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On Estimation of Surface Trigonometric Integrals

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Abstract. In this article new upper bounds for the multiple trigonometric integrals are found when the phase function's gradient defines a non-degenerating mapping.

Key Words and Phrases: multiple trigonometric integrals, surface integrals, phase function, algebraic function.
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1. Introduction

An integral of a view

$$\int_{\Omega} G(\bar{x}) e^{2\pi i F(\bar{x})} d\bar{x} \tag{1}$$

is called a multiple trigonometric integral; here Ω denotes some domain of n dimensional space $\mathbb{R}^n$, and on the functions $G(x)$ and $F(x)$ one imposes definite conditions on boundedness or smoothness. Many investigations (see [1, 2, 3, 4, 7, 8, 9, 10, 11, 18, 19]) were devoted to estimations of trigonometric integrals. The first result in this direction belongs to Van der Corput and E.Landau (see [11]). The result established in the work [4] where the authors have received a non-improvable estimation for trigonometric integrals has important applications. The multidimensional case also was investigated in the literature. Unlike one-dimensional case, estimating of multiple trigonometric integrals of a view (1) in which Ω is some Jordan domain with a smooth boundary and the functions $G(x)$,$F(x)$ are from a certain class of smoothness is much more difficult.

The scheme of finding of estimates for integrals of a view (1) is similar to the scheme of one-dimensional case. After some transformations (see [11]) the integral reduces to the view

$$\int_a^b V(u) e^{2\pi i u} du,$$

where $V(u)$ represents the surface integral depending on parameter u.

Let Ω be a bounded closed domain of n-dimensional space $\mathbb{R}^n$, $n \geq 2$. Let's assume that in Ω an $n-1$-dimensional surface be given by means of a polynomial equation

$$f(\bar{x}) = 0 \tag{2}$$

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with the gradient $\nabla f = (\partial f/dx_1, ..., \partial f/dx_n)$ which has everywhere in Ω a non-vanishing norm. In this article we consider surface trigonometric integrals taken over hypersurface Π given by the polynomial equation (2):

$$\int_\Pi g(\bar{x}) e^{2\pi i F(\bar{x})} ds; \tag{3}$$

here $g(\bar{x})$ is some algebraic function. Such integrals arise after of transformations by using Stokes type formulae. Trivial estimation of integral (3) can be obtained as follows

$$\int_\Pi g(\bar{x}) e^{2\pi i F(\bar{x})} ds \leq \int_\Pi |g(\bar{x})| ds.$$

Non-trivial estimation for the integrals of such type can be useful in applications to the questions connected with the distribution of integral points in multidimensional domains.

2. Auxiliary statements

Let Ω be a bounded closed domain of n-dimensional space $\mathbb{R}^n$, $n > 1$. Let's assume that in Ω some r-dimensional surface be given by means of a system of polynomial equations

$$f_j(\bar{x}) = 0, j = 1, ..., n - r, 0 \leq r \leq n, \tag{4}$$

with a Jacoby matrix

$$J = J(\bar{x}) = \|\frac{\partial f_j}{\partial xi}\|, i = 1, ..., n, j = 1, ..., n - r$$

which has everywhere in Ω a maximal rank.

Let $A_0 = A_0(\bar{x})$ be some functional matrix written down in a form

$$A_0 = A_0 = \|f_{ij}(\bar{x})\|, \; 1 \leq i \leq r, \; 1 \leq j \leq m, rm \geq n$$

with smooth entries. Arranging the entries of columns of this matrix in a line as below

$$f_{11}(\bar{x}), ..., f_{r1}(\bar{x}), f_{12}(\bar{x}), ..., f_{r2}(\bar{x}), ..., f_{1m}(\bar{x}), ..., f_{rm}(\bar{x}),$$

let's take the transposed Jacoby matrix of this system of functions designating it as A_1:

$$A_1 = A_1(\bar{x}) = \begin{Vmatrix} \frac{\partial f_{11}}{\partial x_1} & \cdots & \frac{\partial f_{r1}}{\partial x_1} & \cdots & \frac{\partial f_{1m}}{\partial x_1} & \cdots & \frac{\partial f_{rm}}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_{11}}{\partial x_n} & \cdots & \frac{\partial f_{r1}}{\partial x_n} & \cdots & \frac{\partial f_{1m}}{\partial x_n} & \cdots & \frac{\partial f_{rm}}{\partial x_n} \end{Vmatrix}.$$

Then, entries of columns of this matrix, consequently as above, we arrange in a line, and take the transposed Jacoby matrix $A_2 = A_2(\bar{x}) = A_1'(\bar{x})$ of the received system of functions. Let's continue this procedure while we have not received a matrix $A_k = A_{k-1}'(\bar{x})$ for a given $k \geq 1$. The last matrix defined by such procedure consists of all possible partial derivatives of the same order k of entries of the matrix $A_0 = A_0(\bar{x})$ and has the size $n \times n^{k-1}rm$. Let's assume that $A_j(\bar{x})$ has in Ω a maximal rank equal to n. Let's designate by $G_j(\bar{x})$ the product of the last (smallest) r singular numbers of the matrix $A_j(\bar{x})$, $j = 0, ..., k$. We put

$$E = E(H) = \{\bar{x} \in \Omega | G_0(\bar{x}) \leq H\}, \; H > 0.$$

If $\varphi_{ik}(\bar{x})$ are entries of the matrix $A_j(\bar{x})$ we will accept the following designations

$$L_j(\bar{x}) = \left(\sum_{i,k} |\varphi_{ik}(\bar{x})|^2 \right),$$

$$L = \max_j \max_{\bar{x} \in \Omega} L_j(\bar{x}), \;\; G_j = \min_{\bar{x} \in \Omega} G_j(\bar{x}), j = 0, ..., k.$$

The cases $r = n-1$ and $r = n-2$ we will consider separately. Assume that the domain Ω can be dissected into such parts that on each of them the equation (2) allows one sheeted and one valued solvability, and in every of them one of minors of the matrix $A_j(\bar{x})$ (also one of partial derivatives of the function) has the maximal absolute values among all minors. So, doesn't destroying a generality, we assume that in Ω some of minors, say the minor placed on the first $n-1$ columns of the Jacoby matrix, has positive maximal absolute values. Then, by the theorem on implicit functions ([5, 12, 15, 17]), we may solve the equation (2) with respect to the first $n-1$ variables. Denote by $\bar{\xi} = (\xi_2, ..., \xi_n)$ a vector of independent variables. Then, x_1 is possible to represent as a function $x_1 = x_1(\bar{\xi})$ of these independent variables. Denote by $A_0(\bar{\xi})$ the matrix constructed from the matrix $A_0(\bar{x})$ by replacing of the variable x_1 by the function $x_1 = x_1(\bar{\xi})$. In other words we consider the functional matrix $A_0(\bar{\xi})$ as a matrix depending on $\bar{\xi}$. Denote by $G_{(1)}$ the minimal value of Gram determinant for gradients of entries of the matrix $A_0(\bar{\xi})$ (differentiation is taken with regard to $\bar{\xi}$), i.e.

