12],[5], while for a partial equation the number of initial conditions coincides with the highest order of time derivative contained in the considered equation. As for a local boundary condition (if the number of space variables is greater than a unit with arbitrary boundaries) their number coincides with the half of higher derivatives in space variables contained in the considered equations [9],[6].

Note that linear local boundary conditions with global addends (integrals) are also encountered in the the paper [6], while nonlocal ones in our case [14](with sewing of

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boundary values) for many-dimensional boundary value problems are encountered in [3]. Note that [14] contains over 250 works devoted mainly to boundary value problems.

In [14] one can find boundary value problems both for elliptic (both of even and odd orders), parabolic type equations and also for mixed and composite type equations.

There are also boundary value problems for fractional derivative ordinary and partial equations.

Finally, note that we considered both the Cauchy problem and a boundary value problem for a linear differential equation with continuously alternating order of derivative [1].

2. Problem statement

Let us consider the following boundary value problem:

$$\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} = f(x), \quad x_2 \in (a_k, b_k), \quad k = 1, 2; \tag{1}$$

$$\alpha_{j1}(t)u(a_1 + t(b_1 - a_1), a_2) + \alpha_{j2}(t)(b_1, b_2 + t(a_2 - b_2)) + \alpha_{j3}(t)u(b_1 + t(a_1 - b_1), b_2)$$
$$+\alpha_{j4}(t)u(a_1, a_2 + t(b_2 - a_1)) = \alpha_j(t), \quad j = 1, 2; \; t \in [0, 1], \tag{2}$$

where $x = (x_1, x_2)$, $b_k > a_k > 0$, $k = 1, 2; i = \sqrt{-1}$ $f(x)$ for $x_k \in (a_k, b_k)$, $k = 1, 2$; $\alpha_{jk}(t)$ $\alpha_j(t)$ for $j = 1, 2$; $k = \overline{1, 4}$ are continuous functions and boundary conditions (2) that are linear independent.

Remark 1. *As is seen from the statements of of problems(1)-(2) the Carleman conditions [3] are fulfilled i.e. on the boundary four points move simultaneously and the neighboring points move away from one boundary point or they approach to one boundary point.*

Remark 2. *We show that if simultaneously more than point move along the boundary, i.e. the Carleman conditions are not fulfilled, then the problem is ill-posed, i.e. may have no solution or have a non-unique solution.*

Main relations: As is known, the fundamental solution of the Caushy-Riemann equation (1) has the form ([13]:)

$$U(x - \xi) = \frac{1}{\pi} \frac{1}{x_2 - \xi_2 + i(x_1 - \xi_1)}. \tag{3}$$

For determining the main relation, we multiply equation (1) by fundamental solution (3), integrate with respect to the domain $D = \{x = (x_1, x_2) : x_k \in (a_k, b_k), k = 1, 2$ apply the Ostrogradsky-Gauss formula and have:

$$\int_D \frac{\partial u(x)}{\partial x_2} U(x - \xi) dx + i \int_D \frac{\partial u(x)}{\partial x_1} U(x - \xi) dx = \int_D f(x) U(x - \xi) dx$$

$$\int_\Gamma u(x) U(x - \xi) \left[\cos(\nu, x_2) + i \cos(\nu, x_1)\right] dx$$

$$-\int_D f(x)U(x-\xi)dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2}u(\xi), & \xi \in \Gamma, \end{cases} \tag{4}$$

where $\Gamma = \partial D$ is the boundary of the domain D, ν is the external normal to the boundary Γ of domain D.

The basic relation (4), consists of two parts. The first part corresponding to $\xi \in D$ gives the general solution of equation (1) determined in domain D, the second part corresponding to $\xi \in \Gamma$ is a necessary condition.

For giving necessary conditions at first we write the main relation (4) in the expanded form, i.e.

$$-\frac{1}{2\pi}\int_0^1 \frac{u(a_1+\tau(b_1-a_1),a_2)}{a_2-\xi_2+i(a_1-\tau(b_1-a_1)-\xi_1)}d\tau + \frac{i}{2\pi}\int_0^1 \frac{u(b_1,b_2+\tau(a_2-b_2))}{b_2+\tau(a_2-b_2)-\xi_2+i(b_1-\xi_1)}d\tau$$

$$+\frac{1}{2\pi}\int_0^1 \frac{u(b_1+\tau(a_1-b_1),b_2)}{b_2-\xi_2+i(b_1+\tau(a_1-b_1)-\xi_1)}d\tau - \frac{i}{2\pi}\int_0^1 \frac{u(a_1,a_2+\tau(b_2-a_1))}{a_2+\tau(b_2-a_2)-\xi_2+i(a_1-\xi_1)}d\tau$$

$$-\frac{1}{2\pi}\frac{f(x)}{x_2-\xi_2+i(x_1-\xi_1)}dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2}u(\xi), & \xi \in \Gamma. \end{cases} \tag{5}$$

The necessary conditions:

$$u(a_1+\tau(b_1-a_1),a_2) = \frac{i}{\pi(b_1-a_1)}\int_0^1 \frac{u(a_1+\tau(b_1-a_1),a_2)}{\tau-t}d\tau$$

$$+\frac{i}{\pi}\int_0^1 \frac{u(b_1,b_2+\tau(a_2-b_2))}{(a_2-b_2)(\tau-t)+i(b_1-a_1)(1-t)}d\tau + \frac{1}{\pi}\int_0^1 \frac{u(b_1+\tau(a_1-b_1),b_2)}{b_2-a_2+i(b_1-a_1)(1-\tau-t)}d\tau$$

$$-\frac{i}{\pi}\int_0^1 \frac{u(a_1,a_2+\tau(b_2-a_1))}{(b_2-a_2)\,\tau-i(b_1-a_1)t}d\tau - \frac{1}{\pi}\int_D \frac{f(x)}{x_2-a_2+i(x_1-a_1-t(b_1-a_1))}dx. \tag{6}$$

$$u(b_1,b_2+t(a_2-b_2)) = -\frac{1}{\pi}\int_0^1 \frac{u(a_1+\tau(b_1-a_1),a_2)}{(a_2-b_2)(1-t)+i(a_1-b_1)(1-\tau)}d\tau$$

$$+\frac{i}{\pi(a_2-b_2)}\int_0^1 \frac{u(b_1,b_2+\tau(a_2-b_2))}{\tau-t}d\tau + \frac{1}{\pi}\int_0^1 \frac{u(b_1+\tau(a_1-b_1),b_2)}{(b_2-a_2)\,t+i(a_1-b_1)\tau}d\tau$$

$$-\frac{i}{\pi}\int_0^1 \frac{u(a_1,a_2+\tau(a_2-b_2))}{(a_2-b_2)(1-\tau-t)+i(a_1-b_1)}d\tau - \frac{1}{\pi}\int_D \frac{f(x)}{x_2-b_2-t(a_2-b_2)+i(x_1-b_1)}dx, \tag{7}$$

$$u(b_1+t(a_1-b_1),b_2)=-\frac{1}{\pi}\int_0^1\frac{u(a_1+\tau(b_1-a_1),a_2)}{a_2-b_2+i(a_1-b_1)(1-\tau-t)}d\tau$$

$$+\frac{i}{\pi}\int_0^1\frac{u(b_1,b_2+\tau(a_2-b_2))}{(a_2-b_2)\tau+i(b_1-a_1)t}d\tau-\frac{1}{\pi(a_1-b_1)}\int_0^1\frac{u(b_1+\tau(a_1-b_1),b_2)}{\tau-t}d\tau$$

$$-\frac{i}{\pi}\int_0^1\frac{u(a_1,a_2+\tau(b_2-a_2))}{(a_2-b_2)(1-\tau)+i(a_1-b_1)(1-t)}d\tau-\frac{1}{\pi}\int_D\frac{f(x)}{x_2-b_2+i(x_1-b_1-t(a_1-b_1))}dx, \tag{8}$$

$$u(a_1,a_2+t(b_2-a_2))=-\frac{1}{\pi}\int_0^1\frac{u(a_1+\tau(b_1-a_1),a_2)}{(a_2-b_2)t+i(b_1-a_1)\tau}d\tau$$

$$+\frac{1}{\pi}\int_0^1\frac{u(b_1,b_2+\tau(a_2-b_2))}{(b_2-a_2)(1-\tau-t)+i(b_1-a_1)}d\tau+\frac{1}{\pi}\int_0^1\frac{u(b_1+\tau(a_1-b_1),b_2)}{(b_2-a_2)(1-t)-i(a_1-b_1)(1-\tau)}d\tau$$

$$-\frac{i}{(b_2-a_2)\pi}\int_0^1\frac{u(a_1,a_2+\tau(b_2-a_2))}{\tau-t}d\tau-\frac{1}{\pi}\int_D\frac{f(x)}{x_2-a_2-t(b_2-a_2)+i(x_1-a_1)}dx. \tag{9}$$

This establishes the following statement:

Theorem 1. *If $f(x)$ is a continuous function, then every solution of equation (1), determined in domain D satisfies necessary singular conditions (6)-(9)*

Remark 3. *As was mentioned above, every solution of equation (1) determined in domain D is found from the main relation (5) for $\xi\in D$, i.e. the first expression of the main relation (5)*

Regularization: Proceeding from (6)-(9), we create the following linear combination:

$$\alpha_{j1}(t)(b_1-a_1)u(a_1+t(b_1-a_1),a_2)+\alpha_{j2}(t)(a_2-b_2)u(b_1,b_2+t(a_2-b_2))$$

$$+\alpha_{j3}(t)(b_1-a_1)u(b_1+t(a_1-b_1),b_2)+\alpha_{j4}(t)(a_2-b_2)u(a_1,a_2+t(b_2-a_2))$$

$$=\frac{i}{\pi}\int_0^1[\alpha_{j1}(\tau)u(a_1+\tau(b_1-a_1),a_2)+\alpha_{j2}(\tau)u(b_1,b_2+t(a_2-b_2))$$

$$+\alpha_{j3}(\tau)u(b_1+\tau(a_1-b_1),b_2)+\alpha_{j4}(\tau)u(a_1,a_2+\tau(b_2-a_2))]\,\frac{d\tau}{\tau-t}+...,\tag{10}$$

where, when obtaining (10) it was supposed that

$$\alpha_{jk}(t)\in H^{\mu}(0,1),\quad j=1,2;\quad k=\overline{1,4};\mu\in(0,1),\tag{11}$$

$H^{\mu}(0,1)$ is a Holder class with the exponent $\mu \in (0,1)$, the dots $(\cdots)$ denotes the sum of nonsingular addends.

Taking boundary condition (2) into account in (10), we get

$$\alpha_{j1}(t)(b_1-a_1)u(a_1+t(b_1-a_1),a_2)+\alpha_{j2}(t)(a_2-b_2)u(b_1,b_2+t(a_2-b_2))$$
$$+\alpha_{j3}(t)(b_1-a_1)u(b_1+t(a_1-b_1),b_2)+\alpha_{j4}(t)(a_2-b_2)u(a_1,a_2+t(b_2-a_2))$$
$$=\frac{i}{\pi}\int_0^1 \frac{\alpha_j(\tau)}{q-t}dt+...,j=1,2;\ t\in[0,1]. \qquad (12)$$

As is seen from (12), as the first part does not contain an unknown function, then it exists in the Cauchy sense.

If we suppose

$$\alpha_j(t)\in C^{(1)}(0,1),\quad j=1,2;\quad \alpha_j(0)=\alpha_j(1)\quad j=1,2; \qquad (13)$$

then the integral in the right hand side of (12) exists in the ordinary sence .

This establishes

Theorem 2. *Under conditions of theorem 1, if conditions (11) and (12) hold, then relations (13) are regular.*

Fredholm property: Now combining the given boundary condition (2) with regular expressions (12), we have:

$$\alpha_{j1}(\tau)u(a_1+t(b_1-a_1),a_2)+\alpha_{j2}(t)u(b_1,b_2+t(a_2-b_2))$$
$$+\alpha_{j3}(\tau)u(b_1+t(a_1-b_1),b_2)+\alpha_{j4}(t)u(a_1,a_2+t(b_2-a_2))=\alpha_j(t),\quad j=1,2;\ t\in[0,1],$$
$$\alpha_{j1}(t)(b_1-a_1)u(a_1+t(b_1-a_1),a_2)+\alpha_{j2}(t)(a_2-b_2)u(b_1,b_2+t(a_2-b_2))$$
$$+\alpha_{j3}(t)(b_1-a_1)u(b_1+t(a_1-b_1),b_2)$$
$$+\alpha_{j4}(t)(a_2-b_2)u(a_1,a_2+t(b_2-a_2))=...j=1,2;\ t\in[0,1]. \qquad (14)$$

Let

$$\Delta(t)=\begin{vmatrix} \alpha_{11}(t) & \alpha_{12}(t) & \alpha_{13}(t) & \alpha_{14}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) & \alpha_{23}(t) & \alpha_{24}(t) \\ \alpha_{11}(t)(b_1-a_1) & \alpha_{12}(t)(a_2-b_2) & \alpha_{13}(t)(b_1-a_1) & \alpha_{14}(t)(a_2-b_2) \\ \alpha_{21}(t)(b_1-a_1) & \alpha_{22}(t)(a_2-b_2) & \alpha_{23}(t)(b_1-a_1) & \alpha_{24}(t)(a_2-b_2) \end{vmatrix}\neq 0, \qquad (15)$$

Then from (14) we get a system of normal form of Fredholm integral equations of second kind with nonsingular kernels.

We get the following statement:

Theorem 3. *Let the condition of theorem 3 hold, then if condition (15) is valid, the stated boundary value problem (1)-(2) is Fredholm.*

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On the Completeness of System of Cosines in Weighted Morrey Spaces

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Abstract. In this work the problem of the completeness of the classical system of cosines is considered in a weighted Morrey spaces with a power weight. These spaces, generally speaking, are not separable. Therefore, classical trigonometric systems are not complete in these spaces. Starting from the shift operator, a subspace of Morrey space in which continuous functions are dense is defined. A sufficient condition on the weight function is found, under which the cosine system is complete in this subspace.

Key Words and Phrases: Morrey space, completeness, system of cosines
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1. Introduction

Morrey spaces were introduced by Morrey, see [1], in the setting of partial differential equations, and presented in various books, see [2, 3, 4, 5], survey papers [6, 7, 8] and the references therein. The splash of interest to Morrey-type spaces during the last decade has advances in many areas, which allow to consider the basis properties of systems in such spaces in order to fill the gaps in the theory of Morrey spaces. These problems arise naturally in the solution of many partial differential equations by the Fourier method.

Several authors have studied the basis properties of trigonometric systems in Banach function spaces. Well-known results concerning the basis properties of the systems of exponentials in the case of the Lebesgue space L_p, $(1 < p < \infty)$, can be found in [9, 10, 11]. Babenko [12] has proved that the degenerate system of exponentials $\left\{|t|^{\alpha} e^{int}\right\}_{n\in\mathbb{Z}}$ with $|\alpha| < \frac{1}{2}$ forms a basis for $L_2(-\pi, \pi)$ but does not form a Riesz basis when $\alpha \neq 0$, where $\mathbb{Z}$ is the set of integers. This result has been generalized by Gaposhkin [13]. In [14], the conditions on the weight function ρ, for which the system $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ forms an unconditional basis for the weighted Besov space have been obtained. Similar problems have been studied in [15, 16, 17, 18, 38, 39]. The basicity of the systems of sines and cosines with degenerate coefficients have been widely analyzed. Amongst the Banach spaces where the basicity are known we mention the Lebesgue space L_p, $(1 < p < \infty)$, [19, 20]. Basis properties of the systems of sines, cosines and exponentials with the linear phase in weighted Lebesgue space have been studied in [21, 22, 23]; see also [24, 25, 26].

The basis properties of the systems of sines, cosines and exponentials in Morrey spaces are much less studied. In the paper [27], there were studied the basis properties of the system of exponentials in Morrey space. Also, in [28, 37] the basis properties of the perturbed systems of exponentials in Morrey space have been investigated. On the other hand, some approximation problems have been investigated in Morrey-Smirnov classes in [29].

We will use the standard notation. Denote the set of natural numbers by $\mathbb{N}$ and the set of nonnegative integers by $\mathbb{N}_0$. We denote by $L[M]$ the linear span of the set M. $\overline{M}$ will stand for the closure of the set M. $\|\cdot\|_\infty$ means sup-norm.

Our goal in this paper is the study of completeness of the system $\{\cos nt\}_{n\in\mathbb{N}_0}$ in weighted Morrey space $\mathcal{L}_\nu^{p,\lambda}(0,\pi)$ defined by a product of power weights of the form

$$\nu(t)=\prod_{k=0}^{r}|t-t_k|^{\alpha_k}\,,\quad t\in[0,\pi]\,, \tag{1}$$

where $t_0=0, t_r=\pi$, and t_k are arbitrary finite points in the interval $(0,\pi)$ for all $k=1,2,...,r-1$, and $\alpha_k\in$ R for all $k=0,1,...,r$. Also, we will consider the weighted Morrey space $\mathcal{L}_\nu^{p,\lambda}(-\pi,\pi)$, where

$$\nu(t)=\prod_{k=0}^{r}|t-t_k|^{\alpha_k}\,,\quad t\in[-\pi,\pi]\,, \tag{2}$$

and t_k are arbitrary finite points in the interval $[-\pi,\pi]$ and $\alpha_k\in$ R for all $k=0,1,...,r$.

Although the same properties of trigonometric systems, as well as their pertubations, are well studied in weighted Lebesgue spaces, the situation changes cardinally in Morrey spaces. For instance, since the functional characterization of dual spaces of Morrey spaces is not known, it avoids working with dual spaces. Another difficulty, that frustrates the "usual" attempts is that, the infinitely differentiable functions(even continuous functions) are not dense in Morrey spaces, but we still seek to prove "density" property of trigonometric functions, which are infinily differentiable. For these reasons, unlike the L_p case, here will be used another methods to study the basis properties(especially, completeness and basisness) in weighted Morrey spaces.

In this work the problem of the completeness of the classical system of cosines is considered in a weighted Morrey spaces with a power weight. These spaces, generally speaking, are not separable. Therefore, classical trigonometric systems are not complete in these spaces. Starting from the shift operator, a subspace of Morrey space in which continuous functions are dense is defined. A sufficient condition on the weight function is found, under which the cosine system is complete in this subspace.

2. Preliminaries

2.1. (Weighted) Morrey space on an interval

For $1 < p < \infty$ and $0 \le \lambda < 1$ we define the Morrey space $\mathcal{L}^{p,\lambda}(a,b)$ as the set of functions f on (a,b) such that

$$\|f\|_{p,\lambda} := \|f\|_{\mathcal{L}^{p,\lambda}(a,b)} = \sup_{I \subset (a,b)} \left(\frac{1}{|I|^{\lambda}} \int_I |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty,$$

where $I \subset (a,b)$ is any interval. It is clear that $\mathcal{L}^{p,\lambda}(a,b)$ are Banach spaces. Morrey spaces can be defined in a more general way (see e.g. [5, 8, 29]) but this is enough for our purposes. The $L_p(a,b)$ spaces with the Lebesgue measure correspond with the case $\lambda = 0$. The weighted Morrey space $\mathcal{L}^{p,\lambda}_{\nu}(a,b)$ is defined in the usual way

$$\mathcal{L}^{p,\lambda}_{\nu}(a,b) := \left\{ f : \nu f \in \mathcal{L}^{p,\lambda}(a,b) \right\},$$

with $\|f\|_{p,\lambda;\nu} := \|\nu f\|_{p,\lambda}$. The function ν is called the weight or weight function of this space.

It is evident that the space $\mathcal{L}^{p,\lambda}_{\nu}(a,b)$ contains constant functions if and only if $\nu \in \mathcal{L}^{p,\lambda}(a,b)$. Throughout the paper, unless otherwise stated, we will assume that $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$ and $0 < \lambda < 1$. Also, the letter "c" denotes a positive constant, which is not necessarily the same at each occurance but is independent of the essential variable and quantities. The expression $f \sim g$, $t \to a$ means that in suffeciently small neighborhood O_δ of the point $t = a$, the inequalities $0 < \delta \le \left| \frac{f(t)}{g(t)} \right| \le \delta^{-1} < \infty$ hold in O_δ. If the last inequalities hold on an interval I, we write $f \sim g$ on I. For example $\sin t \sim t(\pi - t)$ on $[0, \pi]$.

We assume here some familiarity with basic concepts of basis theory and we refer to the books of Heil [30], Christensen [31], Singer [32, 33] and Bilalov B.T. [39] for basic definitions such as complete and minimal systems and basis in Banach spaces.

The following lemma has been proved by Samko [34] in the case of Morrey space on a bounded rectifiable curve. In our case it reads

Lemma 1. *The power function* $|t - t_0|^{\alpha}$, $t_0 \in [a,b]$, *belongs to the Morrey space* $\mathcal{L}^{p,\lambda}(a,b)$ *if and only if* $\alpha \in \left[\frac{\lambda - 1}{p}, \infty \right)$.

Direct application of the above lemma implies the following

Proposition 1. *Let* ν *be given as in* (1). *Then*
$\{\cos nt\}_{n \in \mathrm{N}_0} \subseteq \mathcal{L}^{p,\lambda}_{\nu}(0,\pi), 0 < \lambda < 1,$ *if and only if*

$$\alpha_k \in \left[\frac{\lambda - 1}{p}, \infty \right), \ \ for\ all\ \ k = 0, 1, 2, ..., r. \tag{3}$$

Remark 1. *The case* $\lambda > 0$ *differs from the case* $\lambda = 0$*: when* $\lambda = 0$, *conditions* (3) *must be replaced by the conditions*

$$\alpha_k \in \left(-\frac{1}{p}, \infty \right), \ \ for\ \ all\ \ k = 0, 1, 2, ..., r.$$

2.2. Auxiliary propositions

Let us start by considering the space

$$\left(\mathcal{L}^{p,\lambda}\right)' = \left\{ g : \sup_{\|f\|_{p,\lambda}=1} \|f\, g\|_{L_1} < +\infty \right\},$$

with the norm

$$\|g\|_{\left(\mathcal{L}^{p,\lambda}\right)'} = \sup_{f\in\mathcal{L}^{p,\lambda}, \|f\|_{p,\lambda}=1} \|fg\|_{L^1}.$$

It can be proved that $\left(\mathcal{L}^{p,\lambda}\right)'$ is a normed space and the following inequality is satisfied

$$\|fg\|_{L^1} \leq \|f\|_{p,\lambda} \|g\|_{\left(\mathcal{L}^{p,\lambda}\right)'}, \tag{4}$$

for all $f \in \mathcal{L}^{p,\lambda}$ and $g \in \left(\mathcal{L}^{p,\lambda}\right)'$.

The following lemma is true.

Lemma 2. $|t|^{\beta} \in \left(\mathcal{L}^{p,\lambda}(-\pi,\pi)\right)' \Leftarrow \beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right), 0 \leq \lambda < 1, 1 < p < +\infty.$

The following lemma is also true.

Lemma 3. $|t|^{\beta} \in \left(\mathcal{L}^{p,\lambda}(0,\pi)\right)' \Leftarrow \beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right), 0 \leq \lambda < 1, 1 < p < +\infty.$

Proof. Firstly, suppose $\beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right)$. Then, for all $f \in \mathcal{L}^{p,\lambda}(0,\pi)$, we have

$$\int_{-\pi}^{\pi} |t|^{\beta} |f(t)|\, dt = \sum_{k=1}^{\infty} \int_{t\in\left[2^{-k-1}\pi, 2^{-k}\pi\right]} |t|^{\beta} |f(t)|\, dt$$

$$\leq c \sum_{k=1}^{\infty} 2^{-k\beta} \int_{t\in\left[2^{-k-1}\pi, 2^{-k}\pi\right]} |f(t)|\, dt$$

$$\leq c \sum_{k=1}^{\infty} 2^{-k\beta} 2^{-k\left(1-\frac{1}{p}\right)} \left(\int_{t\in\left[2^{-k-1}\pi, 2^{-k}\pi\right]} |f(t)|^p\, dt \right)^{\frac{1}{p}}$$

$$= c \sum_{k=1}^{\infty} 2^{-k\left(\beta+1-\frac{1}{p}+\frac{\lambda}{p}\right)} \|f\|_{p,\lambda} \leq c \|f\|_{p,\lambda}.$$

Then, $|t|^{\beta} \in \left(\mathcal{L}^{p,\lambda}(0,\pi)\right)'$.

