Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

Investigation of Laplace Transforms for Distribution of the First Passage of Zero Level of the Semi-Markov Random Process with Positive Tendency and Negative Jump

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Abstract. One of the important problems of stochastic process theory is to define the Laplace transforms for the distribution of semi-markov random processes. With this purpose, we will investigate the semi-markovrandom processes with positive tendency and negative jump in this article. The first passage of the zero level of the process will be included as a random variable. The Laplace transforms for the distribution of this random variable is defined. The parameters of the distribution will be calculated on the basis of the final results.
Key Words and Phrases: Laplace transforms, semi-Markov random process, random variable, process with positive tendency and negative jumps.
2010 Mathematics Subject Classifications: 60A10, 60J25

1. Introduction

There are number of works devoted to definition of the Laplace transforms for the distribution of the first pas- sage of the zero level. (Borovkov 2004) [1] defined the explicit form of the distribution, while (Klimov 1996) [3] and (Lotov V. I.) [2] indicated implicit form of the distribution of the first passage of zero level. The presented work explicitly defines the Laplace transforms for the unconditional and conditional distribution of the semi-markov random processes with positive tendency and negative jump.

2. Mathematical Statement of the problem

Let a sequence of independent and identically distributed pairs of random variables $\{\xi_k, \zeta_k\}_{k\geq 1}$, $k = \overline{1,\infty}$ defined on a probability space (Ω, F, P) such that ξ_k and ζ_k are independent random variables and $\xi_k > 0$,$\zeta_k > 0$. Using these random variables we will derive the following step processes of semi-Markov random walk:

$$X_z(t) = z + t - \sum_{i=1}^{k-1} \zeta_i \text{ if } \sum_{i=1}^{k-1} \xi_i \leq t < \sum_{i=1}^{k} \xi_i,\ k = \overline{1,\infty}\ z \geq 0$$

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$X_z(t)$ process is the (asymptotic) semi-Markov random processes with positive tendency and negative jump.One of the realizations of the process $X_z(t)$ will be in the following form:

a) $X_z(t) = z + t$ if $t < \xi_1$ (see, Figure 1),
b) $X_z(t) = z + t - \zeta_1$, if $\xi_1 \geq t < \xi_1 + \xi_2$ (see, Figure 2)

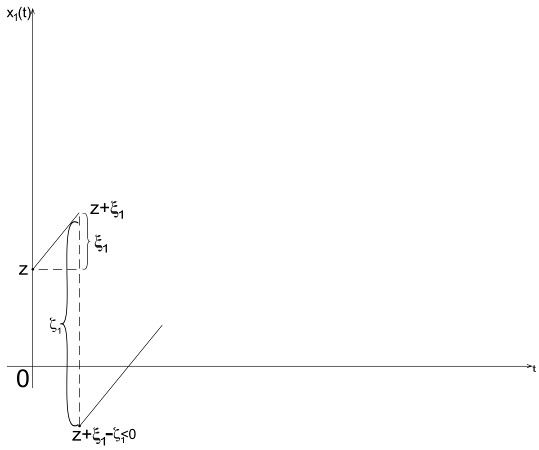

Fig. 1.

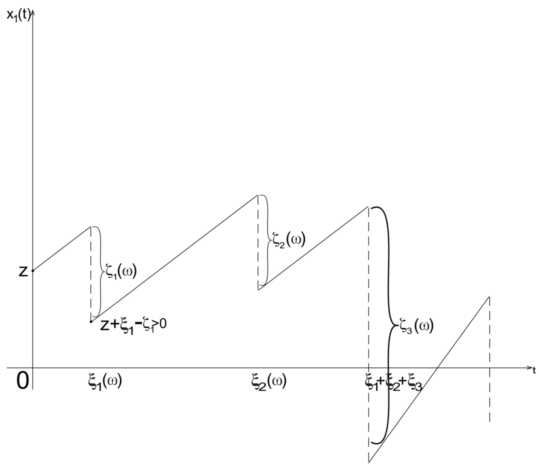

Fig. 2.

Let's include the τ_z^0 random variable defined as below:

$$\tau_z^0 = min\{t : X_z(t) \leq 0\}$$

where τ_z^0, is the time of the first passage of $X(t)$ process. We need to find Laplace transform for distribution of τ_z^0 random variable. Let us set Laplace transform for the distribution of τ_z^0 random variable as $L(\theta)$

$$L(\theta) = Ee^{-\theta\tau_z^0},\ \theta > 0;$$

$$L(\theta|z) = E(e^{-\theta\tau_z^0}|X_z(t) = z),,\ z \geq 0$$

In this case we can express the equation as

$$\tau_z^0 = \begin{cases} \xi_1, & z + \xi_1 - \zeta_1 < 0 \\ \xi_1 + T^0_{z+\xi_1-\zeta_1}, & z + \xi_1 - \zeta_1 > 0 \end{cases}$$

Thus, T and τ_z^0 are evenly distributed random variables.Our goal is to find Laplace transform of relative and non-relative distribution of τ_z^0 random variable.

Theorem 1. *Let a sequence of independent and identically distributed pairs of random variables* $\{\xi_k, \zeta_k\}_{k\geq 1}$, $k = \overline{1,\infty}$, *defined on a probability space* (Ω, F, P) *such that* ξ_k *and* ζ_k *are independent random variables and* $\xi_k > 0, \zeta_k > 0$. *Then an integral equation of Laplace transform of distribution of* τ_z^0 *random variable will be as follows:*

$$L(\theta|z) = \int\limits_{s=0}^{\infty} e^{-\theta s} P\{\zeta_1 > z + s\} P\{\xi_1 \in ds\} -$$

$$- \int\limits_{s=0}^{\infty} e^{-\theta s} \int\limits_{\alpha=0}^{z+s} L(\theta|\alpha) d_\alpha P\{\zeta_1 < z + s - \alpha\} dP\{\xi_1 < s\} \qquad (1)$$

Proof: According to the formula of total probability, we can put it as

$$E\left(e^{-\theta\tau_z^0}|X_z(0) = z\right) = \int\limits_{\Omega} e^{-\theta\tau_z^0} P(d\omega) =$$

$$= \int\limits_{\{\omega: z+\xi_1-\zeta_1<0\}} e^{-\theta\xi_1} P(d\omega) + \int\limits_{\{\omega: z+\xi_1-\zeta_1>0\}} e^{-\theta(\xi_1+T)} P(d\omega)$$

If to consider the following substitution

$$\xi_1 = s;\quad \varsigma_1 = y : T = \beta$$

we derive

$$E\left(e^{-\theta s\tau_z^0}|X_1(0) = z\right) = \int\limits_{s=0}^{\infty} \int\limits_{y=z+s}^{\infty} e^{\theta s} P\{\xi_1 \in ds; \zeta_1 \in dy\} +$$

$$+ \int\limits_{s=0}^{\infty} \int_{y=0}^{z+s} \int\limits_{\beta=0}^{\infty} e^{-\theta(s+\beta)} P\{\xi_1 \in ds; \zeta_1 \in dy; T \in d\beta\} =$$

$$= \int\limits_{s=0}^{\infty} e^{-\theta s} P\{\xi_1 \in ds\} \int\limits_{y=z+s}^{\infty} P\{\zeta_1 \in dy\} +$$

$$+ \int\limits_{s=0}^{\infty} e^{-\theta s} \int\limits_{y=0}^{z+s} dP\{\zeta_1 \in y\} dP\{\xi_1 < s\} L(\theta | z+s-y) =$$

$$= \int\limits_{s=0}^{\infty} e^{-\theta s} P\{\xi_1 \in ds\} P\{\zeta_1 > z+s\} +$$

$$+ \int\limits_{s=0}^{\infty} e^{-\theta s} \int\limits_{\beta=z+s}^{0} L(\theta|\beta) dP\{\zeta_1 < z+s-\beta\} dP\{\xi_1 < s\}$$

or

$$L(\theta|z) = \int\limits_{s=0}^{\infty} e^{-\theta s} P\{\zeta_1 > z+s\} P\{\xi_1 \in ds\} +$$

$$+ \int\limits_{s=0}^{\infty} e^{-\theta s} \int\limits_{y=0}^{z+s} L(\theta | z+s-y) P\{\zeta_1 \in ds\} P\{\xi_1 \in ds\}$$

Let's assume that $z+s-y = \alpha$. In this case we will receive the following integral equation:

$$L(\theta|z) = \int\limits_{s=0}^{\infty} e^{-\theta s} P\{\zeta_1 > z+s\} P\{\xi_1 \in ds\} -$$

$$- \int\limits_{s=0}^{\infty} e^{-\theta s} \int\limits_{\alpha=0}^{z+s} L(\theta|\alpha) d_{\alpha} P\{\zeta_1 < z+s-\alpha\} dP\{\xi_1 < s\}$$

The theorem 1 is proved.

We will solve this integral equation in special case. Let's assume that $\xi_1(\omega)$ random variable has the Erlangian distribution of third construction, while $\zeta_1(\omega)$ random variable has the single construction Erlangian distribution:

$$P\{\xi_1(\omega) < t\} = \left[1 - \left(1 + \lambda t + \frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}\right] \varepsilon(t), \quad \lambda > 0,$$

$$P\{\zeta_1(\omega) < t\} = \left[1 - e^{-\mu t}\right] \varepsilon(t), \quad \mu > 0$$

where $\varepsilon(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$

In this case Equation (1) will be as follows:

$$L(\theta|z)=\frac{\lambda^3e^{-\mu z}}{(\lambda+\mu+\theta)^3}+\frac{\lambda^3\mu e^{-\mu z}}{2}\int\limits_{s=0}^{\infty}S^2e^{-(\lambda+\mu+\theta)s}\int\limits_{\alpha=0}^{z+s}e^{\mu\alpha}L(\theta|\alpha)d\alpha ds \qquad (2)$$

We can derive differential equation from this integral equation. For this purpose, we will multiply both sides of equation (2) by $e^{\mu z}$

$$\mu L(\theta|z)+L'(\theta|z)=\frac{\lambda^3\mu}{2}\int\limits_{s=0}^{\infty}S^2e^{-(\lambda+\theta)s}L(\theta|z+s)ds$$

If to consider the following substitution $x=z+s$, multiply both sides of last equation by $e^{-(\lambda+\theta)z}$ and derive on z we can find the following differential equation:

$$L^{IV}(\theta|z)-[3(\lambda+\theta)-\mu]L'''(\theta|z)+3(\lambda+\theta)(\lambda+\theta-\mu)L''(\theta|z)-$$

$$-(\lambda+\theta)^2(\lambda+\theta-3\mu)L'(\theta|z)-\mu[(\lambda+\theta)^2-\lambda^3]L(\theta|z)=0$$

The general solution of this differential equation will be as follows :

$$L(\theta|z)=C_1(\theta)e^{k_{1(\theta)z}}=\frac{\lambda^3}{[\lambda+\theta-k_1(\theta)]^3}e^{k_{1(\theta)z}}.$$

This expression is the Laplace transform of the conditional distribution of τ_z^0 random variable. Then, we will need to find Laplace transform for the unconditional distribution of τ_z^0 random variable. In accordance with formula of total probability

$$L(\theta)=\int\limits_{z=0}^{\infty}L(\theta|z)\lambda^3z^2e^{-\lambda z}dz=\int\limits_{z=0}^{\infty}c_1(\theta)e^{k_{1(\theta)}z}\lambda^3z^2e^{-\lambda z}dz=$$

$$=c_1(\theta)\lambda^3\int\limits_{z=0}^{\infty}z^2e^{[k_1(\theta-\lambda)]z}dz=\frac{\lambda^3}{[\lambda-k_1(\theta)]^3}C_1(\theta)$$

Therefore

$$L(\theta)=\frac{\lambda^3}{[\lambda-k_1(\theta)]^3}C_1(\theta)$$

Respectively, we will get the following characteristics:

$$E\tau_1^0=-L'(0)=\frac{3(\lambda+\mu)}{\lambda(\lambda-3\mu)}$$

$$E(\tau_z^0|z)=\frac{3(1+z\mu)}{\lambda-3\mu}$$

$$E(\tau_z^0|z)=\frac{3}{(\lambda-3\mu)^2}+\frac{12(3+z\lambda)\mu}{(\lambda-3\mu)^3}$$

3. Conclusions

In this article we have defined Laplace transforms for the unconditional and conditional distribution of the first passage of zero level of semimarkov random processes with positive tendency and negative jump.

Acknowledgements

The research of Ulviyya Y. Karimova and Konul K. Omarova was supported by the Science Development Foundation under the President of the Republic of Azerbaijan-Grant N EIF/MQM/Elm-Tehsil-1- 2016-1(26)-71/19/1.

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Received: 21 December 2018
Accepted: 07 June 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

Methodological Aspects of Cluster Policy Formation in Azerbaijan

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Abstract. The ways to form and develop the country's competitively oriented national economy: 1) the role of clusters in the development of regional and national economies; 2) factors determining the growing impact of the state on clustering processes; 3) work carried out within the framework of cluster policy in Azerbaijan.

Key Words and Phrases: cluster policy, industrial parks, competitive economy, innovation activity.

1. Introduction

The formation and development of a competitively oriented national economy plays the key role in the recognition of the efficiency of a country's economy. As specified in the political programs of many countries, the competitiveness of the national economy is a key development priority.

The Global Competitiveness Index (GCI) of the CIS countries for 2008-2009 has the following rankings: Russia — 4.31 (51st place), Kazakhstan — 4.11 (66^{th} place), Azerbaijan — 4.1 (69^{th} place), Ukraine Belarus — 3.95 (82^{nd} place), Georgia — 3.86 (90^{th} place), Moldova — 3.75 (95^{th} place) [1]. The low level of the global GCI indicators in post-Soviet states being makes the problem of forming a competitively oriented national economy highly relevant.

Today, along with adapting strategic analysis tools to the world practice, there is a great need to develop new approaches to the economic justification of development concepts and strategies. Globalization processes are an objective condition for changing the paradigm of competitive management, which consists in the intensification of international competition that characterizes the world economy, the rejection of the traditional industry policy and the transition to a new, cluster system.

2. Problem statement

Technology clusters, such as technology parks, technopoles, scientific and technological centers, are very popular, well known and studied in the modern economy. Their importance in catalyzing innovation activity is undisputable.

Production cluster is a network of enterprises and organizations (including specialized service providers, manufacturers and buyers) which are concentrated around a science and education center, interconnected through mutual cooperation geographically and have partnership relations with local institutions and governing agencies with the purpose of stepping up the competitiveness of regions and the national economy.

The features of a production cluster that distinguish it from a technology cluster are as follows: manufacturing of a "core" product (the product with the largest share in the cluster production volume that defines the cluster name); cooperation relations with competitors (implementation of joint projects around common interests, such as education, science, marketing); combination of businesses with the completed production cycle (from raw materials to finished products).

International practice proves the importance of clusters in the development of regional and national economies, which is confirmed by the following conditions.

1. Clusters have positive external impacts. External impacts are due to the effect of one company's actions on other companies. The benefits of the cluster are distributed over all contact areas: new manufacturers from other industries accelerate the development of the entire group, stimulating the development of research and development; network cooperation leads to free information exchange through the channels of suppliers or consumers that are in contact with many competitors and through rapid spread of innovation; internal cluster relations create conditions for emerging competition methods that create conditions for innovation.

2. The cluster form of business organization causes a particular kind of innovation to emerge — a "general innovation product". A cluster based on vertical integration forms a specific system of dissemination of new knowledge and technologies rather than a spontaneous concentration of various scientific and technological inventions. Besides, the most important prerequisite for the efficient transformation of inventions into innovations and competitive advantages of innovations is the creation of a robust communication network involving all cluster participants. Within the framework of international technological cooperation, it is particularly important to have ties stimulating the formation of international clusters. Clusters create conditions for the formation of regional innovation systems.

3. Being the "development points" of the domestic market in the economy of the entire country or a region, clusters perform the function of assimilation of the international market. The presence of clusters in many industries accelerates the process of emergence of competitive factors through joint investment (as part of network cooperation and public-private partnerships) directed to the development of technology, information, infrastructure, education.

4. Major manufacturers of the cluster create demand for specialized material and technical resources and services. Intra-cluster relations ensure the development of external sources ("outsourcing"), which accelerates the development of small and medium-sized businesses in the region by small and medium-sized businesses producing products, jobs and services for key subjects of the cluster, thereby increasing their competitiveness [2].

5. Competition among cluster manufacturers leads to deeper specialization in the cluster, a search for new fields and cluster expansion, ultimately giving rise to new businesses that increase the profitability of regional production, solve employment issues and raise the integration potential.

6. Clusters are one of the institutional forms of ensuring frontier co-operation in trade, agriculture, tourism, transport and infrastructure; they facilitate economic development of frontier areas.

7. The development of clusters enhances connections between industries, stimulating economic growth. Entering foreign markets, competitors inside a cluster develop joint marketing programs and ensure an increase in export volumes. All of this contributes in general to socio-economic development and the competitiveness of regions and the national economy.

3. Methodological aspects of cluster policy formation

Foreign experience demonstrates that countries' strategies in cluster policy differ depending on national traditions and culture of their strategy engineering process, as well as on the cluster concept. Analysis of information sources shows that technical and methodological framework for cluster policy formulation have not been clearly and unambiguously studied in science.

The most important methodological document on cluster policy is the European Cluster Memorandum signed by the Member States of the European Union in 2006 [3]. It defined the essence and importance of clusters in innovation development and identified the key provisions of cluster policy.

One of the essential methodological issues is the role of government in the formation of clusters. The following factors can be associated with the state's growing influence on clustering processes:

- market weakness, increase in the volume and value of public goods;
- an objective priority of public interest in the context of globalization;
- the need to protect the national economy in the international economic relations;
- the need for institutional regulations in the national and global economy.

Foreign experience demonstrates that the numerous cluster initiatives running in major developed countries over the last few years have been brought forward by local or regional governments. In relatively small developed countries and in a number of developing countries, the government plays an essential role in the cluster development initiative, especially when local and regional government agencies cannot partner with the private sector. A number of decisions on clusters have been adopted at national level in countries with centralized decision-making process.

A new era in the development of industry began under the leadership of President Ilham Aliyev in 2004. During this period, some of the revenues from the oil and gas sector have been directed to the development of various industries, state programs for the optimization of the industrial structure in the regions have been developed, substantial work has been done to address the energy supply problem, the overall infrastructure has been improved, and numerous projects for the opening of new production facilities have been implemented. The favorable business environment created in the country and the important decisions in the field of business regulation played a significant role in the development of industries. Due to the state support measures in the field of business development carried out in recent years, the share of the private sector in the GDP in 2015 was 81.2 percent. The number of businesses was 677,000, including 100,000 legal entities [4].

Since 2012, innovations in the regions have been supported by the state as part of cluster policy in Azerbaijan. The cluster approach stimulates the growth of territorial and socio-economic development, competitiveness of industries and the region, labor productivity, budget revenues, etc.

As a follow-up to the work done, the year 2014 was declared the "Year of Industry" in the Republic of Azerbaijan by Decree No. 212 of the President of the Republic of Azerbaijan dated 10 January 2014, and the plan of industrial development measures was implemented. Also, the State Program for Industrial Development in the Republic of Azerbaijan for 2015-2020 was approved by Decree No. 964 of the President of the Republic of Azerbaijan dated 26 December 2014. The implementation of the planned state policy has created conditions for the formation of sustainable financial resources in the country and thereby for the development of all industries. The volume of industry has grown almost twice over the last ten years, which is mainly due to the non-oil sector.

Under the leadership of the President of the Republic of Azerbaijan Mr. Ilham Aliyev, complex measures are being implemented to diversify the non-oil sector, to create new production areas based on competitive and export-oriented innovations, to intensify economic activity and to support business activity. The infrastructure required for the efficient business activity is being created and preferential treatment is introduced in industrial parks, which play an important role in the development of industry; the interest of entrepreneurs in industrial zones is growing. As a result of the implementation of the investment promotion mechanism, investment promotion certificates were issued to 182 projects in a short time; as a result of their implementation, over 1.6 billion manat will be invested in the national economy and around 12,000 new jobs will be created. 62% of these projects are industry-related [5].

With the purpose of supporting the activity of small and medium-sized businesses in industry and increasing the employment of the population, the head of state signed the Order on establishing the Hajigabul Industrial Site.

SOCAR Polymer was founded on 16 July 2016 in order to enhance the country's chemical industry. The company's production facilities include two plants, one producing polypropylene (PP) and the other high-density polyethylene (HDPE). These plants are currently being constructed in the grounds of the Sumgayit Chemical Industry Park

(SCIP). The PP and HDPE plants with the capacity of 18,000 and 120,000 t/yr, respectively, will be commissioned in 2018. The reason for founding SOCAR Polymer LLC is that our country currently exports low-density polyethylene and imports high-density polyethylene, and the main purpose of the SOCAR Polymer project is to eliminate imports in this field. Along with the domestic market, the products will be exported to the Turkish and European markets. The work on commissioning the production lines with annual capacity of 180,000 tons for polypropylene based on the Canadian technology and 120,000 tons for high-density polyethylene based on the Austrian technology has been in progress since the beginning of 2018. Propylene and ethylene produced by the Azerkimya Production Association are used as raw material. About 3,000 workers are involved in the construction of the facility. SOCAR Polymer LLC is negotiating on creating clusters around these facilities in the future.

Integration of national clusters into the international cluster network for Azerbaijan enhances the competitiveness of enterprises on international level by increasing the quality and rate of economic growth, raising the level of the national technological base, and the performance of advanced management methods.

The search for sources of financing of innovation activities being crucial for the national economy, the formation and development of clusters can be one of the most effective mechanisms for spurring foreign investment, including foreign economic integration processes.

The functioning and development of clusters in the region creates conditions for improving the competitiveness of the business environment through the opportunities of employing additional kinds of services by establishing mutually beneficial relations between enterprises and organizations, citing them close to each other, cutting transport costs, supplying new ideas, developing innovative infrastructure, collaborating with organizations with developed infrastructure and established domestic and foreign relations.

Based on the above, it should be noted that the cluster policy of innovation development is the main strategy of regional innovation development in the conditions of the modernization of the Azerbaijani economy. Cluster, in its turn, is a kind of regional innovation system that organizes the essential elements, which design, manufacture and implement innovations.

The global economic crisis has raised the issue of Azerbaijan's transition to the path of innovation development to a new level, which, in turn, requires a revision of key sources of strategic directions of competitively oriented economy. It is necessary to change the direction of economic development, accelerating economic development, ensuring economic security and reducing dependence on the world market.

Statistical data analysis shows the presence of a number of problems hindering the efficient implementation of the innovative potential of Azerbaijan. The results of innovative development are far from satisfactory. The share of Azerbaijan in the global market of high technology products is only 0.3%, much smaller than that of developed countries. The efficiency of the high technology sector in Azerbaijan is very low. Apart from that, the share of R&D expenditure in the GDP in Azerbaijan is 1.16%, which is due to the fact that, unlike the Western countries, the main source of funding for science in Azerbaijan is the state, not the private sector. These figures indicate the weak sensitivity of Azerbaijani enterprises to innovation [6].

Thus, we can conclude that the only condition for the integration of the Azerbaijani economy into the global economic community as an equal participant rather than a primary producer is to transition the structure of all sectors of the national economy to the innovative mode involving economical use of raw materials. The first steps have been taken in this direction; however, it is necessary to speed up these processes to maintain the existing scientific and educational potential and create a competitive and beneficial partnership in the field of science.

International practices show that countries with a high innovation potential have an independent position. Innovation is a driving force of all social development, the source that boosts economic growth. Building a national innovation system is one of the main objectives of the Azerbaijani economy.

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Received: 03 March 2019
Accepted: 11 June 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

On the Solution of Generalized Fractional Kinetic Equations Involving Generalized M-Series

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Abstract. This paper refers to further generalizations of fractional kinetic equation. By using the generalized M-Series, solutions of unified fractional kinetic equations are obtained. Solutions are obtained in a compact form containing Wright hypergeometric function by using Laplace transform and Sumudu transform. Certain special cases of our main results are also pointed out.

Key Words and Phrases: generalized fractional kinetic equation, fractional calculus, Laplace transform, Sumudu transform, generalized M-series, special function.

2010 Mathematics Subject Classifications: 26A33, 44A10, 33E12, 34A08

1. Introduction

The M-series was introduced by the mathematician M. Sharma [10], and defined as

$$ {}_p\overset{\alpha}{M}_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;\ z\right) = {}_p\overset{\alpha}{M}_q\left(z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{\Gamma(\alpha\, k+1)}, \qquad (1) $$

where $z,\alpha\in C,\Re(\alpha)>0$ and $(a_i)_k\,,(b_j)_k\,(i=1,\cdots,p;j=1,\cdots,q)$ are the Pochhammer symbol given by $(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$.

The series in (1) is convergent for all z if $p\leq q$, also if $p=q+1$ its convergent absolutely or conditionally when $|z|=1$, and divergent if $p>q+1$.

In 2009, the generalization of (1) was introduced and studied by Sharma and Jain [11], and given as

$$ \underset{p,q}{\overset{\alpha,\beta}{M}}(z) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{\Gamma(\alpha k+\beta)}. \qquad (2) $$

The series in (2) is convergent for all z if $p\leq q+\Re(\alpha)$, also it is convergent for $|z|<\delta=\alpha^{\alpha}$ if $p=q+\Re(\alpha)$ and divergent if $p>q+\Re(\alpha)$.

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Recently, a new generalization of M-series introduced by Faraj *et al.* [2] in the following manner:

$$\underset{\mathrm{p,\,q;\,m,\,n}}{\overset{\alpha,\,\beta}{\mathrm{M}}}(a_1,\cdots,a_p;b_1,\cdots,b_q;\ z)\ =\underset{\mathrm{p,\,q;\,m,\,n}}{\overset{\alpha,\,\beta}{\mathrm{M}}}(z)\ =\ \sum_{\mathrm{k=0}}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{\mathrm{z^k}}{\Gamma(\alpha k+\beta)},\tag{3}$$

where $z,\alpha,\beta\in C,\Re(\alpha)>0$ and m,n are non–negative real number.

The series in (3) is absolutely convergent for all values of z provided that $pm<qn+\Re(\alpha)$, moreover if $pm=qn+\Re(\alpha)$, the series converges for $|z|<\delta=\alpha^{\alpha}$.

For $m=n=1$ *and* $m=n=\beta=1$, equation (3) reduces to generalized M-series $\underset{\mathrm{p,q}}{\overset{\alpha,\beta}{\mathrm{M}}}(\mathrm{z})$ and M-series ${}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}(\mathrm{z})$, respectively (see (1) and (2)).

Further, if we take $p=q=1$, equation (3) reduces to generalized Mittag–Leffler function introduced by Salim and Faraj [4] and given as

$$\underset{1,1\,;\,\mathrm{m,\,n}}{\overset{\alpha,\beta}{\mathrm{M}}}(\mathrm{z})=E\begin{array}{c}a_1,b_1,m\\ \alpha,\beta,n\end{array}(z)=\sum_{\mathrm{k=0}}^{\infty}\frac{(a_1)_{km}}{(b_1)_{kn}}\frac{\mathrm{z^k}}{\Gamma(\alpha k+\beta)}.\tag{4}$$

The generalized Wright hypergeometric function was introduced by Wright [14] and defined as

$${}_p\Psi_q(z)\ =\ {}_p\Psi_q\left[\begin{array}{ll}(\mathrm{a}_i,\ \alpha_i)_{1,\,\mathrm{p}} & \\ (\mathrm{b_j},\ \beta_{\mathrm{j}})_{1,\mathrm{q}} & ;\ \mathrm{z}\end{array}\right]=\sum_{\mathrm{k=0}}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{\mathrm{j=1}}^{q}\Gamma(b_j+\beta_j k)}\frac{\mathrm{z^k}}{\mathrm{k!}},\tag{5}$$

where $z,a_i,b_j\in\mathbb{C}$ and $\alpha_i,\beta_j\in\mathbb{R}-\{0\}\,(i=1,\cdots,p;j=1,\cdots,q)$.

Haubold and Mathai [3] established a fractional differential equation between the rate of change of reaction, the destruction rate and the production rate as follows:

$$\frac{\mathrm{dN}}{\mathrm{dt}}=-\,\mathrm{d}(\mathrm{N_t})+\mathrm{p}(\mathrm{N_t}),\tag{6}$$

where $N=N(t)$ is the rate of reaction, $d(N_t)$ is the rate of destruction, $p(N_t)$ is the rate of production and N_t denotes the function defined by $N_t\,(t^*)=N\,(t-t^*)\,,\ t^*>0$.

A special case of (6), when spatial fluctuations or homogeneities in the quantity $N(t)$ are neglected, is given by the following differential equation:

$$\frac{\mathrm{dN}_i}{\mathrm{dt}}=-\,\mathrm{c}_i\ \mathrm{N}_i(\mathrm{t})\tag{7}$$

with the initial condition that $N_i\,(t=0)=N_0$ is the number of density of species i at time $t=0$ and constant $c_i>0$. If we remove the index i and integrate the standard kinetic equation (7), we have

$$\mathrm{N(t)}-\mathrm{N}_0=-\,c\ {}_0\mathrm{D}_{\mathrm{t}}^{-1}\mathrm{N(t)},\tag{8}$$

where ${}_0\mathrm{D}_{\mathrm{t}}^{-1}$ is the standard integral operator.

Houbold and Mathai [3], obtained the fractional generalization of the standard kinetic equation (7) as

$$\mathrm{N(t)} - \mathrm{N}_0 = -\, c^{\nu}_0 \mathrm{D}_{\mathrm{t}}^{-v} \mathrm{N(t)}, \tag{9}$$

where ${}_0\mathrm{D}_{\mathrm{t}}^{-v}$ is Riemann–Liouville fractional integral operator defined as follows [5]:

$${}_0\mathrm{D}_{\mathrm{t}}^{-v} f(t) = \frac{1}{\Gamma(\nu)} \int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\nu-1} f(\mathrm{s})\,\mathrm{ds}\ (\mathrm{t}>0,\, f(\nu)>0). \tag{10}$$

The solution of equation (8) is given by (See [3])

$$\mathrm{N(t)} = \mathrm{N}_0 \sum_{k=0}^{\infty} \frac{(-1)^{\mathrm{k}}}{\Gamma(\nu\, \mathrm{k}+1)} (\mathrm{ct})^{\nu \mathrm{k}}. \tag{11}$$

Further, Saxena and Kalla [6] considered the following fractional kinetic equation

$$\mathrm{N(t)} - \mathrm{N}_0\, f(\mathrm{t}) = -\, \mathrm{c}^{\nu}_0 \mathrm{D}_{\mathrm{t}}^{-v} \mathrm{N(t)},\ (\Re(\nu) > 0), \tag{12}$$

where $N(t)$ denotes the number of density of a given species at time t, $N_0 = N(0)$ is the number of density of that species at time $t = 0$ and c is a constant.

2. Solution of generalized fractional kinetic equations by using the Laplace transform

In this section, we will establish and derive the solution of the generalized kinetic equations involving the generalized M-series (3) by applying the Laplace transform.

Laplace transform [12] of the function $f(\mathrm{t})$ is defined as

$$\mathrm{L}\{f(t) : \mathrm{s}\} = \int_0^{\infty} \mathrm{e}^{-\mathrm{st}} f(\mathrm{t})\,\mathrm{dt},\ (\Re\,(\mathrm{s}) > 0). \tag{13}$$

and convolution theorem is given by

$$L\{f * g\}(s) = L\left\{\int_0^t f(t-\xi)\, g(\xi)\, \mathrm{d}\xi\right\} = L\{f(s)\}.L\{g(s)\}. \tag{14}$$

Laplace transform of the Riemann–Liouville fractional integral operator given by Erdélyi *et al.* [1] as

$$\mathrm{L}\left\{{}_0\mathrm{D}_{\mathrm{t}}^{-\mathrm{v}}\, \mathrm{N(t)} :\ \mathrm{s}\right\} =\ \mathrm{s}^{-\nu}\, \mathrm{N(s)}, \tag{15}$$

and also

$$\mathrm{L}\{\mathrm{N(t)} :\ \mathrm{s}\} = \mathrm{N(s)}. \tag{16}$$

The following Lemmas are required to prove our main results.

Lemma 1. *For $\Re(\gamma), \Re(\sigma), \Re(s) > 0$, the following Laplace transform of generalized M-series $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ holds true:*

$$L\left\{t^{\gamma-1} \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(t^{\sigma}) : s\right\}$$
$$= s^{-\gamma}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\ {}_{\mathrm{p+2}}\Psi_{\mathrm{q+1}}\left[\begin{matrix}(\mathrm{a}_1,\mathrm{m}),\cdots,(\mathrm{a}_\mathrm{p},\ \mathrm{m}),(\gamma,\sigma),(1,1)\\(b_1,n),\cdots,(b_q,n),(\beta,\alpha)\end{matrix}\ ;s^{-\sigma}\right], \tag{17}$$

where ${}_{\mathrm{p+2}}\Psi_{q+1}(.)$ is given by (5).

Proof. By taking (3) and (13) into account, we can easily obtain the required result (17) after a little simplification.

If we take $\gamma = \beta$ and $\sigma = \alpha$ in (17), then a special case of (17) is given by following lemma.

Lemma 2. *For $\min\{\Re(s), \Re(\alpha), \Re(\beta)\} > 0$, the Laplace transform of (3) is given by*

$$L\left\{t^{\beta-1} \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(t^{\alpha}) : s\right\} = s^{-\beta}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\ {}_{\mathrm{p+1}}\Psi_{\mathrm{q}}\left[\begin{matrix}(\mathrm{a}_1,\mathrm{m}),\cdots,(\mathrm{a}_\mathrm{p},\mathrm{m}),(1,1)\\(b_1,n),\cdots,(b_q,n),\end{matrix}\ ;s^{-\alpha}\right]. \tag{18}$$

Theorem 1. *Let $c,\ w, \nu, \gamma, \sigma \in \mathbb{R}^+; \alpha, \beta, t \in C;\ m,\ n > 0; \Re(\alpha) > 0$ and $pm \le qn + \Re(\alpha)$. Then, the solution of the following generalized fractional kinetic equation*

$$\mathrm{N(t)} - \mathrm{N_0}\ \mathrm{t}^{\gamma-1} \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(wt^{\sigma}) = -\mathrm{c}^{\nu}_0\mathrm{D}_\mathrm{t}^{-v}\mathrm{N(t)} \tag{19}$$

is given by

$$N(t) = \mathrm{N_0}\, t^{\gamma-1}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\sum_{r=0}^{\infty}(-c^{v}t^{v})^{r}$$
$$\times\ {}_{\mathrm{p+2}}\Psi_{\mathrm{q+2}}\left[\begin{matrix}(\mathrm{a}_1,\mathrm{m}),\cdots,(\mathrm{a}_\mathrm{p},\mathrm{m}),\ (\gamma,\sigma),\ (1,1)\\(b_1,n),\cdots,(b_q,n),(\beta,\alpha),\ (\gamma+\nu r,\sigma)\end{matrix}\ ;wt^{\sigma}\right]. \tag{20}$$

Proof. Applying the Laplace transform on both sides of (19). Using (15) and (16) into account, we get

$$L\{N(t)\ :s\} - \mathrm{N_0}\ L\left\{t^{\gamma-1} \underset{p,q;\ m,n}{\overset{\alpha,\beta}{M}}(wt^{\sigma}) : s\right\} = -c^{v}\, L\left\{{}_0D_t^{-\nu}N(t)\ :s\right\}$$

$$N(s) = \frac{\mathrm{N_0}}{1+\left(\frac{c}{s}\right)^{\nu}} L\left\{t^{\gamma-1} \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(wt^{\sigma}) : s\right\}$$

Next, by using (17) and the following binomial series expansion

$$\left[1+\left(\frac{c}{s}\right)^{\nu}\right]^{-1} = \sum_{r=0}^{\infty}\ (-1)^{r}\ \left(\frac{c}{s}\right)^{vr} (c < |s|),$$

we obtain

$$N(s) = \mathrm{N}_0 \sum_{r=0}^{\infty}(-1)^r \left(\frac{c}{s}\right)^{\nu r} \sum_{k=0}^{\infty} \frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}} \frac{\Gamma(\gamma+\sigma\mathrm{k})\ \ \Gamma(\mathrm{k}+1)}{\Gamma(\alpha\mathrm{k}+\beta)} \frac{\mathrm{w}^{\mathrm{k}}}{k!} \frac{1}{\mathrm{s}^{\gamma+\sigma\mathrm{k}}}$$

$$N(s) = \mathrm{N}_0 \sum_{r=0}^{\infty}(-c^{v})^r \sum_{k=0}^{\infty} \frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}} \frac{\Gamma(\gamma+\sigma\mathrm{k})\Gamma(\mathrm{k}+1)}{\Gamma(\alpha\mathrm{k}+\beta)} \frac{\mathrm{w}^{\mathrm{k}}}{\mathrm{k}!} \frac{1}{\mathrm{s}^{\gamma+\nu\mathrm{r}+\sigma\mathrm{k}}}. \tag{21}$$

Now, taking inverse Laplace transform of (21) and using $\mathrm{L}^{-1}\{\ \mathrm{s}^{-\nu}\!:\ \mathrm{t}\} = \frac{\mathrm{t}^{\nu-1}}{\Gamma(\nu)}$, $(\Re\,(\mathrm{v}) > 0)$ and $\mathrm{L}^{-1}\{\mathrm{N}(s)\!:\ \mathrm{t}\} = N(t)$, we arrive at

$$L^{-1}\{N(s):t\} = \mathrm{N}_0 \sum_{r=0}^{\infty}(-c^{v})^r \sum_{k=0}^{\infty} \frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}$$

$$\times \frac{\Gamma(\gamma+\sigma\mathrm{k})\ \ \Gamma(\mathrm{k}+1)}{\Gamma(\alpha\mathrm{k}+\beta)} \frac{\mathrm{w}^{\mathrm{k}}}{\mathrm{k}!}\ \ \mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}^{\gamma+\nu\mathrm{r}+\sigma\mathrm{k}}}\!:\ \mathrm{t}\right\}$$

or

$$N(t) = \mathrm{N}_0 \sum_{r=0}^{\infty}(-c^{\nu})^r \sum_{k=0}^{\infty} \frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}} \frac{\Gamma(\gamma+\sigma\mathrm{k})\ \ \Gamma(\mathrm{k}+1)}{\Gamma(\alpha k+\beta)} \frac{\mathrm{w}^{\mathrm{k}}}{\mathrm{k}!} \frac{\mathrm{t}^{\gamma+\nu\mathrm{r}+\sigma\mathrm{k}-1}}{\Gamma(\gamma+\nu\mathrm{r}+\sigma\mathrm{k})}$$

$$= \mathrm{N}_0\, t^{\gamma-1} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{r=0}^{\infty}(\,-\,c^{v}\mathrm{t}^{\nu})^r$$

$$\times \sum_{\mathrm{k}=0}^{\infty} \frac{\Gamma(\mathrm{a}_1+\mathrm{mk})\cdots\Gamma(\mathrm{a}_\mathrm{p}+\mathrm{mk})}{\Gamma(\mathrm{b}_1+\mathrm{nk})\cdots\Gamma(\mathrm{b}_\mathrm{q}+\mathrm{nk})} \frac{\Gamma(\gamma+\sigma\mathrm{k})\,\Gamma(\mathrm{k}+1)}{\Gamma(\alpha\mathrm{k}+\beta)\,\Gamma(\gamma+\nu\mathrm{r}+\sigma\mathrm{k})} \frac{(\mathrm{wt}^{\sigma})^{\mathrm{k}}}{\mathrm{k}!}.$$

Finally, by using (5), we get the desired result (20).
This complete the proof of Theorem 1.