$$G_{(1)} = \min_{\bar{\xi}} \det \left(A_{1\bar{\xi}} \cdot A^t_{1\bar{\xi}} \right).$$

Thus, $A_{1\bar{\xi}}$ means the matrix of a size $(n-1) \times rm$ received from A_0 by differentiation in regard to $\bar{\xi}$, $A_{1\bar{\xi}} = A_0'(\bar{\xi})$. So, the matrix $A_1(\bar{x})$ being considered as a matrix of $\bar{\xi}$, differs from$A_{1\bar{\xi}}$. Similarly, we can, beginning from the matrix A_{j-1}, form a matrix $A_{j\bar{\xi}}$ assuming that $G_{(j)} > 0$ for all considered $j > 0$. For a positive number $a > 0$ we write $h(a) = a + a^{-1}$. We have $a \leq h(a)$, $h(a) = h(a^{-1})$, and $h(ab) \leq h(a)h(b)$, for $a, b > 0$.

Lemma 1. *Let Π_H be a part of a surface (4) included in $E(H)$, $k > 1$ and $G_{(k)} > 0$. Then under the conditions above we have:*

$$\mu(\Pi_H) \leq KH^{1/k} \cdot G_{(k)}^{-1/k} \cdot \mathrm{Q}_k^n;$$

$$\mathrm{Q}_k = \log \tilde{H}; \; \tilde{H} = \max \left\{ h(H), h(C_{(1)}), ..., h(C_{(k-1)}), h(G_{(k)}), h(L) \right\},$$

and K is a constant, and numbers $C_{(2)}, ..., C_{(k-1)}$ are defined by equalities

$$C_{(1)} = H^{1/2} C_{(2)}^{1/2}, C_{(2)} = H^{1/3} C_{(3)}^{1/3}, ..., G_{(k)} = H^{1/k} C_{(k-1)}^{1/k}.$$

The proof of the lemma 1 is given in [11, 13]. Following lemma is a generalization of this lemma ([11, 13]).

Lemma 2. *Under the conditions of the lemma 1 there exist an absolute constant K_1 such that:*

$$\mu(\Pi_H) \leq K_1 H^{1/k} \cdot G_k^{-1/k} \cdot \tilde{Q}_{\mathrm{k}}^n.$$

Let $F(\bar{x})$ be some polynomial. Let's consider the trigonometric integral (3), in the domain Ω with a boundary consisted of finite number of algebraic surfaces. Gradient of this function is a matrix A_0:

$$A_0 = \nabla F = \left\| \frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n} \right\|.$$

Let everywhere in Ω

$$\|\nabla F\| = \sqrt{\left(\frac{\partial F}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial F}{\partial x_n}\right)^2} \neq 0.$$

We assume that the boundary of the domain Ω is a union of surfaces defined by finite number of algebraic equations of a view $H(\bar{x}) = 0$. Not breaking a generality, we can take this number equal to 1. Assume, also, that the Jacoby matrix of the system of functions F, H has rank 2.

It is clear that the matrix $A_1(\bar{x})$ looks like

$$A_1 = A_1(\bar{x}) = \left\| \begin{array}{c} \frac{\partial^2 F}{\partial x_1^2} \cdots \frac{\partial^2 F}{\partial x_1 \partial x_r} \\ \cdots \quad \cdots \\ \frac{\partial^2 F}{\partial x_r \partial x_1} \cdots \frac{\partial^2 F}{\partial x_r^2} \end{array} \right\|, \tag{5}$$

and the matrix $A_{k-1}(\bar{x})$ is combined of all partial derivatives of order $k \geq 2$ of the function $F(\bar{x})$. Let now $\tilde{G}_{k-1}$ be a minimal value of the product of $n-2$ least singular numbers of the matrix A_{k-1}. Similarly, we can, beginning from the matrix A_{j-1}, form a matrix $A_{j\bar{\xi}}$ assuming that $\tilde{G}_{(j)} > 0$ for all considered $j > 0$. Now we formulate analogs of the lemmas 1 and 2 for the case $r = n-2$ designating the numbers $G_{(j)}$ and G_j as $\tilde{G}_{(j)}$ and $\tilde{G}_j$, respectively.

Lemma 3. *Let Π_H be a part of a surface* (4) *included in $E(H)$ and $\tilde{G}_1 > 0$. Then for the area $\mu(\Pi_H)$ we have the bound*

$$\mu(\Pi_H) \leq C_0 H \tilde{G}_{(1)}^{-1} \tilde{\wp}^r,$$

where

$$\tilde{\wp} = r^2 \log\left[h\left(\tilde{G}_{(1)}\right) h(H)\, h(L)\right],$$

and C_0 is an absolute constant.

Lemma 4. *Let $k \geq 1$ and $\tilde{G}_{(k)} > 0$. Then under the conditions of the lemma 1 we have:*

$$\mu\left(\Pi_H\right) << H^{1/k}\tilde{G}_{(k)}^{-1/k}\wp_k^r;$$

$$\tilde{\wp}_k = 3r^2\log\tilde{H};\ \tilde{H} = \max\left\{h\left(H\right), h\left(\tilde{C}_{(1)}\right), ..., h\left(\tilde{C}_{(k-1)}\right), h\left(\tilde{G}_{(k)}\right), h\left(L\right)\right\},$$

and numbers $\tilde{C}_{(1)}, ..., \tilde{C}_{(k-1)}, \tilde{G}_{(k)}$ are defined by equalities

$$\tilde{C}_{(1)} = H^{1/2}\tilde{C}_{(2)}^{1/2}, ..., \tilde{C}_{(k-1)} = H^{1/k}\tilde{G}_{(k)}^{1/k}$$

Lemma 5. *Let $k \geq 1$ and $\tilde{G}_k > 0$. Then, under the conditions of the lemma 2, one has:*

$$\mu\left(\Pi_H\right) << H^{1/k}\tilde{G}_k^{-1/k}\wp_k^r.$$

Lemma 6. *There exist such a dissection of the domain Ω into the union of no more than finite number of subdomains so that the surface integral $\varphi(u) = \int_{F(\bar{x})=u}\frac{g(\bar{x})ds}{\|\nabla F\|}$, respectively, breaks into the sum of the surface integrals being monotonous functions of a variable u, moreover, the number of addends of the last sum depends on the degree of a polynomial F only.*

Proof. Proof of this lemma we will spend using reasonings of the proof of analogical lemma from the work [11]. Having given to the variable u some increment, we can write

$$\varphi(u+\Delta u) - \varphi(u) = \int_{F(\bar{x})=u+\Delta u}\frac{g(\bar{x})ds}{\|\nabla F\|} - \int_{F(\bar{x})=u}\frac{g(\bar{x})ds}{\|\nabla F\|}.$$