Conversely, suppose that $\beta \notin \left(-\frac{\lambda-1}{p} - 1, \infty\right)$. That is $\beta + \frac{\lambda-1}{p} \leq -1$.

Then, $|t|^{\frac{\lambda-1}{p}} \in \mathcal{L}^{p,\lambda}(0,\pi)$ and

$$\int_0^{\pi} |t|^{\beta} |t|^{\frac{\lambda-1}{p}}\, dt = \int_0^{\pi} |t|^{\beta+\frac{\lambda-1}{p}}\, dt = \infty.$$

Thus $|t|^{\beta} \notin \left(\mathcal{L}^{p,\lambda}\right)'$. This completes the proof. ◄

2.3. Zorko subspace of weighted Morrey space

Denote by $C_0^\infty[-\pi,\pi]$ the set of all infinitely differentiable functions with compact support in $(-\pi,\pi)$. We observe that functions in $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ can not be approximated by functions in $C_0^\infty[-\pi,\pi]$, nor even by continuous functions. That is the set $C_0^\infty[-\pi,\pi]$ is not dense in $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ (*c.f.* [5,35]). This fact still valid in the weighted setting of Morrey space. For example, let ν be given as in (2) under conditions (3). Let $\tau_0 \neq t_k$, $\forall k = \overline{0,r}$, $\tau_0 \in (-\pi,\pi)$ be any points. Then, there exists sufficianly small $\delta_0 > 0$, so that

$$t_k \notin O_{\delta_0} \subset (-\pi,\pi)\,,\ \forall k = \overline{0,r}\,,$$

where $O_{\delta_0} = [\tau_0, \tau_0 + \delta_0]$. Then it's clear that $g_{\delta_0}^{\pm}(t) = \chi_{O_{\delta_0}}(t)\,\nu^{\pm 1}(t)$ is a continuous function on $[-\pi,\pi]$. Consider the function

$$f(t) = |t-\tau_0|^{\frac{\lambda-1}{p}}\,\nu^{-1}(t)\,.$$

It's obvious that $f \in L_\nu^{p,\lambda}(-\pi,\pi)$. Let $g \in C[-\pi,\pi]$ be any function. From (3) it follows that $g \in L_\nu^{p,\lambda}(-\pi,\pi)$. We have

$$\|f-g\|_{L_\nu^{p,\lambda}(-\pi,\pi)} \geq \|f-g\|_{L_\nu^{p,\lambda}(O_{\delta_0})} =$$

$$= \|f\nu - g\nu\|_{L^{p,\lambda}(O_{\delta_0})} = \|F-G\|_{L^{p,\lambda}(O_{\delta_0})}\,,$$

where $F(t) = |t-\tau_0|^{\frac{\lambda-1}{p}} \in L^{p,\lambda}(O_{\delta_0})$, $G = g\nu \in C(O_{\delta_0})$. For the rest one needs to follow the corresponding example of Zorko [5, 35].

Let $f(\cdot)$ be the given function on $[a,b]$. In determining the Zorko type subspace we will assume that the function $f(\cdot)$ is continued to $[2a-b, 2b-a]$ with the following expression (and this function is also denoted by $f(\cdot)$)

$$f(x) = \begin{cases} f(2a-x)\,,\ x \in [2a-b, a)\,, \\ f(2b-x)\,,\ x \in (b, 2b-a]\,. \end{cases}$$

So, following Zorko [35], we consider the subspace

$$\widetilde{\mathcal{L}_\nu^{p,\lambda}}(a,b) := \left\{ f \in \mathcal{L}_\nu^{p,\lambda}(a,b) : \|f(.+\delta) - f(.)\|_{p,\lambda;\nu} \to 0\ as \delta \to 0 \right\},$$

where ν is given as in (2) under conditions (3). We will refer to this subspace as the Zorko subspace of $\mathcal{L}_\nu^{p,\lambda}(a,b)$. Also, we consider the $\mathcal{L}_\nu^{p,\lambda}$-closure of $\widetilde{\mathcal{L}_\nu^{p,\lambda}}(a,b)$ and denote it by $M_\nu^{p,\lambda}(a,b)$. It is easy to see that if $\nu \in \mathcal{L}^{p,\lambda}(a,b)$, then $C[-a,b] \subset M_\nu^{p,\lambda}(a,b)$. In fact, let $f \in C[a,b]$ be an arbitrary function and δ be an arbitrary number (with $|\delta|$sufficiently small). It is obvious that the extended function $f(\cdot)$ is continuous on $[2a-b, 2b-a]$. We have

$$\|f(\cdot+\delta)-f(\cdot)\|_{p,\lambda,\nu}=\sup_{I\subset(a,b)}\left(\frac{1}{|I|^{\lambda}}\int_I|(f(t+\delta)-f(t))\,\nu(t)|^p\,dt\right)^{1/p}\leq$$

$$\leq\sup_{t\in[a,b]}|f(t+\delta)-f(t)|\ \|\nu\|_{p,\lambda}\to 0,\ \ \delta\to 0.$$

Thus we have the following

Lemma 4. *If $\nu\in L^{p,\lambda}(a,b)$, then $C[a,b]\subset M_\nu^{p,\lambda}(a,b)$.*

Since $M_\nu^{p,\lambda}(a,b)$ is a closed subspace of $\mathcal{L}_\nu^{p,\lambda}(a,b)$, it also contains the $\mathcal{L}_\nu^{p,\lambda}$-closure of $C_0^\infty[a,b]$; in fact, $M_\nu^{p,\lambda}(a,b)$ is precisely that closure.

Proposition 2. *Let ν be given as in (2) and the following condition holds*

$$\alpha_k\in\left[-\frac{1-\lambda}{p},-\frac{1-\lambda}{p}+1\right),\ k=\overline{0,r}. \tag{5}$$

Then the set $C^\infty[-\pi,\pi]$ is dense in $M_\nu^{p,\lambda}(-\pi,\pi)$.

We need the following lemma.

Lemma 5. *[Minkowski inequality for integrals in weighted Morrey spaces] Let $(X;X_\sigma;\mu)$ be a measurable space with an σ-additive measure $\mu(\cdot)$ on a set X, $\nu=\nu(t)$ a weight function, dy a linear Lebesgue measure on an interval (a,b) and $F(x,y)$ is $\mu\times dy$-measurable. If $1\leq p<\infty$, then*

$$\left\|\int_X F(x,y)d\mu(x)\right\|_{p,\lambda;\nu}\leq\int_X\|F(x,y)\|_{p,\lambda;\nu}\,d\mu(x).$$

Proof. By using the Minkowski inequality for integrals in $L_p(a,b)$,

$$\left\|\int_X F(x,y)\nu(y)d\mu(x)\right\|_{L_p}\leq\int_X\|F(x,y)\nu(y)\|_{L_p}\,d\mu(x),$$

we have

$$\left(\int_{B_r(x)}\left|\int_X F(x,y)\nu(y)d\mu(x)\right|^p dy\right)^{\frac{1}{p}}\leq\int_X\left(\int_{B_r(x)}|F(x,y)\nu(y)|^p\,dy\right)^{\frac{1}{p}}d\mu(x),$$

where $B_r(x)$ is a ball with a radius $r>0$ and the center at $x\in X$. Then

$$\left(\frac{1}{r^\lambda}\int_{B_r(x)}\left|\int_X F(x,y)\nu(y)d\mu(x)\right|^p dy\right)^{\frac{1}{p}}$$

$$\leq \int_X \left(\frac{1}{r^\lambda} \int_{B_r(x)} |F(x,y)\nu(y)|^p \, dy \right)^{\frac{1}{p}} d\mu(x).$$

The required result follows by taking the supremum over all $x \in (a,b)$ and $r > 0$ in the last inequality. ◄

It is now easy to provide the

Proof of Proposition 2. Let $f \in M_\nu^{p,\lambda}(-\pi,\pi)$, and $\varepsilon > 0$, be a sufficiently small number. Consider the function

$$w_\varepsilon(t) = \begin{cases} c_\varepsilon e^{\left(\frac{-\varepsilon^2}{\varepsilon^2 - t^2}\right)}, & |t| < \varepsilon, \\ 0, & |t| \geq \varepsilon, \end{cases}$$

where c_ε is chosen such that $\int_{-\infty}^{\infty} w_\varepsilon(t)dt = 1$. Define the function $f_\varepsilon(t)$ as

$$f_\varepsilon(t) = \int_{-\infty}^{\infty} w_\varepsilon(s) f(t-s) ds.$$

As $\varepsilon > 0$ is sufficiently small, this definition is correct. Indeed, it is enough to prove that $f \in L_1(-\pi,\pi)$. From $f \in M_\nu^{p,\lambda}(-\pi,\pi)$ it follows that $(f\nu) \in L_{p,\lambda}(-\pi,\pi)$. Let (5) holds. By using Lemma 2 it is easy to prove that $\nu^{-1} \in \left(L^{p,\lambda}(-\pi,\pi)\right)'$. Since $(f\nu) \in L_{p,\lambda}(-\pi,\pi)$, we have $f = (f\nu)\,\nu^{-1} \in L_1(-\pi,\pi)$.

It is clear that $f_\varepsilon(t)$ is infinitely differentiable function on $[-\pi,\pi]$, and

$$\|f_\varepsilon - f\|_{p,\lambda;\nu} = \left\| \int_{-\infty}^{\infty} w_\varepsilon(s) f(t-s) ds - f(t) \right\|_{p,\lambda;\nu}$$

$$= \left\| \int_{-\infty}^{\infty} w_\varepsilon(s) \left[f(t-s) - f(t) \right] ds \right\|_{p,\lambda;\nu}$$

Applying Lemma 5, we get

$$\|f_\varepsilon - f\|_{p,\lambda;\nu} \leq \int_{-\infty}^{\infty} \|w_\varepsilon(s) \left[f(.-s) - f(.) \right]\|_{p,\lambda;\nu} \, ds$$

$$\leq \sup_{|s|<\varepsilon} \|[f(.-s) - f(.)]\|_{p,\lambda;\nu} \int_{-\varepsilon}^{\varepsilon} w_\varepsilon(s) ds$$

$$= \sup_{|s|<\varepsilon} \|[f(.-s) - f(.)]\|_{p,\lambda;\nu} \to 0 \ \ as \ \ \varepsilon \to 0.$$

This completes the proof.

By similar way we can define $M_\nu^{p,\lambda}(0,\pi)$ and prove the following

Proposition 3. *Let ν be given as in* (1) *and the conditions* (5) *be satisfied. Then the set $C^\infty[0,\pi]$, of all infinitely differentiable functions with compact support in $(0,\pi)$, is dense in $M_\nu^{p,\lambda}(0,\pi)$.*

3. Main result

In this section we will establish the completeness of system of cosines in weighted Morrey spaces.

Theorem 1. *The system* $\{\cos nt\}_{n\in N_0}$ *is complete in* $M_\nu^{p,\lambda}(0,\pi), 0<\lambda<1, 1<p<+\infty$, *if conditions*

$$\alpha_0;\alpha_r \in \left(-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p}+1\right),\ \alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p}+1\right),\ k=\overline{1,r-1}, \quad (6)$$

are satisfied.

Proof. First, note that $\{\cos nt\}_{n\in N_0} \subset M_\nu^{p,\lambda}(0,\pi)$. Indeed, by Lemma 1 under (5) we have $\nu \in L^{p,\lambda}(0,\pi)$. Then from Lemma 4 we have $C[0,\pi] \subset M_\nu^{p,\lambda}(0,\pi)$, and as a result $\{\cos nt\}_{n\in N_0} \subset M_\nu^{p,\lambda}(0,\pi)$. Show that under (6) the set $C_0^\infty[0,\pi]$ is also dense in $M_\nu^{p,\lambda}(0,\pi)$. Indeed, from Proposition 3, we have that the set $C^\infty[0,\pi]$ is dense in $M_\nu^{p,\lambda}(0,\pi)$. Let$f \in M_\nu^{p,\lambda}(0,\pi)$ be any function and $\varepsilon>0$ be any number. Then $\exists g \in C^\infty[0,\pi]$:

$$\|f-g\|_{p,\lambda;\nu} < \frac{\varepsilon}{2}.$$

Set $E_\delta^+ = (0,\delta)$, $E_\delta^- = (\pi-\delta,\pi)$. We have

$$\left\|g\chi_{E_\delta^\pm}\right\|_{L_\nu^{p,\lambda}(0,\pi)} = \|g\|_{L_\nu^{p,\lambda}\left(E_\delta^\pm\right)} \le \|g\|_\infty \|\nu\|_{L^{p,\lambda}\left(E_\delta^\pm\right)}.$$

For sufficiently small $\delta>0$ we get

$$\|\nu\|_{L^{p,\lambda}\left(E_\delta^+\right)} \le C\left\|t^{\alpha_0}\right\|_{L^{p,\lambda}\left(E_\delta^+\right)} \to 0,\ \delta\to 0.$$

Analogously we have

$$\|\nu\|_{L^{p,\lambda}\left(E_\delta^-\right)} \le C\left\|(\pi-t)^{\alpha_r}\right\|_{L^{p,\lambda}\left(E_\delta^-\right)} \to 0,\ \delta\to 0.$$

Let $\delta_0 < \frac{1}{2}\min\{t_1;\pi-t_{r-1}\}$ is so that

$$\|\nu\|_{L^{p,\lambda}\left(E_\delta^+\right)} + \|\nu\|_{L^{p,\lambda}\left(E_\delta^-\right)} < \frac{\varepsilon}{4\|g\|_\infty},\ \forall\delta\in(0,\delta_0).$$

Set

$$g_{\delta_0}(t) = \begin{cases} g(t),\ t\in(0,\pi)\setminus\left(E_{\delta_0/2}^+ \bigcup E_{\delta_0/2}^-\right), \\ 0,\ t\in\left(E_{\delta_0/2}^+ \bigcup E_{\delta_0/2}^-\right). \end{cases}$$

Consider

$$G_{\delta_0;\tau}(t) = \int_{-\infty}^{\infty} \omega_\varepsilon(s)\, g_{\delta_0}(t-s)\, ds.$$

It is clear that

$$\|G_{\delta_0;\tau} - g_{\delta_0}\|_{p,\lambda;\nu} \to 0,\ \tau \to 0.$$

Since $g_{\delta_0}(\cdot)$ is finitly supported on $(0,\pi)$, for sufficiently small $\tau > 0$ the function $G_{\delta_0;\tau}$ is also finitly supported on $(0,\pi)$, and as a result $G_{\delta_0;\tau} \in C_0^\infty[0,\pi]$. Let $\tau < \frac{\delta_0}{2}$ be so that

$$\|G_{\delta_0;\tau_0} - g_{\delta_0}\|_{p,\lambda;\nu} < \frac{\varepsilon}{4}.$$

We have

$$\|f - G_{\delta_0;\tau_0}\|_{p,\lambda;\nu} \le \|f-g\|_{p,\lambda;\nu} + \|g - g_{\delta_0}\|_{p,\lambda;\nu} +$$

$$+\|g_{\delta_0} - G_{\delta_0;\tau_0}\|_{p,\lambda;\nu} \le \frac{\varepsilon}{2} + \|g\|_{L_\nu^{p,\lambda}\left(E^+_{\delta_0/2}\bigcup E^-_{\delta_0/2}\right)} + \frac{\varepsilon}{4} < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, from here we get that $C_0^\infty[0,\pi]$ is dense in $M_\nu^{p,\lambda}(0,\pi)$.

So, for every $f \in M_\nu^{p,\lambda}(0,\pi)$ and $\varepsilon > 0$, there exists $f_\varepsilon \in C_0^\infty[0,\pi]$ such that $\|f - f_\varepsilon\|_{p,\lambda;\nu} < \varepsilon$. It is known that the Fourier sine series of f_ε converges uniformly to this function on $[0,\pi]$. That is, if

$$S_m(t) = \sum_{n=1}^{m} c_n(f_\varepsilon)\cos nt,\ m \in \mathrm{N},$$

where $c_n(f_\varepsilon) = \frac{2}{\pi}\int_0^\pi f_\varepsilon(t)\cos nt\, dt$, then there exists $m_0 = m_0(\varepsilon) \in \mathrm{N}$, such that

$$\sup_{t\in 0,\pi]} |f_\varepsilon(t) - S_m(t)| < \varepsilon,\ \ for\ \ all\ \ m \ge m_0.$$

Therefore

$$\|f_\varepsilon - S_m\|_{p,\lambda;\nu} = \sup_{I\subset(0,\pi)}\left(\frac{1}{|I|^\lambda}\int_I |f_\varepsilon(t) - S_m(t)|^p\,|\nu(t)|^p\,dt\right)^{\frac{1}{p}}$$

$$\le \varepsilon \sup_{I\subset(0,\pi)}\left(\frac{1}{|I|^\lambda}\int_I |\nu(t)|^p\,dt\right)^{\frac{1}{p}} = \varepsilon\,\|\nu\|_{p,\lambda}.$$

Then

$$\|f - S_m\|_{p,\lambda;\nu} \le \|f - f_\varepsilon\|_{p,\lambda;\nu} + \|f_\varepsilon - S_m\|_{p,\lambda;\nu} < \left(1 + \|\nu\|_{p,\lambda}\right)\varepsilon.$$

Thus, we arrive at the result since ε was arbitrary. Thus, if the conditions (5) are satisfied, then the system $\{\cos nt\}_{n\in\mathrm{N}_0}$ is complete in $M_\nu^{p,\lambda}(0,\pi)$.

The theorem is proved. ◀

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On Boundedness of Hardy Type Integral Operator in Weighted Lebesgue Spaces

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Abstract. In this paper we proved sufficient conditions for boundedness of Hardy type integral operator in weighted Lebesgue spaces.

Key Words and Phrases: weighted Lebesgue spaces, Hardy operator.

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1. Introduction

Let ϕ be a fixed kernel defined on $(0,\infty)$, i.e. $\phi \in L_1^{loc}(0,\infty)$, then the Hardy type integral operator is defined in the following way

$$H_\phi(f)(x) = \int_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)\, dy. \tag{1}$$

This integral operator (1) is deeply rooted in the study of one-dimensional Fourier analysis and has become an essential part of modern harmonic analysis.In particular, it is closely related to the summability of the classical Fourier series (see [8]). Many important operators in analysis are special cases of the integral operator (1), by taking suitable choice of ϕ.

The considered integral operator (1) has been extensively studied in recent years, particularly its boundedness on the Lebesgue space as well as on the Hardy space(see [2, 3, 4]). We also refer to [5, 6, 7] for some recent work in this vein. Moreover the generalized version of the considered operators on multidimensional Euclidean spaces have been studied (see [2], [8]). About boundedness of Hausdorff operator in different Lebesgue spaces we refer to [1].

In this paper we proved sufficient conditions for boundedness of integral operator (1) in weighted Lebesgue spaces.

2. Main Results

We recall some notation and basic facts about function spaces.

Let ω be a weight function on R_+, i.e $\omega \in L_1^{loc}(R_+)$ and almost everywhere is a positive function. The weighted Lebesgue space $L_{p,\omega}(R_+)$ is the class of all measurable functions f defined on R_+ such that

$$\|f\|_{L_{p,\omega}(R_+)} = \left(\int_0^\infty |f(x)|^p \omega(x)\, dx \right)^{\frac{1}{p}} < \infty.$$

Theorem 1. *Let $1 < p < q < \infty$ and H_ϕ is a Hausdorff operator. Let u be positive non-decreasing weighted function on $(0,\infty)$. Suppose that satisfying the following conditions:*

1) $\int_0^{\frac{1}{2}} \frac{\phi(y)}{y} y^{\frac{1}{p}} dy < +\infty$ and there exists a constant C_1 such that for any $t \geq \frac{1}{2}$ the following inequality holds

$$|\phi(t)| \leq \frac{C_1}{t},$$

2)

$$\sup_{t>0} \left(\int_t^\infty \frac{u(x)}{x^p} dx \right)^{\frac{1}{p}} \left(\int_0^t u(x)^{1-p'} dx \right)^{\frac{1}{p'}} < \infty.$$

Then there exists $C > 0$ for all $f \in L_{p,u}(0,\infty)$ the following inequality holds

$$\left(\int_0^\infty |H_\phi f(x)|^p u(x)\, dx \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty |f(x)|^p u(x)\, dx \right)^{\frac{1}{p}}. \quad (2)$$

Proof: Without loss of generality we may assume that the function u has the form

$$u(t) = u(0) + \int_0^t \psi(\tau)\, d\tau,$$

where $u(0) = \lim_{t\to+0} u(t)$ and ψ is a positive function on $(0,\infty)$. Indeed, for increasing functions on $(0,\infty)$ there exists a sequence of absolutely continuous functions $\varphi_n(t)$ such that $\lim_{n\to\infty} \varphi_n(t) = u(t)$, $0 \leq \varphi_n(t) \leq u(t)$ a.e. $t > 0$ and $\varphi_n(0) = u(0)$. Furthermore the functions $\varphi_n(t)$ are increasing, and besides

$$\varphi_n(t) = \varphi_n(0) + \int_0^t \varphi_n'(\tau)\, d\tau.$$

Where $\lim_{n\to\infty}\varphi_n'(t)=\psi(t)$. Hence, using Fatou's theorem , we obtain estimate (2) for any increasing functions on $(0,\infty)$.

Let us estimate the left –hand side of inequality (2). We have

$$\left(\int_0^\infty |H_\phi f(x)|^p u(x)\,dx\right)^{\frac{1}{p}}=\left(\int_0^\infty |H_\phi f(x)|^p\left(u(0)+\int_0^x \psi(t)\,dt\right)dx\right)^{\frac{1}{p}}.$$

If $u(0)=0$, then $\left(\int_0^\infty |H_\phi f(x)|^p u(x)\,dx\right)^{\frac{1}{p}}=\left(\int_0^\infty |H_\phi f(x)|^p\left(\int_0^x \psi(t)\,dt\right)dx\right)^{\frac{1}{p}}$.