If we set $m = n = 1$, then $\underset{\mathrm{p,q;m,n}}{\overset{\alpha,\beta}{\mathrm{M}}}(z)$ reduces to the generalized M-series $\underset{\mathrm{p,q}}{\overset{\alpha,\beta}{\mathrm{M}}}(z)$ [11], we get the generalized fractional kinetic equation with its solution given as follows:

Corollary 1. *Let* $c,\ w,\nu,\gamma,\sigma \in \mathbb{R}^{+};\alpha,\beta,t \in C;\ m,\ n > 0\ ;\ \Re\,(\alpha) > 0$ *and* $p \leq q + \Re(\alpha)$. *Then, the solution of the equation*

$$N(t) - \mathrm{N}_0\, t^{\gamma-1} \underset{p,q}{\overset{\alpha,\beta}{M}}(wt^{\sigma}) = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{22}$$

is given by

$$N(t) = \mathrm{N}_0\ t^{\gamma-1} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{\mathrm{r}=0}^{\infty}(\,-\,c^{v}\mathrm{t}^{\nu})^r$$

$$\times\ {}_{\mathrm{p}+2}\Psi_{\mathrm{q}+2}\left[\begin{array}{c}(\mathrm{a}_1,1),\cdots,(\mathrm{a}_\mathrm{p},1),(\gamma,\sigma),(1,1)\\(b_1,1),\cdots,(b_q,1),(\beta,\alpha),\ (\gamma+\nu r,\sigma)\end{array};wt^{\sigma}\right]. \tag{23}$$

If we take $\beta = 1$ in (22), the generalized M-series $\underset{\mathrm{p,q}}{\overset{\alpha,\beta}{\mathrm{M}}}(z)$ reduces to the M-series ${}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}(z)$ [10], we arrive at

Corollary 2. *Let* $c,\ w, \nu, \gamma, \sigma \in \mathbb{R}^{+}; \alpha, t \in C;\ m,\ n > 0\ ;\ \Re(\alpha) > 0$. *Then, the solution of the equation*

$$N(t) = \mathrm{N}_0\, t^{\gamma-1}\, {}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}(wt^{\sigma}) = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{24}$$

is given by

$$N(t) = \mathrm{N}_0\, t^{\gamma-1} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{r=o}^{\infty} (-c^{v}t^{v})^{r}$$

$$\times\ {}_{\mathrm{p+2}}\Psi_{\mathrm{q+2}}\left[\begin{matrix}(\mathrm{a}_1, 1), \cdots, (\mathrm{a}_{\mathrm{p}}, 1), (\gamma,\sigma), (1,1) \\ (b_1, 1), \cdots, (b_q, 1), (1,\alpha), (\gamma + \nu r, \sigma)\end{matrix} ; wt^{\sigma}\right]. \tag{25}$$

Further, if we put $p = q = l$, then $\underset{\mathrm{p,q;m,n}}{\overset{\alpha,\beta}{\mathrm{M}}}(z)$ reduces to the generalized Mittag–Leffler function $E\begin{matrix} a_1, b_1, m \\ \alpha, \beta, n\end{matrix}(\mathrm{z})$ [4], we obtain

Corollary 3. *Let* $c,\ w, \nu, \gamma, \sigma \in \mathbb{R}^{+}; \alpha, \beta, t \in C;\ m,\ n > 0\ ;\ \Re(\alpha) > 0$ *and* $m \le n + \Re(\alpha)$. *Then, the solution of the equation*

$$N(t) - \mathrm{N}_0\, t^{\gamma-1} E\begin{matrix} a_1, b_1, m \\ \alpha, \beta, n\end{matrix}(\mathrm{wt}^{\sigma}) = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{26}$$

is given by

$$N(t) = \mathrm{N}_0\, t^{\gamma-1} \frac{\Gamma(b_1)}{\Gamma(a_1)} \sum_{r=0}^{\infty} (-c^{v}t^{v})^{r}\ {}_3\Psi_3\left[\begin{matrix}(\mathrm{a}_1, \mathrm{m}),\ (\gamma,\sigma),\ (1,1) \\ (b_1, n), (\beta,\alpha), (\gamma + \nu r, \sigma)\end{matrix} ; wt^{\sigma}\right]. \tag{27}$$

Theorem 2. *Let* $c,\ w, \nu \in \mathbb{R}^{+}; \alpha, \beta, t \in C;\ m,\ n > 0; \Re(\alpha) > 0$ *and* $pm \le qn + \Re(\alpha)$. *Then, the generalized fractional kinetic equation*

$$N(t) - \mathrm{N}_0\, t^{\beta-1} \underset{\mathrm{p,q;\ m,n}}{\overset{\alpha,\beta}{\mathrm{M}}}(\mathrm{wt}^{\alpha}) = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{28}$$

has the solution

$$N(t) = \mathrm{N}_0\, t^{\beta-1} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{r=0}^{\infty} (-c^{v}t^{v})^{r}$$

$$\times\ {}_{p+1}\Psi_{q+1}\left[\begin{matrix}(\mathrm{a}_1, \mathrm{m}), \cdots, (a_{\mathrm{p}}, m), (1,1) \\ (b_1, n), \cdots, (b_q, n), (\beta + \nu r, \alpha)\end{matrix} ; wt^{\alpha}\right]. \tag{29}$$

Proof. The proof of result asserted by Theorem 2 runs parallel to that of Theorem 1. Here, we make use (18) instead of (17) into account. Therefore, we omit the details of the proof.

If we put $m = n = 1$, then $\underset{\mathrm{p,q;\ m,n}}{\overset{\alpha,\beta}{\mathrm{M}}}(z)$ reduces to $\underset{\mathrm{p,q}}{\overset{\alpha,\beta}{\mathrm{M}}}(z)$, we get the following corollary.

Corollary 4. *Let $c,\ w, \nu \in \mathbb{R}^{+}; \alpha, \beta, t \in C;\ m,\ n > 0; \Re(\alpha) > 0$ and $p \leq q + \Re(\alpha)$. Then, the equation*

$$N(t) - \mathrm{N}_0\, t^{\beta-1} \overset{\alpha,\beta}{\underset{\mathrm{p,q}}{\mathrm{M}}} (\mathrm{wt}^{\alpha}) \ = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{30}$$

has the solution

$$N(t) = \mathrm{N}_0\, t^{\beta-1} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{r=0}^{\infty} (-c^v t^v)^r$$

$$\times\ {}_{p+1}\Psi_{q+1} \left[\begin{array}{c} (\mathrm{a}_1, 1), \cdots, (a_{\mathrm{p}},1),\ (1,1) \\ (b_1, 1), \cdots, (b_q, 1), (\beta + \nu r, \alpha) \end{array} ; wt^{\alpha} \right]. \tag{31}$$

If we take $\beta = 1$ in (30), we have the solution of generalized fractional kinetic equation involving M-series ${}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}}(z)$ as follows:

Corollary 5. *Let $c,\ w, \nu \in \mathbb{R}^{+}; \alpha, t \in C;\ m,\ n > 0; \Re(\alpha) > 0$. Then, the equation*

$$N(t) - \mathrm{N}_0\, {}_{\mathrm{p}}\overset{\alpha}{\mathrm{M}}_{\mathrm{q}} (\mathrm{wt}^{\alpha}) \ = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{32}$$

has the solution

$$N(t) = \mathrm{N}_0 \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{r=0}^{\infty} (-c^v t^v)^r\ {}_{p+1}\Psi_{q+1} \left[\begin{array}{c} (\mathrm{a}_1, 1), \cdots, (a_{\mathrm{p}},1), (1,1) \\ (b_1, 1), \cdots, (b_q, 1), (1 + \nu r, \alpha) \end{array} ; wt^{\alpha} \right]. \tag{33}$$

Further, if we set $p = q = 1$ in (28), then $\overset{\alpha,\beta}{\underset{\mathrm{p,q;\ m,n}}{\mathrm{M}}}(z)$ reduces to $E \begin{array}{c} a_1, b_1, m \\ \alpha, \beta, n \end{array} (\mathrm{z})$ we have the following corollary.

Corollary 6. *Let $c,\ w, \nu \in \mathbb{R}^{+}; \alpha, \beta, t \in C;\ m,\ n > 0; \Re(\alpha) > 0$ and $\mathrm{m} \leq n + \Re(\alpha)$. Then, the equation*

$$N(t) - \mathrm{N}_0\, t^{\beta-1}\, E \begin{array}{c} a_1, b_1, m \\ \alpha, \beta, n \end{array} (\mathrm{wt}^{\alpha}) \ = -\mathrm{c}_0^{\nu}\mathrm{D}_{\mathrm{t}}^{-v} N(t) \tag{34}$$

has the solution

$$N(t) = \mathrm{N}_0\, t^{\beta-1} \frac{\Gamma(b_1)}{\Gamma(a_1)} \sum_{r=0}^{\infty} (-c^v t^v)^r\ {}_2\Psi_2 \left[\begin{array}{c} (\mathrm{a}_1, \mathrm{m}), (1,1) \\ (b_1, n), (\beta + \nu r, \alpha) \end{array} ; wt^{\alpha} \right]. \tag{35}$$

3. Solution of generalized fractional kinetic equations by using the Sumudu transform

In this section, we will discuss the solution of the generalized fractional kinetic equation (18) and (27) involving the generalized M-series [2] by applying another integral transform (i.e. Sumudu transform) technique.

Sumudu transform [13] of the function $f(t)$ is defined as

$$S\{f(t):u\} = \int_0^\infty e^{-t} f(ut)\, dt. \tag{36}$$

The convolution theorem for Sumudu transform is given by

$$S\{f * g : u\} \;=\; u\, S\{f : u\}\, S\{g : u\}. \tag{37}$$

If we apply (37) then, the Sumudu transform of the Riemann–Liouville fractional integral operator (10) is given by

$$S\left\{ {}_0D_t^{-\nu} f(t) : u \right\} = u\, S\left\{ \frac{t^{\nu-1}}{\Gamma(\nu)} \right\} S\{f(t) : u\} \tag{38}$$

and also

$$S\{N(t):u\} = N(u). \tag{39}$$

Now, we begin by stating and proving the following Lemmas.

Lemma 3. *For* $min\{\Re(\gamma), \Re(\sigma), \Re(u)\} > 0$, *the Sumudu transform of the generalized* M*-series* $\underset{p,q;\, m,n}{\overset{\alpha,\beta}{M}}(z)$ *is given by*

$$S\left\{ t^{\gamma-1} \underset{p,q;m,n}{\overset{\alpha,\beta}{M}} (t^{\sigma}) : u \right\}$$
$$= u^{\gamma-1} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\; {}_{p+2}\Psi_{q+1}\left[\begin{matrix} (a_1, m), \cdots, (a_p, m),\ (\gamma,\sigma),\ (1,1) \\ (b_1, n), \cdots, (b_q, n), (\beta, \alpha) \end{matrix} ; u^{\sigma} \right]. \tag{40}$$

Proof. By taking (3) *and* (36) *into account, we can easily obtain* (40) *after a little simplification.*

If we take $\gamma = \beta$ and $\sigma = \alpha$ in (40), then a special case of the above Lemma 3 is given by

Lemma 4. *For* $min\{\Re(\alpha), \Re(\beta), \Re(u)\} > 0$, *the following Sumuda transform of generalized* M*-series* (3) *holds true:*

$$S\left\{ t^{\beta-1} \underset{p,q;m,n}{\overset{\alpha,\beta}{M}} (t^{\alpha}) : u \right\} = u^{\beta-1} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\; {}_{p+1}\Psi_{q}\left[\begin{matrix} (a_1, m), \cdots, (a_p, m), (1,1) \\ (b_1, n), \cdots, (b_q, n), \end{matrix} ; u^{\alpha} \right]. \tag{41}$$

Discussion I. Let $c,\ w, v, \gamma, \sigma \in \mathbb{R}^+$ *and* $\Re(u) > 0$ with $|u| < c^{-1}(c \neq w)$. Also $\alpha, \beta, t \in C$; $m, n > 0$; $\Re(\alpha) > 0$ *and* $pm \leq qn + \Re(\alpha)$. Then, the solution of the generalized fractional kinetic equation (19) is given by (20).

By taking the Sumudu transform on both side of (19). Using (38) and (39), we have

$$\mathrm{S}\{\mathrm{N(t)} :\ \mathrm{u}\} - \mathrm{N}_0\, \mathrm{S}\left\{ \mathrm{t}^{\gamma-1} \underset{\mathrm{p,q;m,n}}{\overset{\alpha,\beta}{\mathrm{M}}} (\mathrm{wt}^{\sigma}) :\ \mathrm{u} \right\} = -\mathrm{c}^{\nu} \mathrm{S}\left\{ {}_0\mathrm{D}_{\mathrm{t}}^{-v} N(t)\ : u \right\}$$

$$N(u) = \frac{\mathrm{N}_0}{1+c^\nu u^\nu}\; S\left\{ \mathrm{t}^{\gamma-1} \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(wt^\sigma) \;: u\right\}$$

. Next, by using (39) and the binomial series expansion $(1+\mathrm{c}^\nu \mathrm{u}^\nu)^{-1} = \sum_{\mathrm{r}=0}^{\infty}(-1)^{\mathrm{r}}(\mathrm{cu})^{\nu \mathrm{r}}$, we obtain

$$\mathrm{N(u)} = \mathrm{N}_0 \sum_{\mathrm{r}=0}^{\infty}(-1)^r(\mathrm{cu})^{\nu r}\sum_{k=0}^{\infty}\frac{(\mathrm{a}_1)_{\mathrm{km}}\cdots(\mathrm{a_p})_{\mathrm{km}}}{(\mathrm{b}_1)_{\mathrm{kn}}\cdots(\mathrm{b_q})_{\mathrm{kn}}}\,\frac{\Gamma(\gamma+\sigma\mathrm{k})\;\Gamma(\mathrm{k}+1)}{\Gamma(\alpha k+\beta)}\,\frac{\mathrm{w^k}}{\mathrm{k!}}\mathrm{u}^{\gamma+\sigma\mathrm{k}-1}$$

$$= \mathrm{N}_0 \sum_{r=0}^{\infty}(-c^\nu)^r\sum_{k=0}^{\infty}\frac{(\mathrm{a}_1)_{\mathrm{km}}\cdots(\mathrm{a_p})_{\mathrm{km}}}{(\mathrm{b}_1)_{\mathrm{kn}}\cdots(\mathrm{b_q})_{\mathrm{kn}}}\,\frac{\Gamma(\gamma+\sigma\mathrm{k})\;\Gamma(\mathrm{k}+1)}{\Gamma(\alpha k+\beta)}\,\frac{\mathrm{w^k}}{\mathrm{k!}}\mathrm{u}^{\gamma+vr+\sigma\mathrm{k}-1}. \tag{42}$$

Now, taking inverse Sumudu transform of (42) and using

$$S^{-1}\left\{u^{v-1}\;:t\right\} = \frac{t^{\nu-1}}{\Gamma(\nu)}, (\min\{\Re(\nu), \Re(u)\} > 0)$$

and $S^{-1}\{N(u)\;:t\} = N(t)$, we get

$$S^{-1}\{N(u):t\} = \mathrm{N}_0 \sum_{\mathrm{r}=0}^{\infty}(-\mathrm{c}^\nu)^r\sum_{\mathrm{k}=0}^{\infty}\frac{(\mathrm{a}_1)_{\mathrm{km}}\cdots(\mathrm{a_p})_{\mathrm{km}}}{(\mathrm{b}_1)_{\mathrm{kn}}\cdots(\mathrm{b_q})_{\mathrm{kn}}}\,\frac{\Gamma(\gamma+\sigma\mathrm{k})\;\Gamma(\mathrm{k}+1)}{\Gamma(\alpha k+\beta)}\,\frac{\mathrm{w^k}}{\mathrm{k!}}$$
$$\times\;\; \mathrm{S}^{-1}\left\{\mathrm{u}^{\gamma+\nu\mathrm{r}+\sigma\mathrm{k}-1}\mathrm{:}\;\mathrm{t}\right\}$$

or

$$N(t) = \mathrm{N}_0 \sum_{r=0}^{\infty}(-c^\nu)^r\sum_{k=0}^{\infty}\frac{(\mathrm{a}_1)_{\mathrm{km}}\cdots(\mathrm{a_p})_{\mathrm{km}}}{(\mathrm{b}_1)_{\mathrm{kn}}\cdots(\mathrm{b_q})_{\mathrm{kn}}}\,\frac{\Gamma(\gamma+\sigma\mathrm{k})\;\Gamma(\mathrm{k}+1)}{\Gamma(\alpha\mathrm{k}+\beta)}\,\frac{\mathrm{w}^k}{\mathrm{k!}}\,\frac{\mathrm{t}^{\gamma+\nu r+\sigma k-1}}{\Gamma(\gamma+\nu r+\sigma k)}.$$

Finally, by using (5), we arrive at the desired result (20).

Discussion II. Let $c, w, v \in \mathbb{R}^+$ and $\Re(u) > 0$ with $|u| < c^{--1}(c \neq w)$. Also $\alpha, \beta, t \in C$; $m, n > 0$; $\Re(\alpha) > 0$ and $pm \leq qn + \Re(\alpha)$. Then, the solution of the generalized fractional kinetic equation (28) is given by (29).
As in the proof of the Theorem 2, we make use Sumudu transform instead of Laplace transform into account, then we can obtain desired result (29).

4. Conclusion

In this paper we have introduced a new fractional generalization of the standard kinetic equation and derived their solutions in view of generalized M-Series, M-series and generalized Mittag–Leffler function. We can also obtain the number of special functions as the special cases of our main results, being of general nature, are shown to be some unification and extension of many known results given, for example Saxena *et al.* [7, 8, 9], Saxena and Kalla [6] etc.

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Received: 22 March 2017
Accepted: 15 June 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

On an Inverse Boundary Value Problem For a Third Order Partial Differential Equation With Non-classical Boundary Conditions

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Abstract. In this work the inverse boundary value problem with unknown time-dependent coefficient for a third-order partial differential equation with non-classical boundary conditions is studied. The definition of the classical solution of the stated problem is given. The essence of the problem is that it is required together with the solution to determine an unknown coefficient. The problem is considered in the rectangular domain. When solving the initial inverse boundary value problem, the transition from the initial inverse problem to some auxiliary inverse problem is performed. With the help of contraction mappings, the existence and uniqueness of the solution of an auxiliary problem are proved. Then the transition to the original inverse problem is made again, and as a result, a conclusion is made about the solvability of the initial inverse problem.
Key Words and Phrases: inverse problem, third order equations, existence and uniqueness of a classical solution.

1. Introduction

In the present work, by the inverse problem for partial differential equations we mean such a problem in which, together with the solution, it is required to determine the right-hand side or (and) one or another coefficient (coefficients) of the equation itself. Inverse problems arise in the most diverse areas of human activity, such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc., which puts them in a series of actual problems of modern mathematics. If in the inverse problem the solution and the right-hand side are unknown, then such as inverse problem will be linear; if the solution and at least one of the coefficients are unknown, then the inverse problem will be nonlinear

Various inverse problems for particular types of partial differential equations have been studied in many papers. We note here, first of all, the works of A.N. Tikhonov [1], M.M. Lavrent'ev [2,3], V.K. Ivanov [4] and their students. For more details, see the monograph by A.M. Denisov [5].

The goal of this paper is to prove the existence and uniqueness of the solution of an inverse boundary value problem for a third order differential equation with nonclassical boundary conditions.

2. Statement of the inverse boundary value problem

Consider an inverse boundary value problem for the equation

$$u_{tt}(x,t) - a(t)u_{txx}(x,t) = p(t)u(x,t) + q(t)u_t(x,t) + f(x,t) \tag{1}$$

in the domain $D_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$ with initial conditions

$$u(x,0) = \varphi(x)\,,\ u_t(x,0)\ = \psi(x)\ \ (0 \le x \le 1)\ , \tag{2}$$

with Dirichlet boundary condition

$$u(0,t) = 0\ \ (0 \le t \le T)\ , \tag{3}$$

with non-classical boundary condition

$$u_x(1,\ t) + du_{xx}(1,\ t) = 0\ \ (0 \le t \le T), \tag{4}$$

and with an additional condition

$$u(x_i,t) = h_i(t)\ \ \ (i = 1,2;\ 0 < x_1, x_2 < 1,\ \ x_1 \ne x_2,\ \ 0 \le t \le T), \tag{5}$$

where $d > 0$ is a given number, $a(t) > 0$, $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h_i(t)$ $(i = 1,2)$-are given functions, $u(x,t)$, $p(t)$ and $q(t)$ are required functions.

Let us introduce the notation

$$\tilde{C}^{2,2}(D_T) = \ \{u(x,t) : u(x,t) \in C^2(D_T),\ u_{txx}(x,t) \in C^2(D_T)\}\,.$$

Definition 1. *Under the classical solution of the inverse problem* (1)-(5) *we mean the triple* $\{u(x,t), p(t)\,, q(t)\}$ *of the functions* $u(x,t)$, $p(t)$, $q(t)$, *if* $u(x,t) \in \tilde{C}^{2,2}(D_T)$, $p(t) \in C[0,T]$, $q(t) \in C[0,T]$ *and relations* (1) - (5) *are satisfied in the usual sense.*

First consider the following spectral problem [6,7] :

$$y''(x) + \lambda\, y\,(x) = 0\ \ (0 \le x \le 1),$$

$$y(0) = 0,\ \ y'(1) = d\lambda\, y\,(1),\ \ d > 0. \tag{6}$$

This problem has only eigenfunctions $y_k(x) = \sqrt{2}\sin\left(\sqrt{\lambda_k}x\right),\ k = 0,\ 1,\ 2,\ ...,$ with positive eigenvalues λ_k from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$. The zero index is assigned to any eigenfunction, and all the others are numbered in ascending order of eigenvalues.

The following theorem is true.

Theorem 1. *Let* $f(x,t) \in C(D_T)$, $\varphi(x)$, $\psi(x) \in C[0,\ 1]$, $h_i(t) \in C^2[0,T]$ $(i = 1,2)$, $h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \ne 0$ $(0 \le t \le T)$, $\varphi'(1) + d\varphi''(1) = 0$,

$$\varphi(1) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 \varphi(x)\sin(\sqrt{\lambda_0}x)dx = 0\,, \tag{7}$$

$$\psi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1\psi(x)\sin(\sqrt{\lambda_0}x)dx=0\,, \tag{8}$$

$$f(1,\,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 f(x,t)\sin(\sqrt{\lambda_0}x)dx=0\ (0\le t\le T), \tag{9}$$

and the conditions of matching are satisfied

$$\varphi(x_i)=h_i(0\,),\ \psi\,(x_i)=h_i'(0)\ \ (i=1,2)\,. \tag{10}$$

Then the problem of finding a classical solution of problem (1) - (5) *is equivalent to the problem of determining the functions* $u(x,t)\in\tilde{C}^{2,2}(D_T)$, $p(t)\in C[0,T]$, $q(t)\in C[0,T]$, *satisfying the equation* (1), *conditions* (2), (3) *and the conditions*

$$u(1,\ t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx=0\ \ (0\le t\le T), \tag{11}$$

$$h_i''(t)-a(t)u_{txx}(x_i,t)=p(t)h_i(t)+q(t)h_i'(t)+f(x_i,t)\ (i=1,\,2;\ 0\le t\le T). \tag{12}$$

Proof. Let $\{u(x,t),p(t)\,,q(t)\}$ be any solution of problem (1) - (5). Then from equation (1), with considering (9), we have:

$$\left[u_{tt}(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{tt}(x,t)\sin(\sqrt{\lambda_0}x)dx\right]-$$

$$-a(t)\left[u_{txx}(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{txx}(x,t)\sin(\sqrt{\lambda_0}x)dx\right]=$$

$$=p(t)\left[u(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]+$$

$$+q(t)\left[u_t(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_t(x,t)\sin(\sqrt{\lambda_0}x)dx\right]\ (0\le t\le T). \tag{13}$$

Integrating in parts twice, in view of (3), with the help of easy transformations we find:

$$u_{xx}(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{xx}(x,t)\sin(\sqrt{\lambda_0}x)dx=\frac{1}{d}\left(u_x\,(1,t)+du_{xx}\,(1,t)\right)-$$

$$-\lambda_0\left[u(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]. \tag{14}$$

Substituting (14) into (13), we get:

$$\left[u_{tt}(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{tt}(x,t)\sin(\sqrt{\lambda_0}x)dx\right]-a(t)\left[\frac{1}{d}\left(u_{tx}\,(1,t)+du_{txx}\,(1,t)\right)\right]=$$

$$=p(t)\left[u(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]+$$

$$+(q(t)-\lambda_0 a(t))\left[u_t(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_t(x,t)\sin(\sqrt{\lambda_0}x)dx\right]\ (0\le t\le T). \qquad (15)$$

From (15), by virtue of (4), we find:

$$\omega''(t)-p(t)\omega(t)-q\,(t)-\lambda_0 a(t))\omega'(t)=0\ \ (0\le t\le T)\ , \qquad (16)$$

where

$$\omega\,(t)\ \equiv u(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\ \ (0\le t\le T)\ . \qquad (17)$$

Further, by virtue of (2) and in view of (7), (8) we find :

$$\omega\,(0)\ =\varphi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 \varphi(x)\sin(\sqrt{\lambda_0}x)dx=0\,,$$

$$\omega'\,(0)\ =\psi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 \psi(x)\sin(\sqrt{\lambda_0}x)dx=0\,. \qquad (18)$$

It is obvious that the problem (16), (18) has only a trivial solution, i.e. $\omega(t)=0\ \ (0\le t\le T)$. Therefore, it is clear from (17) that condition (11) is also satisfied.

Further, from (5) it is clear that

$$u_t(x_i,t)=h_i'(t),\quad u_{tt}(x_i,t)=h_i''(t)\ (i=1,\,2;\ 0\le t\le T)\,. \qquad (19)$$

Supplying $x=x_i\,(i=1,2)$ in equation (1), we have

$$u_{tt}(x_i\,,t)-a(t)u_{txx}(x_i\,,t)=p(t)u(x_i\,,t)+q(t)u_t(x_i\,,t)+f(x_i\,,t)(i=1,2;\ 0\le t\le T). \qquad (20)$$

From here, taking into account (5) and (19), we arrive at the fulfilment of (12).

Now, suppose that $\{u(x,t),p(t),q(t)\}$ is a solution to problem (1) - (3), (11), (12), and the condition of matching (10) is satisfied.

Then from (15), in view of (11) we have:

$$u_{tx}\,(1,t)+du_{txx}\,(1,t)\ =0. \qquad (21)$$

By virtue of (2) and $\varphi'(1)+d\varphi''(1)=0$ it is obvious that

$$u_x\,(1,0)+du_{xx}\,(1,0)\ =\varphi'(1)+d\varphi''(1)=0. \qquad (22)$$

From (21) and (22) we arrive at the fulfilment of (4).

Further, from (12) and (20) we obtain:

$$\frac{d^2}{dt^2}(u(x_i,t)-h_i(t))-q(t)\frac{d}{dt}(u(x_i,t)-h_i(t))$$
$$-p(t)(u(x_i,t)-h_i(t))=0\quad (i=1,2;\ 0\le t\le T). \qquad (23)$$

By virtue of (2) and condition of matching (10), we have:

$$u(x_i,0)-h_i(0)=\varphi(x_i)-h_i(0\,)=0,$$
$$u_t(x_i,0)-h_i'(0)=\psi\,(x_i)-h_i'(0)=0\ (i=1,2). \qquad (24)$$

From (23) and (24) we conclude that condition (5) is satisfied. The theorem is proved.

3. Auxiliary facts

Solving the homogeneous problem corresponding to problem (1) - (3), (11), (12), by the method of separation of variables we arrive at the spectral problem

$$y''(x) + \lambda\, y(x) = 0 \ \ (0 \le x \le 1),$$

$$y(0) = 0\,,\ y(1) + \frac{1}{d \sin \sqrt{\lambda_0}} \int_0^1 y(x) \sin(\sqrt{\lambda_0} x) dx = 0. \tag{25}$$

It is known [6] that the spectral problem (25) is equivalent to the spectral problem (6) without an eigenfunction corresponding to an eigenvalue λ_0. Consequently, the spectral problem (25) has only eigenfunctions $y_k(x) = \sqrt{2} \sin(\sqrt{\lambda_k} x)\,,\ \ k = 1,\ 2,\ ...$ with positive eigenvalues λ_k, defined from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$, numbered in increasing order.

Consequently, the spectral problem (25) has only eigenfunctions $y_k(x) = \sqrt{2} \sin(\sqrt{\lambda_k} x)\,,$ $k = 1,\ 2,\ ...$ with positive eigenvalues λ_k, determined from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$, numbered in increasing order.

The following statements were formulated and substantiated in [6,7].

Lemma 1. *Starting from some number N, the estimate*

$$0 < \sqrt{\lambda_k} - \pi k < (d\pi k)^{-1}\,. \tag{26}$$

Corollary 1. *Let $v_k(x) = \sqrt{2}\, \sin(\sqrt{\mu_k} x)$, where $\sqrt{\mu_k} = \pi k$, $k = 1, 2, 3, ...$ Then the following inequalities are true*

$$\sum_{k=N}^{\infty} \|y_k(x) - v_k(x)\|^2_{L_2(0,1)} \le 1/(9d^2). \tag{27}$$

Lemma 2. *Biorthogonally conjugated system $\{z_k(x)\}$ to the system $\{y_k(x)\},\ k = 1, 2, 3, ...,$ is determined by the formula*

$$z_k(x) = \sqrt{2}(\sin(\sqrt{\lambda_k} x) - \sin\sqrt{\lambda_k}(\sin\sqrt{\lambda_0} x)/(\sin\sqrt{\lambda_0}))/(1 + d \sin^2 \sqrt{\lambda_k}). \tag{28}$$

Theorem 2. *Systems $\{y_k(x)\}$, $k = 1, 2, ...,$ form a Riesz basis for $L_2(0, 1)$.*

Now, let $\eta_k(x) = \sqrt{2}\cos(\sqrt{\lambda_k} x),\ \xi_k(x) = \sqrt{2}\cos(\sqrt{\mu_k} x),\ k = 1, 2, 3, ...$ Then, similarly to (27), the inequalities

$$\sum_{k=N}^{\infty} \|\eta_k(x) - \xi_k(x)\|^2_{L_2(0,1)} \le 1/(9d^2), \tag{29}$$

are true. Suppose that $g(x) \in L_2(0, 1)$. Then, in view of (27), we obtain

$$\left(\sum_{k=1}^{\infty} \left(\int_0^1 g(x) y_k(x) dx \right)^2 \right)^{1/2} \le M\ \|g(x)\|_{L_2(0,1)}\,, \tag{30}$$

where

$$M=\left[\sum_{k=1}^{N}\int_0^1 y_k^2(x)dx+2/\left(9d^2\right)+2\right]^{1/2}. \tag{31}$$

Similar to (30), taking into account (29), we find:

$$\left(\sum_{k=1}^{\infty}\left(\int_0^1 g(x)\eta_k(x)dx\right)^2\right)^{1/2}\le M\ \|g(x)\|_{L_2(0,1)}. \tag{32}$$

Since the functions $\{y_k(x)\}$, $k=1,2,3,...$, form a Riesz basis for space $L_2(0,1)$, then it is known that for any function $g(x)\in L_2(0,1)$ the equality

$$g(x)=\sum_{k=1}^{\infty} g_k y_k(x), \tag{33}$$

is true, where

$$g_k=\int_0^1 g(x)z_k(x)dx\,(k=1,2,....).$$

Further, it is not difficult to see that

$$g_k=\frac{\sqrt{2}}{\alpha_k}\left[\int_0^1 g(x)\sin(\sqrt{\lambda_k}x)dx-\frac{\cos\sqrt{\lambda_k}}{d\sqrt{\lambda_k}\,\sin\sqrt{\lambda_0}}\int_0^1 g(x)\sin\sqrt{\lambda_0}x dx\right], \tag{34}$$

where

$$\alpha_k=1+d\sin^2\sqrt{\lambda_k}>1.$$

Hence, in view of (30) we have:

$$\left(\sum_{k=1}^{\infty} g_k^2\right)^{1/2}\le M_0\ \|g(x)\|_{L_2(0,1)}, \tag{35}$$

where

$$M_0=\left[M+\frac{1}{d\left|\sin\sqrt{\lambda_0}\right|}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{1/2}\right]\sqrt{2}. \tag{36}$$

Assume that $g(x)\in C[0,1]$, $g'(x)\in L_2(0,1)$, $g(0)=0$ and

$$J(g)\equiv g(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 g(x)\sin(\sqrt{\lambda_0}x)dx=0.$$

Then from (34) we have:

$$g_k=\frac{\sqrt{2}}{\alpha_k}\frac{1}{\sqrt{\lambda_k}}\int_0^1 g'(x)\cos\left(\sqrt{\lambda_k}x\right)\,dx. \tag{37}$$

Hence, in view of (29) we obtain:

$$\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\ |g_k|\,)^2\right)^{1/2} \le M\left\|g'(x)\right\|_{L_2(0,1)}. \tag{38}$$

Let $g(x) \in C^1[0,1]$, $g''(x) \in L_2(0,1)$, $g(0) = 0$ and $J(g) = 0$. Then from (37) we obtain:

$$g_k = -\frac{\sqrt{2}}{\alpha_k}\left[\frac{1}{\lambda_k}\int_0^1 g''(x)\sin(\sqrt{\lambda_k}x)dx - \frac{\cos\sqrt{\lambda_k}}{d\lambda_k\sqrt{\lambda_k}}g'\,(1)\right]. \tag{39}$$

Hence, we get:

$$\left(\sum_{k=1}^{\infty}(\lambda_k\ |g_k|\,)^2\right)^{1/2} \le m\left|g'(0)\right| + \sqrt{2}M\left\|g''(x)\right\|_{L_2(0,1)}, \tag{40}$$

where $m = \frac{\sqrt{2}}{d}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{1/2}$.

Now, suppose that $g(x) \in C^2[0,1]$, $g'''(x) \in L_2(0,1)$, $g(0) = 0$, $J(g) = 0$, $g''(0) = 0$ and $dg''(1) + g'(1) = 0$. Then from (39) we have:

$$g_k = -\frac{\sqrt{2}}{\alpha_k}\frac{1}{\lambda_k\sqrt{\lambda_k}}\int_0^1 g'''(x)\cos(\sqrt{\lambda_k}x)dx.$$

Hence, in view of (29) we have :

$$\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\ |g_k|\,)^2\right)^{1/2} \le M\left\|g'''(x)\right\|_{L_2(0,1)}. \tag{41}$$

1. Denote by $B_{2,T}^{\frac{3}{2},\frac{3}{2}}$ [8], the set of all functions $u(x,t)$ of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)y_k(x)\ ,$$

considering in D_T, where each of the functions $u_k(t)$ is continuously differentiable on $[0,T]$ and

$$I(u) \equiv \left\{\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}} + \left\{\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left\|u_k'(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}} < +\infty.$$

The norm on this set is defined as: $\|u(x,t)\|_{B_{2,T}^{\frac{3}{2},\frac{3}{2}}} = I(u)$.

2. By $E_T^{\frac{3}{2},\frac{3}{2}}$ denote the space consisting of the topological product $B_{2,T}^{\frac{3}{2},\frac{3}{2}} \times C[0,T] \times C[0,T]$. Norm of the element $z = \{u,p,q\}$ is defined by the formula

$$\|z\|_{E_T^{\frac{3}{2},\frac{3}{2}}} = \|u(x,t)\|_{B_{2,T}^{\frac{3}{2},\frac{3}{2}}} + \|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]}\,.$$

It is known that $B_{2,T}^{\frac{3}{2},\frac{3}{2}}$ and $E_T^{\frac{3}{2},\frac{3}{2}}$ are Banach spaces.