As the domain Ω is closed, the gradient of functions $F(\bar{x})$ and $g(\bar{x})$ and their partial derivatives of the second order are bounded. Consider the Taylor decomposition of the function $F(\bar{x})$ in a neighborhood of the point $\bar{x}$, lying on the surface $F(\bar{x}) = u$, in the gradient direction:

$$F(\bar{x}+\lambda\nabla F) - F(\bar{x}) = \lambda\nabla F\cdot\nabla F + o(\lambda).$$

Let's pick up λ so that the point $F(\bar{x}+\lambda\nabla F)$ was placing on the surface $F = u+\Delta u$. Then, we get

$$\Delta u = \lambda\nabla F\cdot\nabla F + o(\lambda).$$

When Δu is sufficiently small, the second term on the right part is small also. So,

$$\lambda = \frac{\Delta u}{\nabla F\cdot\nabla F} + o(\Delta u) = \frac{\Delta u}{\|\nabla F\|^2} + o(\Delta u).$$

After of shifting of the argument in the gradient direction the function $\frac{g(\bar{x})}{\|F\|}$ takes on an increment δ which can be written as follows:

$$\delta = \nabla\left(\frac{g(\bar{x})}{\|F\|}\right)\cdot\lambda\nabla F(1+o(\lambda)) =$$

$$= \lambda \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \left(\frac{\partial g/\partial x_i}{\|F\|} - \frac{g}{\|F\|^3} \left(\sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \frac{\partial^2 F}{\partial x_j \partial x_i} \right) \right) + o(\lambda)) =$$

$$= \lambda \frac{\nabla F \cdot \nabla g}{\|F\|} - \lambda g \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \left(\frac{g}{\|F\|^3} \left(\sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \frac{\partial^2 F}{\partial x_j \partial x_i} \right) \right) + o(\lambda)).$$

Using definition of the matrix A_1 given above and denoting $\bar{\nabla} = \nabla F / \|\nabla F\|$, we can rewrite the last equality as below:

$$\delta = \frac{\lambda}{\|F\|} \left(\nabla F \cdot \nabla g - g(A_1 \bar{\nabla}, \bar{\nabla}) \right) = \frac{\Delta u}{\|F\|^3} \left(\nabla g \cdot \nabla F - g(A_1 \bar{\nabla}, \bar{\nabla}) \right).$$

Under the conditions imposed on a gradient, as shown above, the domain Ω may be dissected into finite number of subdomains which pairwisely intersecting by parts of the boundary only, and where the equation $F(\bar{x}) = u$ allows one sheeted solvability with respect to one and the same variable. Let's consider one of them where the mentioned equation is solved with respect, say, to x_1:

$$x_1 = \psi(x_2, ..., x_n);\ (x_2, ..., x_n) \in \omega,$$

and ω is an domain of changing for independent variables. Having fixed any point $\bar{\xi}_0 \in \omega$, we will define the mapping $\bar{\psi}$ in $\omega - \bar{\xi}_0 = \{\Delta y \in \mathbb{R}^{n-1} | \bar{\xi}_0 + \Delta y \in \omega\}$ which puts to each point Δy in correspondence the point $(\psi(\xi_0 + \Delta y), \xi_0 + \Delta y)$ on the surface $F = u$, and will consider tangential linear mapping

$$\Phi : \Delta y \mapsto \psi(\bar{\xi}_0) + \psi(\bar{\xi}_0) \cdot \Delta y; \Delta y \in \omega - \bar{\xi}_0. \tag{6}$$

The image of this mapping is a tangential linear variety (hyper plane) to the surface $F = u$ in the point $(\psi(\bar{\xi}_0), \bar{\xi}_0)$. Let's notice that the point $(\Phi(\bar{\xi}), \bar{\xi})$ of the tangential hyper plane will situated from the corresponding point $(\psi(\bar{\xi}), \bar{\xi})$ on the surface $F = u$ at a distance $o\left(\left|\Phi(\Delta y) - \bar{\psi}(\Delta y)\right|\right)$ which is of order $o(\Delta u)$. At each point $\bar{x}$ of the surface $F = u$ the gradient ∇F is orthogonal to the tangential hyper plane. Really,

$$\nabla F \cdot \Phi'(\vec{\xi}) \Delta x = \left(\frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n} \right) \cdot$$

$$\cdot \left(\left(\frac{\partial F}{\partial x_1} \right)^{-1} \left(-\frac{\partial F}{\partial x_2} \Delta x_2 -, , , -\frac{\partial F}{\partial x_n} \Delta x_n \right), \Delta x_2, ..., \Delta x_n \right) = 0.$$

When λ is defined as above, the point $\bar{x} + \lambda \nabla F$ where $\bar{x} \in \Pi(u)$, belongs to the surface $\Pi(u + \Delta u)$; here by $\Pi(u)$ we designate the surface defined by the equation $F = u$ in a wider open domain $\Omega' \supset \Omega$. For any open domain Ω' the surface $\Pi(u + \Delta u) \bigcap \Omega$ entirely lies in Ω' for all enough small values of $|\Delta u|$. The mapping $\Psi : \Pi(u) \to \Omega'$ defined as $\Psi(\bar{x}) = \bar{x} + \lambda \nabla F$ is one-one mapping when $|\Delta u|$ is sufficiently small. Really,

$$\Psi(\bar{x}) = \bar{x} + \left(\frac{\Delta u}{\|\nabla F\|^2} + o(|\Delta u|) \right) \nabla F = \bar{x} + \Delta u \frac{\Delta F}{\|\nabla F\|^2} + o(|\Delta u|),$$

and at sufficiently small $|\Delta u|$ the Jacoby matrix of this mapping can be represented as a sum of identity matrix and a Jacoby matrix of the mapping

$$\bar{x} \mapsto \Psi(\bar{x}) - \bar{x}.$$

Note that when u and Δu are fixed then we have $\Psi(\bar{x}) - \bar{x} = \lambda(\bar{x})\nabla F(\bar{x})$, and we can define partial derivatives of the function $\lambda(\bar{x})$ from the identity

$$F(\bar{x} + \lambda(\bar{x})\nabla F(\bar{x})) - F(\bar{x}) = \Delta u.$$

If we take partial derivatives both sides of this identity with respect to the variables of $\bar{x}$ then we get the system of linear equations from which we can define required partial derivatives. Since the domain is closed and the matrix $A_1(\bar{x})$ (see (4)) is not degenerating, then as it follows from Cramer's Rule all of obtained partial derivatives will be bounded. So, at sufficiently small values of Δu, determinant of the Jacoby matrix of the mapping $\bar{x} \mapsto \Psi(\bar{x})$ tends to 1 as $\Delta u \to 0$, i.e. this determinant will be distinct from zero everywhere in considered domain. So, Ψ is a bijective mapping for sufficiently small $|\Delta u|$.