However, if $u(0)>0$, then

$$\left(\int_0^\infty |H_\phi f(x)|^p u(x)\,dx\right)^{\frac{1}{p}}\le\left(\int_0^\infty |H_\phi f(x)|^p u(0)\,dx\right)^{\frac{1}{p}}$$

$$+\left(\int_0^\infty |H_\phi f(x)|^p\left(\int_0^x \psi(t)\,dt\right)dx\right)^{\frac{1}{p}}=E_1+E_2.$$

First estimate E_1. By boundedness of integral operator (1) in Lebesgue spaces (see [2, 8]), we get

$$E_1=\left(\int_0^\infty |H_\phi f(x)|^p u(0)\,dx\right)^{\frac{1}{p}}=(u(0))^{\frac{1}{p}}\left(\int_0^\infty |H_\phi f(x)|^p\,dx\right)^{\frac{1}{p}}$$

$$\le C(u(0))^{\frac{1}{p}}\left(\int_0^\infty |f(x)|^p\,dx\right)^{\frac{1}{p}}\le C\left(\int_0^\infty |f(x)|^p u(x)\,dx\right)^{\frac{1}{p}}=C\|f\|_{L_{p,u}(0,\infty)}.$$

Let us estimate the integral E_2. We have

$$E_2=\left(\int_0^\infty |H_\phi f(x)|^p\left(\int_0^x \psi(t)\,dt\right)dx\right)^{\frac{1}{p}}$$

$$=\left(\int_0^\infty |H_\phi f(x)|^p\left(\int_0^\infty \psi(t)\,\chi_{\{x>t\}}(x)\,dt\right)dx\right)^{\frac{1}{p}}$$

$$=\left(\int_0^\infty \psi(t)\left(\int_t^\infty |H_\phi f(x)|^p\,dx\right)dt\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi\left(t\right)\left(\int\limits_t^\infty \left|\int\limits_{2t}^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f\left(y\right)dy\right|^p dx\right)dt\right)^{\frac{1}{p}}$$

$$+2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi\left(t\right)\left(\int\limits_t^\infty \left|\int\limits_0^{2t} \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)dy\right|^p dx\right)dt\right)^{\frac{1}{p}} = E_{21}+E_{22}.$$

We estimate E_{21}. Using Theorem on boundedness of integral operator (1) in Lebesgue space, (see [1,7]) we get

$$E_{21} = 2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi\left(t\right)\left(\int\limits_0^\infty \left|\int\limits_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f\left(y\right)\chi_{\{y>2t\}}\left(y\right)dy\right|^p \chi_{\{x>t\}}\left(x\right)dx\right)dt\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p'}}\left(\int_0^\infty \psi\left(t\right)\left(\int\limits_0^\infty \left|\int\limits_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f\left(y\right)\chi_{\{y>2t\}}\left(y\right)dy\right|^p dx\right)dt\right)^{\frac{1}{p}}$$

$$\leq C_2\left(\int\limits_0^\infty \psi\left(t\right)\left(\int_0^\infty \left|f\left(x\right)\right|^p \chi_{\{y>2t\}}\left(x\right)dx\right)dt\right)^{\frac{1}{p}}$$

$$= C_2\left(\int\limits_0^\infty \left|f\left(x\right)\right|^p\left(\int\limits_0^{\frac{x}{2}} \psi\left(t\right)dt\right)dx\right)^{\frac{1}{p}} \leq C_2\left(\int\limits_0^\infty \left|f\left(x\right)\right|^p u\left(\frac{x}{2}\right)dx\right)^{\frac{1}{p}}$$

$$\leq C_2\left(\int\limits_0^\infty \left|f\left(x\right)\right|^p u\left(x\right)dx\right)^{\frac{1}{p}} = C_2\left\|f\right\|_{L_{p,u}(0,\infty)}.$$

Now we estimate E_{22}. Note that if $x>t, y\leq 2t$, then $\frac{x}{y}\geq\frac{1}{2}$. By virtue of condition 1) of Theorem 1, one has

$$E_{22} = 2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi\left(t\right)\left(\int\limits_t^\infty \left|\int\limits_0^{2t} \frac{\varphi\left(\frac{x}{y}\right)}{y} f\left(y\right)dy\right|^p dx\right)dt\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi\left(t\right)\left(\int\limits_t^\infty \left(\int\limits_0^{2t} \frac{\left|\varphi\left(\frac{x}{y}\right)\right|}{y}\left|f\left(y\right)\right|dy\right)^p dx\right)dt\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p'}} \left(\int_0^\infty \psi(t) \left(\int_t^\infty \left(\int_0^{2t} \frac{|f(y)|}{x} dy \right)^p dx \right) dt \right)^{\frac{1}{p}}$$

$$= 2^{\frac{1}{p'}} \left(\int_0^\infty \psi(t) \left(\int_t^\infty \frac{dx}{x^p} \right) \left(\int_0^{2t} |f(y)| \, dy \right)^p dt \right)^{\frac{1}{p}}.$$

We get following formula in a way that made use of change of variables $\left(t = \frac{z}{2}, dt = \frac{1}{2}dz, 0 < z < \infty\right)$

$$E_{22} = 2^{\frac{1}{p'} - \frac{1}{p}} \left(\int_0^\infty \psi\left(\frac{t}{2}\right) \left(\int_{\frac{t}{2}}^\infty \frac{dx}{x^p} \right) \left(\int_0^t |f(y)| \, dy \right)^p dt \right)^{\frac{1}{p}}.$$

As is well-known, the classical Hardy operator of function $|f|$ is determined by

$$\int_0^t |f(y)| \, dy.$$

We have

$$\int_{2t}^\infty \psi\left(\frac{s}{2}\right) \left(\int_{\frac{s}{2}}^\infty \frac{dx}{x^p} \right) ds = 2 \int_t^\infty \psi(s) \left(\int_s^\infty \frac{dx}{x^p} \right) ds$$

$$= 2 \int_t^\infty \psi(s) \left(\int_0^\infty \chi_{(s,\infty)}(x) \, x^{-p} dx \right) ds = 2 \int_0^\infty \psi(s) \, \chi_{(t,\infty)}(s)$$

$$\times \left(\int_0^\infty \chi_{(s,\infty)}(x) \, x^{-p} dx \right) ds = 2 \int_0^\infty \int_0^\infty \psi(s) \, x^{-p} \chi_{(t,\infty)}(s) \, \chi_{(s,\infty)}(x) \, dx ds$$

$$= 2 \int_t^\infty x^{-p} \left(\int_t^x \psi(s) \, ds \right) dx \leq 2 \int_t^\infty x^{-p} \left(\int_0^x \psi(s) \, ds \right) dx \leq 2 \int_t^\infty x^{-p} u(x) \, dx.$$

From this, we get

$$\int_t^\infty \psi(s) \left(\int_s^\infty \frac{dx}{x^p} \right) ds \leq \int_t^\infty \frac{u(x)}{x^p} dx.$$

Let v and ω is weight functions defined on $(0, \infty)$. Follows by the theory of boundedness of two-weighted Hardy operators, (see [9]) we have

$$\left(Hf\left(x\right) = \int\limits_0^x f\left(t\right)dt: \ H: L_{p,v}(0,\infty) \to L_{p,\omega}(0,\infty) \right) \Leftrightarrow$$
$$\Leftrightarrow A = \sup_{t>0} \left(\int\limits_t^\infty \omega\left(x\right)dx \right)^{\frac{1}{p}} \left(\int\limits_0^t v(x)^{1-p'}dx \right)^{\frac{1}{p'}} < \infty. \tag{3}$$

Thus, from inequality (3), we have

$$\sup_{t>0} \left(\int\limits_t^\infty \psi(s) \left(\int\limits_s^\infty \frac{dx}{x^p} \right) ds \right)^{\frac{1}{p}} \left(\int\limits_0^t u\left(x\right)^{1-p'} dx \right)^{\frac{1}{p'}}$$
$$\leq \sup_{t>0} \left(\int\limits_t^\infty \frac{u(x)}{x^p} dx \right)^{\frac{1}{p}} \left(\int\limits_0^t u\left(x\right)^{1-p'} dx \right)^{\frac{1}{p'}} < \infty. \tag{4}$$

Taking $\omega(x) = \psi\left(\frac{x}{2}\right)x^{1-p}$ and $v(x) = u(x)$ and applying (3) and (4), we have

$$E_{22} \leq C_6 \left(\int\limits_0^\infty \psi\left(\frac{t}{2}\right) \left(\int\limits_{\frac{t}{2}}^\infty \frac{dx}{x^p} \right) \left(\int\limits_0^t \left|f\left(y\right)\right| dy \right)^p dt \right)^{\frac{1}{p}}$$
$$= C_7 \left(\int\limits_0^\infty \omega(t) \left(\int\limits_0^t \left|f\left(y\right)\right| dy \right)^p dt \right)^{\frac{1}{p}} \leq C_8 \left(\int\limits_0^\infty \left|f\left(t\right)\right|^p u\left(t\right) dt \right)^{\frac{1}{p}}.$$

The proof is completed.

Corollary 1. Let $1 < p < \infty$ and H_ϕ - is the classical Hardy operator or Riemann-Liouville operator.

Then these operators satisfy all terms of theorem 1 and these operators are bounded on $L_{p,u}\left(0,\infty\right)$.

Theorem 2. Let $1 < p < \infty$ and $H_\phi-$ Hausdorff operator. Let u be positive non-increasing weighted function on $(0,\infty)$. Suppose that satisfying the following conditions:

1) $\int\limits_0^{\frac{1}{2}} \frac{\phi(y)}{y} y^{\frac{1}{p}} dy < +\infty$ and there exists a constant $C_1 > 0$ such that for any $\forall t \in (0,2)$ the following inequality holds

$$\left|\phi\left(t\right)\right| \leq C_1;$$

$$2) \sup_{t>0} \left(\int\limits_0^t \frac{u\left(x\right)}{x^p} dx \right)^{\frac{1}{p}} \left(\int\limits_t^\infty u\left(x\right)^{1-p'} dx \right)^{\frac{1}{p'}} < \infty.$$

Then there exists $C>0$ for all $f\in L_{p,u}(0,\infty)$ the following inequality holds

$$\left(\int_0^\infty |H_\phi f(x)|^p\, u(x)\, dx\right)^{\frac{1}{p}} \le C\left(\int_0^\infty |f(x)|^p\, u(x)\, dx\right)^{\frac{1}{p}}.$$

The proof of Theorem 2 is also similar to the proof of the corresponding Theorem 1.

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Weak Solvability of the First Boundary Value Problem for a Class of Parabolic Equations with Discontinuous Coefficients in Paraboloid Type Domains

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Abstract. In the paper, weak solvability of the first boundary value problem is proved for a class of parabolic equations with discontinuous coefficients and given in parabolic type domains in Sobolev's weight spaces. The coefficients of these equation bear discontinuity at the vertex of $P-$ domain. At the vertex $P-$ domain touches the characteristics of the equation.

Key Words and Phrases: boundary value problem, weak solvability, parabolic operator.
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1. Introduction

Let E_n and R_{n+1} be $-$ n- dimensional and $(n+1)$ dimensional Euclidean spaces of the points $x = (x_1, \dots, x_n)$ and $(x,t) = (x_1, \dots, x_n, t)$ respectively. D be a bounded domain E_n with a boundary $\partial\, D$, $\quad 0 \in D$, $\quad R^-_{n+1} = R_{n+1} \cap \{(x,t) : t < 0\}$.

The domain $Q \subset \ R^-_{n+1}$ is said to be a paraboloid type domain (or $P-$domain) if its cross section with each hyperplane $t = \tau\ (\tau < 0)$ has the form:

$$\left\{ x : \frac{x}{2\sqrt{-\tau}} \in D \right\}.$$

The domain D is called a foot of the $P-$ domain Q.

Let further $Q_T = Q \cap \{(x,t)\ :\ -T < t < 0\ \}$, $\ S_T = \ \partial Q \cap \{(x,t)\ :\ T < t < 0\ \}$, $D_T = Q \cap \{(x,t)\ :\ t = -T\ \}$, $\Gamma(Q_T)$ be a parabolic boundary of the domain Q_T.

Consider in Q_T the following operator

$$L = \Delta + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} \cdot \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial U}{\partial t},$$

where Δ is the Laplace operator and the number parameter λ satisfies the condition

$$\frac{1}{d^2} < \lambda < \infty. \qquad (1)$$

Here $d = \sup_{y\epsilon D}|y|$. It is easy to see that subject to condition (1) the operator L uniformly parabolic in the domain Q_T. By analogy with the elliptic case, we call the operator L the Gilbarg-Serrin parabolic operator.

Let us agree in the following denotation u_i and u_{ij} are the derivatives of $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$, respectively,

$$u_{xx} = (u_{ij}), \qquad u_x^2 = \sum_{i=1}^{n} u_i^2, \quad u_{xx}^2 = \sum_{i,j=1}^{n} u_{ij}^2 \ ; \qquad i,j = \overline{1,n}.$$

Let the number parameter γ satisfy the condition

$$\gamma\epsilon \left(\frac{n^2 \left(\lambda - \frac{1}{d^2}\right) + 2\lambda n}{2}, \infty \right). \tag{2}$$

$A_0^\infty(Q_T)$ be a space of infinitely differentiable and finite in Q_T functions for which the following integral is finite $\int_{Q_T}(-t)^\gamma u^2 dxdt$, $\qquad L_{2,\gamma}(Q_T)$ be Banach space of measurable functions $u(x,t)$ given on, Q_T with finite norm

$$\|u\|_{L_{2,\gamma}(Q_T)} = \left(\int_{Q_T} (-t)^2 u^2 dxdt \right)^{\frac{1}{2}},$$

$\overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T)$ and $\overset{0}{W}{}_{2,\gamma}^{1,1}(Q_T)$ be Banach spaces of measurable functions $u(x,t)$ given on Q_T with finite norms

$$\|u\|_{W_{2,\gamma}^{1,0}(Q_T)} = \left(\int_{Q_T} (-t)^\gamma (u^2 + u_x^2) dxdt \right)^{\frac{1}{2}},$$

$$\|u\|_{W_{2,\gamma}^{1,1}(Q_T)} = \left(\int_{Q_T} (-t)^\gamma (u^2 + u_x^2 + u_t^2) dxdt \right)^{\frac{1}{2}},$$

respectively.

$\overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T)$ and$\overset{0}{W}{}_{2,\gamma}^{1,1}(Q_T)$ be subspaces of $W_{2,\gamma}^{1,0}(Q_T)$ and $W_{2,\gamma}^{1,1}(Q_T)$, respectively, in which $A_0^\infty(Q_T)$ is a dense set.

In the domain Q_T consider the first boundary value problem

$$Lu = \Delta \mathrm{u} + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f + \sum_{k=1}^{n} \frac{\partial f^k}{\partial x^k} \tag{3}$$

$$u|_{\Gamma(Q_T)} = 0, \tag{4}$$

where $f\epsilon L_{2,\gamma}(Q_T)$, $f^k \epsilon L_{2,\gamma}(Q_T)$; $\quad k = \overline{1,n}$.

Therewith, it is assumed that with regard to number parameters λ and γ, conditions (1) and (2) are fulfilled. At first give definition of the weak solution of the first boundary value problem (3)-(4).

The function $u(x,t)\epsilon W^{1,0}_{2,\gamma}(Q_T)$ is said to be a weak solution of equation (3) in the domain Q_T if for any function $v(x,t)\epsilon \overset{0}{W}{}^{1,1}_{2,\gamma}(Q_T)$the following integral identity is fulfilled.

$$B_{Q_T}(u,v)=\int_{Q_T}(-t)^{\gamma}u\ v_t dxdt-\int_{Q_T}(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)v_iu_jdxdt+$$

$$+\lambda(n+1)\int_{Q_T}(-t)^{\gamma}\sum_{i=1}^{n}\frac{x_i}{4(-t)}uv_idxdt+\frac{\lambda n(n+1)}{4}\int_{Q_T}(-t)^{\gamma}uvdxdt-$$

$$-\gamma\int_{Q_T}(-t)^{\gamma}uvdxdt\ =\int_{Q_T}(-t)^{\gamma}fvdxdt-\int_{Q_T}(-t)^{\gamma}\sum_{k=1}^{n}f^kv_kdxdt, \qquad (5)$$

where δ_{ij} is the Kronecker symbol.

The function $u(x,t)\epsilon \overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T)$ being a weak solution of equation (3) in Q_T is called a weak solution of boundary value problem (3)-(5). Now we show the relation between equation (3) and integral identity (5). At first we represent the Hilbarg-Serrin parabolic operator in the form of a divergent operator with unbounded minor coefficients. We have

$$Lu=\Delta u+\lambda\sum_{i,j=1}^{n}\left(\frac{x_ix_j}{4(-t)}u_i\right)_j-\lambda(n+1)\sum_{i=1}^{n}\frac{x_i}{4(-t)}u_i-u_t\ .$$

Consider the domain $Q_{T,\delta}=Q_T\backslash\overline{Q_\delta}$ multiply the both parts of equation (3) by the function $v(x,t)\epsilon A_0^{\infty}(Q_T)$ and integrate the obtained equality with respect to $Q_{T,\delta}$. We get

$$\int_{Q_{T,\delta}}(-t)^{\gamma}v\ \Delta u\ dx\ dt+$$

$$+\lambda\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\frac{x_{??}x_j}{4(-t)}u_i\right)_j v\ dx\ dt-\lambda(n+1)\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}\frac{x_j}{4(-t)}u_jv\ dx\ dt-$$

$$-\int_{Q_{T,\delta}}(-t)^{\gamma}u_tv\ dx\ dt=\int_{Q_{T,\delta}}(-t)^{\gamma}f\cdot v\ dx\ dt+\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}\frac{\partial f^i}{\partial x_i}v\ dx\ dt. \qquad (6)$$

By Ostrogradskii's formula

$$\int_{Q_{T,\delta}}(-t)^{\gamma}v\ \Delta u\ dx\ dt+\lambda\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\frac{x_ix_j}{4(-t)}u_i\right)_j v\ dx\ dt=$$

$$=-\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)u_iv_j\ dx\ dt. \qquad (7)$$

In what follows we have

$$-\lambda\,(n+1)\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}\frac{x_i}{4\,(-t)}\;u_i v\;dx\;dt=\frac{\lambda n\,(n+1)}{4}\int_{Q_{T,\delta}}(-t)^{\gamma-1}uv\;dx\;dt+$$

$$+\lambda\,(n+1)\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}\frac{x_i}{4\,(-t)}v_i u\;dx\;dt \tag{8}$$

Furthermore

$$\int_{Q_{T,\delta}}(-t)^{\gamma}f\cdot v\;dx\;dt+\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}\frac{\partial f^i}{\partial x_i}v\;dx\;dt=$$

$$=\int_{Q_{T,\delta}}(-t)^{\gamma}f\;v\;dx\;dt-\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}f^i\;v_i\;dx\;dt \tag{9}$$

Let $\Pi_R=\left\{x:\ |x_i|<R,\ \ i=\overline{1,n}\right\}$, $\quad \mathrm{K}_\delta=\Pi_R\times(-T,-\delta)$. For simplicity we will consider that we can continue the function $u(x,t)$ in $\mathrm{K}_\delta\backslash \mathrm{Q}_{T,\delta}$ so that the obtained continuation $\tilde{u}(x,t)$ be the element of the space $W^{1,0}_{2,\gamma}(K_\delta)$. We continue the function $v(x,t)$ by a zero to$\mathrm{K}_\delta\backslash \mathrm{Q}_{T,\delta}$ and denote the obtained continuation again by $v(x,t)$.We have

$$J_\delta=-\int_{K_\delta}(-t)^{\gamma}\tilde{u}_t v\;dx\;dt=-\delta\int_{\Pi_R}?\widetilde{D}\,(x,-\delta)\;\;v\;\;(x,-\delta)\,dx+$$

$$+\int_{K_\delta}(-t)^{\gamma}v_t\tilde{u}\;dx\;dt-\gamma\int_{K_\delta}(-t)^{\gamma-1}\widetilde{u}\;\;v\;dx\;dt=-\delta^{\gamma}\int_{K_\delta}u\,(x,-\delta)\;\;v\;\;(x,-\delta)\;\;dx+$$

$$+\int_{Q_{T,\delta}}(-t)^{\gamma}v_t u dx dt-\gamma\int_{Q_{T,\delta}}(-t)^{\gamma-1}uv dx dt.$$

Hence it follows that

$$\lim_{\delta\to 0+}\;J_\delta\;=\int_{Q_T}(-t)^{\gamma}v_t\;u\;dx\;dt-\gamma\int_{Q_T}(-t)^{\gamma-1}u\;v\;dx\;dt. \tag{10}$$

Now, taking into account (7)-(10) in (6), and tending δ to zero we arrive at integral identity (5).

Theorem 1. *If with respect to number parameters λ and γ conditions* (1) *and* (2) *are fulfilled, then the first boundary value problem* (3)-(4) *is uniquely weakly solvable in the space $\overset{0}{W}{}^{1,0}_{2,\gamma}\,(Q_T)$ for any $f(x,t)\in L_{2,\gamma}(Q_T)$ and $f^k\,(x,t)\in L_{2,\gamma}\,(Q_T)\,;\quad k=\overline{1,n}$.*

Proof. At first prove the existence of the solution. To this end we consider the extending sequence of domains $\{D_m\}\,,\quad m=1,2,\ldots;$ approximating from within the domain D, i.e. $\overline{D_m}\subset D_{m+1},\ \overline{D_m}\subset D, \lim_{m\to\infty}D_m=D$. Therewith we choose D_m so that for any natural $m\quad \partial D_m\in C^2$. Let further Q^m be a $P-$domain whose foot is the domain D_m,

$$Q^m_T=Q^m\cap\{(x,t):t>-T\}\,,\quad Q^m_{T,\delta}=Q^m_T\backslash\overline{Q}^m_\delta\;\;,\quad \delta\in(0,T)\,.$$

Denote by f^h and $f^{k,h}$ the Friedrichs averaged functions, respectively, $k=\overline{1,n}$ with a parameter $h>0$. Consider for $h>0$ and natural m the family of the first boundary value problems

$$\Delta u^{m,h}+\lambda\sum_{i,j=1}^{n}\left(\frac{x_ix_j}{4\,(-t)}u_i^{m,h}\right)_j-\lambda\,(n+1)\sum_{i=1}^{n}\frac{x_i}{4\,(-t)}u_i^{m,h}-u_t^{m,h}=$$

$$=f^h+\sum_{k=1}^{n}\frac{\partial f^{k,h}}{\partial x_k}\ ;\qquad (x,t)\in Q_{T,\delta}^m, \tag{11}$$

$$u^{m,h}\Big|_{\Gamma\left(Q_{T,\delta}^m\right)}=0. \tag{12}$$

As for any natural m and positive h and δ the coefficients and the right side of equation (11) are infinitely differentiable in $\overline{Q}_{T,\delta}^m$ functions, problem (11)-(12) has a unique classic solution $u^{m,h}(x,t)$. Indeed,$u^{m,h}(x,t)$ depends on δ as well, but for brevity of notation we write $u^{m,h}(x,t)$ instead of $u_\delta^{m,h}(x,t)$. Multiply the both sides of equation (11) by the function $(-t)^\gamma u^{m,h}(x,t)$ and integrate the obtained equality with respect to the domain $Q_{T,\delta}^m$.

We get

$$\int_{Q_{T,\delta}^m}(-t)^\gamma\Delta u^{m,h}\cdot u^{m,h}dx\ dt+\int_{Q_{T,\delta}^m}(-t)^\gamma u^{m,h}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{x_ix_j}{4\,(-t)}u_j^{m,h}\right)dx\ dt-$$

$$-\lambda\,(n+1)\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{i=1}^{n}\frac{x_i}{4\,(-t)}u_i^{m,h}u^{m,h}dx\ dt-\int_{Q_{T,\delta}^m}(-t)^\gamma u_i^{m,h}u_t^{m,h}dxdt=$$

$$=\int_{Q_{T,\delta}^m}(-t)^\gamma f^h\cdot u^{m,h}dx\ dt+\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{k=1}^{n}\frac{\partial f^{k,h}}{\partial x_k}u^{m,h}dx\ dt. \tag{13}$$

Further we have

$$\int_{Q_{T,\delta}^m}(-t)^\gamma\Delta u^{m,h}\cdot u^{m,h}dx\ dt+\lambda\int_{Q_{T,\delta}^m}(-t)^\gamma u^{m,h}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{x_ix_j}{4\,(-t)}u_j^{m,h}\right)dx\ dt=$$

$$=-\int_{Q_{T,\delta}^m}(-t)^\gamma\left(u_x^{m,h}\right)^2dx\ dt-\lambda\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{i,j=1}^{n}\frac{x_ix_j}{4\,(-t)}u_i^{m,h}u_j^{m,h}dxdt. \tag{14}$$

Furthermore

$$-\lambda\,(n+1)\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{i=1}^{n}\frac{x_i}{4\,(-t)}u_iu\ dxdt=\frac{\lambda\,(n+1)\cdot n}{2}\int_{Q_{T,\delta}^m}(-t)^\gamma\frac{u^2}{4\,(-t)}dx\ dt, \tag{15}$$

$$\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{k=1}^{n}\frac{\partial f^{k,h}}{\partial x_k}u^{m,k}dx\ dt=-\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{k=1}^{n}f^{k,h}u_k^{m,h}dx\ dt. \tag{16}$$

Finally, by means of arguments similar to ones that were used by deriving integral identity (5), we get

$$-\int_{Q^m_{T,\delta}}(-t)^{\gamma}u\ u_t dxdt=-\frac{\gamma}{2}\int_{Q^m_{T,\delta}}(-t)^{\gamma}u^2dxdt+i_1(\delta)\,, \tag{17}$$

where $\lim\limits_{\delta\to\infty} i_1(\delta)=0$.