4. Solvability of an inverse boundary value problem

Taking into account Lemma 2 and Theorem 2, the first component $u(x,t)$ of the solution $\{u(x,t),p(t),q(t)\}$ of the problem (1) - (3), (11), (12) we will be sought in the form:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)y_k(x) \ , \tag{42}$$

where

$$u_k(t)=\int_0^1 u(x,t)z_k(x)dx \ \ (k=1,2,...).$$

We apply the method of separation of variables to determine the desired functions $u_k(t)$ $(k=1,2,...;)$. Then from (1) and (2) we have:

$$u_k''(t)+\lambda_k a(t)u_k'(t)=F_k(t;u,p,q) \ \ (k=1,2,...;\ 0\le t\le T), \tag{43}$$

$$u_k(0)=\varphi_k, u_k'(0)=\psi_k \ (k=1,2,...), \tag{44}$$

where

$$F_k(t;u,p,q)=f_k(t)+p(t)u_k(t)+q(t)u_k'(t),\ f_k(t)=\int_0^1 f(x,t)z_k(x)dx,$$

$$\varphi_k=\int_0^1\varphi(x)z_k(x)dx, \quad \psi_k=\int_0^1\psi(x)z_k(x)dx \ \ (k=1,2,...).$$

Solving problem (43), (44), we find:

$$u_k(t)=\varphi_k+\psi_k\int_0^t e^{-\lambda_k\int_0^{\tau}a(s)ds}d\tau+\int_0^t F_k(\tau;u,p,q)\left(\int_{\tau}^t e^{-\lambda_k\int_{\tau}^{\zeta}a(s)ds}d\xi\right)d\tau. \tag{45}$$

Differentiating twice (45) we get:

$$u_k'(t)=\psi_k e^{-\lambda_k\int_0^t a(s)ds}+\int_0^t F_k(\tau;u,p,q)e^{-\lambda_k\int_{\tau}^t a(s)ds}d\tau \ (k=1,2,...), \tag{46}$$

$$u_k''(t)=-\lambda_k a(t)\psi_k e^{-\lambda_k\int_0^t a(s)ds}-$$

$$-\lambda_k a(t)\int_0^t F_k(\tau;u,p,q)e^{-\lambda_k\int_{\tau}^t a(s)ds}d\tau \ +F_k(t;u,p,q)(k=1,2,...). \tag{47}$$

After substituting the expression $u_k(t)$ $(k=1,2,\ldots)$ from (45) into (42), we have:

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\varphi_k+\psi_k\int_0^t e^{-\lambda_k\int_0^{\tau}a(s)ds}d\tau+\right.$$

$$\left.+\int_0^t F_k(\tau;u,p,q)\left(\int_{\tau}^t e^{-\lambda_k\int_{\tau}^{\zeta}a(s)ds}d\xi\right)d\tau\right\}y_k(x)\ . \tag{48}$$

Now from (12), in view of (42), we get:

$$p(t) = [h(t)]^{-1} \{h_2'(t)\,(h_1''(t) - f(x_1,t)) - h_1'(t)\,(h_2''(t) - f(x_2,t)) -$$

$$\left. -a(t)\sum_{k=1}^{\infty}\lambda_k u_k'(t)(h_2'(t)y_k(x_1) - h_1'(t)y_k(x_2))\right\}, \tag{49}$$

$$q(t) = [h(t)]^{-1} \{h_1(t)\,(h_2''(t) - f(x_2,t)) - h_2(t)(h_1''(t) - f(x_1,t)) -$$

$$\left. -a(t)\sum_{k=1}^{\infty}\lambda_k u_k'(t)(h_1(t)y_k(x_2) - h_2(t)y_k(x_1))\right\}, \tag{50}$$

where

$$h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \neq 0\,(0 \le t \le T).$$

In order to obtain the equation for the second and third components $p(t)$, $q(t)$ of the solution $\{u(x,t),p(t),q(t)\}$ of the problem (1)-(3), (11), (12) we substitute the expression $u_k'(t)$ from (46) into (49), (50) respectively, we have:

$$p(t) = [h(t)]^{-1} \{h_2'(t)\,(h_1''(t) - f(x_1,t)) - h_1'(t)\,(h_2''(t) - f(x_2,t)) -$$

$$-a(t)\sum_{k=1}^{\infty}\lambda_k(h_2'(t)y_k(x_1) - h_1'(t)y_k(x_2))\times$$

$$\left.\times\left(\psi_k e^{-\lambda_k\int_0^t a(s)ds} + \int_0^t F_k(\tau;u,p,q)e^{-\lambda_k\int_\tau^t a(s)ds}d\tau\right)\right\}, \tag{51}$$

$$q(t) = [h(t)]^{-1} \{h_1(t)\,(h_2''(t) - f(x_2,t)) - h_2(t)(h_1''(t) - f(x_1,t)) -$$

$$-a(t)\sum_{k=1}^{\infty}\lambda_k\,(h_1(t)y_k(x_2) - h_2(t)y_k(x_1))\times$$

$$\left.\times\left(\psi_k e^{-\lambda_k\int_0^t a(s)ds} + \int_0^t F_k(\tau;u,p,q)e^{-\lambda_k\int_\tau^t a(s)ds}d\tau\right)\right\}, \tag{52}$$

Thus, the solution of problem (1)-(3), (11), (12) was reduced to the solution of system (48), (51), (52) with respect to unknown functions, $u(x,t)$,$p(t)$ and $q(t)$.

To study the question of the uniqueness of the solution of problem (1) - (3), (11), (12), the following lemma plays an important role.

Lemma 3. *If* $\{u(x,t),p(t),q(t)\}$ *is any solution of the problem* (1)-(3), (11), (12), *then the functions*

$$u_k(t) = \int_0^1 u(x,t)z_k(x)dx (k = 1,2,...)$$

satisfy on $[0,T]$ *the system* (45).

Lemma 3 implies that the following holds.

Corollary 2. *Let system* (48), (51), (52) *have a unique solution. Then the problem* (1)-(3), (11), (12) *cannot have more than one solution, i.e. if problem* (1)-(3), (11), (12) *has a solution, then it is unique.*

Now consider the operator in space $E_T^{\frac{3}{2},\frac{3}{2}}$

$$\Phi(u,p,q)=\left\{\Phi_1(u,p,q),\Phi_2(u,p,q)\,,\ \Phi_3(u,p,q)\right\},$$

where

$$\Phi_1(u,p,q)=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_k(t)X_k(x),\Phi_2(u,p,q)=\tilde{p}(t),\ \Phi_3(u,p,q)=\tilde{q}(t),$$

and $\tilde{u}_k(t)$ $(k=1,2,...)$, $\tilde{p}(t)$ and $\tilde{q}(t)$ are equal, respectively, right sides (45), (51) and (52).

Using easy transformations, we find that inequalities

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \|\tilde{u}_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\le\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\varphi_k|\right)^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{5}\,T\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\psi_k|\right)^2\right)^{\frac{1}{2}}+\sqrt{5}T\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ |f_k(\tau)|\right)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$+T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$\left.+\,T\,\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u'_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right],\tag{53}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \|\tilde{u}'_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\le2\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\psi_k|\right)^2\right)^{\frac{1}{2}}+$$

$$+2\,T\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ |f_k(\tau)|\right)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$+T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$\left.+\,T\,\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u'_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right],\tag{54}$$

$$\|\tilde{p}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]} \times$$

$$\times\{\left\|h_2'(t)\left(h_1''(t)-f(x_1,t)\right)-h_1'(t)\left(h_2''(t)-f(x_2,t)\right)\right\|_{C[0,T]}+$$

$$+\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1'(t)\right|+\left|h_2'(t)\right)|\right\|_{C[0,T]}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\psi_k|\right)^2\right)^{\frac{1}{2}}+\right.$$

$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ |f_k(\tau)|\right)^2d\tau\right)^{\frac{1}{2}}+$$

$$+T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$\left.+\ T\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\Bigg\}, \tag{55}$$

$$\|\tilde{q}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]} \times$$

$$\times\{\left\|h_1(t)\left(h_2''(t)-f(x_2,t)\right)-h_2(t)(h_1''(t)-f(x_1,t))\right\|_{C[0,T]}+$$

$$+\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\|a(t)(|h_1(t)|+|h_2(t))|\|_{C[0,T]}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ |\psi_k|\right)^2\right)^{\frac{1}{2}}+\right.$$

$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ |f_k(\tau)|\right)^2d\tau\right)^{\frac{1}{2}}+$$

$$+T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$\left.+\ T\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\ \ \left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\Bigg\}, \tag{56}$$

are true. Suppose that the data of the problem (1) - (3), (11), (12) satisfy the following conditions:

1) $\varphi(x)\in C^2\,[0,1]\,,\varphi'''(x)\in L_2(0,1),\ \varphi(0)=0\ ,\ J(\varphi)=0\ ,\varphi''(0)=0\,,$

$$d\varphi''(1)+\varphi'(1)=0\,.$$

2) $\psi(x)\in C^2\,[0,1]\,,\psi''(x)\in L_2(0,1),\ \psi(0)=0\ ,\ J(\psi)=0\ ,\psi''(0)=0\,,$

$$d\psi''(1)+\psi'(1)=0\,.$$

3) $f(x,t), f_x(x,t),\ f_{xx}(x,t) \in C(D_T),\ f_{xxx}(x,t) \in L_2(D_T),\ f\ (0,t) = 0, J\,(f) = 0,$ $f_{xx}(0,t) = 0,$

$$df_{xx}(1,t) + f_x(1,t) = 0\ \ (0 \le t \le T)\,.$$

4) $0 < a(t) \in C[0,T]$, $h_i(t) \in C^1[0,T]$ $(i = 1,2)$,

$$h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \ne 0\,(0 \le t \le T).$$

Then from (53) - (56), in view of (41), respectively, we obtain:

$$\|\tilde u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}} \le A_1(T) + B_1(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]})\,\|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}}\,, \tag{57}$$

$$\|\tilde p(t)\|_{C[0,T]} \le A_2(T) + B_2(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]})\,\|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}}\,, \tag{58}$$

$$\|\tilde q(t)\|_{C[0,T]} \le A_3(T) + B_3(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]})\,\|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}}\,, \tag{59}$$

where

$$A_1(T) = \sqrt{5}M\left\|\varphi'''(x)\right\|_{L_2(0,1)} + (\sqrt{5}T + 2)M\left\|\varphi'''(x)\right\|_{L_2(0,1)} +$$
$$+(\sqrt{5}T + 2)\sqrt{T}M\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}, B_1\,(T) = (\sqrt{5}T + 2)T,$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \times$$
$$\times\{\left\|h_2'(t)\,(h_1''(t) - f(x_1,t)) - h_1'(t)\,(h_2''(t) - f(x_2,t))\right\|_{C[0,T]} +$$
$$+\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1'(t)\right| + \left|h_2'(t))\right|\right\|_{C[0,T]}\left[M\left\|\psi'''(x)\right\|_{L_2(0,1)} +\right.$$
$$\left.+\ \sqrt{T}M\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}\right]\Big\},$$

$$B_2(T) = \sqrt{2}\left\|h^{-1}(t)\right\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1'(t)\right| + \left|h_2'(t))\right|\right\|_{C[0,T]},$$

$$A_3(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \times$$
$$\times\Big\{\left\|h_1(t)\left(f(x_2,t) - a_1(t)h_2'(t)\right) - h_2(t)\left(f(x_1,t) - a_1(t)h_1'(t)\right)\right\|_{C[0,T]} +$$
$$+\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1(t)\right| + \left|h_2'(t))\right|\right\|_{C[0,T]}\left[M\left\|\psi'''(x)\right\|_{L_2(0,1)} +\right.$$
$$\left.+\ \sqrt{T}M\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}\right]\Big\},$$

$$B_3(T) = \sqrt{2}\left\|h^{-1}(t)\right\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left\|a(t)(\left|h_1'(t)\right| + \left|h_2'(t))\right|\right\|_{C[0,T]}.$$

From inequalities (57) - (59) we conclude:

$$\|\tilde{u}(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}} + \|\tilde{p}(t)\|_{C[0,T]} + \|\tilde{q}(t)\|_{C[0,T]} \leq$$

$$\leq A(T) + B(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]}) \|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}} , \tag{60}$$

where

$$A(T) = A_1(T) + A_2(T) + A_3(T), \quad B(T) = B_1(T) + B_2(T) + B_3(T).$$

So, we can prove the following theorem:

Theorem 3. *Let the conditions 1)- 4) be fulfilled and*

$$B(T)(A(T) + 2)^2 < 1. \tag{61}$$

Then the problem (1)-(3), (11), (12) *has the only solution in a ball* $K = K_R(\|z\|_{E^{\frac{3}{2},\frac{3}{2}}_T} \leq R = A(T) + 2)$ *from space* $E^{\frac{3}{2},\frac{3}{2}}_T$.

Proof. In space $E^{\frac{3}{2},\frac{3}{2}}_T$ we consider the equation

$$z = \Phi z, \tag{62}$$

where $z = \{u, p, q\}$, components $\Phi_i(u, p, p)(i = 1, 2, 3)$ of operator $\Phi(u, p, q)$ are defined by the right-hand sides of equations (48), (51), (52), respectively.

Consider the operator $\Phi(u, p, q)$ in the ball $K = K_R(\|z\|_{E^{\frac{3}{2},\frac{3}{2}}_T} \leq R = A(T) + 2)$ from $E^{\frac{3}{2},\frac{3}{2}}_T$. Similarly to (60), we obtain that for any $z, z_1, z_2 \in K_R$ valid the following estimates:

$$\|\Phi z\|_{E^{\frac{3}{2},\frac{3}{2}}_T} \leq A(T) + B(T)(\|p(t)\|_{C[0,T]} + \|q(t)\|_{C[0,T]}) \|u(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}} \leq$$

$$\leq A(T) + B(T)(A(T) + 2)^2 , \tag{63}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^{\frac{3}{2},\frac{3}{2}}_T} \leq B(T)R \Big(\|p_1(t) - p_2(t)\|_{C[0,T]} +$$

$$+ \|q_1(t) - q_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B^{\frac{3}{2},\frac{3}{2}}_{2,T}} \Big) . \tag{64}$$

Then, from estimates (63) and (64), taking into account (61), it follows that the operator Φ acts in a ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$, the operator Φ has a unique fixed point $\{u, p, q\}$, which is the unique solution of equation (62), i.e. is the unique solution of the system (48), (51), (52) in the ball $K = K_R$.

The function $u(x,t)$, as an element of space $B^{\frac{3}{2},\frac{3}{2}}_{2,T}$, is continuous and has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$, $u_{tx}(x,t)$, $u_{txx}(x,t)$ in D_T.

From (43), by virtue of (38), it is not difficult to see that

$$\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left\|u_k''(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{2}\left\|a(t)\right\|_{C[0,T]}\left\{\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left\|u_k'(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \right.$$

$$\left. +M\left\|\left\|f_x(x,t)+p(t)u_x(x,t)+p(t)u_{tx}(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\right\}.$$

It follows that $u_{tt}(x,t)$ is continuous in D_T.

It is easy to verify that equation (1) and conditions (2), (3), (11) and (12) are satisfied in the usual sense. Consequently, $\{u(x,t), p(t), q(t)\}$ is the solution of the problem (1) - (3), (11), (12). By virtue of Corollary 2 of Lemma 3, it is unique in the ball $K = K_R$. The theorem is proved.

Using Theorem 1, we prove the following

Theorem 4. *Let all the conditions of Theorem 3 be satisfied and the conditions of matching*

$$\varphi(x_i) = h_i(0), \ \psi(x_i) = h_i'(0) \ \ (i = 1, 2).$$

Then problem (1) - (5) *has a unique classical solution in ball* $K = K_R(\|z\|_{E_T^{\frac{3}{2},\frac{3}{2}}} \leq R = A(T) + 2)$ *of space* $E_T^{\frac{3}{2},\frac{3}{2}}$.

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Received: 28 January 2019
Accepted: 17 June 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

Properties of Eigenvalues and Eigenfunctions of a Spectral Problem With Discontinuity Point

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Abstract. In this paper we obtain the asymptotics of eigenvalues and eigenfunctions of one spectral problem for a discontinuous second-order differential operator with a spectral parameter in discontinuity conditions which arises by solving the problem on vibrations of a loaded string with fixed ends.

Key Words and Phrases: eigenvalues, eigenfunctions, asymptotic formulae, spectral problem.

2010 Mathematics Subject Classifications: 34B05, 34B24, 34L10, 34L20

1. Introduction

Consider the following boundary value problem

$$-y'' + q(x)\, y = \lambda y \quad , x \in \left(0, \frac{1}{3}\right) \bigcup \left(\frac{1}{3}, 1\right), \tag{1}$$

$$\left.\begin{array}{l} y(0) = y(1) = 0, \\ y(-\frac{1}{3}) = y(+\frac{1}{3}), \\ y'(-\frac{1}{3}) - y'(+\frac{1}{3}) = \lambda m y(\frac{1}{3}), \end{array}\right\} \tag{2}$$

which arises by solving the problem on vibrations of a loaded string with the fixed ends[1-3]. In the case when a load is placed in the middle of the string, this problem was investigated in[4,5]. Similar questions for the problem on vibrations of a loaded string when the load is fixed in one or two ends of a string, are investigated by [8-11].

For the case $q(x) \equiv 0$ the asymptotics of eigenvalues and eigenfunctions, also the basis properties of eigenfunctions were investigated completely in [7].

2. The asymptotic of eigenvalues and eigenfunctions

We denote $\lambda = \rho^2$, $Im\rho = \tau$. Suppose that $q(x)$ is a complex valued summable function on $(-1,1)$. Denote by $y_1(x,\lambda)$ the solution of (1) satisfying the initial conditions

$$\left.\begin{array}{l} y_1(0) = 0, \\ {y_1}'(0) = \rho, \end{array}\right\} \tag{3}$$

and by $y_2(x,\lambda)$ the solution of (1) satisfying the initial conditions

$$\left.\begin{array}{l} y_2(1) = 0, \\ {y_2}'(1) = -\rho. \end{array}\right\} \tag{4}$$

Lemma 1. *The following integral representations hold:*

$$y_1(x,\lambda) = \sin\rho x + \frac{1}{\rho}\int_0^x \sin\rho(x-t)\, q(t)\, y_1(t,\lambda)\, dt, \quad 0 < x < \frac{1}{3}, \tag{5}$$

$$y_1'(x,\lambda) = \rho\cos\rho x + \int_0^x \cos\rho(x-t)\, q(t)\, y_1(t,\lambda)\, dt, \quad 0 < x < \frac{1}{3}, \tag{6}$$

$$y_2(x,\lambda) = \sin\rho(1-x) + \frac{1}{\rho}\int_x^1 \sin\rho(t-x)\, q(t)\, y_2(t,\lambda)\, dt, \quad \frac{1}{3} < x < 1, \tag{7}$$

$$y_2'(x,\lambda) = -\rho\cos\rho(1-x) - \int_x^1 \cos\rho(t-x)\, q(t)\, y_2(t,\lambda)\, dt, \quad \frac{1}{3} < x < 1. \tag{8}$$

Proof. Since $y_1(x,\lambda)$ satisfies (1), then

$$\int_0^x \sin\rho(x-t)\, q(t)\, y_1(t)\, dt = \int_0^x \sin\rho(x-t)\, {y_1}''(t,\lambda)\, dt + \rho^2\int_0^x \sin\rho(x-t)\, y_1(t,\lambda)\, dt.$$

Integrating by part the first integral in the right-hand side of the last equation twice and taking into account (3), we find

$$\int_0^x \sin\rho(x-t)\, q(t)\, y_1(t)\, dt = -\rho\sin\rho x + \rho y_1(x,\lambda),$$

i.e. the equality (5).

The equality (6) is obtained by differentiating the equality (5).

The equalities (7) and (8) are obtained similarly.

Lemma 2. *The following asymptotic formulas hold when $\rho \to \infty$*

$$y_1(x,\lambda) = O\left(e^{|\tau|x}\right), \tag{9}$$

$$y_2(x,\lambda) = O\left(e^{|\tau|(1-x)}\right), \tag{10}$$

more precisely

$$y_1(x,\lambda)=\sin\rho x+O\left(\frac{e^{|\tau|x}}{|\rho|}\right), \tag{11}$$

$$y_2(x,\lambda)=\sin\rho\,(1-x)+O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right). \tag{12}$$

All estimates are satisfied uniformly on x for $y_1(x,\lambda)$ when $0\le x\le\frac{1}{3}$ and for $y_2(x,\lambda)$ when $\frac{1}{3}\le x\le 1$.

The proof repeats that lemma in [6] word for word.

Denote

$$q_1(x)=\frac{1}{2}\int_0^x q(t)\,dt,$$

$$q_2(x)=\frac{1}{2}\int_x^1 q(t)\,dt.$$

Theorem 1. *The spectrum of problem* (1)-(2) *consists of three sequences $\lambda_{i,n}=\rho_{i,n}^2$, $i=1,2,3$; $n=1,2,...$, of asymptotically simple eigenvalues:*

$$\begin{array}{l}\rho_{1,n}=3\pi n+\frac{\alpha_1}{n}+o\left(\frac{1}{n}\right)\\ \rho_{2,n}=3\pi n+\frac{\alpha_2}{n}+o\left(\frac{1}{n}\right)\\ \rho_{3,n}=3\pi n+\frac{3\pi}{2}+\frac{\alpha_3}{n}+o\left(\frac{1}{n}\right),\end{array}$$

where α_i, $i=1,2,3$ are different numbers expressed by the values of the functions $q_1(x)$ and $q_2(x)$ at the point $\frac{1}{3}$.

Proof. Substitute asymptotics for $y_1(x)$ from (11) in the right-hand side of (5):

$$y_1(x)=\sin\rho x+\frac{1}{\rho}\int_0^x\sin\rho\,(x-t)\,q(t)\left[\sin\rho t+O\left(\frac{e^{|\tau|}}{\rho}\right)\right]dt=$$

$$=\sin\rho x+\frac{1}{\rho}\int_0^x\sin\rho\,(x-t)\sin\rho t\cdot q(t)\,dt+$$

$$+\frac{1}{\rho^2}\int_0^x\sin\rho\,(x-t)\,q(t)\,O\left(e^{|\tau|t}\right)dt=$$

$$=\sin\rho x+\frac{1}{2\rho}\int_0^x\left[\cos\rho\,(x-2t)-\cos\rho x\right]q(t)\,dt+$$

$$+\frac{1}{\rho^2}\int_0^x\sin\rho\,(x-t)\,q(t)\,O\left(e^{|\tau|t}\right)=$$

$$=\sin\rho x+\frac{1}{2\rho}\int_0^x\cos\rho\,(x-2t)\,q(t)\,dt-\frac{\cos\rho x}{2\rho}\int_0^x q(t)\,dt+$$

$$+\frac{1}{\rho^2}\int_0^x \sin\rho\,(x-t)\,q\,(t)\,O\left(e^{|\tau|t}\right)dt =$$

$$=\sin\rho x+\frac{1}{2\rho}\int_0^x \cos\rho\,(x-2t)\,q\,(t)\,dt-$$

$$-\frac{1}{\rho}\cos\rho x\left(\frac{1}{2}\int_0^x q\,(t)\,dt\right)+$$

$$++\frac{e^{|\tau|x}}{\rho^2}\int_0^x \frac{\sin\rho\,(x-t)}{e^{|\tau|(x-t)}}dt.$$

Hence,

$$y_1\,(x)=\sin\rho x-\frac{1}{\rho}q_1\,(x)\cos\rho x+\frac{1}{2\rho}\int_0^x \cos\rho\,(x-2t)\,q\,(t)\,dt+O\left(\frac{e^{|\tau|x}}{|\rho|^2}\right) \tag{13}$$

Substitute asymptotics for $y_1\,(x)$ from (11) in the right-hand side of (6):

$$y_1'\,(x)=\rho\cos\rho x+\int_0^x \cos\rho\,(x-t)\,q\,(t)\,y_1\,(t,\lambda)\,dt=$$

$$=\rho\cos\rho x+\int_0^x \cos\rho\,(x-t)\left[\sin\rho t+O\left(\frac{e^{|\tau|t}}{|\rho|}\right)\right]q\,(t)\,dt=$$

$$=\rho\cos\rho x+\frac{1}{2}\int_0^x \left[\sin\rho x+\sin\rho\,(2t-x)\right]q\,(t)\,dt+$$

$$+\int_0^x \cos\rho\,(x-t)\,O\left(\frac{e^{|\tau|t}}{|\rho|}\right)q\,(t)\,dt=\rho\cos\rho x+$$

$$++\frac{1}{2}\int_0^x \sin\rho x q\,(t)\,dt+\frac{1}{2}\int_0^x \sin\rho\,(2t-x)\,q\,(t)\,dt+$$

$$+\int_0^x \cos\rho\,(x-t)\,O\left(\frac{e^{|\tau|t}}{|\rho|}\right)dt=\rho\cos\rho x+q_1\,(x)\sin\rho x+$$

$$+\frac{1}{2}\int_0^x \sin\rho\,(2t-x)\,q\,(t)\,dt+\frac{e^{|\tau|x}}{|\rho|}\int_0^x \frac{\cos\rho\,(x-t)}{e^{|\tau|(x-t)}}O\,(1)\,q\,(t)\,dt$$

$$=\rho\cos\rho x+q_1\,(x)\sin\rho x+\frac{1}{2}\int_0^x \sin\rho\,(2t-x)\,q\,(t)\,dt+$$

$$+O\left(\frac{e^{|\tau|x}}{|\rho|}\right).$$

Hence,

$$y_1'\,(x)=\rho\cos\rho x+q_1\,(x)\sin\rho x+\frac{1}{2}\int_0^x \sin\rho\,(2t-x)\,q\,(t)\,dt+O\left(\frac{e^{|\tau|x}}{|\rho|}\right) \tag{14}$$

The following asymptotic equalities are obtained analogously:

$$\begin{array}{l} y_2(x) = \sin\rho(1-x) - \frac{1}{\rho} q_2(x)\cos\rho(1-x) + \\ +\frac{1}{2\rho}\int_x^1 \cos\rho(2t-x-1)\,q(t)\,dt + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|^2}\right) \end{array} \qquad (15)$$

and

$$\begin{array}{l} y_2'(x) = -\rho\cos\rho(1-x) - q_2(x)\sin\rho(1-x) - \\ -\frac{1}{2}\int_x^1 q(t)\sin\rho(1+x-2t)\,dt + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right). \end{array} \qquad (16)$$

Obviously, for any $\lambda \neq 0$ the solution $y(x,\lambda)$ of the problem (1)-(2) have to be in the form

$$y(x) = \begin{cases} C_1 y_1(x), & for \quad 0 < x < \frac{1}{3}, \\ C_2 y_2(x), & for \quad \frac{1}{3} < x < 1, \end{cases}$$

here C_1 and C_2 are complex numbers. $\lambda \neq 0$ is an eigenvalue of the problem (1)-(2) if and only if

C_1 and C_2 are nontrivial solutions of following homogeneous system of linear equations:

$$\begin{cases} C_1\left(\sin\frac{1}{3}\rho - \frac{1}{\rho}q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho + \frac{1}{2\rho}\int_0^x \cos\rho\left(\frac{1}{3}-2t\right)q(t)\,dt + O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|^2}\right)\right) - \\ -C_2\left(\sin\frac{2}{3}\rho - \frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho + \frac{1}{2\rho}\int_{\frac{1}{3}}^1 \cos\left(2t-\frac{4}{3}\right)q(t)\,dt + O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right)\right) = 0 \\ C_1\left(\rho\cos\frac{1}{3}\rho + q_1\left(\frac{1}{3}\right)\sin\frac{1}{3}\rho + \frac{1}{2}\int_0^{\frac{1}{3}} \sin\rho\left(2t-\frac{1}{3}\right)q(t)\,dt + O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|}\right)\right) - \\ -C_2\left(-\rho\cos\frac{2}{3}\rho - q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho - \frac{1}{2}\int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt + O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right)\right) = \\ = C_1\rho^2 m\left(\sin\frac{1}{3}\rho - \frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho + \frac{1}{2\rho}\int_0^{\frac{1}{3}} \cos\rho\left(\frac{1}{3}-2t\right)q(t)\,dt + O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|^2}\right)\right) \end{cases}$$

To define eigenvalues we obtain following equation

$$\Delta(\lambda) = \begin{vmatrix} a_{11}(\rho) & a_{12}(\rho) \\ a_{21}(\rho) & a_{22}(\rho) \end{vmatrix} = 0,$$

here

$$\begin{array}{l} a_{11}(\rho) = \sin\frac{1}{3}\rho - \frac{1}{\rho}q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho + \frac{1}{2\rho}\int_0^{\frac{1}{3}}\cos\rho\left(\frac{1}{3}-2t\right)q(t)\,dt + O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|^2}\right) \\ a_{12}(\rho) = -\sin\frac{2}{3}\rho + \frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho - \frac{1}{2\rho}\int_{\frac{1}{3}}^1\cos\rho\left(2t-\frac{4}{3}\right)q(t)\,dt - O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right) \\ a_{21}(\rho) = \left(\rho\cos\frac{1}{3}\rho - \rho^2 m\sin\frac{1}{3}\rho\right) + \left(q_1\left(\frac{1}{3}\right)\sin\frac{1}{3}\rho + \rho m q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho\right) + \\ + \left(\frac{1}{2}\int_0^{\frac{1}{3}}\sin\rho\left(2t-\frac{1}{3}\right)q(t)\,dt - \frac{\rho m}{2}\int_0^{\frac{1}{3}}\cos\rho\left(\frac{1}{3}-2t\right)q(t)\,dt\right) + O\left(e^{\frac{1}{3}|\tau|}\right) \\ a_{22}(\rho) = \rho\cos\frac{2}{3}\rho + q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho + \frac{1}{2}\int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt - O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{array}$$

Using that, for any complex number z .

$$|\sin z| \leq e^{|Imz|}$$

and

$$|\cos z| \le e^{|Imz|},$$

we can write

$$\begin{array}{ll} \left|\cos\rho\left(\frac{1}{3}-2t\right)\right| \le e^{\frac{1}{3}|\tau|}, & for \quad 0 \le t \le \frac{1}{3}, \\ \left|\cos\rho\left(2t-\frac{4}{3}\right)\right| \le e^{\frac{4}{3}|\tau|}, & for \quad \frac{1}{3} \le t \le 1, \\ \left|\sin\rho\left(2t-\frac{1}{3}\right)\right| \le e^{\frac{1}{3}|\tau|}, & for \quad 0 \le t \le \frac{1}{3}, \\ \left|\sin\rho\left(\frac{4}{3}-2t\right)\right| \le e^{\frac{4}{3}|\tau|}, & for \quad \frac{1}{3} \le t \le 1. \end{array}$$

Taking into account the last inequalities, for $|\rho| \to \infty$ we obtain:

$$\begin{array}{l} \int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt = O\left(e^{\frac{1}{3}|\tau|}\right), \\ \int_{\frac{1}{3}}^1 q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt = O\left(e^{\frac{4}{3}|\tau|}\right), \\ \int_0^{\frac{1}{3}} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt = O\left(e^{\frac{1}{3}|\tau|}\right), \\ \int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt = O\left(e^{\frac{4}{3}|\tau|}\right). \end{array}$$

From the last asymptotic formulas we obtain that $\Delta(\lambda)$ can be written as the form:

$$\Delta(\lambda) = \begin{vmatrix} \sin\frac{1}{3}\rho & -\sin\frac{2}{3}\rho \\ \rho\cos\frac{1}{3}\rho - \rho^2 m\sin\frac{1}{3}\rho & \rho\cos\frac{2}{3}\rho \end{vmatrix} +$$

$$+ \begin{vmatrix} \sin\frac{1}{3}\rho & \frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho \\ \rho\cos\frac{1}{3}\rho - \rho^2 m\sin\frac{1}{3}\rho & q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho \end{vmatrix} +$$

$$+ \begin{vmatrix} \sin\frac{1}{3}\rho & -\frac{1}{2\rho}\int_{\frac{1}{3}}^1 q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt \\ \rho\cos\frac{1}{3}\rho - \rho^2 m\sin\frac{1}{3}\rho & \frac{1}{2}\int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt \end{vmatrix} +$$

$$+ \begin{vmatrix} \sin\frac{1}{3}\rho & -O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right) \\ \rho\cos\frac{1}{3}\rho - \rho^2 m\sin\frac{1}{3}\rho & O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{vmatrix} +$$

$$+ \begin{vmatrix} -\frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho & -\sin\frac{2}{3}\rho \\ q_1\left(\frac{1}{3}\right)\sin\frac{1}{3} + \rho m q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho & \rho\cos\frac{2}{3}\rho \end{vmatrix} +$$

$$+ \begin{vmatrix} -\frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho & \frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho \\ q_1\left(\frac{1}{3}\right)\sin\frac{1}{3} + \rho m q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho & q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho \end{vmatrix} +$$

$$+ \begin{vmatrix} -\frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho & -\frac{1}{2\rho}\int_{\frac{1}{3}}^1 q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt \\ q_1\left(\frac{1}{3}\right)\sin\frac{1}{3} + \rho m q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho & \frac{1}{2}\int_{\frac{1}{3}}^1 q(t)\sin\left(\frac{4}{3}-2t\right)dt \end{vmatrix} +$$

$$+ \begin{vmatrix} -\frac{q_1\left(\frac{1}{3}\right)}{\rho}\cos\frac{1}{3}\rho & -O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right) \\ q_1\left(\frac{1}{3}\right)\sin\frac{1}{3} + \rho m q_1\left(\frac{1}{3}\right)\cos\frac{1}{3}\rho & O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{vmatrix} +$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt & -\sin\frac{2}{3}\rho \\ \frac{1}{2}\int_0^{\frac{1}{3}1} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt - \frac{\rho m}{2}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt & \rho\cos\frac{2}{3}\rho \end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt & \frac{1}{\rho}q_2\left(\frac{1}{3}\right)\cos\frac{2}{3}\rho \\ \frac{1}{2}\int_0^{\frac{1}{3}1} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt - \frac{\rho m}{2}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt & q_2\left(\frac{1}{3}\right)\sin\frac{2}{3}\rho \end{vmatrix}+$$

$$+\begin{vmatrix} O\left(\frac{e^{\frac{1}{3}|\tau|}}{|\rho|^2}\right) & -\sin\frac{2}{3}\rho \\ O\left(e^{\frac{1}{3}|\tau|}\right) & \rho\cos\frac{2}{3}\rho \end{vmatrix}+O\left(\frac{e^{|\tau|}}{|\rho|}\right)$$

Opening all determinants in the last equality, we obtain the following for the function $\Delta(\lambda)$:

$$\Delta(\lambda)=\cos^3\frac{1}{3}\rho\left(2\rho^2 m-4q_2-4q_1-2mq_1q_2\right)+$$
$$+\sin^3\frac{1}{3}\left(-4\rho-2\rho m q_2-2\rho m q_1+\frac{4}{\rho}q_1q_2\right)+$$
$$+\sin\frac{1}{3}\rho\left(3\rho+\rho m q_2+2\rho m q_1-\frac{3}{\rho}q_1q_2\right)+$$
$$+\cos\tfrac{1}{3}\rho\left(-2\rho^2 m+3q_2+3q_1+mq_1q_2\right)+\sin\tfrac{1}{3}\rho\times$$
$$\times\left(\begin{array}{l} \frac{1}{2}\int_{\frac{1}{3}}^{1} q(t)\sin\left(\frac{4}{3}-2t\right)dt-\frac{\rho m}{2}\int_{\frac{1}{3}}^{1} q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right)- \\ -mO\left(e^{\frac{2}{3}|\tau|}\right)+\frac{1}{2\rho}q_1\int_{\frac{1}{3}}^{1} q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+q_1O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right) \end{array}\right)+$$
$$+\cos\frac{1}{3}\rho\left(\begin{array}{l} \frac{1}{2}\int_{\frac{1}{3}}^{1} q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right)-\frac{1}{2\rho}q_1\int_{\frac{1}{3}}^{1} q(t)\sin\rho\left(\frac{4}{3}-2t\right)dt+ \\ +\frac{m}{2}q_1\int_{\frac{1}{3}}^{1} q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt-q_1O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|^2}\right)+mq_1O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{array}\right)+$$
$$+\sin\frac{1}{3}\rho\cos\frac{1}{3}\rho\left(\begin{array}{l} \int_0^{\frac{1}{3}} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt-\rho m\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt+ \\ +\frac{1}{\rho}q_2\left(\frac{1}{3}\right)\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt \end{array}\right)+$$
$$+\cos\frac{2}{3}\rho\left(\begin{array}{l} \frac{1}{2}\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt-\frac{1}{2\rho}q_2\left(\frac{1}{3}\right)\int_0^{\frac{1}{3}} q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt+ \\ +\frac{m}{2}q_2\left(\frac{1}{3}\right)\int_0^{\frac{1}{3}} q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt+O\left(\frac{e^{\frac{2}{3}|\tau|}}{|\rho|}\right) \end{array}\right)+$$
$$+O\left(\frac{e^{|\tau|}}{|\rho|}\right). \tag{17}$$

Circle the points $\tilde{\rho}_k=3\pi k,\ k=1,2,.$ by the circles with radius $\frac{\pi}{4}$. Out of these circles the inequality

$$|\delta(\rho)|\geq C|\rho|^2 e^{2|\tau|}$$

holds for the function

$$\delta(\rho) = \rho \sin \frac{\rho}{3} \left(-\rho m \sin \frac{2\rho}{3} + 2 \cos \frac{2\rho}{3} + 1 \right)$$

here $C > 0$is a constant. Since modules of remained summands of the right-hand side of equality(17) don't exceed $A|\rho| e^{2|\tau|}$ (here $A > 0$ is a constant), then by Rouchet theorem for sufficiently large k function $\Delta(\lambda)$ possesses exactly three zeroes multiplicity taking into account in $|Im\rho| \leq h$, here h is a positive constant.
Since, all zeroes of $\Delta(\rho^2)$ belong to strip $|Im\rho| \leq h$ in a sequel assume that ρ runs only in this strip. Under this assumptation the following asymptotic equalities are true for $|\rho| \to +\infty$:

$$\left. \begin{array}{l} O\left(\frac{e^{|\tau|}}{\rho}\right) = O\left(\frac{e^{\frac{2}{3}|\tau|}}{\rho}\right) = O\left(\frac{e^{\frac{2}{3}|\tau|}}{\rho}\right) = O\left(\frac{1,}{\rho}\right) \\ O\left(\frac{e^{\frac{2}{3}|\tau|}}{\rho^2}\right) = O\left(\frac{1}{\rho^2}\right), \\ O\left(e^{|\tau|}\right) = O(1) \end{array} \right\} \quad (18)$$

In the other hand, in the strip $|Im\rho| \leq h$

$$\begin{array}{l} \int_0^{\frac{1}{3}} q(t) \cos \rho \left(\frac{1}{3} - 2t\right) dt = \\ = \int_{\frac{1}{3}}^1 q(t) \cos \rho \left(2t - \frac{4}{3}\right) dt = \\ \int_0^{\frac{1}{3}} q(t) \sin \rho \left(2t - \frac{1}{3}\right) dt = \\ \int_{\frac{1}{3}}^1 q(t) \sin \left(\frac{4}{3} - 2t\right) dt = o(1) \end{array} \quad (19)$$

for $|\rho| \to +\infty$.