We put: $D(u) = \{\bar{x} \in \Omega | F(\bar{x}) = u\}$. Then, the surface $D(u + \Delta u)$ tends to $D(u)$ as $\Delta u \to 0$ (pointwisely and uniformly). $\Psi(D(u))$ is a closed subset of $D(u + \Delta u)$. Further, a prototype $D(u + \Delta u)$ of the same mapping we will designate as $D'(u + \Delta u)$. Then, we have:

$$\varphi(u + \Delta u) - \varphi(u) = \int_{D'(u+\Delta u)\bigcap D(u)} \left(\frac{g(\bar{x} + \lambda\nabla F)}{\|\nabla F(\bar{x} + \lambda\nabla F)\|} - \frac{g(\bar{x})}{\|\nabla F(\bar{x})\|} \right) ds +$$

$$+ \int_{D(u+\Delta u)\setminus\Psi(D(u))} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|} - \int_{D(u)\setminus D'(u+\Delta u)} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|}. \tag{7}$$

Substituting the value found above for an increment, we find for the first surface integral the following expression:

$$-\Delta u(1 + o(1)) \int_{F(\bar{x})=u} \frac{\left(\nabla g \cdot \nabla F - g(A_1\bar{\nabla}, \bar{\nabla})\right)}{\|F\|^3} ds.$$

Consider now two remained surface integrals on the right hand side of the equality (6). They will be transformed by one and the same way. The first integral is taken over the surface $D(u + \Delta u)\setminus\Psi(D(u))$ which is included between the boundaries $D(u + \Delta u)$ and $\Psi(D(u))$. It is clear that this piece narrowing, will be pulled off along $n-2$-dimensional surface of an intersection $D(u+\Delta u) \bigcap \partial\Omega$, which tends to the limiting position $D(u) \bigcap \partial\Omega$ (it may be empty), as $\Delta u \to 0$.

Let's denote ω' an $n-1$-dimensional domain being a projection of the $D(u+\Delta u)\setminus\Psi(D(u))$ (we will use designation ψ' instead of ψ for the solution of the equation $F(\bar{x}) = u + \Delta u$). Dissect now the projection of the boundary $D(u + \Delta u) \bigcap \partial\Omega$ into the small parts $E_i, i = 1, ..., N$ with the maximal diameter not exceeding Δu. Now taking any point $(\psi'(\bar{\xi_i}), \bar{\xi_i})$ on E_i draw the ray lying on the tangential hyper plane, being orthogonal to the boundary $D(u + \Delta u) \bigcap \partial\Omega$ and intersecting the last at this point. The set of all such rays set up a surface. We restrict this surface by a such way that the projection of the got piece of the surface was coincide with ω'. This surface, consisted of pieces set up by all restricted rays with top points at E_i. The piece corresponding E_i we denote as $F_i = F_i(u, \Delta u)$. They set up something like a tiled covering for the surface $D(u+\Delta u)\backslash\Psi(D(u))$, area of which differs from the area of the surface $D(u+\Delta u)\backslash\Psi(D(u))$ by a value $o(\Delta u)$. Let $\bar{\xi}_i \in E_i$ be any point, ρ_i be a vector lying on the constructed tangential space to the surface $F = u + \Delta u$ at the point $(\psi'(\bar{\xi}_i), \bar{\xi}_i)$, orthogonal to $D(u+\Delta u) \bigcap \partial\Omega$, and with the endpoint at $\bar{\eta}_i$ of the boundary of corresponding piece $F_i = F_i(u, \Delta u)$. For small Δu we have: $|F_i| = |E_i|\, h_i$ (here $|E_i|$ expresses $n-2$-dimensional volume of E_i), and $h_i = |\rho_i|\,(1 + o(1))$, i.e. h_i plays a role of height of F_i which approximately we take as a cylindroid with the base $\Delta_i = \{(\psi'(\bar{\xi}_i), \bar{\xi}_i) | \bar{\xi}_i \in E_i)\}$ (with an error of order $o(\Delta u)$ for $n-2$-dimensional volume). Then, we have:

$$\int_{D(u+\Delta u)\backslash\Psi(D(\bar{u}))} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|} = \sum_{j=1}^{N} \int_{(\Delta_i)} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|}(1 + o(1)).$$

Intersection of tangential hyper planes, respectively, to $\partial\Omega$ and $D(u+\Delta u)$ at the point $(\psi'(\bar{\xi}_i), \bar{\xi}_i)$ is a tangential $n-2$- dimensional subspace to $D(u+\Delta u) \bigcap \partial\Omega$ at the same point. Let's consider three points: a point $P_i = (\psi'(\bar{\xi}_i), \bar{\xi}_i)$, a point $\bar{\eta}_i$ and a point $\Psi^{-1}(\eta_i)$. Let α_i be an angle between an external normal vector $\bar{n}$ to the boundary of Ω and a gradient ∇F. An angle between the segment $[\bar{\eta}_i, P_i]$ and the gradient ∇F, at small ∇u, differs from the angle α_i by a value $o(\Delta u)$ (or their sum is close to π). From a rectangular triangle we receive (the told above segment $[\Psi^{-1}(\eta_i), P_i]$ is here an hypotenuse):

$$\bar{h}_i = |\lambda| \cdot \|\nabla F\|\, ctg\alpha_i(1 + o(1)) = \frac{\Delta u}{\|\nabla F\|} ctg\alpha_i(1 + o(1)).$$

As $\cos\alpha_i = \bar{n} \cdot \nabla F, ctg\alpha_i = \bar{n} \cdot \nabla F / \sqrt{1 - (\bar{n} \cdot \nabla F)^2}$, then we have:

$$\int_{D(u+\Delta u)\backslash\Psi(D(u))} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|} = \sum_{j=1}^{N} \int_{(\Delta_i)} \frac{g(\bar{x})ds}{\|\nabla F(\bar{x})\|}(1 + o(1)) =$$

$$= \sum_{j=1}^{N} \Delta u \int_{(\Delta_i)} \frac{g(\bar{x})ctg\alpha_i d\sigma}{\|\nabla F(\bar{x})\|^2}(1 + o(1)) = \Delta u(1 + o(1)) \int_Z \frac{g(\bar{x})\bar{\nabla} \cdot \bar{n}}{\sqrt{1 - (\bar{\nabla} \cdot \bar{n})^2}} \frac{d\sigma}{\|\nabla F\|^2},$$

where $d\sigma$ designates $n-2$-dimensional element of the volume, and Z denotes an intersection of surfaces $F = u$ and $\partial\Omega$ (it can consist of several pieces). The similar formula is true for the third surface integral in (6). Therefore, from the formula (6) one can derive:

$$\varphi'(u) = \lim_{\Delta u \to 0} \frac{\varphi(u + \Delta u) - \varphi(u)}{\Delta u} = -\int_{F(\bar{x})=u} \frac{\left(\nabla g \cdot \nabla F - g(A_1\bar{\nabla}, \bar{\nabla})\right)}{\|F\|^3} ds +$$

$$+ \int_Z \frac{g(\bar{x})\bar{\nabla} \cdot \bar{n}}{\sqrt{1 - (\bar{\nabla} \cdot \bar{n})^2}} \frac{d\sigma}{\|\nabla F\|^2}, \tag{8}$$

and the sign before the integral is counted by the scalar product $\bar{\nabla} \cdot \bar{n}$.