Taking into account (13)-(16) in (12) and tending δ to zero, we conclude

$$\int_{Q^m_{T,\delta}}(-t)^{\gamma}\left(\left(u^{m,h}_x\right)^2+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u^{m,h}_i\cdot u^{m,h}_j\right)\ dx\ dt-$$

$$-\frac{\lambda(n+1)-4\gamma}{2}\cdot\int_{Q^m_T}(-t)^{\gamma}\frac{\left(u^{m,h}\right)^2}{4(-t)}\ dx\ dt=$$

$$=\int_{Q^m_T}(-t)^{\gamma}\sum_{k=1}^{n}f^k u^{m,h}_k dx\ dt-\int_{Q^m_T}(-t)^{\gamma}f\ \ u^{m,h}dx\ dt. \tag{18}$$

Here $u^{m,h}(x,t)=\lim_{\delta\to0+}u^{m,h}_{\delta}(x,t)$.

The existence of the pointwise limit is proved in the same as in [1].

If now $\frac{\lambda n(n+1)-4\gamma}{2}\le 0$, i.e. $\gamma\ge\frac{\lambda n(n+1)}{2}$, then from (17) it follows

$$\int_{Q^m_T}(-t)^{\gamma}\left(\left(u^{m,h}_x\right)^2+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u^{m,h}_i\cdot u^{m,h}_j\right)dx\ dt\le$$

$$\le\int_{Q^m_T}(-t)^{\gamma}\sum_{k=1}^{n}f^{k,h}u^{m,h}_k dx\ dt-\int_{Q^m_T}(-t)^{\gamma}f^h u^{m,h}dx\ dt. \tag{19}$$

Note that for ≥ 0 , $\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u^{m,h}_i\cdot u^{m,h}_j\ge 0$. But if $-\frac{1}{d^2}<\lambda<0,$ then

$$\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u^{m,h}_i\cdot u^{m,h}_j\ge\lambda\ d^2\left(u^{m,h}_x\right)^2.$$

Thus, if $\gamma\ge\frac{\lambda n(n+1)}{2}$, then from (18) we get

$$\int_{Q^m_T}(-t)^{\gamma}\left(u^{m,h}_x\right)^2\ dx\ dt\le\quad C_1(\lambda,n,d,\gamma)\,.$$

$$\left(\int_{Q^m_T}(-t)^{\gamma}\sum_{k=1}^{n}f^{k,h}u^{m,h}_k dx\ dt-\int_{Q^m_T}(-t)^{\gamma}f^h u^{m,h}dx\ dt\right). \tag{20}$$

Now consider the case

$$\gamma \in \left(\frac{n^2\left(\lambda - \frac{1}{d^2}\right) + 2\lambda n}{8}, \frac{\lambda n\,(n+1)}{2} \right). \tag{21}$$

According to inequality (17)

$$\frac{\lambda n\,(n+1) - 4\gamma}{2} \int_{Q_T^m} (-t)^\gamma \frac{\left(u^{m,h}\right)^2}{4(-t)}\, dx\ dt \leq$$

$$\leq \frac{2\ \lambda\ n\,(n+1) - 8\gamma}{n^2} \int_{Q_T^m} (-t)^\gamma \sum_{i,j=1}^{n} \frac{x_i x_j}{4\,(-t)} u_i^{m,h} \cdot u_j^{m,h} dx\ dt. \tag{22}$$

But on the other hand, from (20) it follows that there exists $\mu \in (0,1)$ for which

$$\frac{2\ \lambda\ n\,(n+1) - 8\gamma}{n^2} < \frac{1}{d^2} + \lambda - \frac{\mu}{d^2}.$$

So, from (18) and (21) we conclude

$$\int_{Q_T^m} (-t)^\gamma \left(\left(u_x^{m,h}\right)^2 + \frac{\mu - 1}{d^2} \sum_{i,j=1}^{n} \frac{x_i x_j}{4\,(-t)} u_i^{m,h} \cdot u_j^{m,h} dx\ dt \right) \leq$$

$$\leq \int_{Q_T^m} (-t)^\gamma \left(\sum_{i=1}^{n} f^{i,h} u_i^{m,h} - f^h u^{m,h} \right) dx\ dt. \tag{23}$$

As $\mu < 1$, then

$$\frac{\mu - 1}{d^2} \sum_{i,j=1}^{n} \frac{x_i x_j}{4\,(-t)} u_i^{m,h} \cdot u_j^{m,h} \geq (\mu - 1) \left(u_x^{m,h}\right)^2. \tag{24}$$

From (22)-(23) it follows that

$$\mu \int_{Q_T^m} (-t)^\gamma \left(u_x^{m,h}\right)^2 dx\ dt \leq \int_{Q_T^m} (-t)^\gamma \left(\sum_{k=1}^{n} f^{k,h} u_k^{m,h} - f^h u^{m,h} \right) dx\ dt.$$

The last inequality and estimation (19) allows to conclude that for $\gamma \in \left(\frac{n^2\left(\lambda - \frac{1}{d^2}\right) + 2\lambda n}{8}, \infty \right)$ the following inequality is valid

$$\int_{Q_T^m} (-t)^\gamma \left(u_x^{m,h}\right)^2 dx\ dt \leq C_2\,(\lambda, n, d, \gamma) \int_{Q_T^m} (-t)^\gamma \left(\sum_{k=1}^{n} f^{k,h} u_k^{m,h} - f^h u^{m,h} \right) dx\ dt. \tag{25}$$

According to Friedrich's inequality we get

$$\int_{Q_T^m} (-t)^\gamma \left(u^{m,h}\right)^2 dx\ dt \le C_3(\lambda, n, d, \gamma) \int_{Q_T^m} (-t)^\gamma \left(\sum_{k=1}^n f^{k,h} u_k^{m,h} - f^h u^{m,h}\right) dx\ dt. \quad (26)$$

Thus, from (24)-(25) we conclude

$$\int_{Q_T^m} (-t)^\gamma \left(\left(u^{m,h}\right)^2 + \left(u_x^{m,h}\right)^2\right) dx\ dt \le$$

$$leC_4(\lambda, n, d, \gamma) \int_{Q_T^m} (-t)^\gamma \left(\sum_{k=1}^n f^{k,h} u_k^{m,h} - f^h u^{m,h}\right) dx\ dt. \quad (27)$$

Further, for any $\varepsilon > 0$ we have

$$\int_{Q_T^m} (-t)^\gamma \left(\sum_{k=1}^n f^{k,h} u_k^{m,h} - f^h u^{m,h}\right) dx\ dt \le$$

$$\le \frac{\varepsilon}{2} \int_{Q_T^m} (-t)^\gamma \sum_{k=1}^n \left(u_k^{m,h}\right)^2 dx\ dt + \frac{1}{2\varepsilon} \int_{Q_T^m} (-t)^\gamma \sum_{k=1}^n \left(f^{k,h}\right)^2 dx\ dt +$$

$$+ \frac{\varepsilon}{2} \int_{Q_T^m} (-t)^\gamma \left(u^{m,h}\right)^2 dx\ dt + \frac{1}{2\varepsilon} \int_{Q_T^m} (-t)^\gamma \left(u^{m,h}\right)^2 dx\ dt. \quad (28)$$

Now choosing $\varepsilon = \frac{1}{C_4}$ from (26)-(27) we get

$$\int_{Q_T^m} (-t)^\gamma \left(\left(u^{m,h}\right)^2 + \left(u_x^{m,h}\right)^2\right) dx\ dt \le C_5(\lambda, n, d, \gamma) \times$$

$$\times \left(\int_{Q_T^m} (-t)^\gamma \sum_{k=1}^n \left(f^{k,h}\right)^2 dx\ dt + \int_{Q_T^m} (-t)^\gamma \left(f^h\right)^2 dx\ dt\right). \quad (29)$$

Without loss of generality, we can consider that for $f \ne 0;\ f^k \ne 0;\ k = \overline{1,n}$. Therefore from (28) it follows that for rather small $h > 0$

$$\left\|u^{m,k}\right\|_{W_{2,\gamma}^{1,0}(Q_T^m)} \le C_6(\lambda, n, d, \gamma) \left(\|f\|_{L_{2,\gamma}(Q_T)} + \sum_{k=1}^n \left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right). \quad (30)$$

Fix an arbitrary natural m. From inequality (29) it follows that the family of functions $\left\{u^{m,h}(x,t)\right\}$ is weakly compact (with respect to h) in the space $\overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T^m)$. Thus, there exists such a sequence $h_l \to 0$ as $l \to \infty$ and the function $u^m(x,t) \in \overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T^m)$ that the functional sequence $\left\{u^{m,h_l}(x,t)\right\}$ weakly converges to the function $u^m(x,t)$ in $\overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T^m)$

as $l \to \infty$. This means that for any function $u^m(x,t) \in \overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T^m)$ it holds the limit equality

$$\lim_{l\to\infty} B_{Q_T^m}\left(u^{m,h_l}, v\right) = B_{Q_T^m}(u^m, v)\ . \tag{31}$$

But on the other hand

$$B_{Q_T^m}\left(u^{m,h_l}, v\right) = \int_{Q_T^m} (-t)^\gamma f^{h_l}\ v\ dx\ dt - \int_{Q_T^m} (-t)^\gamma \sum_{k=1}^n f^{k,h_l} v_k dx\ dt. \tag{32}$$

Furthermore

$$\lim_{l\to\infty}\left(\int_{Q_T^m} (-t)^\gamma f^{h_l}\ v\ dx\ dt - \int_{Q_T^m} (-t)^\gamma \sum_{k=1}^n f^{k,h_l} v_k\ \ dx\ dt\right) =$$

$$= \int_{Q_T^m} (-t)^\gamma f\ v\ dx\ dt - \int_{Q_T^m} (-t)^\gamma \sum_{k=1}^n f^k v_k dx\ dt. \tag{33}$$

From (30)-(32) we conclude that

$$B_{Q_T^m}(u^m, v) = \int_{Q_T^m} (-t)^\gamma f\ v\ dx\ dt - \int_{Q_T^m} (-t)^\gamma \sum_{k=1}^n f^k v_k dx\ dt.$$

The last equality means that the function $u^m(x,t)$ is a weak solution of equation (3) in the domain Q_T^m. Furthermore, for the function $u^m(x,t)$ the following estimation is valid

$$\|u^m\|_{W^{1,0}_{2,\gamma}(Q_T^m)} \le C_7(\lambda, n, d, \gamma)\left(\|f\|_{L_{2,\gamma}(Q_T)} + \sum_{k=1}^n \left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right). \tag{34}$$

For any natural m we continue the function $u^m(x,t)$ by a zero in $Q_T \backslash Q_T^m$ and denote the obtained continuation again by $u^m(x,t)$. It is easy to see that $u^m(x,t) \in \overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T)$. Therewith, according to (33) the following estimation is valid

$$\|u^m\|_{W^{1,0}_{2,\gamma}(Q_T)} \le C_8\left(\|f\|_{L_{2,\gamma}(Q_T)} + \sum_{k=1}^n \left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right). \tag{35}$$

From (34) it follows that the family of functions $\{u^m(x,t)\}, \ldots .m = 1, 2, \ldots$ is weakly compact in the space $\overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T)$. Thus, there exists such a function $u(x,t) \in \overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T^m)$ and sequence $m_r \to \infty$ as $r \to \infty$ that $u(x,t)$ is a weak limit of $u^{m_r}(x,t)$ as $r \to \infty$in $\overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T^m)$. This means that for any function $v(x,t) \in \overset{0}{W}{}^{1,0}_{2,\gamma}$ the following limit equality is valid:

$$\lim_{r\to\infty} B_{Q_T}(u^{m_r}, v) = B_{Q_T}(u, v)\ .$$

Moreover, using the above arguments, we can show that

$$B_{Q_T}(u,v)=\int_{Q_T^m}(-t)^{\gamma}f\ v\ dx\ dt-\int_{Q_T^m}(-t)^{\gamma}\sum_{k=1}^{n}f^k\ v\ dx\ dt.$$

From the last equality it follows that the function $u(x,t)$ is a weak solution of the first boundary value problem (3)-(4). Furthermore, for the functions $u(x,t)$ the following estimation is valid

$$\|u^m\|_{\overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T^m)}\le C_9\left(\|f\|_{L_{2,\gamma}(Q_T)}+\sum_{k=1}^{n}\left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right). \tag{36}$$

Thereby the existence of the weak solution of the first boundary value problem (3)-(4) is proved. Now prove its uniqueness. It suffices to show that a homogeneous problem has only a trivial solution. Let $u(x,t)$ be the solution of homogeneous problem (3)-(4), i.e. for $f\equiv 0;\quad f^k\equiv 0;\quad k=\overline{1,n}$. Fix an arbitrary $\delta\in(0,T)$ and consider the function $v(x,t)\in\overset{0}{W}{}^{1,1}_{2,\gamma}(Q_{T+\delta})$ vanishing for $t\le T$ and $t\ge-\delta$. Let further $K=\Pi_R\times(-T-\delta,0)$, $\Pi_R=\left\{x:\ |x_i|<R,\ i=\overline{1,n}\right\}$. Continue the function $u(x,t)$ and $v(x,t)$ by zero to $K\backslash Q_T$ and denote the obtained continuations again by $u(x,t)$ and $v(x,t)$, respectively. It is easy to see that $u(x,t)\in\overset{0}{W}{}^{1,1}_{2,\gamma}(K)$, while $v(x,t)\in\overset{0}{W}{}^{1,1}_{2,\gamma}(K)$, Denote for

$$h\in(0,\delta]\ \frac{1}{h}\int_{t-h}^{t}v(x,\tau)\ d\tau\quad\text{by}\qquad v_{\overline{h}}(x,\tau)$$

and put into integral identity (5) instead of the function $u(x,t)$ the function $v_{\overline{h}}(x,\tau)$ and get

$$B_K\left(u,v_{\overline{h}}\right)=0. \tag{37}$$

Taking into account the equalities$\left(v_{\overline{h}}\right)_t=(v_t)_{\overline{h}}$, $\left(v_{\overline{h}}\right)_i=(v_i)_{\overline{h}}\quad i=\overline{1,n}$and also

$$-\int_K(-t)^{\gamma}u\ (v_t)_{\overline{h}}\ dx\ dt=-\int_K((-t)^{\gamma}u)_h v_t dx\ dt=\int_K[((-t)^{\gamma}u)_h]_t v dx\ dt,$$

$$\int_K(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)u_iu_i\,(v_j)_{\overline{h}}\ dx\ dt=\int_K(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}u_i\right)v_j\ dx\ dt,$$

where $u_h(x,t)=\frac{1}{h}\int_t^{t+h}u(x,\tau)\,d\tau$, assuming $v(x,t)=u_h(x,t)$ tending h to zero, from (36) we get

$$\frac{\lambda\ n\,(n+1)-4\gamma}{8}\int_{Q_{T,\delta}}(-t)^{\gamma-1}u^2\ dx\ dt-\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)u_iu_j\ dx\ dt=0. \tag{38}$$

Now, behaving as in deriving estimation (24), we get that if with respect to number parameters λ and γ conditions (1) and (2) are fulfilled, then $\int_{Q_{T,\delta}} (-t)^{\gamma} u_x^2 \, dx \, dt = 0$. The last equality yields

$$\int_{Q_{T,\delta}} (-t)^{\gamma-1} u^2 \, dx \, dt = 0.$$

With regard to arbitrariness of δ we conclude

$$\int_{Q_T} (-t)^{\gamma-1} u^2 \, dx \, dt = 0.$$

Hence it follows that $u\,(x,t) = 0$ almost everywhere in Q_T. ◄

In fact in the course of proof we established the estimation of the weak solution of the first boundary value problem (3)-(4). We formulate this statement in the form of a separate theorem.

Theorem 2. *If the conditions of the previous theorem are fulfilled then for the weak solution of the first boundary value problem* (3)-(4), *estimation* (35) *is valid.*

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Characterization of Parabolic Fractional Integral and Its Commutators in Orlicz Spaces

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Abstract. In this paper, we characterize BMO space in terms of the boundedness of commutators of parabolic maximal operator in Orlicz spaces. As an application of this boundedness, we give necessary and sufficient condition for the boundedness of parabolic fractional integral and its commutators in Orlicz spaces.

Key Words and Phrases: Orlicz space, parabolic fractional integral, commutator, BMO.
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1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have positive real part. Let $A_t = t^P \quad (t > 0)$, and set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that

$$
\begin{aligned}
&(a) \quad \rho(A_t x) = t\rho(x), \quad t > 0, \quad \text{for every} \quad x \in \mathbb{R}^n;\\
&(b) \quad \rho(0) = 0, \quad \rho(x-y) = \rho(y-x) \geq 0\\
&\qquad \text{and} \quad \rho(x-y) \leq k(\rho(x-z) + \rho(y-z));\\
&(c) \quad dx = \rho^{\gamma-1} d\sigma(w) d\rho, \quad \text{where} \quad \rho = \rho(x), w = A_{\rho^{-1}} x
\end{aligned}
$$

and $d\sigma(w)$is a C^∞measure on the ellipsoid $\{w : \rho(w) = 1\}$.

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Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([2, 3]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x,r)| = v_\rho r^\gamma$, where v_ρ is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x,r) = \mathbb{R}^n \setminus \mathcal{E}(x,r)$ be the complement of $\mathcal{E}(x,r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x,r) = B(x,r)$. Note that in the standard parabolic case $P = (1,\ldots,1,2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$. The parabolic maximal function $M^P f$ and the parabolic fractional integral $I_\alpha^P f$, $0 < \alpha < \gamma$, of a function $f \in L_1^{\rm loc}(\mathbb{R}^n)$ are defined by

$$M^P f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y)| dy,$$

$$I_\alpha^P f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\rho(x-y)^{\gamma-\alpha}} dy.$$

If $P = I$, then $M \equiv M_0^I$ is the Hardy-Littlewood maximal operator. It is well known that, the parabolic maximal function and the parabolic fractional integral operators play an important role in harmonic analysis (see [4, 15]).

In this work we present the characterization for parabolic fractional integral operator I_α^P (Theorem 6) and its commutators $[b, I_\alpha^P]$ (Theorem 7) in Orlicz spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. On Young Functions and Orlicz Spaces

Orlicz space was first introduced by Orlicz in [12, 13] as a generalizations of Lebesgue spaces L^p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when L^1 space does not work.

First, we recall the definition of Young functions.

Definition 1. *A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.*

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0, \tag{1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\} & , \quad r \in [0, \infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \qquad r \ge 0$$

for some $C > 1$.

Definition 2. *(Orlicz Space). For a Young function Φ, the set*

$$L^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^{\Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^{\Phi}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$. The space $L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in L^{\Phi}(\mathbb{R}^n)$ for all parabolic balls $\mathcal{E} \subset \mathbb{R}^n$.

$L^{\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let $m(\Omega,\ f,\ t) = |\{x \in \Omega : |f(x)| > t\}|$. In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$.

Definition 3. *The weak Orlicz space*

$$WL^{\Phi}(\mathbb{R}^n) = \{f \in L^1_{\rm loc}(\mathbb{R}^n) : \|f\|_{WL^{\Phi}} < \infty\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi}} = \inf\left\{\lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \ \le 1\right\}.$$

We note that $\|f\|_{WL^\Phi} \le \|f\|_{L^\Phi}$,

$$\sup_{t>0} \Phi(t) m(\Omega,\ f,\ t) = \sup_{t>0} t\, m(\Omega,\ f,\ \Phi^{-1}(t)) = \sup_{t>0} t\, m(\Omega,\ \Phi(|f|),\ t)$$

and

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\Big) dx \le 1, \qquad \sup_{t>0} \Phi(t) m\Big(\Omega,\ \frac{f}{\|f\|_{WL^\Phi(\Omega)}},\ t\Big) \le 1, \tag{2}$$

where $\|f\|_{L^\Phi(\Omega)} = \|f\chi_\Omega\|_{L^\Phi}$ and $\|f\|_{WL^\Phi(\Omega)} = \|f\chi_\Omega\|_{WL^\Phi}$.

The following analogue of the Hölder's inequality is well known (see, for example, [14]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f and g measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_\Omega |f(x)g(x)| dx \le 2\|f\|_{L^\Phi(\Omega)} \|g\|_{L^{\tilde{\Phi}}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 1. *Let Φ be a Young function and $\mathcal{E}$ be a parabolic balls in $\mathbb{R}^n$. Then*

$$\|\chi_{\mathcal{E}}\|_{L^\Phi} = \|\chi_{\mathcal{E}}\|_{WL^\Phi} = \frac{1}{\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)}.$$

By Theorem 1, Lemma 1 and (1) we get the following estimate.

Lemma 2. *For a Young function Φ and for the parabolic balls $\mathcal{E} = \mathcal{E}(x,r)$ the following inequality is valid:*

$$\int_{\mathcal{E}} |f(y)| dy \le 2|\mathcal{E}| \Phi^{-1}\left(|\mathcal{E}|^{-1}\right) \|f\|_{L^\Phi(\mathcal{E})}.$$

In [1] the boundedness of the parabolic maximal operator M^P in Orlicz spaces $L^\Phi(\mathbb{R}^n)$ was obtained.

Theorem 2. *[1] Let Φ any Young function. Then the parabolic maximal operator M^P is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Phi(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ bounded in $L^\Phi(\mathbb{R}^n)$.*

We recall that the space $BMO(\mathbb{R}^n) = \{b \in L^1_{\rm loc}(\mathbb{R}^n)\ :\ \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}| dy < \infty,$$

where $b_{\mathcal{E}(x,r)} = \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} b(y) dy$. We will need the following properties of BMO-functions:

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}|^p dy\right)^{\frac{1}{p}}, \tag{3}$$

where $1 \le p < \infty$, and

$$\left|b_{\mathcal{E}(x,r)} - b_{\mathcal{E}(x,t)}\right| \le C\|b\|_* \ln \frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{4}$$

where C does not depend on b, x, r and t. We refer for instance to [9] and [10] for details on this space and properties.

The commutators generated by $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the parabolic maximal operator M^P is defined by

$$M_b^P(f)(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(x) - b(y)||f(y)|dy.$$

Next, we recall the notion of weights. Let w be a locally integrable and positive function on $\mathbb{R}^n$. The function w is said to be a Muckenhoupt A_1 weight if there exists a positive constant C such that for any ellipsoid $\mathcal{E}$

$$\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} w(x)dx \le C \text{ess}\,\inf_{x\in\mathcal{E}} w(x).$$

Lemma 3. *[6, Chapter 1] Let $\omega \in A_1$, then the reverse Hölder inequality holds, that is, there exist $q > 1$ such that*

$$\left(\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} w(x)^q dx\right)^{\frac{1}{q}} \lesssim \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} w(x)dx$$

for all ellipsoids $\mathcal{E}$.

Lemma 4. *Let Φ be a Young function with $\Phi \in \Delta_2$. Then we have*

$$\frac{1}{2|\mathcal{E}|} \int_{\mathcal{E}} |f(x)|dx \le \Phi^{-1}\big(|\mathcal{E}|^{-1}\big) \|f\|_{L^\Phi(\mathcal{E})} \lesssim \left(\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |f(x)|^p dx\right)^{\frac{1}{p}}$$

for some $1 < p < \infty$.

Proof. The left-hand side inequality is just Lemma 2.