Theorem is proved.

Now lets pass to study the asymptotic behavior of eigenfunctions of the problem (1)-(2).

Theorem 2. *Let the function $q(x)$ satisfies the conditions of the Theorem 1. Then the eigenfunctions $y_{1,n}(x)$ corresponding to eigenvalues $\lambda_{1,n} = (\rho_{1,n})^2$, the eigenfunctions $y_{2,n}(x)$corresponding to eigenvalues $\lambda_{2,n} = (\rho_{2,n})^2$ and the eigenfunctions $y_{3,n}(x)$ corresponding to eigenvalues $\lambda_{3,n} = (\rho_{3,n})^2$ satisfies the following asymptotic equalities:*

$$y_{1,n}(x) = \begin{cases} \sin 3\pi n x + O\left(\frac{1}{n}\right), \ x \in \left[0, \frac{1}{3}\right], \\ \gamma_{1,n} \sin 3\pi n x + O\left(\frac{1}{n}\right), \ x \in \left[\frac{1}{3}, 1\right], \end{cases}$$

$$y_{2,n}(x) = \begin{cases} \sin 3\pi n x + O\left(\frac{1}{n}\right), \ x \in \left[0, \frac{1}{3}\right], \\ \gamma_{2,n} \sin 3\pi n x + O\left(\frac{1}{n}\right), \ x \in \left[\frac{1}{3}, 1\right], \end{cases}$$

$$y_{3,n}(x) = \begin{cases} O\left(\frac{1}{n}\right), \ x \in \left[0, \frac{1}{3}\right], \\ m \cos 3\pi (n + \frac{1}{2}) x + O\left(\frac{1}{n}\right), \ x \in \left[\frac{1}{3}, 1\right]. \end{cases}$$

Proof. From the asymptotic equalities obtained for $\rho_{1,n}$,$\rho_{2,n}$ and $\rho_{3,n}$ and asymptotic expression for $A_{22}(\rho)$ for the sufficiently large n we have

$$a_{22}(\rho_{1,n}) \neq 0, \quad a_{22}(\rho_{2,n}) \neq 0 \quad and \quad a_{22}(\rho_{3,n}) \neq 0.$$

Hence, for the sufficiently large n the eigenfunction of the problem (1)-(2) corresponding to eigenvalue $\lambda_{1,n} = (\rho_{1,n})^2$ will be

$$y_{1,n}(x) = \begin{cases} \frac{1}{\rho_{1,n}} a_{22}(\rho_{1,n}) y_1(x, \lambda_{1,n}), & for \quad x \in \left[0, \frac{1}{3}\right], \\ -\frac{1}{\rho_{1,n}} a_{21}(\rho_{1,n}) y_2(x, \lambda_{1,n}), & for \quad x \in \left[\frac{1}{3}, 1\right], \end{cases}$$

and the eigenfunction corresponding to eigenvalue $\lambda_{2,n} = (\rho_{2,n})^2$ and $\lambda_{3,n} = (\rho_{3,n})^2$ will be

$$y_{2,n}(x) = \begin{cases} \frac{1}{\rho_{2,n}} a_{22}(\rho_{2,n}) y_1(x, \lambda_{2,n}), & for \quad x \in \left[0, \frac{1}{3}\right], \\ -\frac{1}{\rho_{2,n}} a_{21}(\rho_{2,n}) y_2(x, \lambda_{2n}), & for \quad x \in \left[\frac{1}{3}, 1\right], \end{cases}$$

$$y_{3,n}(x) = \begin{cases} \frac{1}{\rho_{3n}} a_{22}(\rho_{3,n}) y_1(x, \lambda_{3,n}), & for \quad x \in \left[0, \frac{1}{3}\right], \\ -\frac{1}{\rho_{3,n}} a_{21}(\rho_{3,n}) y_2(x, \lambda_{3n}), & for \quad x \in \left[\frac{1}{3}, 1\right]. \end{cases}$$

Let $x \in \left[0, \frac{1}{3}\right]$. Since,

$$\begin{array}{l} \cos z = 1 + O\left(z^2\right), \quad z \to 0, \\ \sin z = z + O\left(z^3 = O(z)\right), \quad z \to 0. \end{array}.$$

Then we have:

$$\begin{array}{l} \frac{1}{\rho_{1,n}} a_{22}(\rho_{1,n}) = \cos\frac{2}{3}\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) + \\ +O\left(\frac{1}{n}\right) = \cos\left(2\pi n + \frac{2\alpha_1}{3n} + o\left(\frac{1}{n}\right)\right) + O\left(\frac{1}{n}\right) = \\ = 1 + O\left(\frac{1}{n}\right) \end{array}$$

$$y_1(x, \lambda) = \sin\rho_{1,n}x + O\left(\frac{1}{n}\right),$$

$$\begin{array}{l} \sin\rho_{1,n}x = \sin\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right)x = \\ = \sin\left(3\pi n x + O\left(\frac{1}{n}\right)\right) = \sin 3\pi n x + O\left(\frac{1}{n}\right), \end{array}$$

$$-\frac{1}{\rho_{1,n}} a_{21}(\rho_{1,n}) = -\cos\frac{1}{3}\rho_{1,n} + \rho_{1,n} m \sin\frac{1}{3}\rho_{1n} -$$

$$-mq_1\cos\frac{1}{3}\rho_{1,n} + o(1) = -\left(1 + mq_1\left(\frac{1}{3}\right)\right) \times$$

$$\times \cos\frac{1}{3}\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) +$$

$$+m\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right)\sin\frac{1}{3}\left(3\pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right)\right) + o(1) =$$

$$= -\left(1 + mq_1\left(\frac{1}{3}\right)\right)\cos\left(\pi n + \frac{\alpha_1}{3n} + o\left(\frac{1}{n}\right)\right) +$$

$$+m\left(3\pi n+\frac{\alpha_1}{n}+o\left(\frac{1}{n}\right)\right)\sin\left(\pi n+\frac{\alpha_1}{3n}+o\left(\frac{1}{n}\right)\right)=$$

$$=(-1)^{n+1}m\left(3\pi n+\frac{\alpha_1}{n}+o\left(\frac{1}{n}\right)\right)\left(\frac{\alpha_1}{n}+o\left(\frac{1}{n}+O\left(\frac{1}{n^3}\right)\right)\right)+o\left(1\right)=$$

$$=(-1)^{n+1}\left(1+mq_1\left(\frac{1}{3}\right)-m\alpha_1\pi\right)+o\left(1\right).$$

By the same way as for $y_1\left(x,\lambda_{1,n}\right)$ we can prove, that

$y_2\left(x,\lambda_{1,n}\right)=(-1)^{n+1}\sin 3\pi nx+O\left(\frac{1}{n}\right)$ $y_2\left(x,\lambda_{1,n}\right)=(-1)^{n+1}\sin 3\pi nx+O\left(\frac{1}{n}\right).$

Finally for $x\in\left[\frac{1}{3},1\right]$ we have

$$y_{1,n}\left(x\right)=\gamma_{1,n}\sin 3\pi nx+O\left(\frac{1}{n}\right),$$

here

$$\gamma_{1,n}=\left(1+mq_1\left(\frac{1}{3}\right)-m\alpha_1\pi\right)+o\left(1\right).$$

The following asymptotic equality for the eigenfunction $y_{2,n}\left(x\right)$ proves analogously:

$$y_{2,n}\left(x\right)=\begin{cases}\sin 3\pi nx+O\left(\frac{1}{n}\right),\ x\in\left[0,\frac{1}{3}\right],\\ \gamma_{2,n}\sin 3\pi nx+O\left(\frac{1}{n}\right),\ x\in\left[\frac{1}{3},1\right],\end{cases}$$

here

$$\gamma_{2,n}=1+mq_1\left(\frac{1}{3}\right)-m\alpha_2\pi+o\left(1\right).$$

Now we derive formulae for $y_{3,n}\left(x\right)$. At first let $x\in\left[0,\frac{1}{3}\right]$. In this case we obtain

$$\frac{1}{\rho_{3,n}^2}a_{22}\left(\rho_{3,n}\right)=\frac{1}{3\pi\left(n+\frac{1}{2}\right)+O\left(\frac{1}{n}\right)}\times$$

$$\times\cos\frac{2}{3}\left(3\pi\left(n+\frac{1}{2}\right)+\frac{\alpha_3}{n}+o\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=O\left(\frac{1}{n}\right)$$

Consequently, for $x\in\left[0,\frac{1}{3}\right]$ we obtain

$$y_{3n}\left(x\right)=O\left(\frac{1}{n}\right).$$

Now let $x\in\left[\frac{1}{3},1\right]$. In this case we obtain

$$-\frac{1}{\rho_{3,n}^2}a_{21}\left(\rho_{1,n}\right)=-\frac{1}{\rho_{3,n}}\cos\frac{1}{3}\rho_{3,n}+$$

$$+m\sin\frac{1}{3}\rho_{3,n}-\frac{q_1}{\rho_{3,n}}\sin\frac{1}{3}\rho_{3,n}-$$

$$-\frac{mq_1}{\rho_{3,n}^2}\cos\frac{1}{3}\rho_{3,n}+o\left(1\right)=$$

$$=m\sin\left(\pi n+\frac{\pi}{2}+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=$$

$$=-m\cos\left(\pi n+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=m\left(-1\right)^{n+1}+O\left(\frac{1}{n}\right).$$

$$\begin{array}{l}\sin\rho_{3n}(1-x)=\sin\left(3\pi\left(n+\frac{1}{2}\right)+O\left(\frac{1}{n}\right)\right)(1-x)=\\ =(-1)^{n+1}\cos\left(3\pi\left(n+\frac{1}{2}\right)\right)x+O\left(\frac{1}{n}\right).\end{array}$$

Consequently, for $x\in\left[\frac{1}{3},1\right]$we obtain

$$y_{3,n}\left(x\right)=\left(m\left(-1\right)^{n+1}+O\left(\frac{1}{n}\right)\right)\cdot\left(\left(-1\right)^{n+1}\cos\left(3\pi\left(n+\frac{1}{2}\right)x+O\left(\frac{1}{n}\right)\right)\right)=$$

$$=m\cos 3\pi\left(n+\frac{1}{2}\right)x+O\left(\frac{1}{n}\right).$$

Thus,

$$y_{3,n}\left(x\right)=\left\{\begin{array}{l}O\left(\frac{1}{n}\right),\quad x\in\left[0,\frac{1}{3}\right],\\ m\cos 3\pi\left(n+\frac{1}{2}\right)x+O\left(\frac{1}{n}\right),\ x\in\left[\frac{1}{3},1\right].\end{array}\right.$$

Theorem is proved.

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Received: 12 March 2019
Accepted: 21 June 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

Mathematical Approaches to Ground Objects Classification According to Satellite Data

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Abstract. In modern times, many international organizations have been developing research methods for remote diagnostic of ground objects. Over the last 15-20 years, despite the wide-scale development of computer programs that allow space designs to be of qualitatively new materials (up to 0.5 m accuracy) and to process cosmic images, the problem of using satellite imagery for ground objects classification has not been solved practically.
Comparative mathematical approaches to solving the land classification of satellite data, and comparative mathematical approaches to solution are given. Satisfaction with the application of satellite classification according to the satellite data of the classification and recognition methods.

Key Words and Phrases: satellite data, classification, metric distance, cluster algorithms.

2010 Mathematics Subject Classifications: 94A12

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1. Introduction

Rapid development of computer science and the broad range of software systems (e.g. Matlab, Mathematics, Mapple, etc.) have enabled the satellite data to be used for different authentication and recognition issues, or stimulates the accuracy of solving algorithms. The accuracy of the problem solved depends on many factors. Let's note some of them.

1) The mathematical specificity of the problem, in other words, the mathematical model's realistic relevance, so certain assumptions (restrictions) are taken for the model's probability, which in turn creates possible errors;

2) Mathematical problems typically have inverse issues in nature, and these issues are non-corrupt;

3) The solution of the problem is based on statistical data, the prices of these data are coincidental and vary depending on many factors; Statistical data is insufficient;

4) Based on the solution of the problem, the division of the classroom is broken and the classification criteria are different, and finally the solution depends on the chosen method; and so on.

In general, the task of recognition is as follows. The most common definition of a class is the following: a class is a collection (family) of objects that have some common properties. Information about the properties of an object can be obtained by observations, measurements, assessments, etc. and represented by a set of features, the values of which are expressed in numerical scales. Objects belonging to the same class are considered indistinguishable (equivalent), and each class of objects is characterized by a certain quality that distinguishes it from other classes. Together, all classes must constitute the initial set of objects [1].

Spectral images of objects on the earth's surface are non-stationary, as they depend on many factors, such as topography, soil type, climate, geographical location. To increase the reliability of decisions, it is necessary to use a priori information about the geometry of the survey, on the one hand, and the contextual information of the images themselves - on the other.

For this reason, the application of new approaches and the comparative analysis of the obtained results with known outcomes are of great importance both from the theoretical and the point of view.

2. Methodical basis and calculation methodology

The offered work is devoted to the classification of ground objects (e.g. of soil types, of aerosol-gas compounds) according to satellite data. The issue is as follows.

The classification of objects of the given area D is known in the coordinates (points) $\{P_i : i = \overline{1, m}\} \subset D$. Let us point the known classification of objects, in other words, objects classes with $\{M_k : k = \overline{1, r}\} : M_i \cap M_j = \emptyset, i \neq j$. Thus for $\forall i \in 1 : m = \{1; 2; ...; m\}$, $\exists k(i) \in 1 : r$

$$P_i \subset M_{k(i)},$$

it is true. In addition, at each $P \in D$ point the values of $W_{\lambda_k}(P), k = \overline{1, \chi}$- satellite data are known in the wave lengths $\{\lambda_k : k = \overline{1, \chi}\}$. The problem is to find the class M_k, of $\forall P \in D$ point, in other words, to find $\exists k(p) \in 1 : r$ number for $\forall P \in D$, so that $P \in M_{k(p)}$ is true.

First of all, it should be noted that, according to satellite data, the issue of ground objects classification has particular peculiarities. Thus, the variability of ground objects cover (eg: vegetation - natural or artificial, snow cover, other artificial covers, etc.) causes the variability of satellite data (at the same time). This, in turn, contributes to the distortion of the result. Thus, according to the satellite data, the reflection of object cover is first identified, and in the next step it is necessary to analyze the relationship between the value of reflection and object classification.

Now let's look to the problem solving algorithm.

Let's define the following set of indexes.

$$I_k = \{i \in 1 : m / P_i \in M_k\}, k = \overline{1, r}.$$

It is obvious that,

$$I_i \bigcap I_j = \emptyset, i \neq j$$

is true. The accuracy of the solution depends, of course, on the fact that the measurement data, i.e. enough statistical data.

Another peculiarity of the problem is that statistics are not usually sufficient (e.g. in large areas or in inaccessible areas). In this case, there is a need for new and deeper analysis methods to increase accuracy.

Each $P \in D$ point corresponds to the vector $P(\omega; W_1; ...; W_\kappa)$ where ω is the ground class of point P, where $W_k, k = \overline{1,\kappa}$ - is satellite datas. In case of solving the problem, each vector $(W_1; ...; W_\kappa)$ corresponds to a class. Let's point this argument with π, i.e. $\pi(W_1; ...; W_\kappa) = \omega$. Thus, the mathematical implication of the problem consists in the construction of $\pi : R^\kappa \to \{M_k : k = \overline{1,r}\}$. The values of the π judgment are sets, i.e. clusters. In general, the problem does not have this kind of single solution. Since the solution process is based on statistical data, the value of π is a coincidental number, for each $\overrightarrow{W} = (W_1; ...; W_\kappa)$ vector $\pi\left(\overrightarrow{W}\right)$ is a random quantity, i.e. $\pi\left(\overrightarrow{W}\right) \in M_k, k = \overline{1,r}$ occurs in a probability:

$$\sum_{k=1}^{r} P_k\left(\overrightarrow{W}\right) = 1.$$

Another approach to the study of $\pi(\bullet)$ is the application of phases theory methods. In this case, the value of $\pi\left(\overrightarrow{W}\right)$ can belong to each M_k class by defining an affiliation function. We will use Appendix I and II for the determination of $\pi(\bullet)$ in this case.

I. In this case, it is assumed that the classes are separated by satellite data, otherwise, there is no single solution to the problem. The center of each class is found by:

$$\overrightarrow{W}_0^{(k)} = \frac{1}{|I_k|} \sum_{i \in I_k} \overrightarrow{W}_i, k \in 1 : \kappa$$

here $|I_k|$ - the number of elements in I_k.

Should be find $R_k > 0$ radius, that $O_{R_k}\left(\overrightarrow{W}_0^{(k)}\right)$ ($\overrightarrow{W}_0^{(k)}$center, R_k radius) balls

$$O_{R_i}\left(\overrightarrow{W}_0^{(i)}\right) \bigcap O_{R_j}\left(\overrightarrow{W}_0^{(j)}\right) = \emptyset, i \neq j$$

satisfy the condition and $M_k \subset \min_{R_k > 0} O_{R_k}\left(\overrightarrow{W}_0^{(k)}\right)$.

For random $\overrightarrow{W} \in R^\kappa$ ($\forall P\left(\omega; \overrightarrow{W}\right) \in D$ point) if there is $\exists k_0 \in 1 : \kappa$,

$$\overrightarrow{W} \in O_{R_{k_0}}\left(\overrightarrow{W}_0^{(k_0)}\right)$$

then the corresponding point belongs to the class M_{k_0}. Otherwise there is a need for further analysis. For example

$$k_0 = \min_{1 \le k \le \kappa} \left|\overrightarrow{W} - \overrightarrow{W_0}^{(k)}\right| \tag{1}$$

can be taken to $p \in M_{k_0}$. In the case of (1), if k_0 is uniquely determined value, there is still need for additional analysis.

The closeness of the point or the given $\overrightarrow{W}$ vector to any M_k class can be defined as a mean distance from this vector to the M_k class vectors, i.e.,

$$\rho\left(\overrightarrow{W}; M_k\right) = \frac{1}{|I_k|}\sum_{i\in I_k}\left\|\overrightarrow{W} - \overrightarrow{W_i}\right\|.$$

Then $\overrightarrow{W} \in M_{k_0}$, where

$$k_0 = \min_{k\in 1:\kappa}\rho\left(\overrightarrow{W}; M_k\right), \tag{2}$$

can be accepted. There is a need for further analysis of the case (2), which is uniquely in relation to k_0 number.

II. In this approach, the probability of the point belonging to a particular class can be determined by the degree of closeness to that class compared to all classes. Obviously, the closer the point to the class, the greater the probability of belonging to that class. Thus, the probability that the $\overrightarrow{W}$ vector belongs to the M_k class (P_k):

$$P_k = \frac{1}{r-1}\left(1 - \frac{\rho\left(\overrightarrow{W}; M_k\right)}{\sum_{i\in 1:r}\rho\left(\overrightarrow{W}; M_i\right)}\right), k \in 1:r, \tag{3}$$

can be calculated by formula. According to logic, if the $\overrightarrow{W}$ vector coincides with an element of any class, then the probability that this element belongs to that class must be "1". But according to the formula (3) it is not correct. Nevertheless, in the considered assumptions the probability of being the smallest distance from the point to its class and ultimately related to the class is greater.

Note that in the formulas (2) and (3), the probability of the distance is, for example, the dispersion of the difference of M_k with the $\overrightarrow{W}$ vector and so oncan be taken. In general, the distance is

$$\tilde{\rho}\left(\overrightarrow{W}; M_k\right) = \frac{1}{|I_k|}\left(\sum_{i\in I_k}\alpha_k\left\|\overrightarrow{W} - \overrightarrow{W_i}\right\|^{\rho}\right)^{1/\rho},$$

where $\sum_{\alpha_k\geq 0}\alpha_k = 1$ - weight coefficients, $p \in [1; +\infty)$ - numbers.

At the beginning of the article, the broad possibilities of various software systems (eg Matlab, Mathematics, Mapple, etc.) were noted for the possibility of using satellite data in different identification and recognition issues. On our part, the possibilities of applying the automatic classification of ground object class for satellite data of the classification and recognition methods included in the Matlab software system were investigated. The pdist, the linkage, and the cluster included in the MATLAB packet programs were used to perform calculations [2].

3. The results of calculations

In the present study [4], the remote sensing data shown in his / her scientific work is used. [4] uses a description of the September 13, 2006, satellite of Quickbird American satellite, located in Canibek, a geographical area of 49.35-49.43 ° and a geographical latitude of 46.75-46.84 .The images were taken by blue, green, red and NIR spectral bands. The coverage area is 65 km ^{2}. Surface measurements cover the years 2002-2009 and have been implemented by the author of the thesis. The total area of the survey is 50.6 km^2, of which 16.1 km^2 is the stationary.

As a result of the research, the author carried out the classification of the land at the 271 surface measurement point. It has been shown that there are mixed soil at 101 surface measurement points and no soil type has been identified at these measuring points. Land types have been identified at 172 surface measurement points. Land at the measurement points - A (black earth); B (chestnut soil), C (deserted salt); D (salty) soil types. At the point $P\,(\omega; W_1; ...; W_\kappa)$, the set of values ω is {A, B, C, D}. Here, the elements of the $\overrightarrow{W}$ vector are the indicator of the spectral reflection of the soil at the measuring point and the numerical values mentioned in the spectral channels (blue, green, red and NIR interval, MDVI Index). The number of points that the investigator points to different types of soil is given below.

$$I_A = 23, I_B = 45, I_C = 56, I_D = 45.$$

In the present case, using the metric distances and classification methods, the classification of the above described lands is carried out. The pdist, the linkage, and the cluster included in the MATLAB packet programs were used to perform calculations [2]. The number of points to different types was calculated using the possible variants of the metric distance and the classification algorithms. The metric distance used and the name of the classification algorithms and the results obtained are given in Table 2.

Table 2. Number of points on different types of soil
(according to the selected metric distance and classification algorithms)

The used metric distance and cluster algorithms	I_A	I_B	I_C	I_D
Euclidean distance and "nearest neighbor" algorithm	1	158	1	12
Euclidean distance and "remote neighborhood" algorithm	72	45	12	43
Euclidean distance and "medium" algorithm	37	112	11	12
Euclidean distance and "centralization" algorithm	55	49	12	56
Euclidean distance and "step-by-step" algorithm	37	106	17	12

Table 2 continuation

The used metric distance and cluster algorithms	I_A	I_B	I_C	I_D
Normalized Euclidean distance and "nearest neighbor" algorithm	1	158	1	12
Normalized Euclidean distance and "remote neighborhood" algorithm	2	92	12	66
Normalized Euclidean distance and "medium" algorithm	39	107	14	12
Normalized Euclidean distance and "centralization" algorithm	19	76	12	65
Normalized Euclidean distance and "step-by-step" algorithm	38	108	14	12
Distance from the city neighbourhood and "nearest neighbour" algorithm	1	168	2	1
Distance from city neighbourhood and "remote neighbour" algorithm	38	86	39	9
Distance from city neighnourhood and "mid-contact" algorithm	5	140	14	13
Distance from city neighnourhood and the "centralization" algorithm	9	108	13	42
Distance from city neighnourhood and "step-by-step" algorithm	2	154	3	13

Table 2 continuation

The used metric distance and cluster algorithms	I_A	I_B	I_C	I_D
Mahalanobis distance and "nearest neighbor" algorithm	1	158	1	12
Mahalanobis distance and "remote neighbor" algorithm	72	45	12	43
Mahalanobis distance and the "mid-contact" algorithm	37	112	11	12
Mahalanobis distance and the "centralization" algorithm	55	49	12	56
Mahalanobis distance and "step-by-step" algorithm	37	106	17	12
Distance in Minkowski metric and "nearest neighbor" algorithm	1	158	1	12
Distance in Minkowski metric and "remote neighbor" algorithm	12	16	23	121
Distance in Minkowski metric and the "mid-contact" algorithm	35	114	11	12
Distance in Minkowski metric and the "centralization" algorithm	55	48	12	57
Distance in Minkowski metric and "step-by-step" algorithm	37	112	11	12

It appears from the table that the results obtained from the classifications and methods used in calculations are not consistent with the type of soil types taken by the author of the case.

4. Conclusion

According to satellite data, classification of ground object classification is mathematically correct and there are different mathematical approaches to the solution. The use of standard clustering methods is not satisfactory for the classification of soil according to satellite data. The solution of the problem necessitates the application of non-standard approaches and the comparative analysis of the obtained results with known outcomes.

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Received: 05 February 2019
Accepted: 24 April 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

On the Properties of Operator Generated by the Direct Value of the Derivative of Simple Layer Logarithmic Potential

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Abstract. The existence of the derivative of simple layer logarithmic potential is shown and some properties of the operator generated by the derivative of simple layer logarithmic potential are studied in generalized Hölder spaces.

Key Words and Phrases: Lyapunov curve, derivative of simple layer logarithmic potential, curvilinear singular integral, generalized Hölder spaces.

2010 Mathematics Subject Classifications: 45E05, 31B10

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1. Introduction

As is known (see [1]), the boundary value problems for vector Laplace equations are reduced to a singular integral equation which depends on the derivative of simple layer logarithmic potential

$$V(x) = \int_L \overrightarrow{grad_x} \Phi(x,\, y)\, \rho(y) dL_y, \quad x = (x_1,\, x_2) \in L, \tag{1}$$

where $L \subset R^2$ is a simple closed Lyapunov curve with the index $0 < \alpha \leq 1$, $\rho(y)$ is a continuous function on the curve L, $\Phi(x, y)$ is a fundamental solution of the Laplace equation $\Delta u = 0$, i.e.

$$\Phi(x,\, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}, x, y \in R^2, x \neq y,$$

and Δ is a Laplace operator.

Counterexamples provided by Lyapunov show (see [2]) that the derivatives for the simple and double layer potentials with continuous density do not exist in general. It should be noted that in [3], the boundedness of the operator generated by the direct value of the derivative of simple layer acoustic potential was proved in generalized Hölder spaces, and in [4], the acceptable formula for the calculation of derivative of the double layer acoustic potential was obtained and the basic properties of the operator generated by the derivative of double layer acoustic potential were studied in generalized Hölder spaces. Besides, based on these results, the approximate solutions of integral equations of boundary value problems for the Helmholtz equation were studied in [5, 6, 7, 8]. However, some basic properties of the operator $(A\rho)(x) = V(x),\ x \in L$ in generalized Hölder spaces have not been studied yet. This work is just dedicated to this matter.

2. Main Results

We denote by $C(L)$ a space of all continuous functions on L with the norm $\|\rho\|_\infty = \max_{x\in L} |\rho(x)|$, and we introduce a modulus of continuity of the form

$$\omega(\varphi,\, \delta) = \delta \sup_{\tau\geq\delta} \frac{\bar{\omega}(\varphi,\, \tau)}{\tau}, \quad \delta > 0,$$

for the function $\varphi(x) \in C(L)$, where $\bar{\omega}(\varphi,\, \tau) = \max\limits_{\substack{|x-y|\leq\tau \\ x,y\in L}} |\varphi(x) - \varphi(y)|$.

Theorem 1. *Let L be a simple closed Lyapunov curve with the index $0 < \alpha \leq 1$ and*

$$\int_0^{diam\, L} \frac{\omega(\rho,\, t)}{t} dt < +\infty.$$

Then the integral (1) exists in the sense of the Cauchy principal value, with

$$\sup_{x\in L} |\, V(x)\,| \leq M^* \left(\|\rho\|_\infty + \int_0^{diam\, L} \frac{\omega(\rho, t)}{t} dt \right).$$

Proof. Let $V(x) = (V_1(x), V_2(x))$, where

$$V_m(x) = \int_L \frac{\partial\, \Phi(x,y)}{\partial x_m} \rho(y)\, dL_y, \quad x = (x_1, x_2) \in L \quad (m = 1,\, 2).$$

Simple calculation yields

$$V_m(x) = \frac{1}{2\pi} \int_L \frac{y_m - x_m}{|x-y|^2} \rho(y)\; dL_y.$$

Let $d > 0$ be a radius of a standard circle for L (see [9]), and $\vec{n}(x)$ be an outer unit normal at the point $x \in L$. Then, for every point $x \in L$, the neighborhood $L_d(x) = \{y \in L:\ |y-x| < d\}$ either intersects the line parallel to the normal $\vec{n}(x)$ at one point

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*Hereinafter M denotes a positive constant which can be different in different inequalities.

only or does not intersect it at all, i.e. the set $L_d(x)$ is uniquely projected onto the interval $\Omega_d(x)$ lying on the line $\Gamma(x)$ tangent to L at the point x. On some part of $L_d(x)$, we choose a local rectangular coordinate system (u,v) centered at the point x, where the axis v is directed along the normal $\vec{n}(x)$, and the axis u is directed in the positive direction of the tangent line $\Gamma(x)$. It is known that the coordinates of the point x are (0, 0). Besides, in this coordinate system the neighborhood $L_d(x)$ can be given by the equation $v = f(u)$, $u \in \Omega_d(x)$, where $f \in H_{1,\alpha}(\Omega_d(x))$ and $f(0) = 0$, $f'(0) = 0$. Here $H_{1,\alpha}(\Omega_d(x))$ denotes the linear space of all continuously differentiable functions f on $\Omega_d(x)$, which satisfy the condition

$$\left|f'(u_1) - f'(u_2)\right| \le M_f \left|u_1 - u_2\right|^{\alpha}, \forall u_1, u_2 \in \Omega_d(x),$$

where M_f is a positive constant depending on f, but not on u_1 and u_2. Let $\Gamma_d(x)$ be a part of the tangent line $\Gamma(x)$ at the point $x \in L$ lying inside a circle of radius d centered at x. Besides, let $\tilde{y} \in \Gamma(x)$ be a projection of the point $y \in L_d(x)$. Then (see [10])

$$|x-\tilde{y}| \le |x-y| \le C_1 \, |x-\tilde{y}|\,, \quad mesL_d(x) \le C_2 \, mes\Gamma_d(x),$$

where C_1 and C_2 are positive constants depending only on L, and $mesL_d(x)$ denotes the length of the curve $L_d(x)$.

Obviously,

$$\begin{aligned}\int\limits_L \frac{y_m - x_m}{|x-y|^2}\,\rho\,(y)\; dL_y &= \int\limits_{L \backslash L_d(x)} \frac{y_m - x_m}{|x-y|^2}\rho\,(y)\; dL_y + \\ &+ \int\limits_{L_d(x)} \frac{y_m - x_m}{|x-y|^2}\;(\rho(y) - \rho(x))\; dL_y + \\ &+\rho\,(x) \int\limits_{L_d(x)} \frac{y_m - x_m}{|x-y|^2}\, dL_y\,, \quad x \in L \quad (m = 1,\, 2)\,. \end{aligned} \tag{2}$$

As we can see, the first integral on the right-hand side of the last equality exists as a proper integral, while the second one converges as an improper integral, with

$$\left|\int\limits_{L/\,L_d(x)} \frac{y_m - x_m}{|x-y|^2}\,\rho\,(y)\; dL_y\right| \le M\;\|\rho\|_{\infty}\,, \quad \forall x \in L \qquad (m = 1,\, 2) \tag{3}$$

and

$$\begin{aligned}&\left|\int\limits_{L_d(x)} \frac{y_m - x_m}{|x-y|^2}\;(\rho(y) - \rho(x))\; dL_y\right| \le \\ &\le M \int\limits_0^{diamL} \frac{\omega\,(\rho,\, t)}{t}\, dt < +\infty, \quad \forall x \in L \quad (m = 1,\, 2)\,. \end{aligned} \tag{4}$$

It remains to prove that the integral

$$\int\limits_{L_d(x)} \frac{y_m - x_m}{|x-y|^2}\, dL_y \quad (m = 1,\, 2)$$

exists in the sense of the Cauchy principal value. Let $d_0 = d/C_1$. It is clear that $(-d_0, d_0) \subset \Omega_d(x)$. Using the calculation formula for curvilinear integral, we obtain

$$\int\limits_{L_d(x)} \frac{y_1 - x_1}{|x-y|^2}\, dL_y = \int\limits_{\Omega_d(x)\setminus(-d_0,d_0)} \frac{u\sqrt{1+(f'(u))^2}}{u^2+(f(u))^2}\, du + \int\limits_{-d_0}^{d_0} \frac{du}{u} +$$

$$+ \int\limits_{-d_0}^{d_0} \frac{u\left(\sqrt{1+(f'(u))^2}-1\right)}{u^2+(f(u))^2} du + \int\limits_{-d_0}^{d_0} u\left(\frac{1}{u^2+(f(u))^2} - \frac{1}{u^2}\right) du.$$

Denote the integrals on the right-hand side of the last equality by A_1,A_2, A_3 and A_4, respectively.

As we can see, the integral A_1 exists as a proper integral, while the integral A_2 exists in the sense of the Cauchy principal value and is equal to zero. Besides, taking into account that

$$\left|f'(u)\right| \le M\ |u|^{\,\alpha} \tag{5}$$

(see [9]), we find

$$|A_3| = \left| \int\limits_{-d_0}^{d_0} \frac{u\ (f'(u))^2}{\left(u^2+(f(u))^2\right)\left(1+\sqrt{1+(f'(u))^2}\right)} du \right| \le M \int\limits_{-d_0}^{d_0} |u|^{2\alpha-1}\, du \le M.$$

As

$$|\,f(u)| = |\,f(u)\ - f(0)\,| \le M\ |u|^{\,1+\alpha}\,, \tag{6}$$

we have

$$|\,A_4\,| = \left| \int\limits_{-d_0}^{d_0} \frac{u\ (f(u))^2}{u^2\left(u^2+(f(u))^2\right)}\, du \right| \le M \int\limits_{-d_0}^{d_0} |u|^{2\alpha-1}\, du \le M$$

and

$$\left| \int\limits_{L_d(x)} \frac{y_2 - x_2}{|x-y|^2}\, dL_y \right| = \left| \int\limits_{\Omega_d(x)} \frac{f\,(u)\sqrt{1+(f'\,(u))^2}}{u^2+(f(u))^2}\, du \right| \le M \int\limits_{\Omega_d(x)} |u|^{\alpha-1}\, du \le M.$$

So we obtain

$$\left| \rho(x) \int\limits_{L_d(x)} \frac{y_m - x_m}{|x-y|^2} \, dL_y \right| \le M \, \|\rho\|_\infty \,, \quad \forall x \in L \quad (m = 1,\, 2)\,. \tag{7}$$

Considering the inequalities (3), (4) and (7) in (2), we finish the proof of the theorem.

Now let's show the validity of the Zygmund estimate for the direct value of the derivative of simple layer logarithmic potential.

Theorem 2. *Let L be a simple closed Lyapunov curve with the index $0 < \alpha \le 1$ and*

$$\int\limits_0^{diam\,L} \frac{\omega(\rho,\, t)}{t} dt < +\infty.$$

Then for every $m = \overline{1,2}$ and for any two points x', $x'' \in L$ the following estimates hold:

$$\left| V_m(x') - V_m(x'') \right| \le$$

$$\le M_\rho \left(h^\alpha + \omega(\rho, h) + \int\limits_o^h \frac{\omega(\rho, t)}{t} dt + h \int\limits_h^{diam L} \frac{\omega(\rho, t)}{t^2} dt \right) \quad \textit{if} \quad 0 < \alpha < 1,$$

$$\left| V_m(x') - V_m(x'') \right| \le$$

$$M_\rho \left(h \, |\ln h| + \omega(\rho, h) + \int\limits_o^h \frac{\omega(\rho, t)}{t} dt + h \int\limits_h^{diam L} \frac{\omega(\rho, t)}{t^2} dt \right) \quad \textit{if} \ \alpha = 1,$$

where $h = |\, x' - x'' \,|$, and M_ρ is a positive constant depending only on L and ρ.

Proof. Let $0 < \alpha < 1$ and $m = 1$. Consider any two points x', $x'' \in L$ such that h is sufficiently small. It is not difficult to see that

$$V_1(x') - V_1(x'') = \frac{1}{2\pi} \int\limits_L \left(\frac{(y_1 - x_1') \, (\rho(y) - \rho(x'))}{|x' - y|^2} - \right.$$

$$\left. - \frac{(y_1 - x_1'') \, (\rho(y) - \rho(x''))}{|x'' - y|^2} \right) dL_y +$$

$$+ \left(\frac{\rho(x')}{2\pi} \int\limits_L \frac{y_1 - x_1'}{|x' - y|^2} \, dL_y - \frac{\rho(x'')}{2\pi} \int\limits_L \frac{y_1 - x_1''}{|x'' - y|^2} \, dL_y \right).$$

Denote two terms on the right-hand side of the last equality by $F(x', x'')$ and $G(x', x'')$, respectively.