To apply the Stokes formula ([5, p. 645], [16, p. 261]) to the second integral at the right side of (8), we note that the boundary Z is defined by the system of equations of a view $F = u$, $H = c$. Gram determinant of the functions standing at the left sides of the equations is non-zero. By this reason surface integral is possible to represent as below:

$$\int_Z \frac{g(\bar{x})\bar{\nabla} \cdot \bar{n}}{\sqrt{1 - (\bar{\nabla} \cdot \bar{n})^2}} \frac{d\sigma}{\|\nabla F\|^2} = \int_{\partial D(u)} \frac{g(\bar{x})\bar{\nabla} \cdot \bar{n}}{\sqrt{1 - (\bar{\nabla} \cdot \bar{n})^2}} \frac{\sqrt{G_0}}{|J_0|} \frac{d\xi_3 \cdots d\xi_n}{\|\nabla F\|^2},$$

and the variables $\xi_3, ..., \xi_n$ denote independent variables after of suitable solution of the considered system, say, with respect to the first two variables. So, we get integral of a differential form:

$$\eta = W d\xi_3 \wedge \cdots \wedge d\xi_n; \; W = \frac{g(\bar{x})\bar{\nabla} \cdot \bar{n}}{\sqrt{1 - (\bar{\nabla} \cdot \bar{n})^2}} \frac{\sqrt{G_0}}{|J_0|} \frac{1}{\|\nabla F\|^2}$$

and G_0 is a Gram determinant of considered functions F, H, J_0 is a determinant

$$J_0 = \begin{vmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \\ \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} \end{vmatrix}.$$

Now we have

$$d\eta = \left(\frac{\partial W}{\partial x_1} dx_1 + \frac{\partial W}{\partial x_2} dx_2 + \cdots + \frac{\partial W}{\partial x_n} dx_n \right) \wedge d\xi_3 \wedge \cdots \wedge d\xi_n.$$

Further at the surface $F = u$, after of solving this equation, the variable x_1 stands a function of independent variables $\xi_2, ..., \xi_n$ (we suppose that this is possible, not breaking a generality). Then,

$$d\eta = \left(\frac{\partial W}{\partial x_1} dx_1 + \frac{\partial W}{\partial x_2} dx_2 + \cdots + \frac{\partial W}{\partial x_n} dx_n \right) \wedge d\xi_3 \wedge \cdots \wedge d\xi_n =$$

$$d\eta = \left(\frac{\partial W}{\partial x_1} \left(\frac{\partial x_1}{\partial \xi_2} d\xi_2 + \cdots + \frac{\partial x_1}{\partial \xi_n} d\xi_n \right) + \frac{\partial W}{\partial x_2} dx_2 + \cdots + \frac{\partial W}{\partial x_n} dx_n \right)$$
$$\wedge d\xi_3 \wedge \cdots \wedge d\xi_n =$$
$$= \left(\frac{\partial W}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial W}{\partial x_2} \right) d\xi_2 \wedge d\xi_3 \wedge \cdots \wedge d\xi_n + \cdots +$$
$$+ \left(\frac{\partial x_1}{\partial \xi_n} \frac{\partial W}{\partial x_1} + \frac{\partial W}{\partial x_n} \right) d\xi_n \wedge d\xi_3 \wedge \cdots \wedge d\xi_n =$$
$$= \left(\frac{\partial W}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial W}{\partial x_2} \right) d\xi_2 \wedge d\xi_3 \wedge \cdots \wedge d\xi_n.$$

Now in consent with the Stokes formula (see [12, p. 261]):

$$\int_{\partial D(u)} \eta = \int_{D(u)} d\eta.$$

It is obviously, that right hand side of this relation is possible to represent as a surface integral taken over the surface $F = u$ after of multiplying and dividing by a positive element of area. Then, from (8) we derive:

$$\varphi'(u) = \int_{F(\bar{x})=0} G_1(\bar{x})ds, \tag{9}$$

where

$$G_1(\bar{x}) = \frac{\partial F/\partial x_1}{\|\nabla F\|}\left(\frac{\partial W}{\partial x_1}\frac{\partial x_1}{\partial \xi_2} + \frac{\partial W}{\partial x_2}\right) - \frac{\left(\nabla g \cdot \nabla F - g(A_1\bar{\nabla}, \bar{\nabla})\right)}{\|F\|^3}.$$

It is clear that the function G_1 is an algebraic function in Ω. Now, let's dissect the domain Ω into a finite number of such subdomains Ω_i in every of which the function G_1 keeps own sign invariable. Then, the integral (8) splits into the sum of several surface integrals:

$$\varphi'(u) = \sum \varphi_i'(u),\ \varphi_i'(u) = \int_{\Omega_i, F(\bar{x})=u} G(\bar{x})ds \tag{10}$$

(notice that when we consider the sum of the integrals $\int_{S\subset Z}$ taken on the different sides of the piece S of a surface, the normal vector $\bar{n}$ changes the sign, and consequently, such a sum is equal to zero); the number of domains on the right part of (9) depends on Ω and a degree of the polynomial F. Let's designate, in the consent with (9)

$$\varphi(u) = \sum \varphi_i(u), \varphi_i(u) = \int_{\Omega_i, F(\bar{x})=u} \frac{g(\bar{x})ds}{\|\nabla F\|}.$$

Thus, the equality $\phi'(u) = \sum_i \phi_i'(u) = \sum_i \int_{\Omega_i, F=u} G(\bar{x})\, ds$ is true. Since the function under the surface integral does not change its sign, the function is a monotone function. The lemma 6 is proved.

Lemma 7. *Let Ω be a bounded closed domain of n-dimensional space $\mathbb{R}^n$, $n > 1$. Let's assume that in Ω some r-dimensional surface be given by means of a system of equations*

$$f_j(\bar{x}) = 0, j = 1, ..., n-r, 0 \leq r \leq n,$$

with a Jacoby matrix

$$J = J(\bar{x}) = \|\frac{\partial f_j}{\partial x_i}\|, i = 1, ..., n, j = 1, ..., n-r$$

which has, everywhere in Ω, a maximal rank and smooth entries. Let, further a mapping $\bar{\xi} \mapsto \bar{x}$ maps some domain $\Omega' \subset \mathbb{R}$ into Ω with non-degenerating in Ω' Jacoby matrix

$$Q = Q(\bar{\xi}) = \|\frac{\partial f_j}{\partial xi}\|$$

with continuous entries. Then for any continuous in the Ω function $f(\bar{x})$ the formula

$$\int_M f(\bar{x})\frac{ds}{\sqrt{G}} = \int_{M'} \mid \det Q \mid f(\bar{x}(\bar{\xi}))\frac{d\sigma}{\sqrt{G'}}, G' = \det(JQ \cdot Q^t J^t)$$

holds; here M' denotes a pre-image of the piece of the surface on given surface, $d\sigma$ designates the surface element in coordinates $\bar{\xi}$.