Next we prove the right-hand side inequality. Our idea is based on [8]. Take $g \in L_{\widetilde{\Phi}}$ with $\|g\|_{L_{\widetilde{\Phi}}} \le 1$. Note that $\widetilde{\Phi} \in \nabla_2$ since $\Phi \in \Delta_2$, therefore M is bounded on $L_{\widetilde{\Phi}}(\mathbb{R}^n)$ from Theorem 2. Let $Q := \|M\|_{L_{\widetilde{\Phi}} \to L_{\widetilde{\Phi}}}$ and define a function

$$Rg(x) := \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2Q)^k},$$

where

$$M^k g := \begin{cases} |g| & k = 0, \\ Mg & k = 1, \\ M(M^{k-1}g) & k \ge 2. \end{cases}$$

For every $g \in L_{\widetilde{\Phi}}$ with $\|g\|_{L_{\widetilde{\Phi}}} \le 1$, the function Rg satisfies the following properties:

- $|g(x)| \leq Rg(x)$ for almost every $x \in \mathbb{R}^n$;
- $\|Rg\|_{L_{\widetilde{\Phi}}} \leq 2\|g\|_{L_{\widetilde{\Phi}}}$
- $M(Rg)(x) \leq 2QRg(x)$, that is, Rg is a Muckenhoupt A_1 weight with the A_1 constant less than or equal to $2Q$.

By Lemma 3, there exist positive constants $q > 1$ and C independent of g such that for all ellipsoids $\mathcal{E}$,

$$\left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} Rg(x)^q dx\right)^{\frac{1}{q}} \leq \frac{C}{|\mathcal{E}|}\int_{\mathcal{E}} Rg(x) d\mu(x).$$

By Lemmas 2 and 3, we obtain

$$\|Rg\|_{L^q(\mathcal{E})} = |\mathcal{E}|^{1/q}\left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} Rg(x)^q dx\right)^{\frac{1}{q}} \lesssim |\mathcal{E}|^{1/q}\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} Rg(x) dx$$
$$\lesssim |\mathcal{E}|^{-1/q'}\frac{\|Rg\|_{L_{\widetilde{\Phi}}}}{\Phi^{-1}\big(|\mathcal{E}|^{-1}\big)} \lesssim \frac{|\mathcal{E}|^{-1/q'}}{\Phi^{-1}\big(|\mathcal{E}|^{-1}\big)}.$$

Thus we have

$$\int_{\mathcal{E}} |f(x)g(x)|dx \leq \int_{\mathcal{E}} |f(x)|Rg(x)dx \leq \|f\|_{L_{q'}(\mathcal{E})}\|Rg\|_{L_q(\mathcal{E})}$$
$$\lesssim \left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |f(x)|^{q'} dx\right)^{\frac{1}{q'}} \frac{1}{\Phi^{-1}\big(|\mathcal{E}|^{-1}\big)}.$$

Since the Luxemburg-Nakano norm is equivalent to the Orlicz norm (see, for example [14, p. 61]) we get

$$\|f\|_{L^\Phi(\mathcal{E})} \leq \sup\left\{\left|\int_{\mathcal{E}} f(x)g(x)dx\right| : g \in L_{\widetilde{\Phi}}, \quad \|g\|_{L_{\widetilde{\Phi}}} \leq 1\right\}$$
$$\lesssim \left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |f(x)|^{q'} dx\right)^{\frac{1}{q'}} \frac{1}{\Phi^{-1}\big(|\mathcal{E}|^{-1}\big)}.$$

Consequently, the right-hand side inequality follows with $p = q'$.

We have the following result from (3) and Lemma 4.

Lemma 5. *Let $b \in BMO(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$. Then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-\gamma}\big)\left\|b(\cdot) - b_{\mathcal{E}(x,r)}\right\|_{L^\Phi(\mathcal{E}(x,r))}.$$

The known boundedness statements for the commutator operator M_b^P on Orlicz spaces run as follows, see [5, Corollary 2.3].

Theorem 3. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $b \in BMO(\mathbb{R}^n)$. Then M_b^P is bounded on $L^\Phi(\mathbb{R}^n)$ and the inequality*

$$\|M_b^P f\|_{L^\Phi} \leq C_0\|b\|_*\|f\|_{L^\Phi} \tag{5}$$

holds with constant C_0 independent of f.

3. Parabolic fractional integral and its commutators in Orlicz spaces

For proving our main results, we need the following estimate.

Lemma 6. *If $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$, then for every $x \in \mathcal{E}_0$*

$$c_0 r_0^\alpha < I_\alpha^P \chi_{\mathcal{E}_0}(x),$$

where $c_0 = (2k)^{\alpha-\gamma}|\mathcal{E}(0,1)|$.

Proof. If $x, y \in \mathcal{E}_0$, then $\rho(x-y) \le k(\rho(x-x_0)+\rho(y-x_0)) < 2kr_0$. Since $0 < \alpha < \gamma$, we get $(2kr_0)^{\alpha-\gamma} < \rho(x-y)^{\alpha-\gamma}$. Therefore

$$I_\alpha^P \chi_{\mathcal{E}_0}(x) = \int_{\mathcal{E}_0} \rho(x-y)^{\alpha-\gamma} dy > (2kr_0)^{\alpha-\gamma}|\mathcal{E}_0| = c_0 r_0^\alpha.$$

The known boundedness statement for I_α^P in Orlicz spaces on spaces of homogeneous type runs as follows.

Theorem 4. *[11] Let $\Phi, \Psi \in \mathcal{Y}$ and*

$$\int_r^\infty t^{\alpha-1}\Phi^{-1}\left(t^{-\gamma}\right) dt \lesssim r^\alpha \Phi^{-1}\left(r^{-\gamma}\right) \qquad \text{for } 0 < r < \infty, \tag{6}$$

$$r^\alpha \Phi^{-1}\left(r^{-\gamma}\right) \lesssim \Psi^{-1}\left(r^{-\gamma}\right) \qquad \text{for } 0 < r < \infty. \tag{7}$$

Then I_α^P is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then I_α^P is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theorem 5. *Let $\Phi, \Psi \in \mathcal{Y}$ and I_α^P is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ then condition (7) holds.*

Proof. Let $\mathcal{E}_0 = \mathcal{E}(x_0, r_0)$ and $x \in \mathcal{E}_0$. By Lemmas 6 and 1, we have

$$\begin{aligned} r_0^\alpha &\lesssim \Psi^{-1}(r_0^{-\gamma})\|I_\alpha^P \chi_{\mathcal{E}_0}\|_{WL^\Psi(\mathcal{E}_0)} \lesssim \Psi^{-1}(r_0^{-\gamma})\|I_\alpha^P \chi_{\mathcal{E}_0}\|_{WL^\Psi} \\ &\lesssim \Psi^{-1}(r_0^{-\gamma})\|\chi_{\mathcal{E}_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-\gamma})}{\Phi^{-1}(r_0^{-\gamma})}. \end{aligned}$$

Since this is true for every $r_0 > 0$, we are done.

Combining Theorems 4 and 5 we have the following result.

Theorem 6. *Let $\Phi, \Psi \in \mathcal{Y}$. If (6) holds, then the condition (7) is necessary and sufficient for the boundedness of I_α^P from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition (7) is necessary and sufficient for the boundedness of I_α^P from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Remark 1. *Note that Theorem 6 in the isotropic case $P = I$ were proved in [7].*

The commutators $[b, I_\alpha^P]$, $|b, I_\alpha^P|$ generated by $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and the operator I_α^P are defined by

$$[b, I_\alpha^P]f(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{\rho(x-y)^{\gamma-\alpha}} f(y)dy,$$

$$|b, I_\alpha^P|f(x) = \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|}{\rho(x-y)^{\gamma-\alpha}} f(y)dy, \qquad 0<\alpha<\gamma,$$

respectively.

The following lemma is the analogue of the Hedberg's trick for $[b, I_\alpha]$.

Lemma 7. *If $0<\alpha<\gamma$ and $f, b \in L^1_{\rm loc}(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$ and $r>0$ we get*

$$|b, I_\alpha^P|(\chi_{\mathcal{E}(x,r)}|f|)(x) \lesssim r^\alpha M_b^P f(x).$$

Proof.

$$|b, I_\alpha^P|(\chi_{\mathcal{E}(x,r)}|f|)(x) = \int_{\mathcal{E}(x,r)} \frac{|f(y)|}{\rho(x-y)^{\gamma-\alpha}} |b(x)-b(y)|dy$$

$$= \sum_{j=0}^{\infty} \int_{\mathcal{E}(x,2^{-j}r)\setminus\mathcal{E}(x,2^{-j-1}r)} \frac{|f(y)|}{\rho(x-y)^{\gamma-\alpha}} |b(x)-b(y)|dy$$

$$\lesssim \sum_{j=0}^{\infty} (2^{-j}r)^\alpha (2^{-j}r)^{-\gamma} \int_{\mathcal{E}(x,2^{-j}r)} |f(y)||b(x)-b(y)|dy \lesssim r^\alpha M_b^P f(x).$$

Lemma 8. *If $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$, then*

$$r_0^\alpha |b(x) - b_{\mathcal{E}_0}| \le C|b, I_\alpha^P|\chi_{\mathcal{E}_0}(x)$$

for every $x \in \mathcal{E}_0$.

Proof. The proof is similar to the proof of Theorem 6.

Theorem 7. *Let $0<\alpha<\gamma$, $b \in BMO(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.*

1. *If $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$, then the condition*

$$r^\alpha \Phi^{-1}\big(r^{-\gamma}\big) + \int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1}dt \le C\Psi^{-1}\big(r^{-\gamma}\big) \tag{8}$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha^P]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

2. *If $\Psi \in \Delta_2$, then the condition (7) is necessary for the boundedness of $|b, I_\alpha^P|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

3. *Let $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$. If the condition*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1}dt \le Cr^\alpha\Phi^{-1}\big(r^{-\gamma}\big) \tag{9}$$

holds for all $r>0$, where $C>0$ does not depend on r, then the condition (7) is necessary and sufficient for the boundedness of $|b, I_\alpha^P|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Proof. (1) For arbitrary $x_0 \in \mathbb{R}^n$, set $\mathcal{E} = \mathcal{E}(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2k\mathcal{E}}$ and $f_2 = f\chi_{\complement_{(2k\mathcal{E})}}$, where k is the constant from the triangle inequality.

For $x \in \mathcal{E}$ we have

$$
\begin{aligned}
|[b, I^P_\alpha]f_2(x)| &\lesssim \int_{\mathbb{R}^n} \frac{|b(y)-b(x)|}{\rho(x-y)^{\gamma-\alpha}}|f_2(y)|dy \approx \int_{\complement_{(2k\mathcal{E})}} \frac{|b(y)-b(x)|}{\rho(y-x_0)^{\gamma-\alpha}}|f(y)|dy \\
&\lesssim \int_{\complement_{(2k\mathcal{E})}} \frac{|b(y)-b_{\mathcal{E}}|}{\rho(y-x_0)^{\gamma-\alpha}}|f(y)|dy + \int_{\complement_{(2k\mathcal{E})}} \frac{|b(x)-b_{\mathcal{E}}|}{\rho(y-x_0)^{\gamma-\alpha}}|f(y)|dy \\
&= J_1 + J_2(x),
\end{aligned}
$$

since $x \in \mathcal{E}$ and $y \in {}^{\complement}(2k\mathcal{E})$ implies

$$
\frac{1}{2k}\rho(y-x_0) \le \rho(x-y) \le \Big(k+\frac{1}{2}\Big)\rho(y-x_0).
$$

Let us estimate J_1.

$$
\begin{aligned}
J_1 = \int_{\complement_{(2k\mathcal{E})}} \frac{|b(y)-b_{\mathcal{E}}|}{\rho(y-x_0)^{\gamma-\alpha}}|f(y)|dy &\approx \int_{\complement_{(2k\mathcal{E})}} |b(y)-b_{\mathcal{E}}||f(y)| \int_{\rho(y-x_0)}^{\infty} \frac{dt}{t^{\gamma+1-\alpha}}dy \\
\approx \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)\setminus(2k\mathcal{E})} |b(y)-b_{\mathcal{E}}||f(y)|dy \frac{dt}{t^{\gamma+1-\alpha}} \\
\lesssim \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |b(y)-b_{\mathcal{E}}||f(y)|dy \frac{dt}{t^{\gamma+1-\alpha}}.
\end{aligned}
$$

Applying Hölder's inequality, by (1), (4), (5) and Lemma 2 we get

$$
\begin{aligned}
J_1 &\lesssim \int_{2r}^{\infty} \int_{\mathcal{E}(x_0,t)} |b(y)-b_{\mathcal{E}(x_0,t)}||f(y)|dy \frac{dt}{t^{\gamma+1-\alpha}} \\
&\quad + \int_{2r}^{\infty} |b_{\mathcal{E}(x_0,r)} - b_{\mathcal{E}(x_0,t)}| \int_{\mathcal{E}(x_0,t)} |f(y)|dy \frac{dt}{t^{\gamma+1-\alpha}} \\
&\lesssim \int_{2r}^{\infty} \left\| b(\cdot) - b_{\mathcal{E}(x_0,t)} \right\|_{L_{\widetilde{\Phi}}(\mathcal{E}(x_0,t))} \|f\|_{L_\Phi(\mathcal{E}(x_0,t))} \frac{dt}{t^{\gamma+1-\alpha}} \\
&\quad + \int_{2r}^{\infty} |b_{\mathcal{E}(x_0,r)} - b_{\mathcal{E}(x_0,t)}| \|f\|_{L_\Phi(\mathcal{E}(x_0,t))} \Phi^{-1}\big(|\mathcal{E}(x_0,t)|^{-1}\big) \frac{dt}{t^{1-\alpha}} \\
&\lesssim \|b\|_* \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_\Phi(\mathcal{E}(x_0,t))} \Phi^{-1}\big(|\mathcal{E}(x_0,t)|^{-1}\big) \frac{dt}{t^{1-\alpha}}. \\
&\lesssim \|b\|_* \|f\|_{L^\Phi} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1} dt.
\end{aligned}
$$

A geometric observation shows $2k\mathcal{E} \subset \mathcal{E}(x, \delta)$ for all $x \in \mathcal{E}$, where $\delta = (2k+1)kr$. Using Lemma 7, we get

$$
J_0(x) := |[b, I^P_\alpha]f_1(x)| \lesssim \int_{2k\mathcal{E}} \frac{|b(y)-b(x)|}{\rho(x-y)^{\gamma-\alpha}}|f(y)|dy
$$

$$\lesssim \int_{\mathcal{E}(x,\delta)} \frac{|b(y)-b(x)|}{\rho(x-y)^{\gamma-\alpha}}|f(y)|dy \lesssim r^{\alpha} M_b^P f(x).$$

Consequently, we have

$$J_0(x)+J_1 \lesssim \|b\|_* r^{\alpha} M_b^P f(x) + \|b\|_* \|f\|_{L^\Phi} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-\gamma}\big)\; t^{\alpha-1}dt.$$

Thus, by (8) we obtain

$$J_0(x)+J_1 \lesssim \|b\|_* \left(M_b^P f(x) \frac{\Psi^{-1}(r^{-\gamma})}{\Phi^{-1}(r^{-\gamma})} + \Psi^{-1}(r^{-\gamma})\|f\|_{L^\Phi}\right).$$

Choose $r>0$ so that $\Phi^{-1}(r^{-\gamma}) = \frac{M_b^P f(x)}{C_0\|b\|_*\|f\|_{L^\Phi}}$. Then

$$\frac{\Psi^{-1}(r^{-\gamma})}{\Phi^{-1}(r^{-\gamma})} = \frac{(\Psi^{-1}\circ\Phi)\big(\frac{M_b^P f(x)}{C_0\|b\|_*\|f\|_{L^\Phi}}\big)}{\frac{M_b f(x)}{C_0\|b\|_*\|f\|_{L^\Phi}}}.$$

Therefore, we get

$$J_0(x)+J_1 \le C_1\|b\|_*\|f\|_{L^\Phi}(\Psi^{-1}\circ\Phi)\Big(\frac{M_b^P f(x)}{C_0\|b\|_*\|f\|_{L^\Phi}}\Big).$$

Let C_0 be as in (5). Consequently by Theorem 3 we have

$$\int_{\mathcal{E}} \Psi\left(\frac{J_0(x)+J_1}{C_1\|b\|_*\|f\|_{L^\Phi}}\right)dx \le \int_{\mathcal{E}} \Phi\left(\frac{M_b^P f(x)}{C_0\|b\|_*\|f\|_{L^\Phi}}\right)dx$$
$$\le \int_{\mathbb{R}^n} \Phi\left(\frac{M_b^P f(x)}{\|M_b^P f\|_{L^\Phi}}\right)dx \le 1,$$

i.e.

$$\|J_0(\cdot)+J_1\|_{L^\Psi(\mathcal{E})} \lesssim \|b\|_*\|f\|_{L^\Phi}. \tag{10}$$

In order to estimate J_2, by (5), Lemma 2 and condition (8), we also get

$$\|J_2\|_{L^\Psi(\mathcal{E})} = \left\| \int_{\complement(2k\mathcal{E})} \frac{|b(\cdot)-b_{\mathcal{E}}|}{\rho(y-x_0)^{\gamma-\alpha}}|f(y)|dy \right\|_{L^\Psi(\mathcal{E})}$$
$$\approx \|b(\cdot)-b_{\mathcal{E}}\|_{L^\Psi(\mathcal{E})} \int_{\complement(2k\mathcal{E})} \frac{|f(y)|}{\rho(y-x_0)^{\gamma-\alpha}}dy$$
$$\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{\complement(2k\mathcal{E})} \frac{|f(y)|}{\rho(y-x_0)^{\gamma-\alpha}}dy$$
$$\approx \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{\complement(2k\mathcal{E})} |f(y)| \int_{\rho(y-x_0)}^{\infty} \frac{dt}{t^{\gamma+1-\alpha}}dy$$

$$\approx \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)\setminus(2k\mathcal{E})} |f(y)|dy \frac{dt}{t^{\gamma+1-\alpha}}$$
$$\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2r}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)|dy \frac{dt}{t^{\gamma+1-\alpha}}$$
$$\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi(\mathcal{E}(x_0,t))} \Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1} dt$$
$$\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \|f\|_{L^\Phi} \int_{2r}^{\infty} \Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1} dt$$
$$\lesssim \|b\|_* \, \|f\|_{L^\Phi}.$$

Consequently, we have

$$\|J_2\|_{L^\Psi(\mathcal{E})} \lesssim \|b\|_* \, \|f\|_{L^\Phi}. \tag{11}$$

Combining (10) and (11), we get

$$\|[b, I_\alpha^P]f\|_{L^\Psi(\mathcal{E})} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{12}$$

By taking supremum over $\mathcal{E}$ in (12), we get

$$\|[b, I_\alpha^P]f\|_{L^\Psi} \lesssim \|b\|_* \|f\|_{L^\Phi},$$

since the constants in (12) don't depend on x_0 and r.

(2) We shall now prove the second part. Let $\mathcal{E}_0 = \mathcal{E}(x_0, r_0)$ and $x \in \mathcal{E}_0$. By Lemmas 8, 5 and 1 we have

$$r_0^\alpha \lesssim \frac{\||b, I_\alpha^P|\chi_{\mathcal{E}_0}\|_{L^\Psi(\mathcal{E}_0)}}{\|b(\cdot) - b_{\mathcal{E}_0}\|_{L^\Psi(\mathcal{E}_0)}} \lesssim \Psi^{-1}(r_0^{-\gamma})\||b, I_\alpha^P|\chi_{\mathcal{E}_0}\|_{L^\Psi(\mathcal{E}_0)}$$
$$\lesssim \Psi^{-1}(r_0^{-\gamma})\||b, I_\alpha^P|\chi_{\mathcal{E}_0}\|_{L^\Psi} \lesssim \Psi^{-1}(r_0^{-\gamma})\|\chi_{\mathcal{E}_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-\gamma})}{\Phi^{-1}(r_0^{-\gamma})}.$$

Since this is true for every $r_0 > 0$, we are done.

(3) The third statement of the theorem follows from the first and second parts of the theorem.

Remark 2. *Note that Theorem 7 in the isotropic case $P = I$ were proved in [7].*

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The Stability of Basis Properties of Multiple Systems in a Banach Space With Respect to Certain Transformations

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Abstract. In this paper a method for constructing a basis of a Banach space based on the bases of subspaces is proposed. The completeness, minimality, uniform minimality and basicity with the parentheses of the corresponding systems are also studied. The obtained abstract results are applied to the study of the basis properties of the eigenfunctions of a discontinuous differential operator of second order.

Key Words and Phrases: basis, completeness, minimality, uniformly minimality, discontinuous differential operator

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1. Introduction

The study of the spectral properties of some discrete differential operators leads to the development of new methods for constructing bases. In this regard, many mathematicians have paid attention to the study of basis properties (completeness, minimality, basicity) of systems of functions of special types, often being eigen and associated functions of differential operators. At the same time, various methods for studying these properties were proposed. Among such works are the works of the authors [1-6]. In the case of discontinuous differential operators, from eigenfunctions arise systems that for the study of the basicity the previously known methods are not applicable.

In this work is considered an abstract approach to the problem described above. The stability of the basis properties of multiple systems in a Banach space with respect to certain transformations is studied, a method for constructing a basis for the whole space is proposed, based on the bases of subspaces, which has wide application in the spectral theory of discontinuous differential operators.

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2. Necessary information

Recall the definitions of some notions from the theory of basis in a Banach space. Let X be a Banach space.

Definition 1. *The system $\{x_n\}_{n\in N} \subset X$ is called uniformly minimal in X, if*

$$\exists\delta > 0 : \inf_{\forall u \in L\left[\{x_n\}_{n\neq k}\right]} \|x_k - u\| \geq \delta \|x_k\| \ , \ \forall k \in N.$$

Definition 2. *If there exists a sequence of indexes, such that $\{n_k\}_{k\in N} \subset N : n_k < n_{k+1}$, $\forall k \in N$ and any element $x \in X$ is uniquely represented in the form*

$$x = \sum_{k=0}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} c_j x_j \quad (n_0 = 0),$$

then the system $\{x_n\}_{n\epsilon N} \subset X$ is called a basis with parentheses in X.

For $n_k = k$ the system $\{x_n\}_{n\in N}$ forms a usual basis for X.
We need the following easily proved statements.

Statement 1. *Let the system $\{x_n\}_{n\in N}$ form a basis with parentheses for X. If the system $\{x_n\}_{n\in N}$ is uniformly minimal and the sequence $\{n_{k+1} - n_k\}_{k\in N}$ is bounded, then this system forms a usual basis for X.*

Statement 2. *Let the system $\{x_n\}_{n\in N}$ form a Riesz basis with parentheses for a Hilbert space X. If the sequence $\{n_{k+1} - n_k\}_{n\in N}$ is bounded and the following condition*

$$\sup_n \{\|x_n\| : \|v_n\|\} < \infty$$

holds, where $\{v_n\}_{n\in N}$ is a biorthogonal system, then $\{x_n\}_{n\in N}$ forms a usual Riesz basis for X.

Definition 3. *The basis $\{u_n\}_{n\in N}$ of Banach space X is called a p-basis, if for any $x \in X$ the condition*

$$\left(\sum_{n=1}^{\infty} |\langle x, \vartheta_n\rangle|^p\right)^{\frac{1}{p}} \leq M \|x\|,$$

holds, where $\{\vartheta_n\}_{n\in N}$ - is a biorthogonal system to $\{u_n\}_{n\in N}$.

Definition 4. *The sequences $\{u_n\}_{n\in N}$ and $\{\varphi_n\}_{n\in N}$ of Banach space X are called a p-close, if the condition*

$$\sum_{n=1}^{\infty} \|u_n - \varphi_n\|^p < \infty,$$

holds.

We will also use the following results from [3,5] (see, also [6-8]).