Estimate the integral $F(x',\,x'')$.

$$F(x',\,x'')=\int\limits_{L\backslash L_d(x')}\left(\frac{(y_1-x_1')\,\left(\rho\,(y)-\rho\,(x')\right)}{2\pi\,\left|x'-y\right|^2}-\frac{(y_1-x_1'')\,\left(\rho\,(y)-\rho\,(x'')\right)}{2\pi\,\left|x''-y\right|^2}\right)dL_y+$$

$$+\int\limits_{L_{h/2}(x')}\frac{(y_1-x_1')\,\left(\rho\,(y)-\rho\,(x')\right)}{2\pi\,\left|x'-y\right|^2}dL_y-\int\limits_{L_{h/2}(x'')}\frac{(y_1-x_1'')\,\left(\rho\,(y)-\rho\,(x'')\right)}{2\pi\,\left|x''-y\right|^2}dL_y-$$

$$-\int\limits_{L_{h/2}(x')}\frac{(y_1-x_1'')\,\left(\rho\,(y)-\rho\,(x'')\right)}{2\pi\,\left|x''-y\right|^2}dL_y+\int\limits_{L_{h/2}(x'')}\frac{(y_1-x_1')\,\left(\rho\,(y)-\rho\,(x')\right)}{2\pi\,\left|x'-y\right|^2}dL_y+$$

$$+\int\limits_{L_d(x')\backslash\left(L_{h/2}(x')\bigcup L_{h/2}(x'')\right)}\left(y_1-x_1'\right)\,\left(\rho\,(y)-\rho\,(x')\right)\,\times$$

$$\times\left(\frac{1}{2\pi\,\left|x'-y\right|^2}-\frac{1}{2\pi\,\left|x''-y\right|^2}\right)dL_y+$$

$$+\int\limits_{L_d(x')\backslash\left(L_{h/2}(x')\bigcup L_{h/2}(x'')\right)}\frac{(x_1''-x_1')\,\left(\rho\,(y)-\rho\,(x')\right)}{2\pi\,\left|x''-y\right|^2}\,dL_y+$$

$$+\left(\rho\left(x''\right)-\rho\left(x'\right)\right)\int\limits_{L_d(x')\backslash\left(L_{h/2}(x')\bigcup L_{h/2}(x'')\right)}\frac{y_1-x_1''}{2\pi\,\left|x''-y\right|^2}\,dL_y.$$

Denote the terms on the right-hand side of the last equality by $F_1(x',x'')$, $F_2(x',x'')$, $F_3(x',x'')$, $F_4(x',x'')$, $F_5(x',x'')$, $F_6(x',x'')$, $F_7(x',x'')$ and $F_8(x',x'')$, respectively.

Obviously, $|F_1(x',\,x'')|\leq M\,\left\|\rho\right\|_\infty h$.

Using the calculation formula for curvilinear integral, we have

$$\left|F_2(x',x'')\right|\leq M\int\limits_0^h\frac{\omega\,\left(\rho,t\right)}{t}dt,\quad\left|F_3(x',x'')\right|\leq M\int\limits_0^h\frac{\omega\,\left(\rho,t\right)}{t}dt.$$

Besides, considering the inequalities

$$h/2\leq\left|y-x''\right|\leq 3\,h/2,\quad y\in L_{h/2}(x'),$$

we obtain

$$\left|F_4(x',\,x'')\right|\leq M\,\frac{\omega\,\left(\rho,\,3h/2\right)}{h}mesL_{h/2}(x')\leq M\,\omega\,\left(\rho,\,h\right).$$

Similarly, taking into account the inequality

$$h/2\leq\left|y-x'\right|\leq 3\,h/2,\quad y\in L_{h/2}(x''),$$

we obtain $|F_5(x',\, x'')| \le M\,\omega\;(\rho,\, h)$.

For every $y \in L_d(x')\backslash(L_{h/2}(x')\bigcup L_{h/2}(x''))$ we have

$$\left|x'-y\right| \le \left|x'-x''\right| + \left|x''-y\right| \le 3\,\left|x''-y\right|$$

and

$$\left|x''-y\right| \le 3\,\left|x'-y\right|,$$

then

$$\left|F_6(x',x'')\right| \le Mh\int\limits_h^d \frac{\omega\;(\rho,t)}{t^2}dt,\;\; \left|F_7(x',x'')\right| \le Mh\int\limits_h^d \frac{\omega\;(\rho,t)}{t^2}dt.$$

Let's estimate the term $F_8(x',\, x'')$. To do so, we choose on some part of $L_d(x')$ a local rectangular coordinate system $(u,\, v)$ centered at the point x', where the axis v is directed along the normal $\vec{n}(x')$, and the axis u is directed in the positive direction of the tangent line $\Gamma(x')$. The coordinates of the point x' are $(0,\, 0)$, and the coordinates of the point x'' are denoted by $(u'',\, f(u''))$. Let $h_0 = |u''|$ and $\Omega_{h/2}\,(x', x'')$ denote the projection of the set $L_{h/2}\,(x')\bigcup L_{h/2}\,(x'')$ onto the tangent line $\Gamma(x')$.

By the calculation formula for curvilinear integral, we obtain

$$F_8(x',\, x'') = \frac{\rho\,(x'')-\rho\,(x')}{2\pi}\int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x',x'')} \frac{u\left(\sqrt{1+\left(f'\,(u)\right)^2}-1\right)}{u^2+\left(f(u)\right)^2}\,du+$$

$$+\frac{\rho\,(x'')-\rho\,(x')}{2\pi}\int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x',x'')} \left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)u\,du+$$

$$+\frac{\rho\,(x'')-\rho\,(x')}{2\pi}\int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x',x'')} \frac{du}{u}.$$

Taking into account (5), we find

$$\sqrt{1+\left(f'\,(u)\right)^2}-1 \le M\;|u|^{2\alpha},\;\; \forall u \in \Omega_d(x').$$

Besides, by virtue of (6) we obtain

$$\left|\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right| \le M\;|u|^{2\alpha-2},\;\; \forall u \in \Omega_d(x')\backslash 0.$$

Then

$$\left|\frac{\rho\,(x'')-\rho\,(x')}{2\pi}\int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x',x'')} \frac{u\left(\sqrt{1+\left(f'\,(u)\right)^2}-1\right)}{u^2+\left(f(u)\right)^2}\,du\right| \le M\omega\;(\rho,h)$$

and

$$\left|\frac{\rho\left(x''\right)-\rho\left(x'\right)}{2\pi}\int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x',x'')}\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)u\,du\right|\leq M\omega\ (\rho,h)\,.$$

As

$$\int\limits_{(-d_0,d_0)\,\backslash\,(-2h,2h)}\frac{du}{u}=\int\limits_{-d_0}^{-2h}\frac{du}{u}+\int\limits_{2h}^{d_0}\frac{du}{u}=0,$$

we have

$$\left|\frac{\rho\left(x''\right)-\rho\left(x'\right)}{2\pi}\int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x',x'')}\frac{du}{u}\right|=$$

$$=\left|\frac{\rho\left(x''\right)-\rho\left(x'\right)}{2\pi}\left(\int\limits_{\Omega_d(x')\backslash(-d_0,d_0)}\frac{du}{u}+\int\limits_{(-2h,2h)\backslash\Omega_{h/2}(x',x'')}\frac{du}{u}\right)\right|\leq$$

$$\leq\frac{\omega\left(\rho,h\right)}{2\pi}\left(M+M\int\limits_{h/C_1}^{2h}\frac{du}{u}\right)\leq M\omega\left(\rho,h\right),$$

and, consequently, $|F_8(x',\,x'')|\leq M\,\omega\,(\rho,h)$.

As a result, summing up the estimates obtained above for $F_j(x',\,x''),\ j=\overline{1,\,8}$, we find:

$$\left|F\left(x',\,x''\right)\right|\leq M\left(\|\rho\|_\infty\,h+\omega\left(\rho,h\right)+\int\limits_{o}^{h}\frac{\omega\ (\rho,t)}{t}dt+h\int\limits_{h}^{diamL}\frac{\omega\left(\rho,t\right)}{t^2}dt\right).$$

Now let's estimate the expression $G(x',\,x'')$. It is clear that

$$G(x',\,x'')=\frac{\rho\left(x'\right)-\rho\left(x''\right)}{2\,\pi}\int\limits_{L}\frac{y_1-x_1'}{\left|x'-y\right|^2}\,dL_y+$$

$$+\frac{\rho\left(x''\right)}{2\pi}\left(\int\limits_{L\backslash L_d(x')}\frac{y_1-x_1'}{\left|x'-y\right|^2}\,dL_y-\int\limits_{L\backslash L_d(x')}\frac{y_1-x_1''}{\left|x''-y\right|^2}\,dL_y\right)+$$

$$+\frac{\rho\left(x''\right)}{2\pi}\left(\int\limits_{L_d(x')}\frac{y_1-x_1'}{\left|x'-y\right|^2}\,dL_y-\int\limits_{L_d(x')}\frac{y_1-x_1''}{\left|x''-y\right|^2}\,dL_y\right).$$

Denote the terms on the right-hand side of the last equality by $G_1(x', x'')$, $G_2(x', x'')$ and $G_3(x', x'')$, respectively.

As the integral

$$\int_L \frac{y_1 - x_1'}{|x' - y|^2}\, dL_y$$

converges in the sense of the Cauchy principal value, we have

$$\left|G_1(x',\, x'')\right| \le M\, \omega\,(\rho, h)\,.$$

Besides, it is clear that

$$\left|G_2(x',\, x'')\right| \le M\ \|\rho\|_\infty\, h.$$

As is known, the following relations are true in the sense of the Cauchy principal value:

$$\int_{-d_0}^{d_0} \frac{du}{u} = 0 \ \text{ and } \int_{u''-d_0+h_0}^{u''+d_0-h_0} \frac{du}{u - u''} = 0.$$

Then the term $G_3(x',\, x'')$ can be represented as follows:

$$G_3(x',\, x'') = \frac{\rho(x'')}{2\pi} \left[- \int_{(-d_0,d_0)\setminus(u''-d_0+h_0,u''+d_0-h_0)} \frac{du}{u - u''} + \right.$$

$$+ \int_{\Omega_d(x')\setminus(-d_0,d_0)} \left(\frac{u}{u^2 + (f(u))^2} - \frac{u - u''}{(u - u'')^2 + (f(u) - f(u''))^2} \right) \sqrt{1 + (f'(u))^2} du +$$

$$+ \int_{(-d_0,d_0)\setminus((-h_0/2,h_0/2)\bigcup(u''-h_0/2,u''+h_0/2))} \frac{u'' \left(\sqrt{1 + (f'(u))^2} - 1 \right)}{u^2 + (f(u))^2} du +$$

$$+ \int_{(-d_0,d_0)\setminus((-h_0/2,h_0/2)\bigcup(u''-h_0/2,u''+h_0/2))} \left(\sqrt{1 + (f'(u))^2} - 1 \right) \times$$

$$\times \frac{(u - u'') \left((u - u'')^2 - u^2 + (f(u) - f(u''))^2 - (f(u))^2 \right)}{\left(u^2 + (f(u))^2 \right) \left((u - u'')^2 + (f(u) - f(u''))^2 \right)} du +$$

$$+ \int_{-h_0/2}^{h_0/2} \frac{u \left(\sqrt{1 + (f'(u))^2} - 1 \right)}{u^2 + (f(u))^2} du +$$

$$+\int\limits_{u''-h_0/2}^{u''+h_0/2}\frac{u\left(\sqrt{1+\left(f'(u)\right)^2}-1\right)}{u^2+\left(f(u)\right)^2}du-\int\limits_{-h_0/2}^{h_0/2}\frac{(u-u'')\left(\sqrt{1+\left(f'(u)\right)^2}-1\right)}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}du-$$

$$-\frac{\sqrt{1+\left(f'(u'')\right)^2}-1}{1+\left(f'(u'')\right)^2}\int\limits_{u''-h_0/2}^{u''+h_0/2}\frac{du}{u-u''}-$$

$$-\int\limits_{u''-h_0/2}^{u''+h_0/2}\frac{(u-u'')\left(\sqrt{1+\left(f'(u)\right)^2}-\sqrt{1+\left(f'(u'')\right)^2}\right)}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}du-$$

$$-\left(\sqrt{1+\left(f'(u'')\right)^2}-1\right)\int\limits_{u''-h_0/2}^{u''+h_0/2}\frac{1}{u-u''}\left(\frac{(u-u'')^2}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}-\right.$$

$$\left.-\frac{1}{1+\left(f'(u'')\right)^2}\right)du+\int\limits_{(-d_0,d_0)\backslash(u''-d_0+h_0,u''+d_0-h_0)}\left(u\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)-\right.$$

$$\left.-\left(u-u''\right)\left(\frac{1}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}-\frac{1}{(u-u'')^2}\right)\right)du+$$

$$+\int\limits_{(u''-d_0+h_0,u''+d_0-h_0)\backslash((-h_0/2,h_0/2)\bigcup(u''-h_0/2,u''+h_0/2))}u''\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)du+$$

$$+\int\limits_{(u''-d_0+h_0,u''+d_0-h_0)\backslash((-h_0/2,h_0/2)\bigcup(u''-h_0/2,u''+h_0/2))}\left(\left(\frac{1}{u^2+\left(f(u)\right)^2}-\right.\right.$$

$$\left.\left.-\frac{1}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}\right)+\left(\frac{1}{(u-u'')^2\left(1+\left(f'(u'')\right)^2\right)}-\frac{1}{u^2}\right)\right)\times$$

$$\times\left(u-u''\right)du+\int\limits_{-h_0/2}^{h_0/2}\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)u\,du+$$

$$+\int\limits_{u''-h_0/2}^{u''+h_0/2}\left(\frac{1}{u^2+\left(f(u)\right)^2}-\frac{1}{u^2}\right)u\,du+\int\limits_{-h_0/2}^{h_0/2}\left(\frac{1}{(u-u'')^2\left(1+\left(f'(u'')\right)^2\right)}-\right.$$

$$\left.-\frac{1}{(u-u'')^2+\left(f(u)-f(u'')\right)^2}\right)\left(u-u''\right)du+$$

$$+\int\limits_{u''-h_0/2}^{u''+h_0/2}\left(\frac{1}{(u-u'')^2\left(1+(f'(u''))^2\right)}-\frac{1}{(u-u'')^2+(f(u)-f(u''))^2}\right)(u-u'')\,du\Bigg].$$

As there exists a point $u_* = u'' + \theta\,(u - u'')$ such that

$$f(u) - f(u'') = f'(u_*)\,(u - u''),$$

where $\theta \in (0,\,1)$, it is not difficult to show that

$$\left|G_3(x',\,x'')\right| \le M\ \|\rho\|_\infty\,h^\alpha.$$

Consequently,

$$\left|G(x',\,x'')\right| \le M\ \left(\omega(\rho,\,h) + \|\rho\|_\infty\,h^\alpha\right).$$

Now, taking into account the estimates derived above for $F(x',\,x'')$ and $G(x',\,x'')$, we arrive at the conclusion that if $0<\alpha<1$, then

$$\left|V_1(x') - V_1(x'')\right| \le M_\rho\left(h^\alpha + \omega\,(\rho,h) + \int_o^h \frac{\omega\,(\rho,t)}{t}dt + h\int_h^{diamL}\frac{\omega\,(\rho,t)}{t^2}dt\right).$$

Similarly, it is not difficult to prove that

$$\left|V_2(x') - V_2(x'')\right| \le M_\rho\left(h^\alpha + \omega\,(\rho,h) + \int_o^h \frac{\omega\,(\rho,t)}{t}dt + h\int_h^{diamL}\frac{\omega\,(\rho,t)}{t^2}dt\right).$$

It follows from the proof of the theorem that if $\alpha = 1$, then

$$\left|V_m(x') - V_m(x'')\right| \le M_\rho\left(h\,|\ln h| + \omega\,(\rho,h) + \int\limits_o^h \frac{\omega\,(\rho,t)}{t}dt + \right.$$

$$\left. + h\int\limits_h^{diamL}\frac{\omega\,(\rho,t)}{t^2}dt\right), \quad m=\overline{1,2}.$$

Theorem is proved.

Theorem 3. *Let L be a simple closed Lyapunov curve with the index $0<\alpha\le 1$ and*

$$\int_0^{diam\,L}\frac{\omega(\rho,\,t)}{t}dt < +\infty.$$

Then the following estimates hold:
$\omega\,(V,h) \le M_\rho\left(h^\alpha + \omega\,(\rho,h) + \int_o^h \frac{\omega\,(\rho,t)}{t}dt + h\int_h^{diamL}\frac{\omega(\rho,t)}{t^2}dt\right)$ *if* $0<\alpha<1$,
$\omega\,(V,h) \le M_\rho\left(h\mid\ln h\mid + \omega\,(\rho,h) + \int_o^h \frac{\omega(\rho,t)}{t}dt + h\int_h^{diamL}\frac{\omega\,(\rho,t)}{t^2}dt\right)$ *if* $\alpha = 1$,
where M_ρ is a positive constant depending only on L and ρ.

Proof. Consider the function

$$\psi(h)=\begin{cases} h^{\alpha}+\omega(\rho,h)+\int_o^h \frac{\omega(\rho,t)}{t}dt+h\int_h^{diamL}\frac{\omega(\rho,t)}{t^2}dt, & if\ 0<\alpha<1,\\ h\left|\ln h\right|+\omega(\rho,h)+\int_o^h \frac{\omega(\rho,t)}{t}dt+h\int_h^{diamL}\frac{\omega(\rho,t)}{t^2}dt, & if\ \alpha=1.\end{cases}$$

It is not difficult to show that $\lim_{h\to 0}\psi(h)=0$, the function $\psi(h)$ is non-decreasing, and the function $\psi(h)/h$ is non-increasing. Then, using Theorem 2, we finish the proof of the theorem.

Introduce the following classes of functions on $(0, diamL]$:

$$\chi=\left\{\varphi:\ \ \varphi\uparrow, \lim_{\delta\to 0}\varphi(\delta)=0, \varphi(\delta)/\delta\downarrow\right\},$$

$$J_0(S)=\left\{\varphi\in\chi:\ \ \int_0^{diamL}\frac{\varphi(t)}{t}dt<+\infty\right\}.$$

Also consider the function

$$Z(h,\varphi)=\begin{cases} h^{\alpha}+\varphi(h)+\int_o^h \frac{\varphi(t)}{t}dt+h\int_h^{diamL}\frac{\varphi(t)}{t^2}dt, & if\ \ 0<\alpha<1,\\ h\left|\ln h\right|+\varphi(h)+\int_o^h \frac{\varphi(t)}{t}dt+h\int_h^{diamL}\frac{\varphi(t)}{t^2}dt, & if\ \ \alpha=1.\end{cases}$$

Where there is no misunderstanding, we will sometimes write $Z(h)$, $Z(\varphi)$ instead of $Z(h,\varphi)$. It is clear that $\lim_{h\to 0}Z(h)=0$, the function $Z(h)$ is non-decreasing, and the function $Z(h)/h$ is non-increasing.

Let $\varphi\in\chi$. Denote by $H(\varphi)$ the linear space of all continuous functions ρ on L which satisfy the condition

$$\left|\rho(x)-\rho(y)\right|\leq C_{\rho}\varphi(\left|x-y\right|), x,y\in L,$$

where C_{ρ} is a positive constant depending on L and ρ, but not on x and y. It is known (see [11]) that the space $H(\varphi)$ is a Banach space equipped with the norm

$$\|\rho\|_{H(\varphi)}=\sup_{x\in L}\left|\rho(x)\right|+\sup_{\substack{x,y\in L\\ x\neq y}}\frac{\left|\rho(x)-\rho(y)\right|}{\varphi(\left|x-y\right|)}.$$

Theorem 3 implies

Theorem 4. *Let $\varphi\in J_0(L)$. Then the operator $(A\rho)\ (x)=V(x)$, $x\in L$, acts boundedly from $H(\varphi)$ to $H(Z(\varphi))$, and*

$$\|V\|_{H(Z(\varphi))}\leq M\ \|\rho\|_{H(\varphi)}.$$

Denote by $H_\beta(L)$ the space of all continuous functions f on L which satisfy the Hölder condition

$$|f(x) - f(y)| \leq M_f |x-y|^\beta, \forall x, y \in L,$$

where $0 < \beta \leq 1$ and M_f is a positive constant depending on f, but not on x and y. It is known (see [11]) that the space $H_\beta(L)$ is a Banach space equipped with the norm

$$\|f\|_\beta = \sup_{x \in L} |f(x)| + \sup_{\substack{x,y \in L \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\beta}.$$

Corollary 1. *Let L be a simple closed Lyapunov curve with the index $0 < \alpha \leq 1$ and $\rho \in H_\beta(L)$, $0 < \beta \leq 1$. The following assertions are true:*
(a) *if $\alpha < \beta$, then $V \in H_\alpha(L)$ and $\|V\|_\alpha \leq M\ \|\rho\|_\beta$;*
(b) *if $\beta \leq \alpha < 1$, then $V \in H_\beta(L)$ and $\|V\|_\beta \leq M\ \|\rho\|_\beta$;*
(c) *if $\alpha = 1,\ \beta < 1$, then $V \in H_\beta(L)$ and $\|V\|_\beta \leq M\ \|\rho\|_\beta$;*
(d) *if $\alpha = 1,\ \beta = 1$, then $V \in H_\gamma(L)$ and $\|V\|_\gamma \leq M\ \|\rho\|_1$, where $\gamma \in (0,1)$.*

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Received: 19 February 2019
Accepted: 27 April 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

The Properties of the Eigenvalues and Eigenfunctions of a Vibration Boundary Value Problem

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Abstract. In the present paper we consider the eigenvalue problem for ordinary differential equation of fourth order with a spectral parameter contained linearly in the two of boundary conditions. The basic properties of the eigenvalues and eigenfunctions of this spectral problem are investigated.

Key Words and Phrases: spectral problem, eigenvalue, eigenfunction, oscillatory properties of eigenfunctions.

2010 Mathematics Subject Classifications: 34B09, 34B24, 34C23, 34L10, 47B50

1. Introduction

We consider the following boundary-value problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), \;\; 0 < x < 1, \tag{1}$$

$$y(0) = y'(0) = 0, \tag{2}$$

$$y''(1) - (a_1\lambda + b_1)\, y'(1) = 0 \tag{3}$$

$$Ty(1) - (a_2\lambda + b_2)y(1) = 0, \tag{4}$$

where $\lambda \in \mathbb{C}$ is spectral parameter, $Ty \equiv y''' - qy'$, $q(x)$ is positive and absolutely continuous function on $[0, l]$, a_1, a_2, b_1 and b_2 are real constants such that $a_1 > 0$ and $a_2 < 0$.

The problem (1)-(4) describes small bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, the left end is fixed rigidly, the right end is fixed elastically and in this end is concentrated the inertial mass (see [6, Ch. 8, § 5]).

The spectral properties of the eigenvalue problem (1)-(4) in the case $b_2 = 0$ was investigated in [4] (see also [3]). In these papers, the oscillation properties of eigenfunctions and their derivatives, the basis properties of the system of eigenfunctions in $L_p(0, 1)$, $1 < p < \infty$, are studied. Similar questions in the case when the spectral parameter is contained in one of the boundary conditions are studied in detail in the papers [1, 2, 5, 7].

Recall that boundary-value problem (1)-(4) reduces to a spectral problem for a self-adjoint operator in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$ (see [3, 4]). Hence all eigenvalues of this problem are real and simple.

In this paper, we study the general characteristic of the location of eigenvalues on the real axis, the oscillatory properties of eigenfunctions of problem (1)-(4) and their derivatives.

2. Operator interpretation of the boundary problem (1)-(4)

Let $H = L_2(0,1) \oplus \mathbb{C}^2$ be the Hilbert space with inner product

$$(\hat{u}, \hat{v}) = (\{u, m, k\}, \{v, s, t\}) = \int_0^1 u(x)\overline{v(x)}\, dx + \ |a_1|^{-1} m\bar{s} \ + \ |a_2|^{-1} k\bar{t}.$$

It is well known (see [3, 4]) that the boundary-value problem (1)-(4) reduces to the spectral problem for the linear operator L in the Hilbert space H, where

$$L\hat{y} = L\{u, m, k\} = \{(Ty(x))', \, y''(1) - b_1 y'(1), \, Ty(1) - b_2 y(1)\},$$

is an operator with the domain

$$D(L) = \{\{y\ (x), \, m, \, k\} \ : \ y \in W_2^4(0,1), \ (Ty(x))' \in L_2(0,1), \\ y(0) = y'(0) = 0, \ m = a_1 y'(1), \ n = a_2 y(1)\}\,.$$

It is obvious that the operator L is well defined in H and problem (1)-(4) take the form

$$L\hat{y} = \lambda\hat{y}, \ \hat{y} \in D(L).$$

This means that the eigenvalues λ_{nk}, $nk \in \mathbb{N}$, of problem (1)-(4) and of the operator L coincide, and between the eigenvectors, there is a one-to-one correspondence

$$y_n(x) \leftrightarrow \ \{y_n(x), \, m_n, \, k_n\}, \ \ m_n = a_1 y_n'(1), \ k_n = a_2 y_n\,(1).$$

Theorem 1. *L is a self-adjoint discrete lower-semibounded operator in H. The system of eigenvectors $\{\hat{y}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k(x), \, m_k, \, n_k\}$, $m_k = a_1 y_k'(1)$, $n_k = a_2 y_k(1)$, of the operator L forms forms an orthogonal basis in the space H (forms a Riesz basis (after normalization) in H).*

The proof of this theorem is similar to that of [3, Theorem 5.1].

Corollary 1. *The eigenvalues of the operator L are real, simple and form an unboundedly increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$.*

3. Oscillatory properties of eigenfunctions of the boundary-value problem (1)-(4) and their derivatives

Theorem 2. *For each fixed* $\lambda \in \mathbb{C}$ *there exists a unique nontrivial solution* $y(x,\lambda)$ *of the boundary-value problem* (1), (2), (4) *up to a constant factor. The solution* $y(x,\lambda)$ *for each fixed* $x \in [0,1]$ *is an entire function of* λ.

The proof of this theorem is similar to that of [3, Theorem 3.1].

Remark 1. It follows from [7, Theorem 2.2] that the eigenvalues of the boundary value problem

$$\begin{array}{l} \ell(y)(x) = \lambda y(x),\ x \in (0,1), \\ y(0) = y'(0) = 0, \\ y'(1)\cos\gamma + y''(1)\sin\gamma = 0,\ \gamma \in [0, \frac{\pi}{2}], \\ Ty(1) - (a_2\lambda + b_2)y(1) = 0, \end{array} \tag{5}$$

are real, simple and form an infinitely increasing sequence $\{\lambda_k^{(\gamma)}\}_{k=1}^{\infty}$; $\lambda_k(\gamma) > 0$ for $k \geq 2$ and there for each fixed γ exists $b_2(\gamma) < 0$ such that $\lambda_1(\gamma) > 0$ for $b_2 > b_2(\gamma)$, $\lambda_1(\gamma) = 0$ for $b_2 = b_2(\gamma)$ and $\lambda_1(\gamma) < 0$ for $b_2 < b_2(\gamma)$. Moreover, the eigenfunction $u_k^{(\gamma)}(x)$ corresponding to the eigenvalue $\lambda_k(\gamma)$ for $k \geq 2$ has $k-1$ simple zeros in the interval $(0,1)$; the eigenfunction $u_1^{(\gamma)}(x)$ has no zeros for $b_2 \geq b_2(\gamma)$ and has arbitrary number of zeros in $(0,1)$ for $b_2 < b_2(\gamma)$.

Denote: $B_k = (\lambda_{k-1}(0), \lambda_k(0))$, $k \in \mathbb{N}$, where $\lambda_0(0) = -\infty$.

Clearly, the eigenvalues $\lambda_k(0)$ and $\lambda_k(\pi/2)$, $k \in \mathbb{N}$, of problem (5) for $\gamma = 0$ and $\gamma = \pi/2$ are zeros of the entire functions $y'(1,\lambda)$ and $y''(1,\lambda)$, respectively. It is obvious that the function

$$F(\lambda) = y''(1,\lambda)/y'(1,\lambda)$$

is will defined for

$$\lambda \in B \equiv \left(\bigcup_{k=1}^{\infty} B_k\right) \bigcup (\mathbb{C}\backslash\mathbb{R}),$$

and is meromorphic function of finite order. $\lambda_k(0)$ and $\lambda_k(\pi/2)$, $k \in \mathbb{N}$, are the poles and zeros of the function $F(\lambda)$, respectively.

Lemma 1. *The following relations*

$$\frac{dF(\lambda)}{d\lambda} = -\frac{1}{y'^2(1,\lambda)}\left\{\int_0^1 y^2(x,\lambda)dx - a_2 y^2(1,\lambda)\right\},\ \lambda \in B. \tag{6}$$

$$\lim_{\lambda \to -\infty} F(\lambda) = +\infty, \tag{7}$$

are true.

The proof of this lemma is similar to that of [3, Lemmas 3.3 and 3.4].

It follows from the maximal minimal property of eigenvalues and (6) that

$$\lambda_1\left(\frac{\pi}{2}\right) < \lambda_1(0) < \lambda_2\left(\frac{\pi}{2}\right) < \lambda_2(0) < \dots\ . \tag{8}$$

Note that the eigenvalues of problem (1)-(4) are the roots of the following equation

$$y''(1,\lambda) - (a_1\lambda + b_1)\, y'(1,\lambda) = 0. \tag{9}$$

Lemma 2. *Let λ be an eigenvalue of the boundary-value problem* (1)-(4)*. Then $y'(1,\lambda) \neq 0$.*

Proof. Let λ be an eigenvalue of problem (1)-(4) and $y'(1,\lambda) = 0$. Then it follows from boundary condition (3) that $y''(1,\lambda) = 0$. Hence λ is an eigenvalue of problem (5) for $\gamma = 0$ and $\gamma = \frac{\pi}{2}$, which contradicts to relation (8). Thus $y'(1,\lambda) \neq 0$ if λ is an eigenvalue of problem (1)-(4). The proof of this lemma is complete.

Remark 2. It follows from Lemma 2 that each root of (9) is also root of the equation

$$F(\lambda) = a_1\lambda + b_1,$$

as well.

Theorem 3. *There exists an unboundedly increasing sequence of eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$ of the boundary value problem* (1)-(4)*. Moreover, we have the following location of these eigenvalues on the real axis:*

(i) if $\lambda_1(0) > 0$, then there exists a real number $b_{1,0}$ such that $\operatorname{sign} b_{1,0} = \operatorname{sign} \lambda_1(\pi/2)$ *and $\lambda_1 < 0$ for $b_1 > b_{1,0}$, $\lambda_1 = 0$ for $b_1 = b_{1,0}$, $\lambda_1 > 0$ for $b_1 < b_{1,0}$ and $\lambda_k > 0$ for $k \geq 2$;*

(ii) if $\lambda_1(0) = 0$, then $\lambda_1 < 0$ and $\lambda_k > 0$ for $k \geq 2$;

(iii) if $\lambda_1(0) < 0$, then $\lambda_1 < 0$ and there exists a real number $b_{1,1} > 0$ such that $\lambda_1 < 0$ for $b_1 > b_{1,1}$, $\lambda_1 = 0$ for $b_1 = b_{1,1}$, $\lambda_1 > 0$ for $b_1 < b_{1,1}$ and $\lambda_k > 0$ for $k \geq 3$.

The proof of this theorem is similar to that of first part of [3, Theorem 4.1] with use of Corollary 1, Theorem 2, Lemmas 1, 2 (relations (6) and (7)) and Remarks 1, 2.

Now let us take up the question of the number of zeros contained in the interval $(0,1)$ of the functions $y(x,\lambda)$ and $y'(x,\lambda)$.

Remark 3. Following the corresponding reasoning carried out in [3, Lemmas 3.1, 3.2 and 3.6] we can show that the zeros contained in $(0,1]$ of the functions $y(x,\lambda)$ and $y'(x,\lambda)$ are simple and C^1 functions of λ. Moreover, for $\lambda > 0$ between consecutive zeros of the function $y'(x,\lambda)$ in $(0,1]$, there is exactly one zero of function $y(x,\lambda)$; as $\lambda > 0$ increases the number of zeros contained in $(0,1)$ does not decrease.

We denote by $\epsilon(\lambda)$ and $\varkappa(\lambda)$ the number of zeros contained in $(0,1)$ of the functions $y(x,\lambda)$ and $y'(x,\lambda)$, respectively.

Theorem 4. *The functions $y(x,\lambda)$ and $y'(x,\lambda)$ have the following oscillation properties depending on the parameter $\lambda > 0$:*

(i) if $\lambda_1(0) \geq 0$, then
$\epsilon(\lambda) = \varkappa(\lambda) = 0$ *for* $\lambda \in (0, \lambda_1(0)]$,
$\epsilon(\lambda) = k-2$ *or* $\epsilon(\lambda) = k-1$ *for* $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2))$ *at* $k \geq 2$,
$\epsilon(\lambda) = k-1$ *for* $\lambda \in [\lambda_k(\pi/2), \lambda_k(0)]$ *at* $k \geq 2$,
$\varkappa(\lambda) = k-1$ *for* $\lambda \in (\lambda_{k-1}(0), \lambda_k(0)]$ *at* $k \geq 2$;
(ii) if $\lambda_1(0) < 0$, then
$\epsilon(\lambda) = \varkappa(\lambda) = 0$ *for* $\lambda \in (0, \lambda_2(0)]$,
$\epsilon(\lambda) = k-3$ *or* $\epsilon(\lambda) = k-2$ *for* $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2))$ *at* $k \geq 3$,
$\epsilon(\lambda) = k-2$ *for* $\lambda \in [\lambda_k(\pi/2), \lambda_k(0)]$ *at* $k \geq 3$,
$\varkappa(\lambda) = k-2$ *for* $\lambda \in (\lambda_{k-1}(0), \lambda_k(0)]$ *at* $k \geq 3$.

The proof of this theorem is similar to that of [3, Theorem 3.2] with use of Theorem 3 and Remarks 1, 3.

By virtue of [3, Corollary 3.1] as $\lambda < 0$ varies, the functions $y(x,\lambda)$ and $y'(x,\lambda)$ can lose or gain zeros only by these zeros leaving or entering the interval $[0,1]$ only through the endpoint $x = 0$. If these zeros pass through the point $x = 0$, then $x = 0$ would be a triple zero of function $y(x,\lambda)$, i.e. $y(0,\lambda) = y'(0,\lambda) = y''(0,\lambda) = 0$.

Assume that $\lambda < 0$ and μ is a real eigenvalue of the following spectral problem

$$\begin{aligned} &\ell(y)(x) = \lambda y(x),\ x \in (0,1), \\ &y(0) = y'(0) = y''(0) = 0, \\ &Ty(1) - (a_2\lambda + b_2)y(1) = 0. \end{aligned} \tag{10}$$

The oscillation index of μ which we denote by $i(\lambda)$ is the difference between the number of zeros of the function $y(x,\lambda)$ for $\lambda = \mu - 0$ contained in the interval $(0,1)$ and the number of the same zeros for $\lambda = \mu + 0$ (see [4, 5]). It follows from this definition that the number of zeros of the solution $y(x,\lambda)$ of problem (1), (2), (4) contained in $(0,1)$ is equal to the sum of the oscillation indices of all eigenvalues of the spectral problem (10) contained in the interval $(\lambda, 0)$.

Assume that $i(\mu_k)$ is an oscillation index of the eigenvalue μ_k, $k \in \mathbb{N}$, of problem (10), which is negative and simple [4, Lemma 4.2]. If $\lambda < 0$, then by condition (2) we have

$$\epsilon(\lambda) = \sum_{\mu_k \in (\lambda,0)} i\,(\mu_k), \tag{11}$$

$$\begin{aligned} &\varkappa(\lambda) = \sum_{\mu_k \in (\lambda,0)} i\,(\mu_k) \text{ for } \lambda_1(0) \geq 0, \\ &\varkappa(\lambda) = \sum_{\mu_k \in (\lambda,0)} i\,(\mu_k) + H(\lambda - \lambda_1(0)) \text{ for } \lambda_1(0) < 0. \end{aligned} \tag{12}$$

Theorem 5. *The eigenfunctions $y_k(x)$, $k = 1, 2, \ldots$ of the boundary value problem* (1)-(4) *and their derivatives have the following oscillation properties:*

i) if $\lambda_1(0) > 0$, *then:*

the functions $y_1(x)$ *and* $y_1'(x)$ *have no zeros in the case* $\lambda_1 \geq 0$, *have* $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ *simple zeros in the interval* $(0,1)$ *in the case* $\lambda_1 < 0$,

the function $y_k(x)$ *for* $k \geq 2$ *has either* $k-2$ *or* $k-1$ *simple zeros in the interval* $(0,1)$,

the function $y_k'(x)$ *has* $k-1$ *simple zeros in the interval* $(0,1)$;

ii) if $\lambda_1(0) = 0$, *then:*

the functions $y_1(x)$ *and* $y_1'(x)$ *have* $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ *simple zeros in the interval* $(0,1)$,

the function $y_k(x)$ *for* $k \geq 2$ *has either* $k-3$ *or* $k-2$ *simple zeros in the interval* $(0,1)$,

the function $y_k'(x)$ *has* $k-2$ *simple zeros in the interval* $(0,1)$;

iii) if $\lambda_1(0) < 0$, *then:*

the function $y_1(x)$ *has* $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ *and the function* $y_1'(x)$ *has* $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k) + H(\lambda_1 - \lambda_1(0))$ *simple zeros in the interval* $(0,1)$,

$y_2(x)$ *and* $y_2'(x)$ *have no zeros in the case* $\lambda_2 \geq 0$, *have* $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ *simple zeros in the interval* $(0,1)$ *in the case* $\lambda_2 < 0$,

the function $y_k(x)$ *for* $k \geq 3$ *has either* $k-3$ *or* $k-2$ *simple zeros in the interval* $(0,1)$,

the function $y_k'(x)$ *has* $k-2$ *simple zeros in the interval* $(0,1)$.

The proof of this theorem follows directly from Theorems 3, 4 and formulas (11), (12).