Proof of this lemma is given in [11, p.92].

3. Basic results

Consider now the integral (3):

$$\int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds.$$

Our goal is proving following theorems concerning estimations of surface trigonometric integrals. Let's denote

$$H = \max_{\bar{x}\in\Omega} \|\nabla F\|, g_0 = \max_{\bar{x}\in\Omega} |g(\bar{x})|.$$

Designate by G_{k-2} and $\tilde{G}_{k-2}$ a minimal value of the product of, respectively, $n-1$ and $n-2$ least singular numbers of the matrix A_{k-2}.

Theorem 1. *If $k > 2$ then there exist a positive constant $c_0 = c_0(r, k, \deg F)$ such that*

$$\left| \int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \leq c_0 g_0 \max\left(G_1^{-1}, H^{(n-1)/(k-1)} G_{k-2}^{-1/(k-1)} \cdot \mathrm{Q}_{k-2}^{n-1} \right);$$

$$\mathrm{Q}_{k-2} = \log \tilde{H};\ \tilde{H} = \max\left\{ h(H), h(G_{(1)}), ..., h(G_{(k-2)}), h(L) \right\}.$$

Theorem 2. *Suppose that the Jacoby matrix Λ_0 of the system of functions $f(\bar{x})$, $F(\bar{x})$ has a rank 2. If $k > 2$ and $n \geq 3$ then there exist a positive constant $c_1 = c_1(r, k, \deg F)$ such that*

$$\left| \int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \leq c_1 g_0 \max\left(\tilde{G}_1^{-1}, H^{(n-3)/(k-1)} \tilde{G}_{k-2}^{-1/(k-1)} \wp_{k-2}^{n-2} \right);$$

$$\tilde{\wp}_{k-2} = \log \tilde{H};\ \tilde{H} = \max\left\{ h\left(H\right), h\left(\tilde{G}_{(1)}\right), ..., h\left(\tilde{G}_{(k-2)}\right), h\left(L\right) \right\},$$

Note. When $k = 2$ estimations of these theorems remains valid if to take the first expression in the sign of maximum.

Proofs of the theorems. Using the formula of the lemma 1 of the work [13] we can represent the integral

$$\int_{\Pi} g(\bar{x}) e^{2\pi i F(\bar{x})} ds$$

as a limit

$$\lim_{h\to 0} \frac{1}{2h} \int_{|f(\bar{x})|\leq h, \bar{x}\in\Omega} g(\bar{x}) \|\nabla f\| e^{2\pi i F(\bar{x})} d\bar{x}. \tag{11}$$

For every $h > 0$ the condition $|f(\bar{x})| \leq h$ defines some closed subdomain in Ω. We suppose, in agree with the lemma 6 above, that in the considered domain the surface integral

$$\int_{F(\bar{x})=u} \frac{\|\nabla f\| g(\bar{x}) ds}{\|\nabla F\|}$$

is a monotone function of u. We can apply the reasonings of the work [13] to transform the integral under the limit (11) as follows

$$\int_{|f(\bar{x})|\le h, \bar{x}\in\Omega} g(\bar{x}) \|\nabla f\| e^{2\pi i F(\bar{x})} d\bar{x} = \int_m^M e^{2\pi i u} \left(\int_{|f(\bar{x})|\le h, F(\bar{x})=u} \frac{\|\nabla f\| g(\bar{x}) ds}{\|\nabla F\|} \right) du.$$

So, we have:

$$\int_\Pi g(\bar{x}) e^{2\pi i F(\bar{x})} ds = \lim_{h\to 0} \frac{1}{2h} \times$$

$$\times \int_m^M \left(\int_{F(\bar{x})=u, |f(\bar{x})|\le h} \frac{\|\nabla f\| g(\bar{x}) ds}{\|\nabla F\|} \right) (\cos 2\pi u + i \sin 2\pi u)\, du.$$

Applying of the lemma 3, [13] allows us to pass to the limit under the sign of integration. Then we get:

$$\int_\Pi g(\bar{x}) e^{2\pi i F(\bar{x})} ds = \int_m^M (\cos 2\pi u + i \sin 2\pi u) \times$$

$$\times \lim_{h\to 0} \frac{1}{2h} \left(\int_{F(\bar{x})=u, |f(\bar{x})|\le h} \frac{\|\nabla f\| g(\bar{x}) ds}{\|\nabla F\|} \right) du.$$

Using the known method of estimation of this integral (see [2]), one may get a following bound

$$\left| \int_\Pi g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \le 2 \max_u \left| \lim_{h\to 0} \frac{1}{2h} \left(\int_{F(\bar{x})=u, |f(\bar{x})|\le h} \frac{\|\nabla f\| g(\bar{x}) ds}{\|\nabla F\|} \right) \right| \le$$

$$\le 2 g_0 \max_u \left(\int_{\Pi, F(\bar{x})=u} \frac{ds}{\|\nabla F\|} \right). \tag{12}$$

Assume that $K \le H = \max_{\bar{x}\in\Omega} \|\nabla F\|$. As the norm of the gradient is a square root of the polynomial $\|\nabla F\|^2$, then the subset of the domain Ω where $\|\nabla F\| = 0$, as a closed subset, is a Jourdan set with zero measure. Then writing $\Omega' = \{\bar{x} \in \Omega | \, \|\nabla F\| > 0\}$ we find

$$\left| \int_{\Pi \bigcap \Omega} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| = \left| \int_{\Pi \bigcap \Omega'} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| =$$

$$= \sum_{j=1}^{\infty} \lim_{h\to 0} \frac{1}{2h} \left| \int_{|f(\bar{x})|\le h, \bar{x}\in\Omega^{(j)}} g(\bar{x}) \|\nabla f\| e^{2\pi i F(\bar{x})} d\bar{x} \right|; \tag{13}$$

here the subdomains $\Omega^{(j)}$ defined as below

$$\Omega^{(j)} = \{\bar{x} \in \Omega | 2^{-j} K \le \|\nabla F\| \le 2^{1-j} K\}.$$

To estimate the integral over $\Omega^{(j)}$ firstly let's make change of variables $\Phi : \bar{x} \mapsto \nabla F(\bar{x})$:

$$u_1 = \frac{\partial F}{\partial x_1}, ..., u_r = \frac{\partial F}{\partial x_r}.$$