Theorem 1. *[3] Let $\{x_n\}_{n\in N}$ form a q-basis for a Banach space X, and the system $\{y_n\}_{n\in N}$ is p-close to $\{x_n\}_{n\in N}$, where $\frac{1}{p}+\frac{1}{q}=1$. Then the following properties are equivalent:*

i) $\{y_n\}_{n\in N}$-is complete in X;

ii) $\{y_n\}_{n\in N}$-is minimal in X;

iii) $\{y_n\}_{n\in N}$-forms an isomorphic basis to $\{x_n\}_{n\in N}$ for X.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system and $\left\{\hat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ be its biorthogonal system:

$$\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm}); \quad \hat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm}).$$

Let $J = \{n_1, ..., n_m\}$ be some set of m natural numbers. Suppose

$$\delta = \det \|\beta_{n_i j}\|_{i,j=\overline{1,m}}.$$

The following theorem is true.

Theorem 2. *[5] Let the system $\{\hat{u}_n\}_{n\in N}$ form a basis for X_1. In order to the system $\{u_n\}_{n\in N_J}$, where $N_J = N\backslash J$ form a basis for X it is necessary and sufficient that the condition $\delta \neq 0$ be satisfied. In this case the biorthogonal system to $\{u_n\}_{n\in N_J}$ is defined by*

$$\vartheta_n^* = \frac{1}{\delta}\begin{vmatrix} \vartheta_n & \vartheta_{n1} & \dots & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & \dots & \beta_{n_m 1} \\ \dots & \dots & \dots & \dots \\ \beta_{nm} & \beta_{n_1 m} & \dots & \beta_{n_m m} \end{vmatrix}.$$

In particular, if $X$is a Hilbert space and the system $\{u_n\}_{n\in N}$ forms a Riesz basis for X_1,then under the condition $\delta \neq 0$, the system $\{u_n\}_{n\in N_J}$ also forms a Riesz basis for X. For $\delta = 0$ the system $\{u_n\}_{n\in N_J}$is not complete and is not minimal in X.

3. Stability of the basis properties of systems

Suppose that the direct decomposition $X = X_1 \oplus \ldots \oplus X_m$ holds, where X_i,$i = \overline{1,m}$ are Banach spaces. For convenience, the elements of X are identified with vectors: $x \in X \Leftrightarrow x = (x_1; ...; x_m)$, where $x_k \in X_k$, $k = \overline{1,m}$. The norm in X is defined by the formula $\|x\|_X = \sqrt{\sum_{i=1}^m \|x_i\|_{X_i}^2}$. It is clear that $X^* = X_1^* \oplus \ldots \oplus X_m^*$ and for $f \in X^*$ and $x \in X$ it holds $< x; f >= \sum_{i=1}^m < x_i; f_i >$ ($< \cdot\, ; \, \cdot >$ $-$is the value of the functional), where $f = (f_1, ..., f_m)$, $f_k \in X_k^*$, $k = \overline{1,m}$. For $x_k \in X_k$ let us denote by $\tilde{x}_k$ the element from X, which is defined by the formula $\tilde{x}_k = \Bigg(\underbrace{0, ..., x_k}_{k}, ..., 0\Bigg)$.

Suppose that a system $\{u_{in}\}_{n\in N}$ is given in each space X_i, $i=\overline{1,m}$, Consider the following system in X:

$$\hat{u}_{in}=(a_{i1}^{(n)}u_{1n},...,a_{im}^{(n)}u_{mn}), i=\overline{1,m}, n\in N, \tag{1}$$

where $a_{ij}^{(n)}$ −are some numbers. Let $A_n=\left(a_{ij}^{(n)}\right)_{i,\,j=\overline{1,\,m}}$; $\Delta_n=\det A_n$.

The following theorem is proved.

Theorem 3. *Let the system* $\{u_{in}\}_{n\in N}$ *be complete (minimal) in* $X_i, i=\overline{1,m}$. *If* $\Delta_n\neq 0\,, \forall n\in N$, *then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *is also complete (minimal) in* X.

Proof. Let the system $\{u_{in}\}_{n\in N}$ be complete (minimal) in X_i, $i=\overline{1,m}$. If for any $\vartheta\in X^*$

$$<\hat{u}_{in},\,\vartheta>=0\,,\;\; i=\overline{1,\,m}, n\in N,$$

then from the representation $X^*=X_1^*\dot{\oplus}\ldots\oplus X_m^*$ and $\vartheta=(\vartheta_1,...,\vartheta_m)^t$, $\vartheta_i\in X_i^*$, $i=\overline{1,m}$, implies

$$\sum_{j=1}^{m}a_{ij}<u_{jn},\,\vartheta_j>=0\,,\quad i=\overline{1,\,m}. \tag{2}$$

Since $\Delta_n=\det\left(a_{ij}^{(n)}\right)\neq 0$, $n\in N$, then (2) has only trivial solution for each $n\in N$:

$$<u_{jn},\,\vartheta_j>=0,\;\; j=\overline{1,\,m}\,,\;\; n\in N.$$

Then from the completeness of the system $\{u_{jn}\}_{n\in N}$ in X_j implies that $\vartheta_j=0$, $j=\overline{1,\,m}$, i.e. $\vartheta=0$.

Now let the system $\{u_{in}\}_{n\in N}$ be minimal in X_i, and $\{\vartheta_{in}\}_{n\in N}\subset X_i^*$ be conjugate-biorthogonal system. Consider the following system in X^*

$$\hat{\vartheta}_{in}=\left(b_{1i}^{(n)}\vartheta_{1n};b_{2i}^{(n)}\vartheta_{2n};...;b_{mi}^{(n)}\vartheta_{mn}\right)=\sum_{s=1}^{m}b_{si}^{(n)}\tilde{\vartheta}_{sn},\quad i=\overline{1,\,m}\,,\;\; n\in N,$$

where the numbers $b_{ji}^{(n)}$− are the elements of the inverse matrix A_n^{-1}. We obtain

$$<\hat{u}_{in},\,\hat{\vartheta}_{lk}>=\sum_{j=1}^{m}\sum_{s=1}^{m}a_{ij}^{(n)}b_{sl}^{(k)}<\tilde{u}_{jn},\,\tilde{\vartheta}_{sk}>=$$

$$=\sum_{j=1}^{m}a_{ij}^{(n)}b_{jl}^{(k)}<u_{jn},\vartheta_{jk}>=\sum_{j=1}^{m}a_{ij}^{(n)}b_{jl}^{(k)}\delta_{nk}=\sum_{j=1}^{m}a_{ij}^{(n)}b_{jl}^{(n)}\delta_{nk}=\delta_{il}\delta_{nk}\,,\, i;l=\overline{1,m};n;k\in N.$$

The last expressions mean that the system $\left\{\hat{\vartheta}_{in}\right\}_{i=\overline{1,m};n\in N}$ is conjugated to the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$, i.e. the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ is minimal in X.

Theorem is proved.

Theorem 4. *Let the system* $\{u_{in}\}_{n\in N}$ *be minimal in* X_i, $i=\overline{1,m}$. *If* $\exists n_0 \in N$, $\Delta_{n_0}=0$ *then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *is not minimal in* X.

Proof. Let for any $n_0 \in N$, $\Delta_{n_0}=0$. We will show that the system $\{\hat{u}_{in_0}\}_{i=\overline{1,m}}$ is linear dependent. From the condition $\det\left(a_{ij}^{(n_0)}\right)=0$ implies that, there are numbers c_i, $i=\overline{1,\, m}$, which not all equal to zero and such that

$$\sum_{i=1}^{m} a_{ij}^{(n_0)} c_i = 0\,,\; j=\overline{1,m}.$$

Then

$$\sum_{i=1}^{m} c_i \hat{u}_{in_0} = \sum_{i=1}^{m} c_i \sum_{j=1}^{m} a_{ij}^{(n_0)} \tilde{u}_{jn_0} =$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{m} a_{ij}^{(n_0)} c_i \right) \tilde{u}_{jn_0} = 0\,.$$

Thus, the system $\{\hat{u}_{in_0}\}_{i=\overline{1,m}}$ is linear dependent, consequently, all of the systems $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ are linear dependent and especially are not minimal. Theorem is proved.

Theorem 5. *Let the system* $\{u_{in}\}_{n\in N}$ *be complete and minimal in* X_i *, for each* $i \in 1:m$. *If* $\exists n_0 \in N$, $\Delta_{n_0}=0$, *then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *is not complete and is not minimal in* X.

Proof. Non-minimality of the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ in X implies from the previous theorem. We will show that, it is not complete in X. From the condition $\Delta_{n_0} = \det\left(a_{ij}^{(n_0)}\right)=0$ implies that, there are numbers c_j, $j=\overline{1,\, m}$, which not all are equal to zero such that

$$\sum_{j=1}^{m} a_{ij}^{(n_0)} c_j = 0\,,\; j=\overline{1,m}.$$

Suppose

$$\tilde{u}_{jn} = \left(\underbrace{0,\, \ldots,\, u_{jn}}_{j},\, \ldots,\, 0 \right) \in X,\;\; j=\overline{1,\, m}.$$

Then the system $\{\tilde{u}_{jn}\}_{j=\overline{1,m};\;\; n\in N}$ is complete and minimal in X, and its conjugated system is in the following form

$$\tilde{\vartheta}_{jn} = \left(\underbrace{0,\, \ldots,\, \vartheta_{jn}}_{j},\, \ldots,\, 0 \right), \quad j=\overline{1,m};\;\; n\in N,$$

where $\left\{\vartheta_{jn}\right\}_{n\in N} \subset X_j^*$−is conjugate system to $\left\{u_{jn}\right\}_{n\in N}$. Consider the following functional

$$\vartheta_0 = \sum_{s=1}^{m} c_s \tilde{\vartheta}_{sn_0}.$$

It is clear that $\vartheta_0 \in X^*$ and $\vartheta_0 \neq 0$. We will show that the functional , ϑ_0 annuls the system $\{\hat{u}_{in}\}$. Indeed, for $n = n_0$ we obtain

$$< \hat{u}_{in_0}, \vartheta_0 >= \sum_{j=1}^{m} a_{ij}^{(n_0)} < \tilde{u}_{jn_0}, \vartheta_0 >= \sum_{j=1}^{m} a_{ij}^{(n_0)} \sum_{s=1}^{m} c_s < \tilde{u}_{jn_0}, \tilde{\vartheta}_{sn_0} >=$$

$$= \sum_{j=1}^{m} a_{ij}^{(n_0)} \sum_{s=1}^{m} c_s \delta_{js} = \sum_{j=1}^{m} a_{ij}^{(n_0)} c_j = 0.$$

For $n \neq n_0$ we have

$$< \tilde{u}_{in}, \vartheta_0 >= \sum_{j=1}^{m} a_{ij}^{(n)} \sum_{s=1}^{m} c_s < \tilde{u}_{jn}, \tilde{\vartheta}_{sn_0} >= 0.$$

Thus, the system $\{\hat{u}_{in}\}_{i=\overline{1,m};\ n\in N}$ is not complete in X. Theorem is proved.

Theorem 6. *If all* $\Delta_n = \det\left(a_{ij}^{(n)}\right) \neq 0$*,* $n \in N$*, and for each* $i \in 1 : m$ *the system* $\{u_{in}\}_{n\in N}$ *forms a basis in* X_i *, then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *forms a basis with parentheses in* X*. If, the conditions*

$$\sup_n \{\|u_{in}\|; \|\vartheta_{in}\|\} < +\infty, i = \overline{1,m}, \sup_n \left\{\|A_n\|, \left\|A_n^{-1}\right\|\right\} < +\infty, \tag{3}$$

also hold, then the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *forms a usual basis in* X*.*

Proof. Let us present the system $\{\hat{u}_{in}\}$ in the following form

$$\hat{u}_{in} = \sum_{j=1}^{m} a_{ij}^{(n)} \tilde{u}_{jn}\,,\ i = \overline{1,m}\,;\ n \in N. \tag{4}$$

As shown above, the conjugated system is in the following form

$$\hat{\vartheta}_{in} = \sum_{j=1}^{m} b_{li}^{(n)} \tilde{\vartheta}_{l\,n}, l = \overline{1,m}; n \in N, \tag{5}$$

where the numbers b_{ji} are the elements of the inverse matrix A^{-1}. Hence we get (for $x \in X$)

$$\sum_{i=1}^{m} < x, \hat{\vartheta}_{in} > \hat{u}_{in} = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=1}^{m} a_{ij}^{(n)} b_{li}^{(n)} < x, \tilde{\vartheta}_{l\,n} > \tilde{u}_{jn} =$$

$$=\sum_{j=1}^{m}\sum_{l=1}^{m}\left(\sum_{i=1}^{m}b_{li}^{(n)}a_{ij}^{(n)}\right)< x,\,\tilde{\vartheta}_{l\,n} > \tilde{u}_{jn} =$$

$$=\sum_{j=1}^{m}\sum_{l=1}^{m}\delta_{lj} < x,\,\tilde{\vartheta}_{l\,n} > \tilde{u}_{jn} = \sum_{j=1}^{m} < x,\,\tilde{\vartheta}_{jn} > \tilde{u}_{jn}.$$

Consequently

$$S_N\left(x\right)=\sum_{n=1}^{N}\sum_{i=1}^{m} < x,\,\hat{\vartheta}_{in} > \hat{u}_{in} = \sum_{n=1}^{N}\sum_{j=1}^{m} < x,\,\tilde{\vartheta}_{jn} > \tilde{u}_{jn} =$$

$$=\sum_{j=1}^{m}\sum_{n=1}^{N} < x,\,\tilde{\vartheta}_{jn} > \tilde{u}_{jn} \to x, \quad \text{as } N\to\infty.$$

Thus, the system $\{\hat{u}_{in}\}_{i=\overline{1,m}\,;\,n\in N}$ forms a basis with parentheses in X.

Now let us assume that the condition (3) be fulfilled. Then

$$\sup_{i,\,n}\left\{\|\tilde{u}_{in}\|\ ;\ \left\|\tilde{\vartheta}_{in}\right\|\right\} < +\infty, i=\overline{1,m},$$

And from the representations (4) and (5) we obtain

$$\sup_{i,\,n}\left\{\|\hat{u}_{in}\|\ ;\ \left\|\hat{\vartheta}_{in}\right\|\right\} < +\infty.$$

Consequently, the system $\{\hat{u}_{in}\}$ is uniformly minimal and by Statement 1 it forms a usual bases in X.

Theorem 7. *If $X_i-$are Hilbert spaces, and $\{u_{in}\}_{n\in N}$ is a Riesz basis in X_i, $i=\overline{1,m}$, then for $\Delta_n\neq 0$, $n\in N$, the system $\{\hat{u}_{in}\}_{i=\overline{1,m}\,;\,n\in N}$ forms Riesz basis with parentheses in X, and under the condition* (3) *it forms a usual Riesz basis in X.*

Proof of the theorem implies from the Theorem 6 and Statement 2. Note that, in particular, when the matrixes A_n do not depend on n: $A_n=A,\ n\in N$, the similar results were obtained in [9,10].

4. Application to discontinuous differential operators

Consider the following model spectral problem for a second-order discontinuous differential operator

$$-y''\left(x\right)+q\left(x\right)y=\lambda y\left(x\right),\ x\in(-1,\,0)\bigcup(0,\,1)\,, \tag{6}$$

with boundary conditions

$$y\left(-1\right)=y\left(1\right)=0,$$

$$y(-0) = y(+0), \tag{7}$$
$$y'(-0) - y'(+0) = \lambda m y(0).$$

where $m \neq 0-$ is any complex number, $q(x)$ - summable complex-valued function. Such spectral problems arise when the problem of vibrations of a loaded in the middle of the string with fixed ends is solved by applying the Fourier method [11,12]. The justification of the Fourier method requires the study of the basis properties of the eigenfunctions of the spectral problem in the appropriate spaces of functions (as a rule, in Lebesgue or Sobolev spaces). Such questions for the problem (6),(7) studied by another method in [13,14]. Following two theorems are proved in [13].

Theorem 8. *[13] Let*

$$d = 4 + (mq_2(0))^2 + (mq_1(0))^2 + 8mq_2(0) - 2m^2 q_2(0)\, q_1(0) \neq 0,$$

where

$$q_1(0) = \frac{1}{2}\int_{-1}^{0} q(t)\, dt$$

and

$$q_1(0) = \frac{1}{2}\int_{-1}^{0} q(t)\, dt.$$

Then the spectral problem (6), (7) *has two series asymptotically simple eigenvalues* $\lambda_{1,n} = \rho_{1,n}^2$, $n = 1, 2, \ldots$ *and* $\lambda_{2,n} = \rho_{2,n}^2$, $n = 1, 2, \ldots$, *where* $\rho_{1,n}$ *and* $\rho_{2,n}$ *have asymptotics*

$$\rho_{1,n} = \pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)$$

and

$$\rho_{2,n} = \pi n + \frac{\alpha_2}{n} + o\left(\frac{1}{n}\right)$$

respectively, and the numbers α_1 *and* α_2 *are different complex numbers and are defined as follows:*

$$\alpha_1 = \frac{-(2mq_2(0) + mq_1(0)) + \sqrt{d}}{-2m\pi},$$
$$\alpha_2 = \frac{-(2mq_2(0) + mq_1(0)) - \sqrt{d}}{-2m\pi},$$

where $0 \leq \arg\sqrt{d} < \pi$.

Theorem 9. *[13] Let the function* $q(x)$ *satisfy the condition of the Theorem 8. Then the eigen functions* $y_{1,n}(x)$ *of the problem* (6),(7), *corresponding to eigen values* $\lambda_{1,n} = (\rho_{1,n})^2$ *and the eigen functions* $y_{2,n}(x)$, *which correspond to eigen values* $\lambda_{2,n} = (\rho_{2,n})^2$ *have the following asymptotics:*

$$y_{1,n}(x) = \begin{cases} \sin\pi n x + O\left(\frac{1}{n}\right), & x \in [-1, 0], \\ \gamma_{1,n}\sin\pi n x + O\left(\frac{1}{n}\right), & x \in [0, 1], \end{cases} \tag{8}$$

$$y_{2,n}(x)=\begin{cases}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[-1,0],\\ \gamma_{2,n}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[0,1],\end{cases} \tag{9}$$

where the numbers $\gamma_{1,n}$ $\gamma_{2,n}$ *are defined by the formula*

$$\gamma_{1,n}=1+mq_1(0)-m\alpha_1\pi+O\left(\frac{1}{n}\right),$$

$$\gamma_{2,n}=1+mq_1(0)-m\alpha_2\pi+O\left(\frac{1}{n}\right).$$

By $W_p^k(-1,0)\oplus(0,1)$ we denote a space of functions whose constrictions on segments $[-1,0]$ and $[0,1]$ belong to Sobolev spaces $W_p^k(-1,0)$ and $W_p^k(0,1)$, respectively. Let's define the operator L in $L_p(-1,1)\oplus C$ as follows :

$$\begin{array}{l} D(L)=\left\{\hat{u}\in L_p(-1,1)\oplus C:\hat{u}=(u;\,mu(0)),\ u\in W_p^2(-1,0)\bigcup(0,1),\right.\\ \qquad\left. u(-1)=u(1)=0,\ \ u(-0)=u(+0)\right\}\end{array} \tag{10}$$

and for $\hat{u}\in D(L)$

$$L\hat{u}=\left(-u''+q(x)\,u;\ \ u'(-0)-u'(+0)\right). \tag{11}$$

Lemma 1. *Operator* L*, defined by the formulas* (10), (11) *is a linear closed operator with dense definitional domain in* $L_p(-1,1)\oplus C$*. Eigenvalues of the operator* L *and of the problem* (6), (7) *coincide, and* $\{\hat{y}_k\}_{k=0}^{\infty}$ *are eigenvectors of the operator* L*, where* $\hat{y}_{2n-1}=(y_{2n-1}(x);\,my_{2n-1}(0))$, $\hat{y}_{2n}=(y_{2n}(x);\,my_{2n}(0))$.

Proof. To prove the first part of the lemma we take $\hat{y}=(y;\,\alpha)\in L_p(-1,1)\oplus C$ and we define the functional $F(\hat{y})$ as follows:

$$F(\hat{y})=my(+0)-\alpha.$$

Let us assume

$$U_\nu(\hat{y})=U_\nu(y),\nu=1,2,3,$$

where

$$U_1(y)=y(-1),\quad U_2(y)=y(1),\quad U_3(y)=y(-0)-y(+0).$$

Then F, $U_v, v=1,2,3,$ are bounded linear functionals on $W_p^2(-1,0)\bigcup(0,1)\oplus C$, but unbounded on $L_p(-1,1)\oplus C$. Therefore , (see, e.g. [15, pp. 27-29]) the set

$$D(L)=\left\{\hat{y}=(y;\,\alpha),\,y\in W_p^2(-1,0)\bigcup(0,1),\ \mathrm{F}(\hat{y})=U_\nu(\hat{y})=0,\ \ \nu=1,2,3\right\}$$

is dense everywhere in $L_p(-1,1)\oplus C$, and L is a closed operator as constriction of corresponding closed maximal operator.

The second part of the lemma is verified directly.

The lemma is proved.

Theorem 10. *In conditions of the Theorem 8 eigenvectors and conjugate vectors of the operator L, linearized problem* (6), (7) *form basis in* $L_p(-1,\,1)\oplus C$,*and for* $p=2$ *this basis is a Riesz basis.*

Proof. From the Lemma 1 implies that , L is a dense defined closed operator with compact resolvent. Then the system $\{\hat{y}_n\}_{n=0}^{\infty}$ of eigenvectors of the operator L is minimal in $L_p(-1,\,1)\oplus C$, and its conjugate system $\left\{\hat{\vartheta}_n\right\}_{n=0}^{\infty}$ is the system of eigenvectors of the conjugate operator L^* and is in the form

$$\hat{\vartheta}_n=(\vartheta_n,\bar{m}\vartheta_n(0))\ ,\ n=0,1,...,$$

here $\vartheta_n(x)\ ,\ n=0,1,...,$ are eigenfunctions of the conjugate spectral problem

$$-\vartheta''+\overline{q(x)}\vartheta=\lambda\vartheta, \tag{12}$$

$$\vartheta(-1)=\vartheta(1)=0\,;\ \vartheta(-0)=\vartheta(+0)\ ;\ \vartheta'(-0)-\vartheta'(+0)=\lambda\bar{m}\vartheta(0)\ . \tag{13}$$

By the similar way, for the problem (12), (13) we obtain, that for $\vartheta_n(x)$ hold following formulas:

$$\vartheta_{1,n}(x)=\begin{cases}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[-1,\,0]\,,\\ \mu_{1,n}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[0,\,1]\,,\end{cases} \tag{14}$$

$$\vartheta_{2,n}(x)=\begin{cases}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[-1,\,0]\,,\\ \mu_{2,n}\sin\pi nx+\mathrm{O}\left(\frac{1}{n}\right), & x\in[0,\,1]\,,\end{cases} \tag{15}$$

where $\mu_{1,n},\mu_{2,n}$ are the normalization numbers and for which holds

$$\mu_{1,n}=a_1+O\left(\frac{1}{n}\right),\quad \mu_{2,n}=a_2+O\left(\frac{1}{n}\right),$$

and $a_1a_2\neq 0$. Denote

$$e_{1,n}(x)=\begin{cases}\sin\pi nx, & x\in[-1,\,0]\,,\\ \gamma_{1,n}\sin\pi nx, & x\in[0,\,1]\,,\end{cases} \tag{16}$$

$$e_{2,n}(x)=\begin{cases}\sin\pi nx, & x\in[-1,\,0]\,,\\ \gamma_{2,n}\sin\pi nx, & x\in[0,\,1]\,,\end{cases} \tag{17}$$

and consider the system $\{\hat{e}_n\}_{n=0}^{\infty}$, where

$$\hat{e}_0=(0;1)\,,\hat{e}_{2n}=(e_{2,n};0)\,,\hat{e}_{2n-1}=(e_{1,n};0)\,,n\in N.$$

Then $\{\hat{e}_n\}_{n=0}^{\infty}$ is basis in $L_p(-1,\,1)\oplus C$, besides for $1<p\leq 2$, from the formulas (16),(17) implies, that according to inequality Hausdorf-Young for trigonometric system (see., for example, [16]) for each $\hat{f}\in L_p(-1,1)\oplus C$ the inequality

$$\left(\sum_{B=0}^{\infty}\left|\left\langle\hat{f},\hat{e}_n\right\rangle\right|^q\right)^{\frac{1}{q}}\leq c\left\|\hat{f}\right\|_{L_q\oplus C},$$

is fulfilled and from the formulas (8),(9) implies that

$$\sum_n \|\hat{y}_n - \hat{e}_n\|^p_{L_p\oplus C} < \infty.$$

Then by Theorem 1 the system $\{\hat{y}_n\}_{n=0}^{\infty}$ also forms a basis in $L_p\,(-1,\,1)\oplus C$ isomorphic to $\{\hat{e}_n\}_{n=0}^{\infty}$. If $p > 2$ $(1 < q < 2)$, then in this case from the formulas (14),(15) implies that, the system $\left\{\hat{\vartheta}_n\right\}_{n=0}^{\infty}$ is q- close to $\{\hat{e}_n\}_{n=0}^{\infty}$:

$$\sum_n \left\|\hat{\vartheta}_n - \hat{e}_n\right\|^q_{L_q\oplus C} < \infty,$$

and for each $\hat{g} \in L_q\,(-1,\,1)\oplus C$

$$\left(\sum_{B=0}^{\infty} |\langle g, \hat{e}_n\rangle|^p\right)^{\frac{1}{p}} \le c\,\|\hat{g}\|_{L_q\oplus C}\,,$$

and by Theorem 1 the system $\left\{\hat{\vartheta}_n\right\}_{n=0}^{\infty}$ forms a basis in $L_q\,(-1,\,1)\oplus C$ and consequently, the system $\{\hat{y}_n\}_{n=0}^{\infty}$ forms a basis in $L_p\,(-1,\,1)\oplus C$ isomorphic to $\{\hat{e}_n\}_{n=0}^{\infty}$.