Remark 4. Using oscillation Theorems 3, 5 and applying the technique carried out in [3], it is possible to establish sufficient conditions for the subsystems of eigenfunctions of problem (1)-(4) to form a basis in the space $L_p(0,1)$, $1 < p < \infty$.

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Received 18 February 2019
Accepted 20 May 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

On Some Class of Extremal Manifolds

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Abstract. In this paper it is studied some class of extremal manifolds given by a system of smooth functions. V. G. Sprindzuk in [11] put question on obtaining the conditions in which a manifold is extremal. In this paper it is given such a condition in the terms of convergence exponent for some improper integrals like the special integral of Terry's problem.

Key Words and Phrases: measure theory, extremal manifold, transcendental number, Lebesgue measure, measurable function.

2010 Mathematics Subject Classifications: 26D15, 28A35, 57R35

1. Introduction.

In 1932 Mahler K. had advanced a conjecture about S-numbers. To formulate this conjecture let's introduce some notations. We shall denote by Π a following set of polynomials with integral coefficients of degree not exceeding n:

$$\Pi = \{f(x) = \sum_{i=0}^{n} a_i x^i \neq 0 | a_i \in \mathbb{Z}\}.$$

The number

$$H(f) = max(|a_0|, |a_1|, ..., |a_n|)$$

is called to be the height of the polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

with real coefficients. Let α be a transcendental number. Then $f(\alpha) \neq 0$. Consider some real number $H > 0$, and take all polynomials from the class Π with the heights not exceeding $H(f) \leq H$. Mahler had proven that the inequality

$$\|f(\alpha)\| > H^{-n\kappa}; h(f) \leq H$$

is satisfied for all polynomials in the class Π with the height not exceeding H for almost all real transcendental numbers, in the Lebesgue sense. The value firstly established for

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the constant κ by Mahler was $4+\varepsilon$, with arbitrarily small positive constant ε. Mahler had conjectured that it is possible to take $\kappa = 1+\varepsilon$. This conjecture was proven by Sprindzuk V. G. in 1965 by the method of essential and non-essential domains (see [11]).

For a given real number $H > 0$ the number of polynomials with heights doesn't exceeding H is finite. Denote by $\omega_n(\alpha)$ the supremum of that positive numbers $\gamma > 0$, for which the inequality

$$|f(\alpha)| < H^{-\gamma};\ H = H(f) \tag{1}$$

is satisfied for infinite number of polynomials from Π, when $H \to \infty$. It means that for arbitrary $\varepsilon > 0$ there is a non-bounded from above sequence $H_1, H_2, ...$ such that (1) is satisfied for all such H_m with

$$\gamma = \omega_n(\alpha) + \varepsilon.$$

This number is defined for every given n, and, by this reason, one can define the number (finite or infinite)

$$g = \overline{\lim_{n\to\infty}} \frac{\omega_n(\alpha)}{n}.$$

Note that for transcendental numbers due to Dirichlet's principle we always have $\omega_n(\alpha) \geq n$ and therefore, $g \geq 1$. The Mahler hypothesis is consisted in the statement that $\omega_n(\alpha) = n$ for almost all transcendental numbers α.

Consider now the system of inequalities

$$\max\left(\|\alpha_1 q\|, \|\alpha_2 q\|, ..., \|\alpha_n q\|\right) < q^{-u}, u > 0. \tag{2}$$

Let $u(\alpha_1, ..., \alpha_n)$ be defined as a *sup* of such $u > 0$ for which (2) is satisfied for infinite set of natural numbers q. It is not difficult to show that $u(\alpha_1, ..., \alpha_n) \geq 1/n$ (see [10]). From this definition it follows that the inequality (2) is satisfied for infinitely many natural numbers q when $u < 1/n$. When $u(\alpha_1, ..., \alpha_n) = 1/n$ for almost all points of the variety $(\alpha_1, ..., \alpha_n) \in R^n$ of less dimension, then we call this manifold as an *extremal manifold.* By Khintchine's Transference Principle (see [5, 9]), mentioned above Mahler hypothesis is equivalent to the hypothesis on extremality of the variety $(x, x^2, \ldots, x^n)$.

In 1993 Karatsuba A. A. advanced an opinion that the question on extremality of some algebraic varieties could be investigated by using of results on convergence exponent in the Tarry's problem (about the problem see [1]). This hypothesis was proven in [7].

Let we are given with some continuously differentiable n-dimensional manifold $\Gamma = (f_1(\bar{x}), ..., f_N(\bar{x}))$, $\bar{x} \in \Omega \subset \mathbb{R}^n, n < N$. In this work we continue consideration of conditions supplying the extremality of the manifold. Consider the integral (for some integral $h > 0$)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\int_{\Omega} e^{2\pi i(\alpha_1 f_1(\bar{x})+\alpha_2 f_2(\bar{x})\cdots+\alpha_n f_n(\bar{x}))}d\bar{x}\right|^{2h} d\alpha_1 d\alpha_2\cdots d\alpha_n.$$

The number γ is called to be the convergence exponent for the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\int_{\Omega} e^{2\pi i(\alpha_1 f_1(\bar{x})+\alpha_2 f_2(\bar{x})\cdots+\alpha_n f_1(\bar{x}))}dx\right|^{2h} d\alpha_1 d\alpha_2\cdots d\alpha_n,$$

if this integral is convergent when $2h > \gamma$ and divergent when $2h < \gamma$. In the section 3 we prove the extremality of above manifold if the last integral has finite exponent of convergence.

Authors express their sincere thanks to the professor M. Bayramoglu for useful discussions concerning the result of the article.

2. Auxiliary statements

Following lemma is known as Borel-Kantelly's lemma and plays an important role in the questions concerning extremality of manifolds (see[10]).

Lemma 1. Let $A_q\,(q = 1, 2, ...)$ be a sequence of measurable sets in $\mathbb{R}^n$, and

$$\sum_{q=1}^{\infty} \mathrm{m}esA_q < \infty.$$

Then the measure of a set E of such points $x \in R^n$ which fall into infinite number of sets A_q equals to zero.

Proof. For every $x \in E \subset R^n$ and natural n there is a natural number $m > n$ for which $x \in A_m$. Then for any $x \in E$ and natural number $n \in N$

$$x \in \bigcup_{k=n}^{\infty} A_k.$$

So,

$$E \subset \bigcup_{k=n}^{\infty} A_k.$$

Since the series of measures is convergent, then for arbitrary $\varepsilon > 0$ there exist a number n such that

$$mes \bigcup_{k=n}^{\infty} A_k \le \sum_{k=n}^{\infty} mesA_k < \varepsilon.$$

From the said above we deduce that $mesE = 0$. Lemma 1 is proven.

Below we will use the symbol $<<$ introduced by Vinogradov I. M. For two quantities A and B we write $A << B$ if one can find a fixed contant c such that $A \le cB$.

Following lemma belongs to Kavalevskaja E. I. (see [4,8,10]).

Lemma 2. Let *m, n, q* be natural numbers, $f_j(\bar{x}), j = 1, ..., N$ be a real measurable functions defined in the cube $\Omega = [0, 1]^r, 1 \le r \le N$. Denote by $\mu(q)$ the measure of a set of that $\bar{x} \in \Omega = [0, 1]^r$ for which

$$\|f_j(\bar{x})\| < q^{-r_j}\,(1 \le j \le N).$$

Then,

$$\mu(q) << q^{-r} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \int_{\Omega} e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} d\bar{x} \right|;$$

here $r = r_1 + \cdots + r_N$, and the constant in the symbol $<<$ depends on N only.

Let we are given with some continuously differentiable n-dimensional manifold $\Gamma = (f_1(\bar{x}), ..., f_N(\bar{x}))$, $\bar{x} \in \Omega = [0,1]^n, n < N$. Taking natural number h such that $nh > N$ consider the map

$$\varphi_j : \Omega^h \to R^N$$

defined by the equalities

$$\varphi_j(\bar{x}) = \varphi_j(\bar{x}_1, ..., \bar{x}_h) = f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h);\ \bar{x}_s = (x_{s1}, ..., x_{sm}).$$

Let the Jacoby matrix of the map $(\bar{x}_1, ..., \bar{x}) \mapsto (\varphi_1(\bar{x}), ..., \varphi_h(\bar{x}))$, i. e. the matrix composed of the gradients of the functions $\varphi_1(\bar{x}), ..., \varphi_h(\bar{x})$, be the matrix of maximal rank. It is easy to see that the Jacoby matrix has a view

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_{11}} & \cdots & \frac{\partial \varphi_1}{\partial x_{hn}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_N}{\partial x_{11}} & \cdots & \frac{\partial \varphi_N}{\partial x_{hn}} \end{pmatrix}.$$

In the work [3] there was proven the following result.

Lemma. If the Jacoby matrix of the map $(\bar{x}_1, ..., \bar{x}_h) \mapsto (\varphi_1(\bar{x}), ..., \varphi_h(\bar{x}))$ has a maximal rank for some natural h then the differentiable manifold Γ is extremal.

3. Main results.

Theorem 1. Let $g(\bar{x}) = \sum_{i=1}^{N} \alpha_i f_i(\bar{x})$. Then, in the conditions of the lemma the following formula is fair

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\int_{\Omega} e^{2\pi i g(\bar{x})} d\bar{x} \right)^{2h} d\alpha_1 \cdots d\alpha_N = \int_{\Pi} \frac{ds}{\sqrt{G_0}},$$

where the surface integral at the right side of the equality is taken over the surface defined by system of the equations

$$\begin{gathered} f_1(\bar{x}_1) + f_1(\bar{x}_2) + \cdots + f_1(\bar{x}_h) - f_1(\bar{x}_{h+1}) - f_1(\bar{x}_{h+2}) - \cdots - f_1(\bar{x}_{2h}) = 0, \\ f_2(\bar{x}_1) + f_2(\bar{x}_2) + \cdots + f_2(\bar{x}_h) - f_2(\bar{x}_{h+1}) - f_2(\bar{x}_{h+2}) - \cdots - f_2(\bar{x}_{2h}) = 0, \\ \cdots \quad \cdots \quad \cdots \\ f_j(\bar{x}_1) + f_j(\bar{x}_2) + \cdots + f_j(\bar{x}_h) - f_j(\bar{x}_{h+1}) - f_j(\bar{x}_{h+2}) - \cdots - f_j(\bar{x}_{2h}) = 0, \qquad (3) \\ \cdots \quad \cdots \quad \cdots \\ f_N(\bar{x}_1) + f_N(\bar{x}_2) + \cdots + f_N(\bar{x}_h) - f_N(\bar{x}_{h+1}) - f_N(\bar{x}_{h+2}) - \cdots - f_N(\bar{x}_{2h}) = 0 \end{gathered}$$

in Ω^{2h}, G_0 is a Gram determinant of gradients of functions standing on the left parts of equations from the system (3).

Remark. We can describe G_0 more explicitly. Let's designate

$$F_j(\bar{x}) = f_j(\bar{x}_1) + f_j(\bar{x}_2) + \cdots + f_j(\bar{x}_h) -$$

$$- f_j(\bar{x}_{h+1}) - f_j(\bar{x}_{h+2}) - \cdots - f_j(\bar{x}_{2h})$$

with $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_{2h}) \in R^{2hn}$. It is easy to see that the gradient vector for the function $F_j(\bar{x})$ has a view

$$\nabla F_j(\bar{x}) = (\nabla f_j(\bar{x}_1), \nabla f_j(\bar{x}_2), ..., \nabla f_j(\bar{x}_h),$$

$$- \nabla f_j(\bar{x}_{h+1}), - \nabla f_j(\bar{x}_{h+2}), ..., - \nabla f_j(\bar{x}_{2h})).$$

Now we put

$$A_0 = \begin{pmatrix} \nabla F_1(\bar{x}) \\ \vdots \\ \nabla F_N(\bar{x}) \end{pmatrix}.$$

Then one can write $G_0 = \det(A_0 A_0^T)$.

Proof of the theorem 1. Performing easy calculations we get

$$\left(\int_\Omega e^{2\pi i g(\bar{x})} d\bar{x} \right)^h = \int_\Omega \cdots \int_\Omega e^{2\pi i (g(\bar{x}_1) + \cdots + g(\bar{x}_h))} d\bar{x}_1 \cdots d\bar{x}_h, \tag{4}$$

where the function $g(\bar{x})$ stands for a linear combination of the functions $f_1(\bar{x}), ..., f_1(\bar{x})$:

$$g(\bar{x}) = \alpha_1 f_1(\bar{x}) + \cdots + \alpha_N f_N(\bar{x})$$

with real coefficients. Consider now the functions

$$\varphi_j(\bar{x}) = u_j = f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h),\ j = 1, ..., N,$$

with $\bar{x}_s = (x_{s1}, ..., x_{sn})$. Since the considered functions are continuous and the domain Ω is closed, then there exists a positive number $\eta > 0$ such that $G \geq \eta$. Applying the consequence of the lemma 1 from the work [2,6], we can represent the integral (4) as below

$$\int_\Omega \cdots \int_\Omega e^{2\pi i (\alpha_1 (f_1(\bar{x}_1) + \cdots + f_1(\bar{x}_h)) + \cdots + \alpha_N (f_N(\bar{x}_1) + \cdots + f_N(\bar{x}_h)))} d\bar{x}_1 \cdots d\bar{x}_h =$$

$$= \int_{m_1}^{M_1} \cdots \int_{m_n}^{M_n} \left(\int_\Pi \frac{ds}{\sqrt{G}} \right) e^{2\pi i (\alpha_1 u_1 + \cdots + \alpha_n u_n)} du_1 \cdots du_n, \tag{5}$$

designating by $\Pi = \Pi(\bar{u})$ the surface defined by the system of equations

$$f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h) = u_j,\ j = 1, ..., N,$$

and here the numbers m_j, M_j stand for the minimal and maximal values of the function $\varphi_j(\bar{x})$. Then, considering the last integral as a Fourier transformation, we will have by Parseval identity:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_\Omega e^{2\pi i (\alpha_1 f_1(\bar{x}) + \cdots + \alpha_N f_N(\bar{x}))} d\bar{x} \right|^{2h} d\alpha_1 \cdots d\alpha_N =$$

$$= (2\pi)^N \int_{m_1}^{M_1} \cdots \int_{m_n}^{M_n} \left(\int_\Pi \frac{ds}{\sqrt{G}} \right)^2 du_1 \cdots du_n, \tag{6}$$

and the equality is understood in the sense that from the convergence of one of its two parts the convergence of other part follows, and the corresponding values are equal.

Now we will use (6) to prove the statement of the main theorem. Let's assume that the right side part of the equality (6) is convergent. Applying the lemma 1, we have:

$$\int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} = \lim_{h\to 0} \frac{1}{(2\delta)^N} \int_{u_j-\delta<\varphi_j<u_j+\delta} d\bar{x}. \tag{7}$$

Therefore, designating the left part of (6) $\varphi_{\mathrm{D}}(\bar{u})$, we can, represent the last integral by the lemma 1 and its corollary write

$$\int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \left(\int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} \right)^2 d\bar{u} = \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u})\varphi_D(\bar{u}) d\bar{u} =$$

$$= \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u}) \lim_{h\to 0} \frac{1}{(2\delta)^N} \int_{u_j-\delta<\varphi_j<u_j+\delta} d\bar{x} d\bar{u}.$$

Applying the lemma 3 under integral on the right part it is possible to rearrange the orders of integration and passing to the limit. For this purpose we put $\delta = \delta_n$ with $\delta_n \to 0$ and apply the specified lemma to our integral, when $\delta = \delta_n$:

$$\int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u})\varphi_D(\bar{u}) d\bar{u} =$$

$$= \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u}) \lim_{n\to \infty} \frac{1}{(2\delta_n)^N} \int_{u_j-\delta_{\mathrm{n}}<\varphi_j<u_j+\delta_{\mathrm{n}}} d\bar{x} d\bar{u} =$$

$$= \lim_{n\to 0} \frac{1}{(2\delta_n)^N} \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \varphi_D(\bar{u}) \int_{u_j-\delta_{\mathrm{n}}<\varphi_j<u_j+\delta_{\mathrm{n}}} d\bar{x} d\bar{u} =$$

$$= \lim_{h\to 0} \frac{1}{(2\delta)^N} \int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \left(\int_{\Pi'(\bar{u})} \frac{ds'}{\sqrt{G'}} \right) \int_{\substack{u_j - \delta < \varphi_j < u_j + \delta \\ j = 1, ..., N}} d\bar{x} d\bar{u}, \tag{8}$$

where ds' means an element of the area of the surface defined in Ω by the system of equations $f_j(\bar{x}'_1) + \cdots + f_j(\bar{x}'_h) = u_j$, $j = 1, ..., N$, and is a Gram determinant for the system of functions standing at the left side of this system of equations. For the points $\bar{x}' \in \Omega^h$ we introduce the function $f(\bar{x}')$ defining its value at $\bar{x}' \in \Pi'(\bar{u})$ to be equal to the inner integral:

$$f(\bar{x}') = \int_{\substack{u_j - \delta < \varphi_j < u_j + \delta \\ j = 1, ..., N}} d\bar{x}.$$

Let's consider, at fixed δ, inner integral in the last chain of equalities (8), i.e. the integral

$$\int_{\Pi'(\bar{u})}\left(\int_{\substack{u_j-\delta<\varphi_j<u_j+\delta\\ j=1,...,N}} d\bar{x}\right)\frac{ds'}{\sqrt{G'}}=\int_{\Pi'(\bar{u})} f(\bar{x}')\frac{ds'}{\sqrt{G'}}.$$

Let's prove that the function $f(\bar{x}')$ is continuous in Ω^h. Let $\bar{x}'_1,\,\bar{x}'_2\in\Omega^h, \bar{x}'_1=(\bar{x}_{11},...,\bar{x}_{1h})$, $\bar{x}'_2=(\bar{x}_{21},...,\bar{x}_{2h})$; $\bar{x}_{ij}=(x^1_{ij},...,x^n_{ij})\in R^n, i=1,2,$ and

$$\sum_j\sum_s(x^s_{1j}-x^s_{2j})^2\le\varepsilon$$

for given $\varepsilon>0$. Then, denoting $u^1_j=\varphi_j(\bar{x}'_1)$, $u^2_j=\varphi_j(\bar{x}'_2)$ (here we use top indexing) we in accordance with the formula on finite increments have:

$$\left|u^1_j-u^2_j\right|=\left|\sum_{1\le s\le n}\sum_{1\le i\le h}\left(\frac{\partial f_j(\bar{x}'_i+\bar{\theta})}{\partial x_i{}^s}({x'_{1i}}^s-{x'_{2i}}^s)\right)\right|\le M\sqrt{nh\varepsilon}$$

for some $\bar{\theta}$, if $\sum_s\sum_i({x'_{1i}}^s-{x'_{2i}}^s)^2\le\varepsilon$, and M stands for maximal value of partial derivatives of the functions $f_j(\bar{x})$ in the considered domain. Therefore, recalling the definition of the function $f(\bar{x}')$, we find:

$$\left|f(\bar{x}'_1))-f(\bar{x}'_2)\right|=\Big|\int_{\substack{u^1_j-\delta<\varphi_j<u^1_j+\delta\\ j=1,...,N}} d\bar{x}-$$

$$-\int_{\substack{u^2_j-\delta<\varphi_j<u^{21}_j+\delta\\ j=1,...,N}} d\bar{x}\ \Big|\,.$$

The integrals at the right side of this equality express volumes of pre-images of two cubes with sufficiently close centers, when ε is small enough. From geometric representations it is clear that the difference between these volumes coincides with the sum of volumes of pre-images of parallelepipeds including lateral sides of the two initial cubes. Since the number of lateral sides is not exceeding $2N$, then we have

$$\left|f(\bar{x}'_1)-f(\bar{x}'_2)\right|\le 2N\max_j\int_{u^1_j-\delta-M\sqrt{nh\varepsilon}<\varphi_j<u^1_j-\delta+M\sqrt{nh\varepsilon}} d\bar{x}+$$

$$+2N\max_j\int_{u^2_j+\delta-M\sqrt{nh\varepsilon}<\varphi_j<u^2_j+\delta+M\sqrt{nh\varepsilon}} d\bar{x}.$$

These integrals can be bounded by a similar way. Estimate first of them. We have

$$\int_{u^2_j+\delta-M\sqrt{nh\varepsilon}<\varphi_j<u^2_j+\delta+M\sqrt{nh\varepsilon}} d\bar{x}=$$

$$\int_{u_j^2+\delta-M\sqrt{nh\varepsilon}}^{u_j^2+\delta+M\sqrt{nh\varepsilon}} du_1 \int_{m_2}^{M_2} du_2 \cdots \int_{m_{\mathrm{N}}}^{M_{\mathrm{N}}} du_N \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} \leq$$

$$\leq 2\prod_{k=2}^{N}(M_k-m_k)\sqrt{\frac{nh\varepsilon}{\eta}}\Pi_0;\ \Pi_0=\max_{\bar{u}}\int_{\Pi(\bar{u})}\frac{ds}{\sqrt{G}}$$

Since the domain Ω is bounded and the functions are continuous the last expression tends to 0 as $\varepsilon \to 0$. Therefore, the function $f(\bar{x}')$ is continuous. Applying the consequence to the lemma 1 of the work [6], we find:

$$\int_{m_1}^{M_1}\cdots\int_{m_N}^{M_N}\int_{\Pi(\bar{u})} d\bar{u} \int_{\substack{u_j-\delta<\varphi_j<u_j+\delta \\ j=1,...,N}} d\bar{x}\frac{ds'}{\sqrt{G'}} =$$

$$\int_{m_1}^{M_1}\cdots\int_{m_N}^{M_N} d\bar{u}\int_{\Pi(\bar{u})} f(\bar{x}')\frac{ds'}{\sqrt{G'}} = \int_{\substack{-\delta<\varphi_j-\varphi_j'<\delta \\ j=1,...,N}} d\bar{x}d\bar{x}'.$$

So, from the equality (8) we derive

$$\int_{m_1}^{M_1}\cdots\int_{m_N}^{M_N}\varphi_D(\bar{u})\varphi_D(\bar{u})d\bar{u} =$$

$$= \lim_{h\to 0}\frac{1}{(2\delta)^N}\int_{\substack{-\delta<\varphi_j-\varphi_j'<\delta \\ j=1,...,N}} d\bar{x}d\bar{x}' = \int_{\Pi_0}\frac{ds}{\sqrt{G_0}},$$

where G_0 is defined above.

The left part of the received equality under condition of existence of the right or left part of (8) coincides with the integral on the right part (8). It is clear that the all of reasonings performed above can be made in opposite direction. So, the theorem 1 is proven.

Theorem 2. Let the conditions of the theorem 1 be satisfied. If the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\int_{\Omega} e^{2\pi i(\alpha_1 f_1(\bar{x})+\alpha_2 f_2(\bar{x})\cdots+\alpha_n f_n(\bar{x}))}dx\right|^{2h} d\alpha_1 d\alpha_2\cdots d\alpha_n,$$

has finite exponent of convergence, then the manifold $\Gamma = (f_1(\bar{x}), f_2(\bar{x}), \cdots, f_n(\bar{x}))$ is extremal.

This theorem is an easy consequence of the theorem 1.

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Received: 03 March 2019
Accepted: 27 May 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

On Strong Law of Large Numbers for the Family of First Passage Times for the Level in Random Walk Described by a Non-Linear Function of Autoregression Process of Order One $(AR\,(1))$

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Abstract. In the paper we prove strong law of large numbers for the family of first passage times for the level in random walk described by a non-linear function of autoregression process of order one $(AR\,(1))$.

Key Words and Phrases: strong law of large numbers, autoregression process, first passage times, random walk.

2010 Mathematics Subject Classifications: 60F05

1. Introduction

Let on some probability space (Ω, F, P) we are given the sequence of independent identically distributed random variables $\xi_n = \xi_n\,(\omega)\,, n \geq 1, \omega \in \Omega$.

As is known ([1]-[9]), autoregression process of order one is determined as the solution of the equation

$$X_n = \beta X_{n-1} + \xi_n, \qquad n \geq 1$$

where β is some fixed number and the initial value of the process X_0 is independent of the innovation $\{\xi_n\}$.

Assume

$$T_n = \sum_{k=1}^{n} X_n X_{k-1} \quad \text{and} \quad \overline{T}_n = \frac{T_n}{n}, \qquad n \geq 1.$$

A number of asymptotic properties of distributed sums T_n, $n \geq 1$ were studied in the paper [1].

Let us consider the family of the first exit times

$$t_a = \inf\left\{n \geq 1 : n\Delta\left(\overline{T}_n\right) > a\right\} \tag{1}$$

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for the level $a \geq 0$, where $\Delta(x)$, $x \in R = (-\infty, \infty)$ is some Borel function.

The family of the stoppage time t_a, $a \geq 0$ of the form (1) play a significant roll in applied fields of theory of probability and mathematical statistics ([1-6]). Note that the boundary value problems related to the family of the first passage time

$$\tau_a = \inf \left\{ n \geq 1 : n\Delta\left(\frac{S_n}{n}\right) > a \right\},$$

where

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad n \geq 1$$

(see [7], [10]) are on the base of classic theory of nonlinear renewal.

In the case $\Delta(x) = x$ the limit theorems for the family of the first exit times t_a of the form (1) were studied in the monograph [10].

In the present paper we prove a theorem on strong law of large numbers for the family τ_a, $a \geq 0$.

2. Formulation and proof of the main result

For the function $\Delta(x)$ we will suppose that it is positive and twice continuously-differentiable in R.

In the paper [1] (see also [9], it was proved that under the continuous $E\xi_1 = 0$, $D\xi_1 = 1$, $|\beta| < 1$ and $EX_0^2 < \infty$ it holds the strong law of large numbers for the sequence of the sums $T_n, n \geq 1$:

$$\frac{T_n}{n} \overset{a.s.}{\to} \frac{\beta}{1-\beta^2} = \lambda \quad \text{as} \quad n \to \infty. \tag{2}$$

By the made assumptions for the function $\Delta(x)$ we have

$$n\Delta\left(\overline{T}_n\right) = n\Delta(\lambda) + u\Delta'(\lambda)\left(\overline{T}_n - \lambda\right) +$$

$$+\frac{n}{2}\Delta''(\lambda_n)\left(\overline{T}_n - \lambda\right)^2 = n\Delta(\lambda) + \Delta'(\lambda)(T_n - n\lambda) +$$

$$+\frac{1}{2}\Delta''(\lambda)\left(\frac{T_n - n\lambda}{\sqrt{n}}\right)^2,$$

where λ_n is an intermediate point between λ and $\overline{T}_n$, $n \geq 1$.

Assume

$$Z_n = n\Delta(\lambda) + n\Delta(\lambda) + \Delta'(\lambda)(T_n - n\lambda) = \sum_{k=1}^{n} \eta_k,$$

$$\eta_k = \Delta(\lambda) + \Delta'(\lambda)(X_k X_{k-1} - \lambda)$$

and

$$\varepsilon_n = \frac{1}{2}\Delta''(\lambda_n)\left(\frac{T_n - n\lambda}{\sqrt{n}}\right)^2$$

$$H_n = n\Delta\left(\overline{T}_n\right).$$

Then we have

$$H_n = Z_n + \varepsilon_n. \tag{3}$$

By (2),

$$\frac{Z_n}{n} \overset{a.s.}{\to} \Delta(\lambda) \quad \text{and} \quad \frac{\varepsilon_n}{n} \overset{a.s.}{\to} o \tag{4}$$

as $n \to \infty$, by continuity

$$\Delta''(\lambda_n) \overset{a.s.}{\to} \Delta''(\lambda), \quad n \to \infty.$$

Then from (3) and (4) it follows that

$$\frac{H_n}{n} \overset{a.s.}{\to} \Delta(\nu) \quad \text{as} \quad n \to \infty. \tag{5}$$

It holds

Theorem 1. *Let* $|\beta| < 1$, $E\xi_1 = 0$, $D\xi_1 = 1$ *and* $EX_0^2 < \infty$. *Assume that the above mentioned conditions are fulfilled for the functions* $\Delta(x)$, *moreover* $\Delta(\lambda) > 0$.

Then

$$\frac{t_a}{a} \overset{a.s.}{\to} \frac{1}{\Delta(\lambda)}, \quad a \to \infty.$$

Proof. From (5) it follows that $\sup\limits_n H_n = \infty$. Hence, by definition of the variable t_a it follows that $P(t_a < \infty) = P\left(\sup\limits_n H_n > a\right) = 1$ for all $a \geq 0$. Show that

$$t_a \overset{a.s.}{\to} \infty \quad \text{as} \quad a \to \infty$$

Indeed, by definition of the variable t_a it increases as a function of a. Therefore

$$P\left(t_\infty = \lim_{a\to\infty} t_a \leq \infty\right) = 1.$$

We have

$$P(t_\infty \leq n) = P\left(\lim_{a\to\infty} t_a \leq n\right) =$$

$$= \lim_{a\to\infty} P(t_a \leq n) = \lim_{a\to\infty} P\left(\max_{k\leq n} H_k > a\right) = o$$

for all $n \geq 1$.

This means that for all $n \geq 1$

$$P(t_\infty > n) = 1.$$

Hence it follows that $P(t_\infty = \infty) = 1$.

Thus, we have

$$P\left(\lim_{a\to\infty} t_a = \infty\right) = 1. \tag{6}$$

Prove that from (5) and (6) it follows that

$$\frac{Ht_a}{t_a} \overset{a.s.}{\to} \Delta(\nu) \quad \text{as} \quad a \to \infty. \tag{7}$$

Denote

$$A = \left\{ \omega : \frac{H_n}{n} \to \Delta(\nu) \right\}$$

$$B = \{ \omega : t_a \to \infty \}$$

$$C = \left\{ \omega : \frac{H_{t_a}}{t_a} \to \Delta(\nu) \right\}.$$

It is clear that

$$A \cap B \subset C. \tag{8}$$

Taking into account $P(A) = P(B) = 1$, we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1$$

hence

$$P(A \cup B) = 1.$$

Then from (8) it follows that $P(C) = 1$. Thus, (7) is proved. By (7) the statement of the theorem follows from the following two-sided inequality

$$\frac{H_{t_a - 1}}{t_a} \le \frac{a}{t_a} < \frac{H_{t_a}}{t_a},$$

whose validity follows from the definition of the first exit time t_a of the form (1) .

From the proved theorem and the well known theorem on convergence of a sequence of identically integrable random variables (see e.i. [10]) it follows the following result.

Corollary 1. *Let the theorem conditions be fulfilled and the family* $\frac{t_a}{a}$, $a > 0$ *be identically integrable. Then*

$$\frac{Et_a}{a} \to \frac{1}{\Delta(\nu)}, a \to \infty.$$

Remark 1. *Note that the statement of the Corollary in the case* $\Delta(x) = x$ *was proved in the paper [4], where the sufficient condition was found for identically integrable family* $\frac{t_a}{a}$, $a > 0$.

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Received: 18 December 2018
Accepted: 29 May 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

Calculating Steady-state Probabilities of the G/M/n/m Queueing Systems

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Abstract. This article proposes a method for calculating steady-state probabilities of the $G/M/n/m$ queueing systems. The approach based on the use of fictitious phases and hyper-exponential approximations with parameters of the paradoxical and complex type by method of moments. The obtained results are verified using simulation models.

Key Words and Phrases: non-Markovian queueing system, hyperexponential approximation, fictitious phases, method of moments.
2010 Mathematics Subject Classifications: 60G10, 60J28, 60K25, 93B40

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1. Introduction

For the study of non-Markovian process in queueing systems, phase-type distributions are used with exponential distributions of delays in the phases [2, 5, 8]. In the case of fixing the number of the phase, the states of the system has a Markov property that makes it possible to represent the transitions between them in the form of a discrete Markov process with continuous time. The order of approximation is the number of retained initial moments of the original distribution.

Recently, interest in the hyperexponential distribution has increased since its use showed its high performance in solving problems of summation of recurrent flows [4], in computing characteristics of queuing systems with impatient customers [3] and Jackson's networks of queueing [6], and also in analyzing stock management systems [1].

Article [8] shows that the use of hyperexponential approximation (H_l) makes it possible to determine with high accuracy the steady-state probabilities of non-Markovian single-channel queuing systems. These probabilities are determined using solutions of a system of linear algebraic equations obtained by the method of fictitious phases. To find parameters of the H_l-approximation of a certain distribution it is sufficient to solve the system of equations of the moments method. For the values $V < 1$ of the variation coefficient, roots of this system are complex-valued or paradoxical (i.e., negative or with probabilities that exceed the boundaries of the interval [0, 1]) but in most cases as a result of summation of probabilities of microstates, their complex-valued and paradoxical parts are annihilated.

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The purpose of the paper is to use of the hyperexponential approximation method for calculating steady-state probabilities of the $G/M/n/m$ queueing systems. The obtained results are verified using simulation models. We also indicate ways to evaluate the accuracy of approach the obtained steady-state distribution to the true distribution without the need to use simulation models.

2. Equations for steady-state probabilities of the $H_l/M/n/m$ system

The hyperexponential distribution of order l is a phase-type distribution and provides for choosing one of l alternative phases by a random process. With probability α_s, the process is at the sth phase and is in it during an exponentially distributed time with a parameter λ_s.

Suppose that the times elapsed between two consecutive arrivals are independent random variables distributed according to the hyperexponential law H_l $(l \geq 2)$ with probabilities α_s and parameters λ_s $(1 \leq s \leq l)$ and the service time of each customer is distributed exponentially with parameter μ. Let n and m denote the number of channels in the system and limit on the queue length respectively.

Let us enumerate the $H_l/M/n/m$ system's states as follows: $x_{0(s)}$ corresponds to the empty system and the time interval until the arrival of the first customer is in the phase s $(1 \leq s \leq l)$; $x_{k(s)}$ is the state, when there are k customers in the system $(1 \leq k \leq n+m)$, the time interval until the arrival of the next customer is in the phase s $(1 \leq s \leq l)$. We denote by $p_{0(s)}$ and $p_{k(s)}$ respectively, steady-state probabilities that the system is in the each of these states. To calculate $p_{0(s)}$ and $p_{k(s)}$ we obtain the system of linear equations:

$$\begin{aligned}
-\lambda_s p_{0(s)} + \mu p_{1(s)} &= 0, \quad 1 \leq s \leq l; \\
-(\lambda_s + k\mu) p_{k(s)} + \alpha_s \sum_{u=1}^{l} \lambda_u p_{k-1(u)} + (k+1)\mu p_{k+1(s)} &= 0, \;\; 1 \leq k \leq n-1, \;\; 1 \leq s \leq l; \\
-(\lambda_s + n\mu) p_{k(s)} + \alpha_s \sum_{u=1}^{l} \lambda_u p_{k-1(u)} + n\mu p_{k+1(s)} &= 0, \;\; n \leq k \leq n+m-1, \;\; 1 \leq s \leq l; \\
-(\lambda_s + n\mu) p_{n+m(s)} + \alpha_s \sum_{u=1}^{l} \lambda_u \left(p_{n+m-1(u)} + p_{n+m(u)}\right) &= 0, \quad 1 \leq s \leq l; \\
\sum_{k=0}^{n+m} \sum_{u=1}^{l} p_{k(u)} &= 1.
\end{aligned} \tag{1}$$

Solving the system (1), we find the steady-state probabilities p_k of the presence in the queueing system of k customers using the formulas

$$p_k = \sum_{u=1}^{l} p_{k(u)}, \quad 0 \leq k \leq n+m. \tag{2}$$

3. Features of finding probabilities p_k in the case of complex-valued or paradoxical parameters of H_l-approximation

We calculate the approximate steady-state probabilities p_k for the $G/M/n/m$ system using solutions of equations (1), written for the $H_l/M/n/m$ system, considering the order of approximation l from 2 to 6.

To find parameters of H_l-approximation of a certain distribution with a given coefficient of variation it is sufficient to solve the system of equations of the moments method only for the case of any one given mean value of this distribution since roots of the equations of the moments method are invariant with respect to the scale transformation.

The system of equations of the moments method for approximating the distribution of some random variable X using a random variable Y_l distributed by law of H_l is of the form

$$\sum_{s=1}^{l} \frac{\alpha_s}{\lambda_s^i} = \frac{m_i}{i!}, \qquad 0 \le i \le 2l-1; \qquad \sum_{s=1}^{l} \alpha_s = 1, \tag{3}$$

where $m_i = E(X^i)$ is the initial moment of order i of the random variable X. The dependence of the nature of the roots of system (3) on values of the variation coefficient V for the original gamma distributions and Weibull distributions is described in [8]. For the values $V < 1$ of the variation coefficient, some of the roots of system (3) are complex-valued but in most cases as a result of summation of probabilities of microstates the steady-state probabilities p_k are real-valued.