Then we have:

$$\left| \int_{\Pi \bigcap \Omega^{(j)}} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| =$$

$$= \lim_{h \to 0} \frac{1}{2h} \left| \int_{\substack{\left|f(\Phi^{-1}(\bar{u}))\right| \leq h \\ 2^{-j} K \leq \|\bar{u}\| \leq 2^{1-j} K}} g(\Phi^{-1}(\bar{u})) \|\nabla f\| e^{2\pi i F(\Phi^{-1}(\bar{u}))} (\det A_1)^{-1} d\bar{u} \right| \leq$$

$$\leq \lim_{h \to 0} \frac{1}{2h} \left| \int_{\substack{f(\Phi^{-1}(\bar{u})) = 0 \\ 2^{-j} K \leq \|\bar{u}\| \leq 2^{1-j} K}} g(\Phi^{-1}(\bar{u})) \|\nabla f\| (\det A_1)^{-1} d\bar{u} \right| =$$

$$= \int_{f(\Phi^{-1}(\bar{u}))=0, 2^{-j}K \leq \|\bar{u}\| \leq 2^{1-j}K} \frac{\|\nabla f\| g(\Phi^{-1}(\bar{u}))(\det A_1)^{-1} ds}{\left\|A_1^{-1}(\nabla f)\right\|} \leq$$

$$\leq g_0 R \int_{f(\Phi^{-1}(\bar{u}))=0, 2^{-j}K \leq \|\bar{u}\| \leq 2^{1-j}K} ds; \tag{14}$$

here

$$R = \max_{\bar{x} \in \Omega} \frac{\|\nabla f\| (\det A_1)^{-1}}{\left\|A_1^{-1}(\nabla f)\right\|}.$$

It is easy to note that

$$\left\|A_1^{-1}(\nabla f)\right\| \geq \lambda_1^{-1} \|\nabla f\|,$$

where λ_1 is a maximal singular number of the matrix A_1. Then we realize that

$$R \leq G_1^{-1},$$

and G_1 is a minimal value of the product of all singular numbers of the matrix A_1, with exception of λ_1.

Consider now the surface integral at last chain of (14). The algebraic equation

$$f(x_1, x_2, ..., x_n) = 0$$

has a set of solutions consisted of finite number of connected hypersurfaces (see [12]) of a view $x_1 = \varphi(x_2, ..., x_n)$. This connected sets will be mapped one-valudely to connected $n-1$ dimensional manifolds of a view $\bar{u} = \Phi(\bar{x}) = (\varphi_1(\bar{x}), ..., \varphi_n(\bar{x}))$ with

$$\varphi_i(\bar{x}) = \frac{\partial F}{\partial x_i}(\varphi(x_2, ..., x_n), x_2, ..., x_n)$$

Then these manifolds are defined by the equation

$$f(\Phi^{-1}(\bar{u})) = 0. \tag{15}$$

From the compactness it follows that the set of solutions of this equation decomposes into n subsets every of which is a finite union of open simple connected components. In every component partial derivatives of the left hand side of the equation (15) takes maximal absolute values with respect to one of the variables $u_1, u_2, ..., u_n$. Since the mapping Φ is one to one mapping then all of open components is possible to include into one subset. Then, surface integral splits into the union of n integrals of following view:

$$\int_{2^{-j}K \le \|\bar{u}\| \le 2^{1-j}K} du_1 ... du_{n-1} \le c_0 \left(2^{1-j}K\right)^{n-1}.$$

So, summing this estimation for all $j = 1, 2, ...$, we get the estimation

$$\left| \int_{\Pi \bigcap \Omega} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \le 4 c_0 g_0 K^{n-1} G_1^{-1}. \tag{16}$$

Taking some parameter $T > 0$ we estimate the part of the integral over the subset $\Pi \bigcap \Omega_1$, where $G_1 \ge T$, as below

$$\left| \int_{\Pi \bigcap \Omega_1} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \le 4 c_0 g_0 K^{n-1} T^{-1}.$$

The integral over remaining part of the surface where $G_1 \le T$ we estimate applying the lemma 2 as follows:

$$\left| \int_{\Pi \bigcap \Omega_1} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| << T^{1/k-2} \cdot G_{k-2}^{-1/(k-2)} \cdot \tilde{Q}_{k-2}^{n-1}.$$

Define now the parameter T from the equality

$$K^{n-1} T^{-1} = T^{1/(k-2)} G_{k-2}^{-1/(k-2)} \Rightarrow T = K^{\frac{(k-2)(n-1)}{k-1}} G_{k-2}^{1/(k-1)}.$$

Then we find:

$$\int_{\Pi \bigcap \Omega} g(\bar{x}) e^{2\pi i F(\bar{x})} ds << K^{\frac{n-1}{k-1}} G_{k-2}^{-1/(k-1)} \cdot \tilde{Q}_{k-2}^{n-1}.$$

Theorem 1 is proven.

Consider now the estimation of the integral under the limit (11) by another method. We have

$$\left| \int_{\Pi \bigcap \Omega^{(j)}} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \le 2 g_0 \max_u \left(\int_{\Pi \bigcap \Omega^{(j)}, F(\bar{x}) = u} \frac{ds}{\|\nabla F\|} \right) \le$$

$$\le 2 g_0 K^{-1} \max_u \left(\int_{\Pi \bigcap \Omega^{(j)}, F(\bar{x}) = u} ds \right). \tag{17}$$

Now we apply the lemma 7, and make change of variables $u_1 = \frac{\partial F}{\partial x_1}, ..., u_r = \frac{\partial F}{\partial x_r}$. Then this surface will be transformed into the surface defined by the system of equations

$$f(\Phi^{-1}(\bar{u})) = 0,\ F(\Phi^{-1}(\bar{u})) = 0. \tag{18}$$

By the conditions of the theorem the Jacoby matrix Λ_0 of the system of functions $f(\bar{x})$, $F(\bar{x})$ has a rank 2. Applying the lemma 7, we get

$$\int_{\Pi \bigcap \Omega^{(j)}, F(\bar{x})=u} ds \leq \int_{2^{-j}K \leq \|\nabla F\| \leq 2^{1-j}K} 1\times$$

$$\times \frac{\sqrt{\det(\Lambda_0 \cdot \Lambda_0^t)}}{|\det A_1| \sqrt{\left|\det(\Lambda_0 A_1^{-1} \cdot (A_1^t)^{-1} \Lambda_0^t)\right|}} d\sigma; \tag{19}$$

here $d\sigma$ is an surface element at the transformed surface (18), and the sign ^{t} over the matrix means a transposition. Consider square root of the determinant at the denominator of the expression under integral. There is an integral representation (see [13, p. 131) for it:

$$\frac{1}{\sqrt{\left|\det(\Lambda_0 A_1^{-1} \cdot (A_1^t)^{-1} \Lambda_0^t)\right|}} =$$

$$= \pi \iint_{\left\|(A_1^t)^{-1} \Lambda_0^t \begin{pmatrix} x \\ y \end{pmatrix}\right\| \leq 1} dxdy = \frac{\pi}{\sqrt{\det(\Lambda_0 \cdot \Lambda_0^t)}} \int_{\left\|(A_1^t)^{-1} \bar{u}\right\| \leq 1} ds;$$