As noted in the Theorem 8, $\alpha_1 \neq \alpha_2$, because, although one of these numbers does not equal zero. With this in mind and applying the Theorem 2 and 7, we obtain, that right is next

Theorem 11. *If $\alpha_1 \neq 0$, then for sufficiently great values of n_0 we eliminate $y_{1,\,n_0}\,(x)$, and if $\alpha_2 \neq 0$, then for sufficiently great values of n_0 we eliminate $y_{2,\,n_0}\,(x)$ from the system of the eigen and conjugate functions of the problem (6), (7) we obtain a basis in $L_p\,(-1,\,1)$, and for $p = 2$ we obtain a Riesz basis in $L_2\,(-1,\,1)$.*

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Fractal Conception Evaluation of Blood-Vessel System State in the Anterior Part of an Eye

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Abstract. The paper deals with parametrization of graphical representation in the anterior part of an eye, analysis of the systems that perform statistical analysis, information processing, diagnostics and data bank creation. Information about fractal conception, the effect of its application and complex of necessary theoretical and practical works in this field are given. The use of the fractal size of the eye vessels as an information wrapping method and the perspective of linear regression or the least square methods are studied. The efficiency of the use of fractal concept for the preservation and processing of graphical representation of the blood-vessel system of the anterior part of an eye is shown.

Key Words and Phrases: blood-vessel system of an eye, image recognition, simulation of chaotic structures, fractal structures, simulation.

2010 Mathematics Subject Classifications: 34L10, 41A58, 46A35

1. Introduction

Numerous historical facts from ancient medicine can be considered as important facts of eye-diagnostics information. There is also scientific evidence that there is a correlation between the visual analysis of changes in the anterior part of an eye and internal diseases and even some psychological disorder symptoms.

Today, there is a great believance that the functional changes and hormonal shortcomings that occur in interior organs and other objective factors are revealed. The importance given to the perspective of the eye that the blood vessel structure or changes in it can be instrumental in early prediction of internal diseases is developing on a day-to-day. One of the reasons of this tendency is related to the achievements of electronic appliances and medical devices in medicine, and the other reason is the high achievement of modern information technologies and the broad range of mathematical-cybernetic methods.

As the symptomatic factors as the appearance of yellow strains in the eye is usually more common in liver pathology, swelling in the eyes in the blood-vessel cardiological diseases, redness in the eyes in hypertension have been known for a long time, now there are more complex requirements based on external showings of the eye and they consider

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to study pathalogical changes more thouroughly. Local vein blockage in the eye is one of the factors that can dramatically alter the appearance of the anterior part of the eye. Today the intuitive conviction based on subjective observation of the people as "Eyes are the light of life", "The eyes are the mirror of the heart" needs to be transferred to a more serious and objective evaluation. Analysis of the existing scientific literature in this field shows that analysis of an eye-vascular system based on mathematical-cybernetic methods can be considered as a perspective direction that creates important steps in the field of medical diagnostics [2].

2. The most important apparatus for the automatic examination of the eye's blood-vascular system image

A positive solution to the problem of objective parametrization (characterization) of the eye's blood-vessel system can refer to the medical examination tools available in the current eye care. A positive solution to the problem of symptomatic diagnostic problem is undoubtedly dependent on the level of improvement of the analyzers and measurement systems currently used in the eye examination and treatment. The analysis of survey reviewer of technical literature, which we have undertaken in this field, does not cover the issue, but it can to a certain extent give the most important idea of the general situation.

Some of the apparatus manufactured by leading companies of advanced countries in the field of medical devices can play a major role in the diagnostics, prophylaxis and treatment prophylaxis and treatment of eye diseases and enable to carry out fundamental researches in the field of examination of blood-vascular system of the eye. Many of these equipments were designed to examine the anterior part the eye, the ultrasound examination of the eyeground, the visual field and the visual acuity. The last model apparatus manufactured in Japan, Korea, the USA, Central Europe and other countries are widely used in modern medicine. The Japanese "TOPCON" company's test, measurement, treatment complex is successfully used in the diagnostics, prevention and treatment of eye diseases. The "FOROPTER" complex is a computerized intellectual system, has the function of evaluating the patien's vision area and sharpness and is widely used in clinical practice. Germanc "VOLK" and "Reister" systems also examine the eyeground, "ALKON", "LAUREAT" systems are able to carry out the most recent cataract surgery. The "Quantel" system of France is used in the prevention and treatment of degenerative diseases of retina.

Of course it is impossible to claim that mathematical-cybernetic methods such as statistical analysis, image recognition and mathematical simulation are used as a fundamental necessity in addressing the broad spectrum of mentioned systems. On the other hand, there is no need to note that the functional issues such as simulation with differential functions, arising in operational control in the dynamics of processes during the patient's examination, and the analysis of transition state in the patient are widely used. For example, the German-made "Zeiss" brand equipment examines the activity of the blood stream in retinal vessels. For this examination, the patient is administrated intravenously 2ml-25 % or 5ml-10 % fluorescein sodium salt. Fluorescein enters ocular circulation from the internal carotid artery through the eye artery. Fluoresein first enters the choroidal vessels,

Figure 1: A picture of the unique structure of the eye's blood-vessel system received from optic devices

then to arteries and veins of retina. The drug is injected within 6 seconds and appears in fluorescein vision nerve and choroid within 8 to 11 seconds. The duration of fluorescein entering the veins depends on the age of the patient, the patient's cardiovascular status, and the rate of fluorescein entering. White and black images are taken within 10 seconds after injection, 1 image in every second during 20 seconds. Then the patient is offered 10-minute rest. Then 5-10 images are taken. In some cases, the pictures taken in 15 minutes can also be used as useful diagnostic information [9].

Thus, it can be shown from this example that during the examination, both the recording of the reactions arising from the dynamic effects in the body and the effects of intermediate stages of examination require to have and storage the graphical description by parametrizing it. With regard to the problem of blood vessel examination of the eye, it should be noted that for both diagnostic and statistical analysis, reduction of optic at and ultrasound examination results as physical measure information to a compact form, in other words, parametrization does not manifests itself to day.

3. Possible folding methods for the eye's blood-vasculus system image. Fractal reflection

The picture shows a blood-vessel-eye system's photo surrounded by different size square networks. The distinctive thickness of vessel and formation of their dendrites with random character forms draws attention.

If one or more dendrite aggregates are taken within each check, each of them can be regarded as an implementation of a random function, and we can suggest that we can centralize multiple implementations on square networks. If this implementation is carried out on a basis of any methods, we can obtain folding of the image either in the form of any numerical characteristic of characteristic function. Such statement of the problem first of all focuses on the Fourier spectral analysis that requires complex, graphical processing processes such as separately analyzing the image of the dendrite aggregate falling on each network, subsequent centering and defecting the spectral composition. It should be noted that this issue itself will create an important stage and will require independent algorithmic works to find its solution.

The another direction may be calculation of general ℓlength of vessels with respect to δin arbitrary $\delta + d\delta$ interval within the network of the given size. The Fourier transform of this distribution function gives a complex variable characteristic function, and in principle, such a substitution can be used as an image folding method.

It should be noted that as a mathematical description of dendritic structures, there is a scientific study showing that fractal notation of such structures is far more effective [7,8]. In this study, the total volume of chaotic pores channel, surface area and effective perimetry concepts were introduced and they were used to parametrize the dendritric channel system. Fig. 2.

Fractal is a fraction - dimensional object, i.e. a whole consisting of its own parts Benua Mandelbrot proposed initial guidelines for geometrical measurement or calculation of such structures [5]. These rules and formulas are now also available.

According to Mandelbrot, the fractal is a structure recurring in scale changes from the biggest to the smallest and where any geomterical object is considered. When ordinary geomterical objects (full size) are broken down into similar parts the resulting size for example total length, total surface area or total volume remain unchanged. In fractal objects (fractional dimensional) this dimension varies with the fact that the fractal dimension differs from the topological dimension. In the quantitative sense fractal property is reflected just in the mentioned difference. It should be noted that the size of the fractality may be both greater or less than the topological dimension.

Divide sequentially the identified sides of any D- dimensional geometrical structure into Mequal parts. After divisions as each iteration the original (or any element obtained from dividend) turns into N number identical element, we can write:

$$N = M^D \tag{1}$$

It is obvious that in objects with no fractality, D equals 1, 2 or 3. Now in addition to some variants of fractality that have become classic examples, the expanding diversity is successfully applied not only to non-traditional computer graphics, but to modeling of non-traditional objects, too.

The most common formula for determining the fractal dimension can be written using formula

$$D = \frac{\ln M}{\ln M} \tag{2}$$

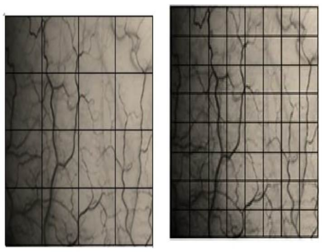
Here D is a fractal dimension [5,6].

At present, there are numerous algorithms based on different approaches to determine the dimension of fractals in nature [3,7,8].

It is clear from Fig. 1 that vascular system in the eye forms one natural fractal system. Here, the main problem is the determination of the regular fractal structure of the system equivalent to it in this or other point of view and the solution of parametrization problem on this basis. In our study, we aimed to determine the fractal dimension of blood vessels in the eye and the length of blood vessels to parametrize the eye. At first, in order to determine the length of blood vessels, we measure by taking two points of the vessel and determine the length along any straight line. But, the result obtained at this time may be considered as an approximate result. If the distance between the points is taken smaller, the size will be adjusted. We can define the total length of blood vessels in the eye based on the formula used by Lewis Fry Richardson in measuring the geographical borders of the countries and the algorithm suggested below. We can also define the fractal dimension according to the length of blood vessels in the eye, i.e. by the length we can determine the fractal dimension. Lewis Fry Richardson has shown that the L length of the borders of the countries varies depending on the δ scale of the map, $L = L(\delta)$. We can use this formula for the blood vessel system we consider [8].

$$L(\delta) = A\delta^{1-D}. \tag{3}$$

Here A is a constant, D is a fractal dimension.



a) b)

Figure 2: Statistical processing of eye-vascular system image based on reflection on different dimensional networks

E. Feder has shown on the basis of different experiments that the following formula can be used to determine any δ –dependent length [6]:

$$m_i\delta_i = A\delta_i^{1-D}. \tag{4}$$

Here $L(\delta) = m\delta$, where m sizes for different scale δ were obtained. Since we do not have a scale in the graphical description of the considered blood vascular system, if we place on the graphic representation of a square grid with the side of length δ_1 (fig 2. a)), then a quadratic grid with side of length δ_2 (fig. 2. b)), and in this sequence a quadratic grid with side of length δ_n , then for each δ_1 δ_2.....δ_n we obtain the length of vessels, m_1 m_2 .....m_n. It is appropriate to take $n \geq 5$. Here m_i is the sum length of the parts of vessels that the quadratic grid cuts. Here each δ value has a specific $m\delta$ length, and if we draw a graph of these values, we can see their linear dependence. For each $\lg(\delta)$ we get certain $\lg(m\delta)$. If based on

$$x = \lg \delta; y = \lg(m\delta), \tag{5}$$

we draw a graph, we can see linear dependence in the form of $y = ax + b$ [8]. So, by the least square method we can find the constants a and b

$$a = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{n \sum x_i^2 - \sum x_i \sum x_i};$$

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \sum x_i \sum x_i}.$$

From the known $L(\delta) = m\delta$ and formula (5) we get $y = \lg(m\delta) = \lg L(\delta)$. Then we can write $y = ax + b$ in the form $\lg L = a \lg \delta + b$. Hence we get

$$10^{\lg L} = 10^{a \lg \delta + b};\ L = 10^b \delta^a. \tag{6}$$

Taking (3) into account, we find $A = 10^b$; $a = 1 - D$. From the expression $D = 1 - a$ we can determine the fractal dimension with respect to the length of the blood vessel according to the graphic description of the anterior of the eye. Using the quantities D and A being the found fractal dimension in length by means of the formula $L(\delta) = A\delta^{1-D}$ (3), we can determine L length of the vessels according to the graphic description of the anterior part of the eye for any $\delta \times \delta$- dimensional square grid.

The carried out investigations enable to determine two unique values, the length and fractal dimension of the vascular system of the anterior part of the eye. Thus, by the unique value corresponding to each graphic description of the anterior part of the eye, we can parametrize this description.

4. Conclusion

Modern development of information technology shows that parametrization of graphic descriptions in automation of medical diagnostics is more likely to provide possible information. The studies have shown that as the analysis of graphic description complicates, the focus is on the parametrization of description in studying diagnostic descriptive bank, statistical analysis and the dynamics of the description. Today, fractal description analysis as a separate field of science involves professionals in the field of information technology and is used in many fields including graphic descriptions. Parametrization of the graphic description of the anterior part of the eye using the fractal analysis apparatus may be used to examine the anterior part of the eye and to study the dynamics of pathological conditions. The achievements in the field of finding these parameters with certain accuracy can eventually be an important step in ensuring a positive solution to the problem of computerized diagnostics.

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Global Bifurcation From Infinity in Nonlinear Elliptic Problems with Indefinite Weight

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Abstract. In this paper we consider global bifurcation of solutions in nonlinear eigenvalue problems for semi-linear elliptic partial differential equations with indefinite weight function. We prove the existence of two pairs of unbounded continua of solutions bifurcating from the points in $\mathbb{R}\times\{\infty\}$ corresponding to the positive and negative principal eigenvalues of the linear problem and such that the continua of each pair consists of positive and negative functions, respectively, in the neighborhood of these points.

Key Words and Phrases: nonlinear eigenvalue problem, bifurcation point, global continua, principal eigenvalue, indefinite weight function

2010 Mathematics Subject Classifications: 35J15, 35J65, 35P05, 35P30, 47J10, 47J15

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1. Introduction

In this paper, we consider the following nonlinear eigenvalue problem

$$\begin{aligned} &Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)\,u = \lambda a(x)u + g(x,u,\nabla u,\,\lambda) \text{ in } \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n})$ and λ is a real parameter. We assume that L is uniformly elliptic in $\overline{\Omega}$ and that the $a_{ij}(x) \in C^1(\bar{\Omega})$, $a_{ij}(x) = a_{ji}(x)$ for $x \in \overline{\Omega}$, $c(x) \in C(\bar{\Omega})$, $c(x) \geq 0$ for $x \in \overline{\Omega}$. Let $a(x) \in C(\overline{\Omega})$ such that $|\Omega_a^\sigma| > 0$ for $\sigma \in \{+,-\}$, where $\Omega_a^\sigma = \{x \in \Omega : \sigma a(x) > 0\}$ and $|\Omega_a^\sigma| = \text{meas}\{\Omega_a^\sigma\}$. Moreover, the nonlinear term $g \in C(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R})$ and satisfies the following condition:

$$g(x,u,v,\lambda) = o(|u|+|s|) \;\text{ as }\; |u|+|v| \to \infty, \tag{2}$$

uniformly in $x \in \overline{\Omega}$ and $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

Problem (1) with $a(x) > 0$, $x \in \overline{\Omega}$, and all the coefficients and the nonlinear terms are smooth was considered by Rabinowitz [9] in a more general case, where, in particular, it was shown that there exist two unbounded continua of solutions emanating from asymptotically bifurcation point corresponding to the first eigenvalue of the linear problem obtained from (1) by setting $g \equiv 0$ and contained in the classes of positive and negative functions in near of this point. In the future, Przybycin [8] and Rynne [10] extended the results of Rabinowitz [9] to the class of nonlinearizable eigenvalue problems for elliptic partial differential equations with a definite weight.

In the papers [3, 4], problem (1) was studied in the case when the nonlinear term g satisfies a $o(|u| + |\nabla u|)$ condition at $u = 0$. For such a problem, the authors show the existence of two pairs of unbounded continua of solutions bifurcating from points of the line of trivial solutions corresponding to the positive and negative principal eigenvalues of linear problem, and such that the continua of each pair are contained in the classes of positive and negative functions, respectively.

The purpose of the present paper is extend the result of Rabinowitz concerning the existence of branches of positive and negative solutions, [9], to the nonlinear problem (1) with indefinite weight function $a(x)$.

2. The classes P_σ^μ and principal eigenvalues of the corresponding linear problem

For $k \in \mathbb{N}$, and $\alpha \in (0,1)$ let $C^{k,\alpha}(\overline{\Omega})$ denote the Banach space of the functions in $C^k((\overline{\Omega})$ having all their derivatives of order k Hölder continuous with exponent α. We let $|\cdot|_k$ and $|\cdot|_{k,\alpha}$ denote the standard sup-norms on spaces $C^k((\overline{\Omega})$ and $C^{k,\alpha}(\overline{\Omega})$, respectively. For $p > 1$, let $W^{k,p}(\overline{\Omega})$ denote the standard Sobolev space of functions whose distributional derivatives, up to order k, belong to $L^p(\Omega)$. We let $||\cdot||_p$ and $||\cdot||_{k,p}$ denote the norm on $L_p(\Omega)$ and $W^{k,p}(\Omega)$, respectively.

It is known (see [1]) that, if $p > N$, then there exists a constant γ such that

$$||u||_{C^{1,1-n/p}} \leq \gamma\, ||u||_{W^{2,p}} \text{ for all } u \in W^{2,p}(\Omega).$$

Now let $\alpha \in (0,1)$ be the given number and p be a real number such that $p > n$ and $\alpha < 1 - n/p$. Then $W^{2,p}(\Omega)$ is compactly embedded in $C^{1,\alpha}(\overline{\Omega})$.

Let $E = \{u \in C^{1,\alpha}\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega\}$ be the Banach space with the norm $||\cdot||_{C^{1,\alpha}}$. A pair (λ, u) is said to be a solution of problem (1) if $u \in W^{2,p}(\Omega)$ and (λ, u) satisfies (1). By virtue of compactly embedding $W^{2,p}(\Omega)$ in $C^{1,\alpha}(\overline{\Omega})$ we conclude that every solution of the nonlinear problem (1) belongs to $\mathbb{R} \times E$. Thus we may consider the structure of the set of solutions of problem (1) in $\mathbb{R} \times E$. Let $P_\sigma^+ = \{u \in E : u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega,\ \sigma \int\limits_\Omega a u^2 dx > 0\}$, where $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on $\partial\Omega$.

Remark 1. It follows from the definition that for each $\sigma \in \{+, -\}$ the sets P_σ^+, $P_\sigma^- = -P_\sigma^+$ and $P_\sigma = P_\sigma^+ \cup P_\sigma^-$ are open subsets of E; for each $\sigma \in \{+, -\}$ the sets P_σ^+ and

P_σ^-, and for each $\nu \in \{+\,,\,-\}$ the sets P_+^ν and P_-^ν are disjoint. Moreover, if $u \in \partial P_\sigma^\nu$, $\sigma \in \{+\,,\,-\}$, $\nu \in \{+\,,\,-\}$, then the function u has either an interior zero in Ω or $\frac{\partial u}{\partial n} = 0$ at some point on $\partial\Omega$ or $\int\limits_\Omega au^2 dx = 0$ [4].

Now we consider the linear eigenvalue problem obtained from (1) by setting $h \equiv 0$, i.e. the following spectral problem

$$\begin{aligned} &Lu = \lambda\, a(x)\, u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3}$$

It should be noted that if the weight function $a(x)$ does not change sign in $\overline{\Omega}$, then (3) admits one principal eigenvalue [7], and if $a(x)$ changes sign in $\overline{\Omega}$, then problem (3) admits two principal eigenvalues; one positive and the other negative [3].

In [3] the authors obtained the following properties of the eigenfunctions corresponding to the principal eigenvalues of problem (3).

Theorem 1. (see [3, Lemmas 2.1-2.4, Theorems 2.1, 2.2 and Remark 2.1]) *The linear eigenvalue problem* (3) *have positive and negative principal eigenvalues* λ_1^+ *and* λ_1^-, *respectively, which are simple and given by the relations*

$$\lambda_1^\sigma = \inf\{R(u) : u \in H_0^1(\Omega),\ \sigma \int\limits_\Omega au^2 dx > 0\} \text{ for } \sigma \in \{+\,,\,-\},$$

where $H_0^1(\Omega) = \{u \in W^{1,2}(\Omega) : u = 0$ *on* $\partial\Omega\}$ *and* $R(u)$ *is the Rayleigh quotient* [2] *defined as follows:*

$$R(u) = \frac{\int\limits_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx + \int\limits_\Omega cu^2 dx}{\int\limits_\Omega au^2 dx}.$$

Moreover, the corresponding eigenfunction $u_1^\sigma(x)$, $x \in \overline{\Omega}$, $\sigma \in \{+\,,\,-\}$, *can be chosen so that* $u_1^\sigma(x) > 0$ *for all* $x \in \Omega$ *and* $\frac{\partial u_1^\sigma(x)}{\partial n} < 0$ *for all* $x \in \partial\Omega$.

Remark 2. It follows from Theorem 1 that $u_1^\sigma \in P_\sigma^+$ for each $\sigma \in \{+\,,\,-\}$. It should be noted that u_1^σ is made unique by taking $||u_1^\sigma||_{C^{1,\alpha}} = 1$.

3. Global bifurcation of solutions of problem (1) from infinity

The closure of the set of nontrivial solutions of (1) will be denoted by $\mathcal{L}$. We say $(\lambda, \infty) \in \mathbb{R} \times \{\infty\}$ is a bifurcation point for problem (1) if any neighborhood of this point contains solutions of problem (1), i.e. there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathcal{L}$ such that $\lambda_n \to \lambda$ and $|u_n|_{1,\alpha} \to \infty$ as $n \to \infty$ [6].

The main result of this paper is the following theorem.

Theorem 2. *For each $\sigma \in \{+, -\}$ and each $\nu \in \{+, -\}$ there exists a component $C^{\nu}_{1,\sigma}$ of $\mathcal{L}$ which contains $(\lambda_1^{\sigma}, \infty)$ and satisfies the conclusions of Theorem 1.6 and Corollary 1.8 from [9]. Moreover, the neighborhood Q of [9, Corollary 1.8] can be chosen such that*

$$(\mathcal{C}^{\nu}_{1,\sigma} \cap Q) \subset (\mathbb{R} \times P^{\nu}_{\sigma}) \cup \{(\lambda_1^{\sigma}, \infty)\}.$$

Proof. Step 1. We assume that $a_{ij} \in C^2(\overline{\Omega})$, c, $a \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$.