To illustrate this fact, we consider the solutions of system (1) for complex-valued parameters α_s and λ_s, limited to the case when $l = 2$, $n = 1$ and $m = 1$. In this case, using the solutions of system (1) and formula (2), we obtain

$$\begin{aligned} p_0 &= \frac{\mu^2}{\Delta} \big((\alpha_2\lambda_1 + \alpha_1\lambda_2)\mu^2 + \alpha_2\lambda_1^3 + \alpha_1\lambda_2^3 + \\ &\quad + \left(\alpha_2(\alpha_1 + 2\alpha_2)\lambda_1^2 + 2\alpha_1\alpha_2\lambda_1\lambda_2 + \alpha_1(2\alpha_1 + \alpha_2)\lambda_2^2\right)\mu \big), \\ p_1 &= \frac{\lambda_1\lambda_2\mu}{\Delta} \left(\mu^2 + 2(\alpha_2\lambda_1 + \alpha_1\lambda_2)\mu + \alpha_2\lambda_1^2 + \alpha_1\lambda_2^2\right), \qquad p_2 = 1 - p_0 - p_1, \\ \Delta &= (\alpha_2\lambda_1 + \alpha_1\lambda_2)\left(\mu^4 + ((\alpha_1 + 2\alpha_2)\lambda_1 + (2\alpha_1 + \alpha_2)\lambda_2)\,\mu^3 + (\lambda_1 + \lambda_2)^2\mu^2 + \right. \\ &\quad \left. + ((2\alpha_1 + \alpha_2)\lambda_1 + (\alpha_1 + 2\alpha_2)\lambda_2)\,\lambda_1\lambda_2\mu + \lambda_1^2\lambda_2^2\right). \end{aligned} \tag{4}$$

If parameters α_s and λ_s $(s = 1, 2)$ are complex-valued, then they can only be complex conjugate, and all possible cases of alternation of signs before the imaginary unit can be reduced to such two:

$$\begin{aligned} &1)\ \alpha_1 = a + ib,\ \ \lambda_1 = c + id; \quad \alpha_2 = a - ib,\ \ \lambda_2 = c - id; \\ &2)\ \alpha_1 = a + ib,\ \ \lambda_1 = c - id; \quad \alpha_2 = a - ib,\ \ \lambda_2 = c + id. \end{aligned} \tag{5}$$

In each of these cases, the imaginary parts in expressions (4) for p_k $(k = 0, 1, 2)$ are reduced, because the expressions

$$\begin{aligned} &\lambda_1 + \lambda_2,\ \ \lambda_1\lambda_2,\ \ \alpha_1\alpha_2,\ \ \lambda_1^2 + \lambda_2^2,\ \ \alpha_2\lambda_1 + \alpha_1\lambda_2,\ \ \alpha_1\lambda_1 + \alpha_2\lambda_2, \\ &\alpha_2\lambda_1^2 + \alpha_1\lambda_2^2,\ \ \alpha_2\lambda_1^3 + \alpha_1\lambda_2^3,\ \ \alpha_2^2\lambda_1^2 + \alpha_1^2\lambda_2^2 \end{aligned}$$

of which consist p_k, are real-valued.

In the case of complex-valued or paradoxical roots α_s and λ_s of system (3), let us name the function $F_{H_l}(t) = 1 - \sum\limits_{s=1}^{l} \alpha_s e^{-\lambda_s t}$ $(t \geq 0)$ the distribution pseudo-function by law of H_l. Let us show that the function $F_{H_l}(t)$ is a real-valued function if α_s and λ_s $(1 \leq s \leq l)$ are roots of system (3).

In fact, if some of the roots of system (3) are complex-valued, then they can only be complex conjugate, and all possible cases of alternation of signs before the imaginary unit can be reduced to two cases presented in (5). In each of these cases, the imaginary parts in the expression for $F_{H_l}(t)$ are reduced, so the result is the real-valued function:

$$1)\ \ \alpha_1 e^{-\lambda_1 t} + \alpha_2 e^{-\lambda_2 t} = 2\, e^{-ct} \left(a \cdot \cos(dt) + b \cdot \sin(dt)\right);$$
$$2)\ \ \alpha_1 e^{-\lambda_1 t} + \alpha_2 e^{-\lambda_2 t} = 2\, e^{-ct} \left(a \cdot \cos(dt) - b \cdot \sin(dt)\right).$$

The absolute deviation of the function of distribution by law G from a function $F_{H_l}(t)$ which parameters are roots of system (3), we will evaluate with the help of integral

$$\Delta_l(F) = \int\limits_0^{\infty} |F_{H_l}(t) - F_G(t)| dt,$$

where $F_G(t)$ is the probability distribution function by law G.

Let $\Gamma(V)$, $W(V)$ and $U[a,\, b]$ denote the gamma distribution, Weibull distribution with coefficients of variation V, and uniform distribution on the interval $[a,\, b]$ respectively.

Table 1 gives deviation values of $\Delta_l(F)$ for $l = 2, \ldots, 6$, calculated by results of approximation of different distributions with means 1. With increasing order of H_l-distribution, the value of deviation $\Delta_l(F)$ decreases, and with the increase of the coefficient of variation for $V > 1$ the deviation increases, much faster for the Weibull distribution compared with the gamma distribution. For distributions $W(0.7)$, $W(0.8)$, $W(0.9)$ and $W(0.95)$ for some values of l the deviation $\Delta_l(F) = \infty$. In each of these cases, one of roots λ_s of system (3) is real, but negative. Therefore, for the corresponding distribution pseudo-function, the limit relation $\lim\limits_{t \to \infty} F_{H_l}(t) = \infty$ is valid. For these values of l, the steady-state probabilities p_k, obtained using solutions of equations (1), written for the $H_l/M/n/m$ system, can be paradoxical.

Table 1: Values of the absolute deviation $\Delta_l(F)$ for different distributions

Distribution name	$\Delta_2(F)$	$\Delta_3(F)$	$\Delta_4(F)$	$\Delta_5(F)$	$\Delta_6(F)$
$\Gamma(0.001)$	0.3629	0.2605	0.2092	0.1773	0.1549
$U[0,\, 2]$	0.1139	0.0632	0.0411	0.0295	0.0224
$\Gamma(0.7)$	0.0007	$7.2 \cdot 10^{-5}$	$1.4 \cdot 10^{-5}$	$3.7 \cdot 10^{-6}$	$1.2 \cdot 10^{-6}$
$W(0.7)$	0.0071	0.0026	0.0006	∞	$6.1 \cdot 10^{-5}$
$W(0.8)$	0.0043	∞	0.0004	0.0001	∞
$W(0.9)$	0.0049	0.0005	∞	0.0001	$4.8 \cdot 10^{-5}$
$W(0.95)$	0.0031	0.0005	0.0001	∞	$3.5 \cdot 10^{-5}$
$\Gamma(4)$	0.3146	0.1412	0.0787	0.0497	0.0340
$W(3)$	0.3973	0.2790	0.2170	0.1786	0.1524

Calculations show that properties of the solutions of system (1) almost repeats the form of the roots α_s $(1 \le s \le l)$ of system (3). Let's show it on examples of $U[0, 0.25]/M/n/m$ and $\Gamma(0.7)/M/n/m$ queueing systems.

For the order of approximation l from 2 to 6 the roots of system (3) for uniform distribution on the interval [0, 0.25] are as follows:

$$
\begin{aligned}
l = 2: \quad & \alpha_{1,2} = 0.5 \pm 0.86603i, \quad \lambda_{1,2} = 12 \pm 6.92820i; \\
l = 3: \quad & \alpha_1 = 2.65193, \quad \alpha_{2,3} = -0.82596 \pm 0.60435i, \\
& \lambda_1 = 18.57748, \quad \lambda_{2,3} = 14.7113 \pm 14.03505i; \\
l = 4: \quad & \alpha_{1,2} = -0.58906 \mp 0.89679i, \quad \alpha_{3,4} = 1.08906 \pm 4.95602i, \\
& \lambda_{1,2} = 16.83032 \pm 21.25934i, \quad \lambda_{3,4} = 23.16968 \pm 6.93787i;
\end{aligned}
$$

$$
\begin{aligned}
l = 5: \quad & \alpha_1 = 15.24547, \quad \alpha_{2,3} = 1.02783 \mp 0.49426i, \quad \alpha_{4,5} = -8.15056 \pm 2.37119i, \\
& \lambda_1 = 29.17391, \quad \lambda_{2,3} = 18.59739 \pm 28.56818i, \quad \lambda_{4,5} = 26.81565 \pm 13.94129i; \\
l = 6: \quad & \alpha_{1,2} = 0.31983 \pm 1.17903i, \quad \alpha_{3,4} = -3.40926 \mp 12.71978i, \\
& \alpha_{5,6} = 3.58943 \pm 36.22605i, \\
& \lambda_{1,2} = 20.12746 \pm 35.94138i, \quad \lambda_{3,4} = 29.88567 \pm 21.01018i; \\
& \lambda_{5,6} = 33.98688 \pm 6.94008i.
\end{aligned}
$$

For $l = 2$ solutions of the corresponding system (1) are complex conjugate with positive real parts; for $l = 3$ $p_{k(1)} > 0 \; \forall \; k$, $p_{k(2)}$ and $p_{k(3)}$ are complex conjugate with negative real parts for most values of k. For $l = 4$ we have two pairs of complex conjugate solutions with negative real parts for most values of k in the first pair and with positive real parts $\forall \; k$ in the second pair. For $l = 5$ $p_{k(1)} > 0 \; \forall \; k$, and for $s = 2, 3$ and $s = 4, 5$ we have two pairs of complex conjugate solutions $p_{k(s)}$ with positive real parts in the first pair and with negative real parts in the second pair for most values of k. For $l = 6$ we have three pairs of complex conjugate solutions $p_{k(s)}$ with negative real parts in the second pair and with positive real parts in the first and third pairs.

For the order of approximation l from 2 to 6 the roots of system (3) for $\Gamma(0.7)$ distribution with mean 0.125 are as follows:

$$
\begin{aligned}
l = 2: \quad & \alpha_{1,2} = 0.5 \pm 6.18520i, \quad \lambda_{1,2} = 16 \pm 1.31077i;\\
l = 3: \quad & \alpha_1 = 0.02814, \quad \alpha_{2,3} = 0.48592 \pm 15.70863i,\\
& \lambda_1 = 39.57700, \quad \lambda_{2,3} = 16.21150 \pm 0.53937i;\\
l = 4: \quad & \alpha_1 = 0.00548, \quad \alpha_2 = 0.08597, \quad \alpha_{3,4} = 0.45428 \pm 29.37436i,\\
& \lambda_1 = 69.97401, \quad \lambda_2 = 25.49286, \quad \lambda_{3,4} = 16.26656 \pm 0.29607i;\\
l = 5: \quad & \alpha_1 = 0.00186, \quad \alpha_2 = 0.01855, \quad \alpha_3 = 0.16685, \quad \alpha_{4,5} = 0.40637 \pm 47.30434i,\\
& \lambda_1 = 108.92822, \quad \lambda_2 = 37.04067, \quad \lambda_3 = 21.45221,\\
& \lambda_{4,5} = 16.28945 \pm 0.18743i;\\
l = 6: \quad & \alpha_1 = 0.00081, \quad \alpha_2 = 0.00679, \quad \alpha_3 = 0.03738, \quad \alpha_4 = 0.26943,\\
& \alpha_{5,6} = 0.34279 \pm 69.58145i,\\
& \lambda_1 = 156.50494, \quad \lambda_2 = 51.28750, \quad \lambda_3 = 27.95129, \quad \lambda_4 = 19.65369,\\
& \lambda_{5,6} = 16.30125 \pm 0.12940i.
\end{aligned}
$$

For l from 2 to 6 properties of solutions $p_{k(s)}$ of system (1) in the sense of their signs and whether they are real or complex, completely coincide with the properties of the roots α_s $(1 \le s \le l)$ of system (3) with the same numbers.

4. Numerical results

Let us present the results of calculating steady-state probabilities on examples of the $U[0, 0.25]/M/10/15$, $U[0, 0.125]/M/20/15$ systems and $\Gamma(V)/M/n/15$, $W(0.9)/M/n/15$ systems for $n = 10$, 20 and $V = 0.001$, 0.7, 4.

Let $E(T_\lambda)$ denote the mean of the times elapsed between two consecutive arrivals. We take $E(T_\lambda) = 0.125$ and $E(T_\lambda) = 0.0625$ for $n = 10$ and $n = 20$ respectively, and $\mu = 1$ is the parameter of exponential distribution of service times.

The obtained results are verified using simulation models constructed with the help of the GPSS World tools [7]. The results obtained using GPSS World slightly differ from one another for different numbers of library random-number generators used for simulating random variables, i.e., times elapsed between two consecutive arrivals and service times. Therefore, we use averaged results obtained using simulation models with different values of random-numbers generators that take on values of natural numbers from 6 to 10.

Table 2. Results of the calculation of steady-state characteristics of the $G/M/10/15$ and $G/M/20/15$ systems with different G-distributions

G-distribuion name, value of n	Characte-ristic name	Method of calculation and values of characteristics					
		H_2	H_3	H_4	H_5	H_6	GPSS World
$\Gamma(0.001)$, $n=10$	N	8.4783	8.4809	8.4809	8.4809	8.4809	8.4759
	$\Delta_{l(sim)}$	0.0036	0.0026	0.0026	0.0026	0.0026	–
	$\Delta_{l,l-1}$	–	$3.38 \cdot 10^{-3}$	$3.02 \cdot 10^{-5}$	$3.64 \cdot 10^{-7}$	$3.49 \cdot 10^{-9}$	–
$\Gamma(0.001)$, $n=20$	N	16.2230	16.2245	16.2245	16.2245	16.2245	16.1875
	$\Delta_{l(sim)}$	0.0102	0.0095	0.0095	0.0095	0.0095	–
	$\Delta_{l,l-1}$	–	$2.21 \cdot 10^{-3}$	$1.07 \cdot 10^{-5}$	$9.55 \cdot 10^{-8}$	$1.24 \cdot 10^{-9}$	–
$U[0,\ 1/8]$, $n=10$	N	8.8186	8.8206	8.8206	8.8206	8.8206	8.8138
	$\Delta_{l(sim)}$	0.0034	0.0023	0.0023	0.0023	0.0023	–
	$\Delta_{l,l-1}$	–	$2.73 \cdot 10^{-3}$	$4.64 \cdot 10^{-5}$	$1.16 \cdot 10^{-6}$	$3.57 \cdot 10^{-8}$	–
$U[0,\ 1/16]$, $n=20$	N	16.4409	16.4422	16.4422	16.4422	16.4422	16.4356
	$\Delta_{l(sim)}$	0.0038	0.0038	0.0038	0.0038	0.0038	–
	$\Delta_{l,l-1}$	–	$1.81 \cdot 10^{-3}$	$1.93 \cdot 10^{-5}$	$4.05 \cdot 10^{-7}$	$1.36 \cdot 10^{-8}$	–
$\Gamma(0.7)$, $n=10$	N	8.9531	8.9531	8.9531	8.9531	8.9531	8.9605
	$\Delta_{l(sim)}$	0.0017	0.0017	0.0017	0.0017	0.0017	–
	$\Delta_{l,l-1}$	–	$2.11 \cdot 10^{-5}$	$9.39 \cdot 10^{-8}$	$1.21 \cdot 10^{-9}$	$2.55 \cdot 10^{-11}$	–
$\Gamma(0.7)$, $n=20$	N	16.5339	16.5339	16.5339	16.5339	16.5339	16.5306
	$\Delta_{l(sim)}$	0.0027	0.0027	0.0027	0.0027	0.0027	–
	$\Delta_{l,l-1}$	–	$1.41 \cdot 10^{-5}$	$4.20 \cdot 10^{-8}$	$4.46 \cdot 10^{-10}$	$1.08 \cdot 10^{-11}$	–
$W(0.9)$, $n=10$	N	9.2464	9.2463	–	9.2463	9.2463	9.2428
	$\Delta_{l(sim)}$	0.0019	0.0019	–	0.0019	0.0019	–
	$\Delta_{l,l-1}$	–	$1.97 \cdot 10^{-4}$	–	–	$4.55 \cdot 10^{-9}$	–
$W(0.9)$, $n=20$	N	16.7440	16.7440	–	16.7440	16.7440	16.7426
	$\Delta_{l(sim)}$	0.0027	0.0027	–	0.0027	0.0027	–
	$\Delta_{l,l-1}$	–	$1.40 \cdot 10^{-4}$	–	–	$2.04 \cdot 10^{-9}$	–
$\Gamma(4)$, $n=10$	N	10.1667	9.7778	9.7522	9.7532	9.7536	9.7521
	$\Delta_{l(sim)}$	0.0784	0.0216	0.0094	0.0052	0.0032	–
	$\Delta_{l,l-1}$	–	0.0680	0.0146	$5.41 \cdot 10^{-3}$	$2.25 \cdot 10^{-3}$	–
$\Gamma(4)$, $n=20$	N	16.5677	16.2055	16.1864	16.1874	16.1876	16.1773
	$\Delta_{l(sim)}$	0.0652	0.0183	0.0092	0.0062	0.0048	–
	$\Delta_{l,l-1}$	–	0.0576	0.0117	$4.41 \cdot 10^{-3}$	$1.83 \cdot 10^{-3}$	–

Let us introduce the designation: N is the average number of customers in a queueing system, and

$$\Delta_{(l,l-1)} = \sum_{k=0}^{n+15} |p_{k(l)} - p_{k(l-1)}|, \quad \Delta_{l(sim)} = \sum_{k=0}^{n+15} |p_{k(l)} - p_{k(sim)}|,$$

$$p_{k(sim)} = \frac{1}{5}\sum_{i=6}^{10} p_{k(sim,i)}, \qquad 0 \le k \le n+15, \quad 2 \le l \le 6.$$

Here $p_{k(l)}$ are values of probabilities p_k obtained using the H_l-approximation, $p_{k(sim)}$ is the average value of probabilities $p_{k(sim,i)}$, obtained by means of the simulation model using the number i of random-numbers generator for $6 \le i \le 10$. Thus, the quantities $\Delta_{l(sim)}$ are measures of deviations of the distributions $\{p_{k(l)}\}$ from distribution $\{p_{k(sim)}\}$, and the quantities $\Delta_{(l,l-1)}$ give an opportunity to estimate the deviation of distributions $\{p_{k(l)}\}$ from distributions $\{p_{k(l-1)}\}$.

In Table 2 we present the results of calculation of steady-state characteristics of the $G/M/10/15$ and $G/M/20/15$ systems with the considered gamma, Weibull and uniform distributions. The values of deviations $\Delta_{l(sim)}$ and $\Delta_{(l,l-1)}$ decrease with increasing order of H_l-distributions in approximations, and it means that the values of distribution $\{p_{k(l)}\}$ with each step getting closer to a true distribution $\{p_k\}$. With the growth of the variation coefficient of distributions after the value of $V > 1$, as expected taking into account the behavior of deviations $\Delta_l(F)$, the values of the absolute deviations $\Delta_{l(sim)}$ and $\Delta_{(l,l-1)}$ also increase. For the distribution $W(0.9)$ the deviation $\Delta_4(F) = \infty$ and, consequently, some values of "probabilities" of the distribution $\{p_{k(4)}\}$ go beyond the interval [0, 1].

Presented results show that increasing the number of channels of the $G/M/n/m$ system has no significant effect on accuracy of calculating the steady-state probabilities.

Testing the proposed method on the $M/G/1/m$ systems, for which we can find exact values of the steady-state distribution $\{p_k\}$, shows that in cases where the deviation $\Delta_{(6,5)}$ is less than 10^{-2}, the deviation of the distribution $\{p_{k(l)}\}$ from the true distribution $\{p_k\}$ and deviation $\Delta_{(l+1,l)}$ are numbers of the same order, and at the same time the deviations of distribution $\{p_{k(sim)}\}$ from the distribution $\{p_k\}$ usually no less than 10^{-4}. Thus, in most cases we can use values $\Delta_{(l,l-1)}$ to evaluate accuracy of the approximation of the distribution $\{p_{k(l-1)}\}$ to the true $\{p_k\}$ for $3 \le k \le 6$. In cases where $\Delta_{(l,l-1)} < 10^{-4}$, we can argue that the distribution $\{p_{k(l-1)}\}$ is more accurate approximation than $\{p_{k(sim)}\}$.

5. Conclusions

This paper shows that the application of hyperexponential approximation of distributions the times elapsed between two consecutive arrivals allows us to calculate steady-state probabilities of the $G/M/n/m$ queueing systems with high accuracy (higher than in the case of using simulation models). We find these probabilities using solutions of a system of linear algebraic equations obtained by the method of fictitious phases.

To obtain parameters of H_l-approximation of a certain distribution it is necessary to solve the system of equations of the moments method. For the values $V < 1$ of the variation coefficient, some of the roots of this system are complex-valued or, having a sense of probabilities, go beyond the interval [0, 1], but in most cases the final result is close to the desired distribution $\{p_k\}$.

Computing deviations $\Delta_{(l,l-1)}$ allows us to track the accuracy of approaching distributions $\{p_{k(l-1)}\}$ to the true distribution $\{p_k\}$ without the need to use simulation models.

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Received: 05 February 2019
Accepted: 30 May 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

On the 3D Dynamic Normal Stress Field on the Interface of the Bi-layered Hollow Cylinder Under Action a Moving Load in the Interior of That

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Abstract. The paper studies normal stress field on the interface surface of the bi-layered hollow cylinder under action on the interior of that the moving load in the 3D state with utilizing the exact equations and relations of the elastodynamics. It is assumed that in the interior of the cylinder the point located with respect to the cylinder axis moving forces act and the distribution of these forces is non-axisymmetric and is located within a certain central angle. To solve the corresponding mathematical problem the moving coordinate system is used and the Fourier transform of with respect to the axial coordinate is employed. These transforms are presented in the Fourier series form with respect to the circumferential coordinate and the coefficients of these series are found analytically from the corresponding field equations and relations. The inverses of the mentioned transforms are determined numerically as a result of which normal radial stress acting on the interface surface between the layers of the cylinder is analyzed. It is examined the influence of the problem parameters such as moving load velocity, the thicknesses ration of the cylinder's layers, the ration of the inner layer thickness to the external radius of the cross-section of this layer and material properties of the layers to the stress response to the moving load.

1. Introduction

In the paper [1, 2] studied the corresponding 3D dynamic problem for the system consisting of the hollow cylinder and surrounding elastic medium and the review of the related other investigations were considered in the papers [1 − 4]. Consequently, the present paper attempt to develop the investigations started in the paper [1] for the bi-layered hollow cylinder.

Note that detailed consideration of the dynamics of the bi-material elastic systems has been made in the monograph [5] from which and from the other reviews made in the papers [1- 4] follows that up to now the regarding investigations have been made mainly for axisymmetric cases (except the study carried out in the papers [1, 2]). Therefore, each investigation on the 3D dynamics of the cylindrical bi-material systems can be taken as new knowledge in this field which has not only theoretical and application sense.

Taking the foregoing discussion into consideration, in the present paper it is made the attempt to investigate, within the scope of the 3D elastodynamics, normal interface stress on the interface surface of the bi-layered hollow cylinder in the case wherein the interior of the cylinder the moving load acting within a certain arc and point located with respect to the axial coordinate moving load acts.

2. Formulation of the problem

We introduce to the consideration a bi-layered hollow cylinder the sketch of which is illustrated in Fig. 1 and assume that the thicknesses of the walls of the inner and outer cylindersare$h^{(2)}$and $h^{(1)}$respectively, and the external radius of the cross section of the inner cylinder is$R$. We denote by the upper index (2) (by the upper index (1) the values related to the inner (outer) layer of the cylinder and associate the cylindrical system of coordinates $Orz\theta$(Fig. 1a) with the axis of the cylinder. Moreover, we assume that in the interior of the inner hollow cylinder a point located with respect to the cylinder axis and that non-uniformly distributed in the circumferential direction (Fig. 1b) moving normal forces act and these forces move with constant velocity V in the Oz axis direction. Thus, within these framework we attempt to investigate the non-axisymmetric dynamic response of the bi-layered hollow cylinder to the moving forces and analyze the response of the interface normal stress to these forces.

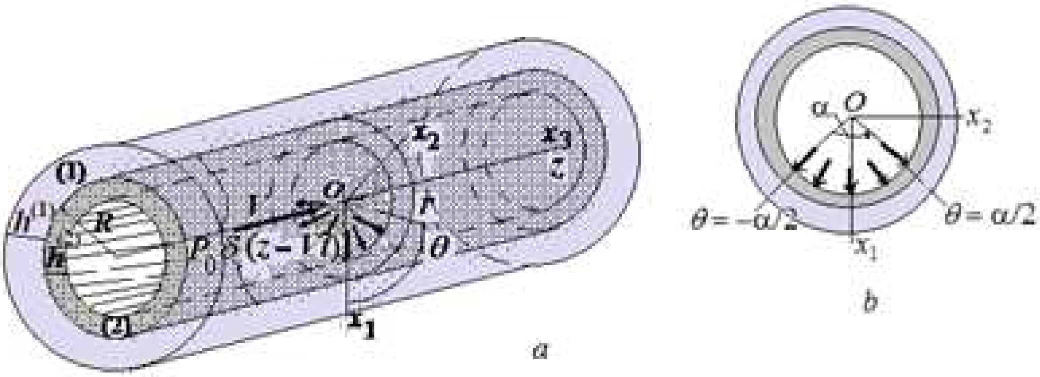


Fig. 1.The sketch of the considered system (a) and the sketch of the distribution of the non-axisymmetric normal forces (b)

We write the following complete system of field equations of the 3D elastodynamics, as well as the corresponding boundary and contact conditions within the framework of which the present investigation will be made.

Equations of motion:

$$\frac{\partial \sigma_{rr}^{(m)}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}^{(m)}}{\partial \theta}+\frac{\partial \sigma_{rz}^{(m)}}{\partial z}+\frac{1}{r}(\sigma_{rr}^{(m)}-\sigma_{\theta\theta}^{(m)})=\rho^{(m)}\frac{\partial^2 u_r^{(m)}}{\partial t^2}$$

$$\frac{\partial \sigma_{r\theta}^{(m)}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}^{(m)}}{\partial \theta}+\frac{\partial \sigma_{z\theta}^{(m)}}{\partial z}+\frac{2}{r}\sigma_{r\theta}^{(m)}=\rho^{(m)}\frac{\partial^2 u_\theta^{(m)}}{\partial t^2}$$

$$\frac{\partial\sigma_{rz}^{(m)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{z\theta}^{(m)}}{\partial\theta}+\frac{\partial\sigma_{zz}^{(m)}}{\partial z}+\frac{1}{r}\sigma_{rz}^{(m)}=\rho^{(m)}\frac{\partial^2u_z^{(m)}}{\partial t^2}. \tag{1}$$

Elasticity relations:

$$\sigma_{rr}^{(m)}=(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_r^{(m)}}{\partial r}+\lambda^{(m)}\frac{1}{r}(\frac{\partial u_\theta^{(m)}}{\partial r}+u_r^{(m)})+\lambda^{(m)}\frac{\partial u_z^{(m)}}{\partial z},$$

$$\sigma_{\theta\theta}^{(m)}=\lambda^{(m)}\frac{\partial u_r^{(m)}}{\partial r}+(\lambda^{(m)}+2\mu^{(m)})\frac{1}{r}(\frac{\partial u_\theta^{(m)}}{\partial r}+u_r^{(m)})+\lambda^{(m)}\frac{\partial u_z^{(m)}}{\partial z},$$

$$\sigma_{zz}^{(m)}=\lambda^{(m)}\frac{\partial u_r^{(m)}}{\partial r}+\lambda^{(m)}\frac{1}{r}(\frac{\partial u_\theta^{(m)}}{\partial r}+u_r^{(m)})+(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_z^{(m)}}{\partial z},$$

$$\sigma_{r\theta}^{(m)}=\mu^{(m)}\frac{\partial u_\theta^{(m)}}{\partial r}+\mu^{(m)}(\frac{1}{r}\frac{\partial u_r^{(m)}}{\partial\theta}-\frac{1}{r}u_\theta^{(m)}),$$

$$\sigma_{z\theta}^{(m)}=\mu^{(m)}\frac{\partial u_\theta^{(m)}}{\partial z}+\mu^{(k)}\frac{\partial u_z^{(m)}}{r\partial\theta},\sigma_{zr}^{(k)}=\mu^{(k)}\frac{\partial u_r^{(k)}}{\partial z}+\mu^{(k)}\frac{\partial u_z^{(k)}}{\partial r}. \tag{2}$$

The conventional notation is use in equations (1) and (2).

The corresponding boundary and contact conditions for the case under consideration can be formulated as follows.

$$\sigma_{rr}^{(2)}\Big|_{r=R-h^{(2)}}=\begin{cases}-P_\alpha\delta(z-Vt) \text{ for } -\alpha/2\le\theta\le\alpha/2\\ 0 \text{ for } \theta\in([-\pi,+\pi]-[-\alpha/2,\alpha/2])\end{cases},$$

$$\sigma_{r\theta}^{(2)}\Big|_{r=R-h^{(2)}}=0,\quad \sigma_{rz}^{(2)}\Big|_{r=R-h^{(2)}}=0,$$

$$\sigma_{rr}^{(1)}\Big|_{r=R+h^{(1)}}=0,\quad \sigma_{r\theta}^{(1)}\Big|_{r=R+h^{(1)}}=0,\quad \sigma_{rz}^{(1)}\Big|_{r=R+h^{(1)}}=0, \tag{3}$$

$$\sigma_{rr}^{(1)}\Big|_{r=R}=\sigma_{rr}^{(2)}\Big|_{r=R},\quad \sigma_{r\theta}^{(1)}\Big|_{r=R}=\sigma_{r\theta}^{(2)}\Big|_{r=R},\quad \sigma_{rz}^{(1)}\Big|_{r=R}=\sigma_{rz}^{(2)}\Big|_{r=R},$$

$$u_r^{(1)}\Big|_{r=R}=u_r^{(2)}\Big|_{r=R},\quad u_\theta^{(1)}\Big|_{r=R}=u_\theta^{(2)}\Big|_{r=R},\quad u_z^{(1)}\Big|_{r=R}=u_z^{(2)}\Big|_{r=R}, \tag{4}$$

$$\left|\sigma_{rr}^{(1)}\right|;\ \left|\sigma_{\theta\theta}^{(1)}\right|;\ \left|\sigma_{zz}^{(1)}\right|;\ \left|\sigma_{r\theta}^{(1)}\right|;\ \left|\sigma_{rz}^{(1)}\right|;\ \left|\sigma_{\theta z}^{(1)}\right|;$$

$$\left|u_r^{(1)}\right|;\ \left|u_\theta^{(1)}\right|;\ \left|u_z^{(1)}\right|\to 0 \text{ as } \sqrt{(z-Vt)^2}\to+\infty, \tag{5}$$

where in (3)P_α is determined from the following relation

$$\int_{-\alpha/2}^{+\alpha/2}P_\alpha(R-h)\cos\theta d\theta=(R-h)P_0=const\Rightarrow\ P_\alpha=P_0/(2\sin(\alpha/2)). \tag{6}$$

Thus, the investigation of the response of the interface normal stress to the moving load is reduced to the boundary-contact problem (1) – (5) for solution to which the method developed in the papers [1,2] is employed. Now we consider some fragments of the application of the mentioned method for the problem under consideration.

3. Method of solution

As in the papers [1, 2] for solution to the foregoing mathematical problem, according to [6], we use the following representation:

$$u_r^{(m)} = \frac{1}{r}\frac{\partial}{\partial\theta}\Psi^{(m)} - \frac{\partial^2}{\partial r\partial z}\mathrm{X}^{(m)}, \quad u_\theta^{(m)} = -\frac{\partial}{\partial r}\Psi^{(m)} - \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}\mathrm{X}^{(m)},$$

$$u_z^{(m)} = (\lambda^{(m)} + \mu^{(m)})^{-1}\left((\lambda^{(m)} + 2\mu^{(m)})\Delta_1 + \mu^{(m)}\frac{\partial^2}{\partial z^2} - \rho^{(m)}\frac{\partial^2}{\partial t^2}\right)\mathrm{X}^{(m)},$$

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, m = 1, 2 \tag{7}$$

In (7) the functions $\Psi^{(m)}$ and $\mathrm{X}^{(m)}$ are the solutions of the equations

$$\left(\Delta_1 + \frac{\partial^2}{\partial z^2} - \frac{\rho^{(k)}}{\mu^{(k)}}\frac{\partial^2}{\partial t^2}\right)\Psi^{(m)} = 0, \left[\left(\Delta_1 + \frac{\partial^2}{\partial z^2}\right)\left(\Delta_1 + \frac{\partial^2}{\partial z^2}\right) - \right.$$

$$-\rho^{(m)}\frac{\lambda^{(m)} + 3\mu^{(m)}}{\mu^{(m)}(\lambda^{(m)} + 2\mu^{(m)})}\times$$

$$\left.\times(\Delta_1 + \frac{\partial^2}{\partial z^2})\right)\frac{\partial^2}{\partial t^2} + \frac{(\rho^{(m)})^2}{\mu^{(m)}(\lambda^{(m)} + 2\mu^{(m)})}\frac{\partial^4}{\partial t^4}\Bigg]\mathrm{X}^{(m)} = 0. \tag{8}$$

We introduce a moving cylindrical coordinate system $O'r'\theta'z'$ which is connected with the reference cylindrical coordinate system $Or\theta z$ through the following relations:

$$r' = r, \quad \theta' = \theta, \quad z' = z - Vt. \tag{9}$$

As a result of the employing of the moving coordinate system (9), the operators $\partial^2/\partial t^2$ and $\partial^4/\partial t^4$ in the foregoing equations are replaced with the operators $V^2\partial^2/\partial z'^2$ and $V^4\partial^4/\partial z'^4$, respectively, and in this way, equations rewritten in the moving coordinate system, are obtained. The exponential Fourier transform $f_F = \int_{-\infty}^{+\infty} f(z')e^{isz'}dz'$ with respect to the moving coordinate z' (where s is a transformation parameter) is applied to all the equations and relations rewritten with the moving coordinates.

Below, we will make all mathematical operations with the moving coordinates and will omit the upper primes over them.

According to the problem statement,we use the following presentations for the originals of the sought values.

$$\left\{\sigma_{rr}^{(m)};\ \sigma_{\theta\theta}^{(m)}; \sigma_{zz}^{(m)}; \sigma_{r\theta}^{(m)}; u_r^{(m)}; u_\theta^{(m)};\ \Psi^{(m)}\right\} =$$

$$\frac{1}{\pi}\int_0^{+\infty}\left\{\sigma_{rrF}^{(m)};\ \sigma_{\theta\theta F}^{(m)}; \sigma_{zzF}^{(m)}; \sigma_{r\theta F}^{(m)}; u_{rF}^{(m)}; u_{\theta F}^{(m)};\ \Psi_F^{(m)}\right\}\cos(sz)ds,$$

$$\left\{\sigma_{\theta z}^{(m)};\ \sigma_{rz}^{(m)}; u_z^{(m)}; \mathrm{X}^{(m)}\right\} = \frac{1}{\pi}\int_0^{+\infty}\left\{\sigma_{\theta zF}^{(m)};\ \sigma_{rzF}^{(m)}; u_{zF}^{(m)};\ \mathrm{X}_F^{(m)}\right\}\sin(sz)ds. \tag{10}$$

Substituting the expressions in Eq. (10) into the equations in (8) and into the rewritten relations in the moving coordinate system, it is obtained the following equations for the functions $\Psi_F^{(m)}$ and $X_F^{(m)}$:

$$\left(\Delta_1 - s^2(1 - \frac{\rho^{(k)}}{\mu^{(k)}}V^2)\right)\Psi_F^{(m)} = 0,$$

$$\left[\left(\Delta_1 - s^2\right)\left(\Delta_1 - s^2\right) - \rho^{(m)}\frac{\lambda^{(m)} + 3\mu^{(m)}}{\mu^{(m)}(\lambda^{(m)} + 2\mu^{(m)})}\times\right.$$

$$\left.(\Delta_1 - s^2))(-s^2V^2) + \frac{(\rho^{(m)})^2}{\mu^{(m)}(\lambda^{(m)} + 2\mu^{(m)})}s^4V^4\right]X_F^{(m)} = 0. \quad (11)$$

According to the periodicity of the problem under consideration with respect to the circumferential coordinate θ, the Fourier transform of the functions $\Psi_F^{(m)}$ and $X_F^{(m)}$can be presented in the Fourier series form as follows.

$$\Psi_F^{(m)}(r,s,\theta) = \sum_{n=1}^{\infty}\Psi_{Fn}^{(m)}(r,s)\sin n\theta, \quad X_F^{(m)}(r,s,\theta) = \frac{1}{2}X_{F0}^{(m)}(r,s) + \sum_{n=1}^{\infty}X_{Fn}^{(m)}(r,s)\cos n\theta. \quad (12)$$

In this way, we obtain from expressions in (12) andequations in (11) the following equation:

$$\left(\Delta_{1n} - (\zeta_1^{(m)})^2\right)\psi_{Fn}^{(m)} = 0, \quad \left(\Delta_{1n} - (\zeta_2^{(m)})^2\right)\left(\Delta_{1n} - (\zeta_3^{(m)})^2\right)X_{Fn}^{(m)} = 0,$$

$$\Delta_{1n} = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\mathrm{d}}{r\mathrm{d}r} - \frac{n^2}{r^2}, \quad (13)$$

where

$$(\zeta_1^{(m)})^2 = s^2\left(1 - \frac{\rho^{(m)}V^2}{\mu^{(m)}}\right) \quad (14)$$

$(\zeta_2^{(m)})^2$ and $(\zeta_3^{(m)})^2$ in (13) are determined from the solutions of the following equation.

$$\mu^{(m)}(\zeta^{(m)})^4 - s^2(\zeta^{(m)})^2\left[-\rho^{(m)}V^2 - (\lambda^{(m)} + 2\mu^{(m)}) +\right.$$

$$\left.+\frac{\mu^{(m)}}{\lambda^{(m)} + 2\mu^{(m)}}\left(-\rho^{(m)}V^2 - \mu^{(m)}\right) + \frac{(\lambda^{(m)} + \mu^{(m)})^2}{\lambda^{(m)} + 2\mu^{(m)}}\right] +$$

$$s^4\left(\frac{-\rho^{(m)}V^2}{\lambda^{(m)} + 2\mu^{(m)}} - 1\right)\left(-\rho^{(m)}V^2 - \mu^{(m)}\right) = 0. \quad (15)$$

The solutions to equations in (13) are determined as follows:

$$\psi_{Fn}^{(m)} = A_{1n}^{(m)}I_n(\zeta_1^{(m)}r) + B_{1n}^{(m)}K_n(\zeta_1^{(m)}r), \quad \chi_{Fn}^{(m)} = A_{2n}^{(m)}I_n(\zeta_2^{(m)}r) + A_{3n}^{(m)}I_n(\zeta_3^{(m)}r) +$$

$$B_{2n}^{(m)} K_n(\zeta_2^{(m)} r) + B_{3n}^{(m)} K_n(\zeta_3^{(m)} r), \quad m = 1, 2. \tag{16}$$

Using (16), (12), (7) and (2) it is completely determined the Fourier transforms of the sought values.Finally, using the algorithm developed and applied in the papers [1-4] the originals of these values are determined. Note that one of the main procedures of this algorithm is the determination of the unknown constants $A_{1n}^{(m)}$, $B_{1n}^{(m)}$, $A_{2n}^{(m)}$, $B_{2n}^{(m)}$, $A_{3n}^{(m)}$ and $B_{3n}^{(m)}$ for which it is obtained a complete system of algebraic equations from the boundary and contact conditions in (3) and (4) respectively.

This completes the consideration of the solution method more detail version of which is given in the papers [1, 2].

4. Numerical results

In the present paper, we will consider numerical results related to the interface normal stress acting on the interface surface between the layers of the cylinder. The algorithm for obtaining numerical results are detailed in the works [1-5] and therefore do not consider here again that. Nevertheless, we note that under obtaining numerical results we take twenty terms in the series in (12). Moreover, we note that these results are obtained for the following two cases:

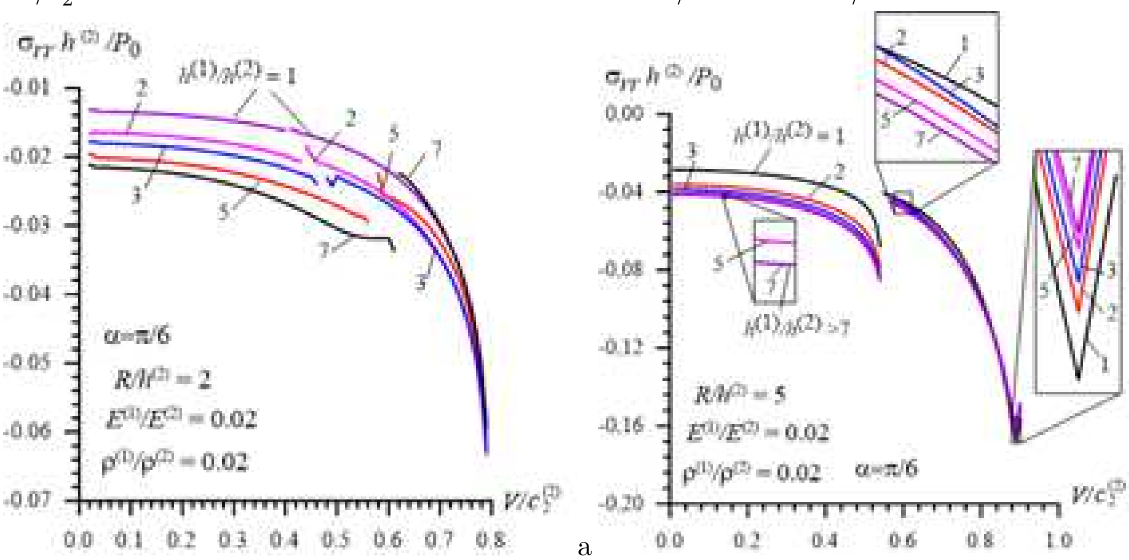
Case 1. $E^{(1)}/E^{(2)} = 0.02, \quad \rho^{(1)}/\rho^{(2)} = 0.01, \quad \nu^{(1)} = \nu^{(2)} = 0.25,$

Case 2. $E^{(1)}/E^{(2)} = 0.5$, $\rho^{(2)}/\rho^{(2)} = 0.5$, $\nu^{(1)} = \nu^{(2)} = 0.3$.

Assume that $\theta = 0$, $z/h = 0$ and $\alpha = \pi/6$, and consider the graphs of the dependencies between

$$\sigma_{rr} = \sigma_{rr}^{(1)}\Big|_{r=R} = \sigma_{rr}^{(2)}\Big|_{r=R} \tag{17}$$

and $V/c_2^{(2)}$ constructed for various values of the ratios $R/h^{(2)}$ and $h^{(1)}/h^{(2)}$.



a b

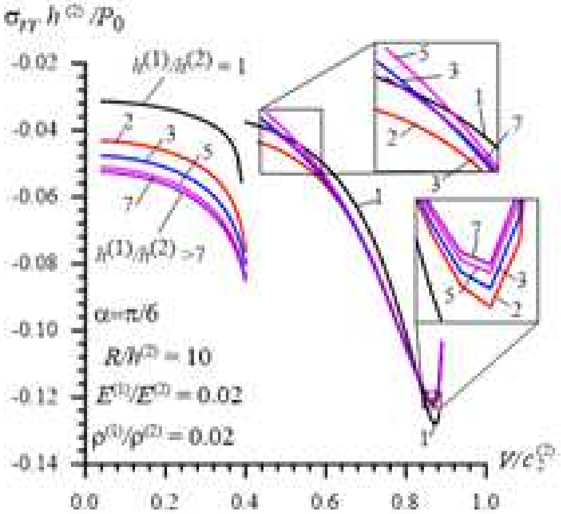


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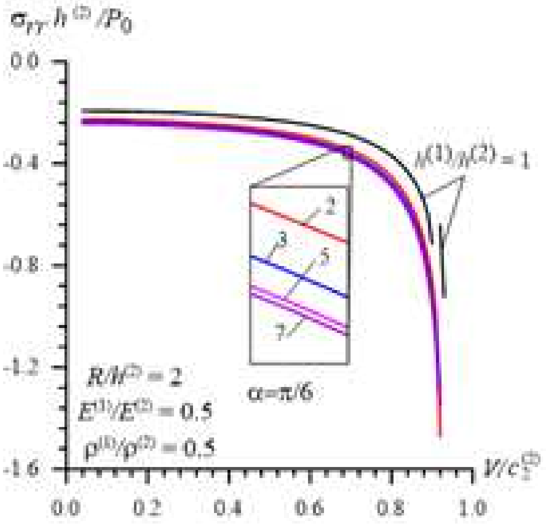
Fig.2. Response of the interface normal stress to the moving load velocity in Case 1 under $R/h^{(2)} = 2$(a), 5 (b) and 10 (c)



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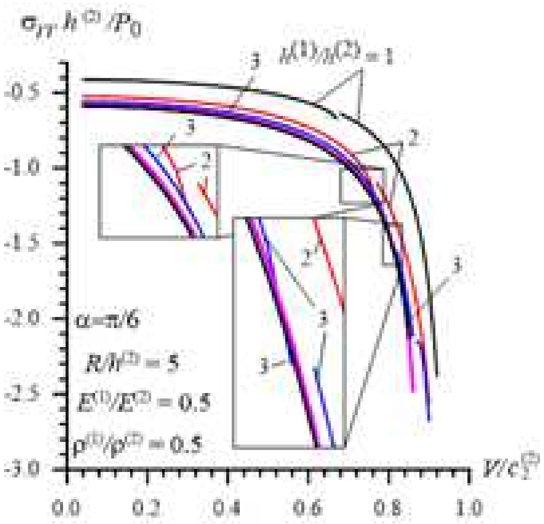

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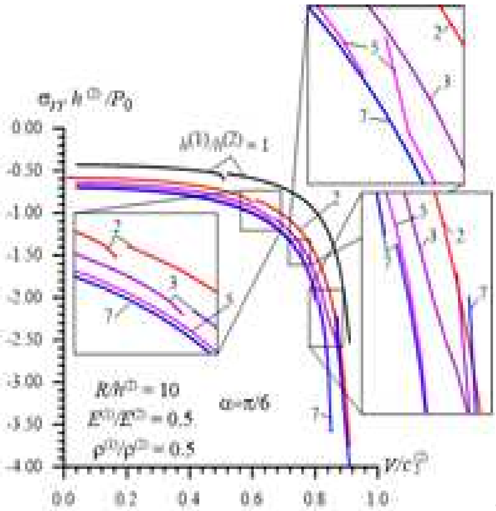

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Fig.3. Response of the interface normal stress to the moving load velocity in Case 2 under $R/h^{(2)} = 2$(a), 5 (b) and 10 (c)

The mentioned graphs are presented in Figs. 2 and 3 for Case 1 and Case 2 respectively for various values of the ratio $h^{(1)}/h^{(2)}$ under $R/h^{(2)} = 2$ (a), 5 (b) and 10 (c). Note that these graphs have a discontinuity at certain values of the dimensionless moving velocity $V/c_2^{(2)}$ which indicates the corresponding critical velocities. Moreover note that, in general, in 3D moving load problems in the subsonic regime there exist two critical velocities, however, in the axisymmetric moving load problems one.

Thus, it follows from the graphs that before the first critical velocity the absolute values of the interface dimensionless normal stress $\sigma_{rr}h^{(2)}/P_0$ increase monotonically with $V/c_2^{(2)}$. At the same time, an increase in the values of the ratio $h^{(1)}/h^{(2)}$ also causes to increase the absolute values of the stress and in the cases under consideration for $h^{(1)}/h^{(2)} \geq 7$ coincide with the corresponding ones obtained in the paper [1], i.e. with the corresponding results which were obtained for the "hollow cylinder + surrounding medium" system. This statement confirms the validity of the calculation algorithm and PC programs used under obtaining of the present results. Moreover, this statement agrees with the well-known physicomechanical and engineering considerations.

Comparison of the results obtained for Case 1 (Fig. 2) with corresponding ones obtained for Case 2 (Fig. 3) shows that the absolute values of the stress obtained in Case 2 is greater significantly than those obtained in Case 1. This situation can be established with the relation $\left(E^{(1)}/E^{(2)}\right)_{Case1} \ll \left(E^{(1)}/E^{(2)}\right)_{Case2}$which also agrees with the engineering consideration.

With this, we restrict ourselves to consideration of the numerical results related to the interface normal stresses obtained for problem under consideration and note that this consideration will be continued in the further works by the author.

5. Conclusions

Thus, in the present paper, the 3D dynamic problem of the moving load acting in the interior of the bi-layered hollow cylinder is studied with employing 3D exact equations of elastodynamics and the numerical results on the response of the interface normal stress to the moving load velocity are presented and discussed. It is assumed that the forces acting in the interior of the inner layer of the cylinder is point located with respect to the axial coordinate and is distributed along a certain arc within the corresponding central angle.

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Received: 26 March 2019
Accepted: 01 June 2019

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 7, No 1, 2019, July
ISSN 1560-4055

The System of Convolution Equations in Concrete Banach Space

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Abstract. The regularity properties of degenerate abstract convolution-elliptic equations are investigated. We prove that the corresponding convolution-elliptic operator is $R-$sectorial and is also a negative generator of an analytic semigroup. These results permit us to, show the separability of the differential operators in a $E-$valued weighted spaces. By using these results integro-differential equations in concrete weighted Banach space $L_{p,\gamma}(R^n; l_q)$ are obtained.

Key Words and Phrases: $R-$sectorial operators, abstract weighted spaces, operator-valued multipliers, convolution equations, integro-differential equations.
2010 Mathematics Subject Classifications: 35J70, 42B35, 46E35, 46E40

1. Introduction, notations and background

Regularity properties of differential operator equations, especially elliptic and parabolic type have been studied extensively e.g in [1], [2], [4], [7-8], [12], [16-18], [21-22] and the references therein. Moreover, convolution-differential equations (CDEs) have been treated e.g. in [4], [15]. Convolution operators in Banach-valued functıon spaces studied e.g. in [3], [10], [13], [16], [17], [18]. However, the convolution-differential operator equations (CDOEs) are relatively less investigated subject. In [4] the parabolic type CDEs with bounded operator coefficient was investigated. In [18] regularity properties of degenerate CDOEs are studied. The main aim of the present paper is to study the maximal L_p-regularity properties of the following degenerate integro-differential equations

$$\sum_{|\alpha|\leq l} a_\alpha * D^{[\alpha]} u_m + \sum_{m=1}^{\infty} d_m * u_m = f_m, \tag{1.1}$$

in concrete weighted Banach space $L_{p,\gamma}(R^n; l_q)$, where l is a natural number, $a_\alpha = a_\alpha(x)$ are complex-valued functions, $d_j = d_j(x)$, $u_j = u_j(x)$, $f_m = f_m(x)$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, α_k are nonnegative integers, $|\alpha| = \sum_{k=1}^{n} \alpha_k$, λ is a complex parameter and $A = A(x)$ is a linear operator in a Banach space E for $x \in R^n$.

In this paper, first we establish the uniform separability properties of the linear CDOEs and the uniform maximal regularity of the infinite system of degenrate integro-differential equations (1.1). Moreover, we prove that the operator generated by problem linear CDOEs is $R-$sectorial. Since the equation (1.1) has an unbounded operator coefficient, some difficulties occur. This fact is derived by using the representation formula for the solution of corresponding convolution equation and operator valued multipliers in $E-$ valued weighted L_p- spaces.

We start by giving the notation and definitions to be used in paper.

Let E be a Banach space and $\gamma=\gamma(x)$, $x=(x_1,x_2,...,x_n)$ be a positive measurable weighted function on a measurable subset $\Omega\subset R^n$. Let $L_{p,\gamma}(\Omega;E)$ denote the space of strongly $E-$valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}}=\|f\|_{L_{p,\gamma}(\Omega;E)}=\left(\int\limits_{\Omega}\|f(x)\|_E^p\,\gamma(x)dx\right)^{1/p},\ 1\le p<\infty,$$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)}=ess\sup_{x\in\Omega}\left[\gamma(x)\,\|f(x)\|_E\right].$$

The weight $\gamma=\gamma\left(x\right)$ satisfy an A_p condition, i.e., $\gamma\in A_p$, $p\in(1,\infty)$ if there is a positive constant C such that

$$\sup_Q\left(\frac{1}{|Q|}\int\limits_Q\gamma(x)dx\right)\left(\frac{1}{|Q|}\int\limits_Q\gamma^{-\dfrac{1}{p-1}}(x)dx\right)^{p-1}\le C$$

for all cubes $Q\subset R^n$ (see e.g [11, Ch.9]).

The result [20] implies that the space l_q for $q\in(1,\infty)$ satisfies multiplier condition with respect to $p\in(1,\infty)$ and the weight functions $\gamma\left(x\right)=\prod\limits_{k=1}^{n}|x_k|^{\nu}$ for $-\frac{1}{n}<\nu<\frac{1}{n}\left(p-1\right)$.

Here, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}$ denotes the set of real numbers. Let $\mathbb{C}$ be the set of complex numbers and

$$S_\varphi=\left\{\lambda\in\mathbb{C},\quad|\arg\lambda|\le\varphi\right\}\cup\left\{0\right\},\quad 0\le\varphi<\pi.$$

Let E_1 and E_2 be two Banach spaces and let $B\left(E_1,E_2\right)$ denote the space of bounded linear operators from E_1 to E_2. For $E_1=E_2=E$ we denote $B\left(E,E\right)$ by $B\left(E\right)$.

Let $D\left(A\right)$, $R\left(A\right)$ denote the domain and range of the linear operator in E, respectively. Let Ker A denote a null space of A.

A closed linear operator A is said to be $\varphi-$ sectorial (or sectorial for $\varphi=0$) in a Banach space E with bound $M>0$ if Ker $A=\{0\}$, $D\left(A\right)$ and $R\left(A\right)$ are dense on E, and $\left\|\left(A+\lambda I\right)^{-1}\right\|_{B(E)}\le M\left|\lambda\right|^{-1}$ for all $\lambda\in S_\varphi$, $\varphi\in[0,\ \pi)$, where I is an identity operator in E. Sometimes $A+\lambda I$ will be written as $A+\lambda$ and will be denoted by A_λ. It is known (see e.g. [19, §1.15.1]) that the fractional powers of the operator A are well defined.

Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graph norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|_E^p + \left\|A^\theta u\right\|_E^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \quad -\infty < \theta < \infty.$$

Note that the above norms are equivalent for $p \in [1, \infty)$.

Here, $S = S(R^n; E)$ denotes the $E-$valued Schwartz class, i.e. the space of $E-$valued rapidly decreasing smooth functions on R^n, equipped with its usual topology generated by seminorms. $S(R^n; \mathbb{C})$ will be denoted by just S.

Let $S'(R^n; E)$ denote the space of all continuous linear operators, $L : S \to E$, equipped with topology of bounded convergence. Recall $S(R^n; E)$ is norm dense in $L_{p,\gamma}(R^n; E)$ when $1 < p < \infty$, $\gamma \in A_p$.

Let Ω be a domain in R^n. $C(\Omega, E)$ and $C^{(m)}(\Omega; E)$ will denote the spaces of $E-$valued uniformly bounded strongly continuous and $m-$times continuously differentiable functions on Ω, respectively.

Here, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, where α_i are integers. An $E-$valued generalized function $D^\alpha f$ is called a generalized derivative in the sense of Schwartz distributions of the function $f \in S(R^n; E)$ if

$$\langle D^\alpha f, \varphi\rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi\rangle$$

holds for all $\varphi \in S$.

Let F denote the Fourier transform. Throughout this section the Fourier transformation of a function f will be denoted by $\widehat{f}$ and $F^{-1} f = \check{f}$. It is known that

$$F(D_x^\alpha f) = (i\xi_1)^{\alpha_1} ... (i\xi_n)^{\alpha_n} \widehat{f}, \quad D_\xi^\alpha (F(f)) = F\left[(-ix_1)^{\alpha_1} ... (-ix_n)^{\alpha_n} f\right]$$

for all $f \in S^{'}(R^n; E)$.

Suppose E_1 and E_2 are two Banach spaces. A function $\Psi \in L_\infty(R^n; B(E_1, E_2))$ is called a Fourier multiplier from $L_{p,\gamma}(R^n; E_1)$ to $L_{p,\gamma}(R^n; E_2)$ for $p \in (1, \infty)$ if the map $u \to Tu = F^{-1}\Psi(\xi) Fu$, $u \in S(R^n; E_1)$ is well defined and extends to a bounded linear operator

$$T : L_{p,\gamma}(R^n; E_1) \to L_{p,\gamma}(R^n; E_2).$$

A Banach space E is called a UMD space (see e.g [5], [6]) if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x - y} dy$$

is initially defined on $S(R; E)$ and is bounded in $L_p(R; E)$, $p \in (1, \infty)$ (see e.g. [6], [8]). UMD spaces include e.g. L_p, l_p spaces and Lorentz spaces L_{pq}, $p, q \in (1, \infty)$.

A set $K \subset B(E_1, E_2)$ is called $R-$bounded (see e.g [7], [21]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_m \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \le C \int_0^1 \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1;1\}$ −valued random variables on $[0,1]$. The smallest C for which the above estimate holds is called the $R-$bound of K and denoted by $R(K)$.

A Banach space E is said to be a space satisfying the multiplier condition with respect to weighted function γ and $p \in (1,\infty)$ (or multiplier condition with respect to $p \in (1,\infty)$ when $\gamma(x) \equiv 1$) if for any $\Psi \in C^{(n)}(R^n \setminus \{0\}; B(E))$ the $R-$boundedness of the set

$$\left\{ |\xi|^{|\beta|} D_\xi^\beta \Psi(\xi) : \xi \in R^n \setminus \{0\},\ \beta = (\beta_1, \beta_2, ..., \beta_n),\ \beta_k \in \{0,1\} \right\}$$

implies that Ψ is a Fourier multiplier in $L_{p,\gamma}(R^n; E)$.

Note that, if E is UMD space then it satisfies the multiplier condition with respect to $p \in (1,\infty)$ (see e.g. [7], [10], [21]).

A sectorial operator $A(x)$, $x \in R^n$ is said to be uniformly $R-$sectorial in a Banach space E if there exists a $\varphi \in [0, \pi)$ such that

$$\sup_{x \in R^n} R\left(\left\{ \left[A(x)(A(x) + \xi I)^{-1} \right] : \xi \in S_\varphi \right\} \right) \le M.$$

Note that, in Hilbert spaces every norm bounded set is $R-$bounded. Therefore, in Hilbert spaces all sectorial operators are $R-$sectorial.

Let $A = A(x)$, $x \in R^n$ be closed linear operator in E with domain $D(A)$ independent of x. The Fourier transformation of $A(x)$ is a linear operator with the domain $D(A)$ defined as

$$\hat{A}(\xi)\, u(\varphi) = A(x)\, u(\hat{\varphi}) \ \text{ for } u \in S'(R^n; D(A)),\ \varphi \in S(R^n).$$

(For details see e.g [2, Section 3]).

Let E_0 and E be two Banach spaces, where E_0 is continuously and densely embedded into E. Let l be a natural number. $W_{p,\gamma}^l(R^n; E_0, E)$ denotes the space of all functions from $S'(R^n; E_0)$ such that $u \in L_{p,\gamma}(R^n; E_0)$ and the generalized derivatives $D_k^l u = \frac{\partial^l u}{\partial x_k^l} \in L_{p,\gamma}(R^n; E)$ with the norm

$$\|u\|_{W_{p,\gamma}^l(R^n; E_0, E)} = \|u\|_{L_{p,\gamma}(R^n; E_0)} + \sum_{k=1}^n \left\| D_k^l u \right\|_{L_{p,\gamma}(R^n; E)} < \infty.$$

It is clear that

$$W_{p,\gamma}^l(R^n; E_0, E) = W_{p,\gamma}^l(R^n; E) \cap L_{p,\gamma}(R^n; E_0).$$

$W_{p,\gamma}^{[l]}\left(R^n;E_0,E\right)$ denotes the space of all functions from $S'\left(R^n;E_0\right)$ such that $u\in L_p\left(R^n;E_0\right)$ and $D_k^{[l]}u\in L_p\left(R^n;E\right)$ with the norm

$$\|u\|_{W_{p,\gamma}^{[l]}(R^n;E_0,E)}=\|u\|_{L_p(R^n;E_0)}+\sum_{k=1}^{n}\left\|D_k^{[l]}u\right\|_{L_p(R^n;E)}<\infty.$$

Note that if $l\geq 2$, E is a space satisfying the multiplier condition with respect to weighted function γ and $p\in(1,\infty)$, then the above definitions are equivalent with usual definitions, i.e.

$$\|u\|_{W_{p,\gamma}^{l}(R^n;E_0,E)}\simeq\|u\|_{L_{p,\gamma}(R^n;E_0)}+\sum_{|\alpha|\leq l}\|D^\alpha u\|_{L_{p,\gamma}(R^n;E)},$$

$$\|u\|_{W_{p,\gamma}^{[l]}(R^n;E_0,E)}\simeq\|u\|_{L_p(R^n;E_0)}+\sum_{|\alpha|\leq l}\left\|D^{[\alpha]}u\right\|_{L_p(R^n;E)}.$$

In a similar way as [7, Theorem 3.25] we obtain:

Proposition A. Let E be a UMD space and $\gamma\in A_p$. Assume Ψ_h is a set of operator functions in $C^{(n)}\left(R^n\backslash\{0\};B(E)\right)$ depending on the parameter $h\in Q\in\mathbb{R}$ and there exists a positive constant K such that

$$\sup_{h\in Q}R\left(\left\{|\xi|^{|\beta|}D^\beta\Psi_h(\xi):\ \xi\in R^n\backslash\{0\},\ \beta_k\in\{0,1\}\right\}\right)\leq K.$$

Then the set Ψ_h is a uniformly bounded collection of Fourier multipliers in $L_{p,\gamma}\left(R^n;E\right)$.

2. Convolution-elliptic equations

The main aim of the present section is to study the maximal L_p-regularity properties of the degenerate linear CDOEs

$$\sum_{|\alpha|\leq l}a_\alpha*D^{[\alpha]}u+(A+\lambda)*u=f(x),\ x\in R^n, \tag{2.1}$$

in $E-$ valued weighted L_p- spaces, where l is a natural number, $a_\alpha=a_\alpha(x)$ are complex-valued functions, $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$, α_k are nonnegative integers, $|\alpha|=\sum_{k=1}^{n}\alpha_k$, λ is a complex parameter and $A=A(x)$ is a linear operator in a Banach space E for $x\in R^n$.

Here, the convolutions $a_\alpha*D^{[\alpha]}u,\ A*u$ are defined in the distribution sense (see e.g. [2]). $\gamma=\gamma(x)$ is a positive measurable function on $\Omega\subset R^n$ and

$$D^{[\alpha]}=D_{x_1}^{[\alpha_1]}D_{x_2}^{[\alpha_2]}...D_{x_n}^{[\alpha_n]},\ D_{x_i}^{[\alpha_i]}=\left(\gamma(x)\frac{\partial}{\partial x_i}\right)^{\alpha_i}.$$

First we consider the following nondegenerate CDOE

$$\sum_{|\alpha|\leq l} a_\alpha * D^\alpha u + (A+\lambda) * u = f, \tag{2.2}$$

where λ are parameters, a_α are complex-valued functions defined in (2.1) and A is a linear operator in a Banach space E.

Condition 2.1. Suppose the following are satisfied:

(1) $L(\xi) = \sum\limits_{|\alpha|\leq l} \widehat{a}_\alpha(\xi)(i\xi)^\alpha \in S_{\varphi_1}$, $\varphi_1 \in [0,\pi)$ for $\xi \in R^n$,

$|L(\xi)| \geq C \sum\limits_{k=1}^{n} \left|\widehat{a}_{\alpha(l,k)}\right| |\xi_k|^l$, $\alpha(l,k) = (0,0,...,l,0,0,...,0)$ i.e.$\alpha_i = 0$, $i \neq k$, $\alpha_k = l$;

(2) $\widehat{a}_\alpha \in C^{(n)}(R^n)$ and $|\xi|^{|\beta|} \left|D^\beta \widehat{a}_\alpha(\xi)\right| \leq C_1$, $\beta_k \in \{0,1\}$, $0 \leq |\beta| \leq n$;

(3) for $0 \leq |\beta| \leq n$, $\xi, \xi_0 \in R^n \setminus \{0\}$, $\left[D^\beta \widehat{A}(\xi)\right] \widehat{A}^{-1}(\xi_0) \in C(R^n; B(E))$,

$|\xi|^{|\beta|} \left\| \left[D^\beta \widehat{A}(\xi)\right] \widehat{A}^{-1}(\xi_0)\right\| \leq C_2$.

In a similar way as [16, Theorem 2.7] we obtain:

Theorem 2.1. Assume that Condition 2.1 holds and E is a Banach space satisfying the multiplier condition with respect to weighted function $\gamma \in A_p$ and $p \in (1,\infty)$. Let $\hat{A}$ be a uniformly $R-$sectorial operator in E with $\varphi \in [0, \pi)$, $\lambda \in S_{\varphi_2}$ and $0 \leq \varphi + \varphi_1 + \varphi_2 < \pi$. Then, problem (2.2) has a unique solution u and the coercive uniform estimate holds

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_{L_{p,\gamma}(R^n;E)} + \|A * u\|_{L_{p,\gamma}(R^n;E)} + |\lambda| \|u\|_{L_{p,\gamma}(R^n;E)} \leq C \|f\|_{L_{p,\gamma}(R^n;E)} \tag{2.3}$$

for all $f \in L_{p,\gamma}(R^n; E)$.

Let O be an operator in $L_{p,\gamma}(R^n; E)$ generated by problem (2.2) for $\lambda = 0$, i.e.

$$D(O) \subset W^l_{p,\gamma}(R^n; E(A), E), \ Ou = \sum_{|\alpha|\leq l} a_\alpha * D^\alpha u + A * u.$$

From Theorem 2.1 we have:

Result 2.1. Assume that the all conditions of Theorem 2.1 hold. Then, for all $\lambda \in S_{\varphi_2}$ the following uniform coercive estimate holds

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\|a_\alpha * D^\alpha (O+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;E))} +$$

$$\left\|A * (O+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;E))} + \left\|\lambda (O+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;E))} \leq C.$$

Result 2.2. Theorem 2.1, particularly implies that the operator O is uniformly sectorial in $L_{p,\gamma}(R^n; E)$; moreover, if $\hat{A}$ is uniformly $R-$sectorial for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, then the operator O is a negative generator of an analytic semigroup in $L_{p,\gamma}(R^n; E)$ (see e.g. [19, §1.14.5]).

From Theorem 2.1 and Proposition A we obtain:

Result 2.3. Let conditions of Theorem 2.1 hold for $E \in UMD$. Then the assertions of Theorem 2.1 are valid.

We find sufficient conditions that guarantee the separability of the problem (2.1). For this purpose we need the following

Remark 2.1. Consider the following substitution

$$y_k = \int_0^{x_k} \gamma^{-1}(z)dz,\ k = 1, 2, ..., n. \tag{2.4}$$

It is clear that, under the substitution (2.4), $D^{[\alpha]}u$ transforms to $D^{\alpha}u$. Moreover, the spaces $L_p(R^n; E)$, $W_{p,\gamma}^{[l]}(R^n; E(A), E)$ are mapped isomorphically onto the weighted spaces $L_{p,\gamma}(R^n; E)$ and $W_{p,\gamma}^{l}(R^n; E(A), E)$ respectively where,

$$\gamma = \widetilde{\gamma}(y) = \gamma(x(y)) = \gamma(x_1(y_1), x_2(y_2), ..., x_n(y_n)).$$

Moreover, under (2.4) the degenerate problem (2.1) considered in $L_p(R^n; E)$ is transformed into the non degenerate problem (2.2) in $L_{p,\gamma}(R^n; E)$, where

$$a_\alpha = a_\alpha(y) = a_\alpha(\widetilde{\gamma}(y)),\ u = u(y) = \tilde{u}(y) = u(\widetilde{\gamma}(y)),$$

$$A = A(y) = \tilde{A}(y) = A(\widetilde{\gamma}(y)),\ f = f(y) = \widetilde{f}(y) = f(\widetilde{\gamma}(y)).$$

Let

$$\tilde{X} = L_p(R^n; E),\ \tilde{Y} = W_{p,\gamma}^{[l]}(R^n; E(A), E),\ p \in (1, \infty).$$

In this section we show the following result:

Theorem 2.2. Assume that Condition 2.1 holds for $a_\alpha = a_\alpha(y)$ and E is a Banach space satisfying the multiplier condition with respect to weighted function $\gamma \in A_p$ and $p \in (1, \infty)$. Let $\hat{A}$ be a uniformly $R-$sectorial operator in E with $\varphi \in [0, \pi)$, $\lambda \in S_{\varphi_2}$ and $0 \leq \varphi + \varphi_1 + \varphi_2 < \pi$ for $A = A(y)$. Then for all $f \in \tilde{X}$ there is a unique solution of the problem (2.1) and the following coercive uniform estimate holds:

$$\sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * D^{[\alpha]}u \right\|_{\tilde{X}} + \|A * u\|_{\tilde{X}} + |\lambda| \|u\|_{\tilde{X}} \leq C \|f\|_{\tilde{X}}. \tag{2.5}$$

Proof. By Remark 2.1, the degenerate problem (2.1) is transformed into the non degenerate problem (2.2) considered in the weighted space $L_{p,\gamma}(R^n; E)$. Then in view of Theorem 2.1 we obtain the estimate (2.5).

3. Degenerate convolution equations in the space $L_{p,\gamma}(R^n; l_q)$

Consider the following system of convolution equations

$$\sum_{|\alpha|\le l} a_\alpha * D^{[\alpha]}u_m + \sum_{m=1}^{\infty} d_m * u_m = f_m, \tag{3.1}$$

in the concrete Banach space $L_{p,\gamma}(R^n; l_q)$, where l is a natural number, $a_\alpha = a_\alpha(x)$ are complex-valued functions, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, α_k are nonnegative integers, $d_m = d_m(x)$, $u_m = u_m(x), f_m = f_m(x), x \in R^n$. The convolutions $a_\alpha * D^{[\alpha]}u$, $d_m * u_m$ are defined in the distribution sense and

$$D^{[\alpha]} = D_{x_1}^{[\alpha_1]} D_{x_2}^{[\alpha_2]} ... D_{x_n}^{[\alpha_n]},\ D_{x_k}^{[\alpha_k]} = \left(\gamma(x)\frac{\partial}{\partial x_k}\right)^{\alpha_k}.$$

$$\gamma(x) = \prod_{k=1}^{n} |x_k|^\gamma,\ \ -\frac{1}{n} < \gamma < \frac{p-1}{n},$$

is a positive measurable weighted function.

For $1 < q < \infty$ we set

$l_q = \left\{ \xi; \xi = \{\xi_i\}_{i=1}^\infty;\ \|\xi\|_{l_q} = \left(\sum_{i=1}^\infty |\xi_i|^q\right)^{1/q} < \infty,\ \xi_i - \text{complex numbers} \right\}.$

Moreover, if $\gamma(x)$ is a positive measurable function, and if $1 < p < \infty$, then

$L_{p,\gamma}(R^n; l_q) = \left\{ f; f = \{f_i(x)\}_{i=1}^\infty, \|f\|_{L_{p,\gamma}(R^n;l_q)} = \left(\int_{R^n} \|\{f_i(x)\}\|_{l_q}^p \gamma(x)dx\right)^{1/p} < \infty \right\}.$

Clearly, $L_{p,\gamma}(R^n; l_q)$ is a Banach space. It is known that

$\|f\|_{L_{p,\gamma}(R^n;l_q)} = \left(\int_{R^n} \left(\sum_{i=1}^\infty |f_i(x)|^q\right)^{\frac{p}{q}} \gamma(x)dx\right)^{\frac{1}{p}}.$

Let $d(x) = \{d_m(x)\}$, $d_m > 0$, $u = \{u_m\}$, $d * u = \{d_m * u_m\}$, $l_q(d) =$

$$\left\{ u \in l_q,\ \|u\|_{l_q(d)} = \left(\sum_{m=1}^\infty |d_m(x) * u_m|^q\right)^{\frac{1}{q}} < \infty \right\},\ 1 < q < \infty,$$

$$X = L_p(R^n; l_q),\ Y = W_{p,\gamma}^{[l]}(R^n; l_q(d), l_q),\ B = B(X),$$

and Q denote the differential operator in $L_p(R^n; l_q)$ generated by (3.1), i.e., $D(Q) =$

$W_{p,\gamma}^{[l]}(R^n; l_q(d), l_q)$, $Qu = \sum_{|\alpha|\le l} a_\alpha * D^{[\alpha]}u + d * u$

Condition 3.1. Assume that there exist positive constants C_1 and C_2 such that for $\{d_m(x)\}_1^\infty \in l_q$ for all $x \in R^n$ and some $x_0 \in R^n$,

$$C_1 |d_m(x_0)| \le |d_m(x)| \le C_2 |d_m(x_0)|.$$

Suppose $\hat{a}_\alpha,\ \hat{d}_m \in C^{(n)}(R^n)$ and there exist positive constants M_1 and M_2 such that

$$|\xi|^{|\beta|}\left|D^\beta \hat{a}_\alpha(\xi)\right| \leq M_1,\ |\xi|^{|\beta|}\left|D^\beta \hat{d}_m(\xi)\right| \leq M_2,$$

$$\xi \in R^n \setminus \{0\},\ \beta_k \in \{0,1\},\ 0 \leq |\beta| \leq n.$$

Applying Theorem 2.2. we have:

Theorem 3.1. Suppose Condition 3.1 and the (1) part of Condition 2.1 are satisfied. Then:

(a) for all $f(x) = \{f_m(x)\}_1^\infty \in L_p(R^n; l_q(d))$, for $\lambda \in S_\varphi$, $\varphi \in [0,\pi)$ problem (3.1) has a unique solution $u = \{u_m(x)\}_1^\infty$ that belongs to Y and the following coercive estimate holds

$$\sum_{|\alpha|\leq l}\left(\int_{R^n}\left(\sum_{m=1}^\infty \left|a_\alpha * D^{[\alpha]}u_m\right|^q\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} + \left(\int_{R^n}\left(\sum_{m=1}^\infty |d_m * u_m|^q\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

$$\leq C\left(\int_{R^n}\left(\sum_{m=1}^\infty |f_m|^q\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}.$$

(b) For $\lambda \in S_\varphi$ there exists a resolvent $(Q+\lambda)^{-1}$ and

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha * \left[D^{[\alpha]}(Q+\lambda)^{-1}\right]\right\|_B +$$

$$\left\|d*(Q+\lambda)^{-1}\right\|_B + \left\|\lambda(Q+\lambda)^{-1}\right\|_B \leq C.$$

Proof. In fact, let $E = l_q$ and $A = [d_m(x)\delta_{jm}]$, $m,\ j = 1,2,...\infty$, where δ_{jm} is the Kronecker symbol ($\delta_{jm} = 1$ for $j = m$, $\delta_{jm} = 0$ for $j \neq m$). Then it is easy to see that $\hat{A}(\xi) = \left[\hat{d}_m(\xi)\delta_{jm}\right]$ is uniformly $R-$sectorial in l_q and the all conditions of Theorem 2.2 hold. Moreover, by [20] we get that the space l_q satisfies the multiplier condition with respect to power weighted function $\gamma(x) = |x|^\gamma$, $-\frac{1}{n} < \gamma < \frac{p-1}{n}$ and $p \in (1,\infty)$. Therefore, by virtue of Theorem 2.2 we obtain the $\sum_{|\alpha|\leq l}\left\|a_\alpha * D^{[\alpha]}u\right\|_X + \|d*u\|_X \leq C\|f\|_X$. From this we get that assertion (a). Taking into account Theorem 2.2 and Remark 2.1 we have for all $\lambda \in S_\varphi$ there exist the resolvent of operator Q and has the estimate

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha * D^{[\alpha]}(Q+\lambda)^{-1}\right\|_{B(X)} + \left\|d*(Q+\lambda)^{-1}\right\|_{B(X)} + \left\|\lambda(Q+\lambda)^{-1}\right\|_{B(X)} \leq C.$$

This means that the assertion (b) is obtained.

Remark 3.1. There are a lot of sectorial operators in concrete Banach spaces. Therefore, putting in (2.1) concrete Banach spaces instead of E and concrete sectorial differential, pseudo differential operators, or finite, infinite matrices, etc. instead of A, by virtue of Theorem 2.2 we can obtain the maximal regularity properties of different class of convolution equations.

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Received: 17 November 2018
Accepted: 05 June 2019