It follows from the L_p theory for uniformly elliptic partial differential equations [2] that there exists a unique $v = G(\lambda, u)$ satisfying

$$\begin{aligned} &Lv = \lambda a(x)\, u + g(x, u, \nabla u,\, \lambda)) \quad \text{in } \Omega, \\ &v = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Since E is compactly embedding in $W_0^{2,p}(\Omega) = W^{2,p}(\Omega) \cap \{u : u = 0 \text{ on } \partial\Omega\}$ the Arzela-Ascoli Theorem imply that G is compact on $\mathbb{R} \times E$.

Denote by $w = \mathcal{L}u \in W_0^{2,p}(\Omega)$ the solution of the following problem

$$\begin{aligned} &Lw = a(x)\, u \;\text{ in }\; \Omega, \\ &w = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Then from the above reasoning imply that $\mathcal{L}$ is a compact linear map on E. By the Theorem 1 it follows that λ_1^+ and λ_1^- are simple principal characteristic values of operator $\mathcal{L}$.

Suppose that $\mathcal{G}(\lambda, u) = G(\lambda, u) - \lambda \mathcal{L} u$. From the our above remarks it follows that (1) is equivalent to the following nonlinear eigenvalue problem

$$u = \lambda \mathcal{L} u + \mathcal{G}(\lambda, u). \tag{4}$$

Following the corresponding reasoning carried out in the proof of Theorem 2.28 from [9], we see that $\mathcal{G}(\lambda, u) = o(|u|_{1,\alpha})$ as $|u|_{1,\alpha} \to \infty$, uniformly on bounded λ intervals and $|u|^2_{1,\alpha} \mathcal{G}\left(\lambda, \frac{u}{|u|^2_{1,\alpha}}\right)$ is compact. Thus [9, Theorem 1.6 and Corollary 1.8] are applicable here. The verification of the last statement of this theorem follows as in [9, Theorem 2.4].

Step 2. To complete the proof of this theorem, we approximate equation (1) by "smoothed equations", as in [5, Section 4], and apply standard elliptic regularity results for elliptic operators [2].

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On the Parametric Resonance Cases of the System Consisting of the Circular Cylinder and Surrounding Elastic Medium Under Action in the Interior of the Cylinder Time-Harmonic Oscillating Moving Load

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Abstract. The paper studies the parametric resonance cases which appear under the action of the oscillating moving ring load on the interior of the hollow cylinder surrounded by an elastic medium. The axisymmetric stress-strain state is considered and it is assumed that the perfect contact conditions satisfy on the interface between the cylinder and surrounding elastic medium and the equations of motion for the cylinder and surrounding elastic medium are written separately and these equations are exact the so-called 3D equations of the elastodynamics. Numerical results on the interface stresses are presented and according to the analyses of these results, it is established the existence of the parametric resonance in certain values of the moving velocity of the oscillating load.

1. Introduction

The detailed review of the related investigations are given in the papers [1-4] and in the monograph [5] therefore we do not consider here this review again. Nevertheless, we note here some particularities of the recent results which have been obtained with the participation of the author of the present paper. We begin this notation with the paper [1] in which it was shown that under the forced vibration of the system consisting of the hollow cylinder and of the surrounding elastic medium under the action time-harmonic axisymmetric ring forces on the interior of the cylinder the resonance phenomenon does not appear.

In this case, the dependence between the frequency and amplitudes of the quantities characterizing the stress-strain state in the aforementioned system appearing as a result of the time-harmonic ring load has non-monotonic character. In other words, there exists such value of the frequency of the external forces under which the absolute values of the mentioned quantities have their maximum. In other words, there exists such value of the frequency of the external forces under which the absolute values of the mentioned quantities have their maximum. However, in the paper [2] it was established that in the case where on the interior of the cylinder act corresponding non-axisymmetric forces, according to which it was solved the relating three-dimensional problem the noted above dependencies have more complicated character and nevertheless the resonance phenomenon does not observe in the 3D case also. At the same time, the paper [3] establishes that if on the interior of the cylinder the axisymmetric moving constant ring load acts then under certain values moving velocity of this load the resonance type phenomenon takes place and the velocity regarding this case is called the critical velocity.

The question "what kind of the response of the foregoing system to the time-harmonic ring forces acting on the interior of the cylinder appears in the case where these forces move with the constant velocity and this velocity is less than the corresponding critical velocity", is the subject of the investigation of the present paper. As a result of this investigation, it is established that there exist such value of the velocity of the moving load under which the resonance cases appear as a result of the oscillation of the external forces.

2. Formulation of the problem

Consider the aforementioned "hollowcylinder + surrounded elastic medium" system the sketch of which is illustrated in Fig. 1 and assume the thickness of the wall of the cylinder is h and the external radius of the cross section of that is R. Moreover, we assume that on the inner surface of this cylinder normal time-harmonic ring forces act and these forces move along the cylinder axis with constant velocity V. We associate with the central axis of the cylinder the cylindrical system of coordinates $Or\theta z$ and within this framework we attempt to investigate the stress-strain state in the system under consideration with utilizing the following field equations of elastodynamics.

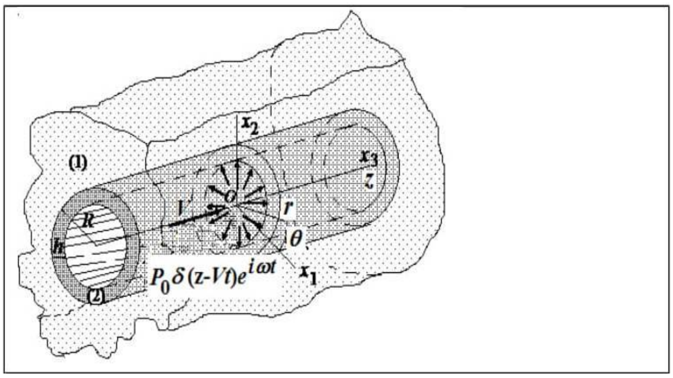

Figure 1: The sketch of the system under consideration and the oscillating moving ring load

Equations of motion:

$$\frac{\partial\sigma_{rr}^{(k)}}{\partial r}+\frac{\partial\sigma_{rz}^{(k)}}{\partial z}+\frac{1}{r}(\sigma_{rr}^{(k)}-\sigma_{\theta\theta}^{(k)})=\rho^{(k)}\frac{\partial^2u_r^{(k)}}{\partial t^2},\ \frac{\partial\sigma_{rz}^{(k)}}{\partial r}+\frac{\partial\sigma_{zz}^{(k)}}{\partial z}+\frac{1}{r}\sigma_{rz}^{(k)}=\rho^{(k)}\frac{\partial^2u_z^{(k)}}{\partial t^2}. \quad (1)$$

Elasticity relations:

$$\sigma_{nn}^{(k)} = \lambda^{(k)}(\varepsilon_{rr}^{(k)} + \varepsilon_{\theta\theta}^{(k)} + \varepsilon_{zz}^{(k)}) + 2\mu^{(k)}\varepsilon_{nn}^{(k)},\ nn = rr;\ \theta\theta;\ zz,\ \sigma_{rz}^{(k)} = 2\mu^{(k)}\varepsilon_{rz}^{(k)}. \qquad (2)$$

Strain – displacement relations:

$$\varepsilon_{rr}^{(k)} = \frac{\partial u_r^{(k)}}{\partial r},\ \varepsilon_{\theta\theta}^{(k)} = \frac{u_r^{(k)}}{r},\ \varepsilon_{zz}^{(k)} = \frac{\partial u_z^{(k)}}{\partial z},\ \varepsilon_{rz}^{(k)} = \frac{1}{2}(\frac{\partial u_z^{(k)}}{\partial r} + \frac{\partial u_r^{(k)}}{\partial z}). \qquad (3)$$

In equations (1), (2) and (3) the conventional notation of the theory of elasticity is used and through the upper index (k) it is indicated the belonging of the quantities to the cylinder under $k = 2$ and to the surrounding elastic medium under $k = 1$.

Consider also formulation of the corresponding boundary and contact conditions which can be written as follows.

$$\sigma_{rr}^{(2)}\Big|_{r=R-h} = -P_0\delta(z - Vt)e^{i\omega t},\ \sigma_{rz}^{(2)}\Big|_{r=R-h} = 0, \qquad (4)$$

$$\sigma_{rr}^{(1)}\Big|_{r=R} = \sigma_{rr}^{(2)}\Big|_{r=R},\ \sigma_{rz}^{(1)}\Big|_{r=R} = \sigma_{rz}^{(2)}\Big|_{r=R},\ u_r^{(1)}\Big|_{r=R} = u_r^{(2)}\Big|_{r=R}.\ u_z^{(1)}\Big|_{r=R} = u_z^{(2)}\Big|_{r=R} \qquad (5)$$

$$\left|\sigma_{rr}^{(1)}\right|;\ \left|\sigma_{\theta\theta}^{(1)}\right|;\ \left|\sigma_{zz}^{(1)}\right|;\ \left|\sigma_{rz}^{(1)}\right|;\ \left|u_r^{(1)}\right|;\ \left|u_z^{(1)}\right| \to 0, \quad \text{as } \sqrt{r^2 + z^2} \to \infty. \qquad (6)$$

Thus, the investigation of the problem is reduced to the boundary-contact problem (1) – (6) for solution to which the method developed in the papers [1-4] is employed. Now we consider some fragments of the application of this method for the problem under consideration.

3. Method of solution

For solution of the equations (1)-(3). We use the well-known, classical Lame (or Helmholtz) decomposition (see, for instance, the monograph [6] and others listed therein) for solution of the above formulated problem:

$$u_r^{(k)} = \frac{\partial \Phi^{(k)}}{\partial r} + \frac{\partial^2 \Psi^{(k)}}{\partial r \partial z},\ u_z^{(k)} = \frac{\partial \Phi^{(k)}}{\partial z} + \frac{\partial^2 \Psi^{(k)}}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi^{(k)}}{\partial r}, \qquad (7)$$

where $\Phi^{(k)}$ and $\Psi^{(k)}$ satisfy the following equations:

$$\nabla^2\Phi^{(k)} - \frac{1}{(c_1^{(k)})^2}\frac{\partial^2\Phi^{(k)}}{\partial t^2} = 0\,,\ \nabla^2\Psi^{(k)} - \frac{1}{(c_2^{(k)})^2}\frac{\partial^2\Psi^{(k)}}{\partial t^2} = 0, \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \qquad (8)$$

Here the notation $c_1^{(k)} = \sqrt{(\lambda^{(k)} + \mu^{(k)})/\rho^{(k)}}$ and $c_2^{(k)} = \sqrt{\mu^{(k)}/\rho^{(k)}}$ is used.

We introduce the moving coordinate system

$$r' = r, z' = z - Vt, \qquad (9)$$

which moves with the ring load. Representing all the sought values as $g(r,z',t) = \bar{g}(r,z')e^{i\omega t}$ (below, the over bar and upper prime will be omitted) and rewriting the Eq. (8) with the coordinates r' and z' determined in (9), we obtain:

$$\begin{aligned} &\nabla^2\Phi^{(k)} - \frac{1}{(c_1^{(k)})^2}\left(V^2\frac{\partial^2\Phi^{(k)}}{\partial z^2} - 2i\omega V\frac{\partial\Phi^{(k)}}{\partial z} - \omega^2\Phi^{(k)}\right) = 0,\\ &\nabla^2\Psi^{(k)} - \frac{1}{(c_2^{(k)})^2}\left(V^2\frac{\partial^2\Psi^{(k)}}{\partial z^2} - 2i\omega V\frac{\partial\Psi^{(k)}}{\partial z} - \omega^2\Psi^{(k)}\right) = 0. \end{aligned} \tag{10}$$

During the foregoing transformations, the first condition in (4) transforms to the following one:

$$\sigma_{rr}^{(2)}\Big|_{r=R-h} = -P_0\delta(z), \tag{11}$$

but the other relations and conditions in (1) – (6) remain valid for the amplitudes of the sought values.

Below we will use the dimensionless coordinates $\bar{r} = r/h$ and $\bar{z} = z/h$ instead of the coordinates r and z, respectively and the over-bar in $\bar{r}$ and $\bar{z}$ will be omitted.

Further, we employ the exponential Fourier transform $f_F = \int_{-\infty}^{+\infty} f(z)e^{isz}dz$, according to which, the functions $\Phi^{(k)}$ and $\Psi^{(k)}$, and the amplitudes of the sought values can be presented as follows:

$$\left\{\Phi^{(k)};\Psi^{(k)};u_z^{(k)};u_r^{(k)};\sigma_{nn}^{(k)};\sigma_{rz}^{(k)};\varepsilon_{nn}^{(k)};\varepsilon_{rz}^{(k)}\right\}(r,z) =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\Phi_F^{(k)};\Psi_F^{(k)};u_{zF}^{(k)};u_{rF}^{(k)};\sigma_{nnF}^{(k)};\sigma_{rzF}^{(k)};\varepsilon_{nnF}^{(k)};\varepsilon_{rzF}^{(k)}\right\}(r,s)e^{-isz}ds, nn = rr;\ \theta\theta;\ zz. \tag{12}$$

Substituting the expressions in (12) into the foregoing equations, relations and contact and boundary conditions, we obtain the corresponding ones for the Fourier transformations of the sought values. After this transform the relation (2), the first and second relation in (3), the second condition in (4) and all the conditions in (5) and (6) also remain valid for their Fourier transforms. Nevertheless, the third and fourth relation in (3), the condition (11) and the relations in (7) transform to the following ones:

$$\begin{aligned} &\varepsilon_{zzF}^{(k)} = isu_{zF}^{(k)}, \varepsilon_{rzF}^{(k)} = \frac{1}{2}\left(\frac{\partial u_{zF}^{(k)}}{\partial r} - isu_{rF}^{(k)}\right), \sigma_{rrF}^{(2)}\Big|_{r=R-h} = -P_0 u_{rF}^{(k)} = \frac{\partial\Phi_F^{(k)}}{\partial r} - is\frac{\partial\Psi_F^{(k)}}{\partial r},\\ &u_z^{(k)} = -is\Phi_F^{(k)} + \frac{\partial^2\Psi_F^{(k)}}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi_F^{(k)}}{\partial r} \end{aligned} \tag{13}$$

where, according to (8), the functions $\Phi_F^{(k)}$ and $\Psi_F^{(k)}$ are determined from the equations:

$$\begin{aligned} &\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(s^2 - \frac{W^2(c_2^{(2)})^2}{(c_1^{(k)})^2}\right)\right]\Phi_F^{(k)} = 0,\\ &\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(s^2 - \frac{W^2(c_2^{(2)})^2}{(c_2^{(k)})^2}\right)\right]\Psi_F^{(k)} = 0, \end{aligned} \tag{14}$$

where

$$W = \Omega - sc,\ \Omega = \frac{\omega h}{c_2^{(2)}},\ c = \frac{V}{c_2^{(2)}}. \tag{15}$$

Taking into consideration the conditions in (6), the solution to the equations in (14) are found as follows:

$$\begin{gathered}\Phi_F^{(2)} = A_1 H_0^{(1)}(r_1) + A_2 H_0^{(2)}(r_1), \Psi_F^{(2)} = B_1 H_0^{(1)}(r_2) + B_2 H_0^{(2)}(r_2), \\ \Phi_F^{(2)} = C_2 H_0^{(2)}(r_{11}), \Psi_F^{(2)} = D_2 H_0^{(2)}(r_{21}),\end{gathered} \tag{16}$$

where $H_0^{(1)}(x)$ and $H_0^{(2)}(x)$ are the Hankel functions of the first and second kinds, respectively and

$$\begin{gathered}r_1 = r\sqrt{W^2\delta_1^2 - s^2}, \delta_1 = \frac{c_2^{(2)}}{c_1^{(2)}},\ r_2 = r\sqrt{W^2 - s^2}, \\ r_{11} = r\sqrt{W_1^2\delta_2^2 - s^2},\ W_1 = W\frac{c_2^{(2)}}{c_2^{(1)}}, r_{21} = r\sqrt{W_1^2 - s^2}.\end{gathered} \tag{17}$$

Substituting the expressions in (17) into (13) and the Fourier transforms of the expressions in (2) it is obtained the analytic expressions for the Fourier transforms of the sought values which contain the unknown constants A_1, A_2, B_1, B_2, C_2 and D_2. Using the Fourier transforms of the contact and boundary conditions (4) and (5) the system of algebraic equations are obtained for these unknowns. Thus, solving this system of equations the Fourier transforms of the sought values are determined completely.

The originals of the aforementioned transforms are determined numerically the algorithm for which are proposed and discussed in the papers [1-5]. Therefore we do not consider here the algorithm and their testing which are used under obtaining numerical results which are discussed below.

4. Numerical results and their discussions

First of all, we note that the numerical results which will be considered below are obtained in the following three cases.

Case 1.

$$E^{(1)}\big/E^{(2)} = 0.35,\ \rho^{(1)}\big/\rho^{(2)} = 0.1,\ \nu^{(1)} = \nu^{(2)} = 0.25. \tag{18}$$

Case 2.

$$E^{(1)}\big/E^{(2)} = 0.05,\ \rho^{(1)}\big/\rho^{(2)} = 0.01,\ \nu^{(1)} = \nu^{(2)} = 0.25. \tag{19}$$

Case 3.

$$E^{(1)}\big/E^{(2)} = 0.5,\ \rho^{(1)}\big/\rho^{(2)} = 0.5,\ \nu^{(1)} = \nu^{(2)} = 0.3. \tag{20}$$

We consider of the frequency response of the interface normal stress

$$\sigma_{rr}(z) = \sigma_{rr}^{(1)}(R, z) = \sigma_{rr}^{(2)}(R, z) \tag{21}$$

in the foregoing cases (18)-(20) for various values of dimensionless moving velocity $c = V/c_2^{(2)}$. The graphs of these responses are illustrated in Figs. 2, 3 and 4 for the cases (18), (19) and (20), respectively.

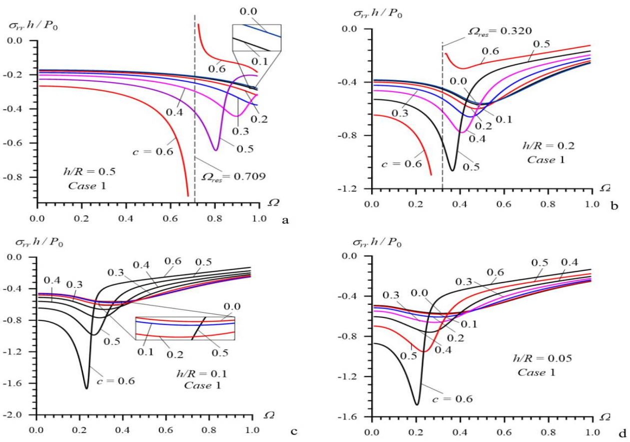

Figure 2: Frequency response of the interface normal stress σ_{rr} obtained for various values of the load moving velocity under $h/R = 0.5$ (a), 0.2 (b), 0.1 (c) and 0.05 (d) in Case 1

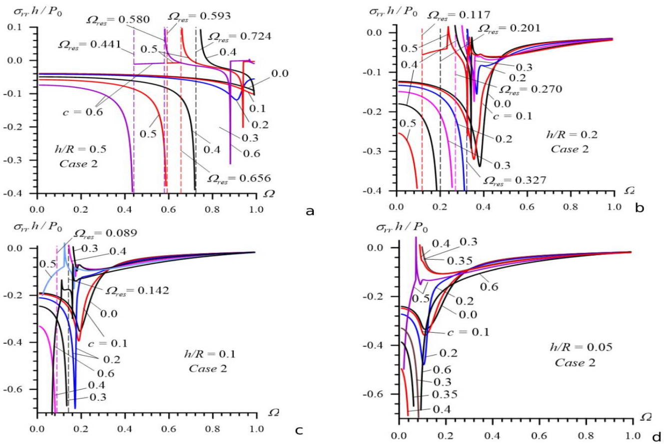

Figure 3: Graphs indicated in Fig. 2 caption and constructed in Case 2

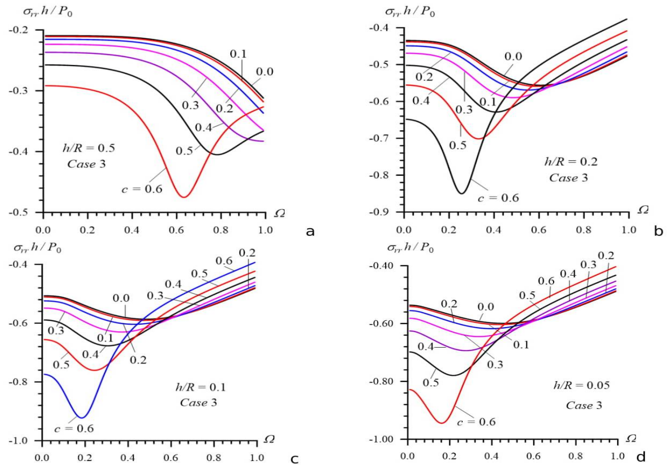

Figure 4: Graphs indicated in Fig. 2 caption and constructed in Case 3

It follows from Figs.2, 3 and 4 that in the all cases under consideration (except the case where $h/R = 0.5$ and $0 \leq c \leq 0.3$ in Case 1 and Case 3, and $0 \leq c \leq 0.1$ in Case 3 under which the absolute values of the stress increase with Ω in the considered change range) the frequency responses have non-monotonic character, i.e. there are such values of Ω (denote this value of Ω by $\Omega*$) before which the absolute value of the stress σ_{rr} becomes maximum and this maximum increases with the moving velocity of the ring load. At the same time, it follows from the results that the values of $\Omega*$ decrease monotonically with c. Moreover, Figs. 2a, 2b, 3a, 3b, 3c and 3d show that there may be cases where an increase in the values of c leads to resonance cases. Such resonance cases, and the corresponding resonance frequencies are indicated in these figures.

The above-noted resonances can be estimated as a parametric resonance and as a parameter it can be taken as the load moving velocity. Consequently, under oscillating moving load action of the ring load, resonance type accidents appear not only under critical moving velocities of this load but also under the foregoing type of parametric resonances. Analyses of the foregoing results also show that the absolute maximum values of the stress under consideration increase with decreasing of the ratio h/R. Moreover, comparison of the results obtained for Case 1, Case 2 and Case 3 with each other shows that the responses of the interface normal stress to the moving velocity of the ring load and its vibration depend not only on the values of this velocity and frequency, but also depend significantly on the ratio of the mechanical properties of the selected pairs of materials, as indicated in (18) – (20) for the hollow cylinder and surrounding elastic medium. At the same time, the latter dependence has not only quantitative, but also qualitative character.

5. Conclusions

Thus, in the present paper the parametric resonance of the system consisting of the hollow cylinder and surrounding elastic medium under action of the time-harmonic oscillating moving ring load acting in the interior of the cylinder is studied. The study is made within the scope of the exact equations and relations of the elastodynamics in the axisymmetric stress-state case. It is described the problem formulation and solution method for this problem.

Numerical results are presented for certain cases which are determined with the ratio of the mechanical constants of the constituents. As a result of the analyses of these results, it is established that there exist the cases under which in the certain values of the velocity of the moving load the oscillation of the moving load causes the resonance of the bi-material elastic system under consideration. The appearance of the resonance cases depends also on the ratio of the cylinder thickness to the cylinder external radius.

The obtained results and their discussions show that the investigations of the problem under consideration have not only theoretical but also the application significance under construction of underground structures. Therefore, it can be concluded that it is necessary to develop such type investigations for the other related problems.

Finally, we note that the results obtained in the present paper have been presented in the 6-th International Conference on Control and Optimization with Industrial Application and the related summary has been published in [7].

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