here the last integral is a surface integral taken over the two-dimensional subspace of $\mathbb{R}^n$ which is a linear span of the gradient vectors of the functions $f(\bar{x})$, $F(\bar{x})$. If we substitute this surface integral by maximal its value taken over all two dimensional subspaces, we get, in accordance with the theorem 6, §11, ch. 7 (in the suitable form) of the book [6, p.148] (see also [14, 20]), exactly the product of inverted minimal singular numbers of the matrix A_1^{-1}, i. e. maximal singular numbers of the matrix A_1. So, the integral at the right hand side of the equality (19) can be represented as follows:

$$\int_{2^{-j}K \leq \sqrt{u_3^2 + \cdots + u_n^2} \leq 2^{1-j}K} \frac{d\sigma}{\Sigma_{n-2}(A_1)},$$

where $\Sigma_{n-2}(A_1)$ means the product of least $n-2$ singular numbers of the matrix A_1. Hence, we have the bound

$$\int_{2^{-j}K \leq \sqrt{u_3^2 + \cdots + u_n^2} \leq 2^{1-j}K} \frac{d\sigma}{\Sigma_{n-2}(A_1)} \leq$$

$$\leq C_n^2 \frac{\Gamma(1 + (n-2)/2)}{\pi^{(n-2)/2}} (2^{1-j}K)^{n-2} \tilde{G}_1^{-1} << K^{n-2} \tilde{G}_1^{-1}.$$

here $\tilde{G}_1 = \min\limits_{\bar{x}\in\Omega} \Sigma_{n-2}(A_1)$ denotes the minimal value of product of last $n-2$ (smallest) singular numbers of the matrix A_1. Therefore, we have

$$\left| \int_{\|\nabla F\| \le 2^{1-j}K} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \le 2^{n-1} \frac{\Gamma(1+(n-2)/2)}{\pi^{(n-2)/2}} g_0 K^{-1} (2^{1-j}K)^{n-2} \tilde{G}_1^{-1}.$$

Summarizing over all $j = 1, 2, ...,$ we obtain:

$$\left| \int_{\Pi \bigcap \Omega} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| \le C g_0 K^{n-3} \tilde{G}_1^{-1}; \tag{20}$$

$$C = 2^{2n} \frac{\Gamma(1+(n-2)/2)}{\pi^{(n-2)/2}}.$$

This estimation is got using constraints over the gradient and the matrix A_1. Applying the lemma 4 we can prove the estimation in the terms of high order derivatives. This lemma can be applied by following way. Denote by Ω_1 subdomain in Ω for all points of which the condition $\tilde{G}_1 \le T$ is satisfied. We have, in consent with the lemma 4, the bound

$$\mu\left(\Pi_H \bigcap \Omega_1\right) << T^{1/(k-2)} \tilde{G}_{k-2}^{-1/(k-2)} \wp_{k-2}^{n-2};$$

$$\tilde{\wp}_{k-2} = 3(n-2)^2 \log \tilde{H};\ \tilde{H} = \max\left\{h\left(H\right), h\left(G_1\right), ..., h\left(G_{k-2}\right), h\left(L\right)\right\}.$$

The value of the parameter T can be defined by the condition

$$K^{n-3} T^{-1} = T^{1/(k-2)} \tilde{G}_{k-2}^{-1/(k-2)}.$$

We have:

$$T = K^{\frac{(k-2)(n-3)}{k-1}} \tilde{G}_{k-2}^{1/(k-1)}.$$

So, we find when $n \ge 2$:

$$\left| \int_{\Pi \bigcap \Omega} g(\bar{x}) e^{2\pi i F(\bar{x})} ds \right| << K^{\frac{n-3}{k-1}} \tilde{G}_{k-2}^{-1/(k-1)} \wp_{k-2}^{n-2}. \tag{21}$$

Theorem 2 is now proven.

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Examples of the Discrete Additive Derivative of the Second-order Discrete Multiplicative Derivative

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Abstract. As is known, in a discrete analysis there are investigated either discrete processes or continuous processes with discrete analog. In a discrete process various properties of sequences are studied. They can be arithmetic, geometric progressions and the Fibonacci sequence. Determination of the general term of an arithmetic progression leads us to the Cauchy problem for first-order equation with discrete additive derivative; determining the general term of a geometric progression brings us again to the Cauchy problem for a first-order equation with discrete multiplicative derivatives. Finally, the definition of the general term leads us to the Fibonacci sequence for the Cauchy problem for second-order equations with discrete additive derivatives.The main objective of the discrete analysis is the discretization of mathematical models derived from the continuous analysis and study of the resulting discrete model. Multiplicative derivative and integrated, integral, compact and simple properties are given in three or four pages in [8]. Thus, the multiplicity properties were expected here rather than additivity. It is shown that "derivation of derivatives, derivatives" and "production of integers are integral". The distinctive (marking) of the integral belongs to us.

Key Words and Phrases: discrete additive analysis, discrete multiplicative analysis, additivo-multiplicative and multiplicative-additive equations.
2010 Mathematics Subject Classifications: 35J25, 35B45,42B20, 47B38

1. Introduction

The derivative taught in "Algebra and the beginning of analysis" in secondary school and "Mathematical Analysis" course in Higher School is mainly additive derivative [1, 2]. Although the multiplicative derivative has been created for around nearly a century [3], problems for the multiplicative derivative equations have been considered recently [4, 5]. Here we will talk about the discrete cases of these additive and multiplicative derivatives [6, 7, 8]. We began to look at the problems for ordinary discrete additivo-multiplicative and multiplicativo - additive derivative equations, [9, 10, 11]. It should be noted that the markings for discrete derivatives and integrals also belong to us [12].

Here we look at Cauchy and boundary value problems for a two-dimensional third-order equation (the second-order discrete additive relative to one variable, which holds a discrete multiplicative derivative relative to the other variable).

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2. Solution of the problem

Such third-order equations as

$$\left(y_n^{('')}\right)^{[']} = f_n, n \geq 0, \tag{1}$$

are considered. Here also f_n $n \geq 0$ is the given sequence and y_n $n \geq 0$ is the search sequence. Using the definitions of derivatives, we'll get

$$y_{n+3} = f_n \cdot \frac{y_{n+2}^2}{y_{n+1}} + y_n \frac{y_{n+2}^3}{y_{n+1}^3}, n \geq 0, \tag{2}$$

Here, emphasizing "n", it becomes possible to determine all y_ns beginning from y_3 with the help of y_0, y_1 and y_2 (dependence from the f_n is also available). But it is impossible to give the analytical note for the general solving of (2).

3. The solution is solved by integration

That is why, returning to the (1) and using the discrete additive derivative:

$$y_{n+1}^{[']} - y_n^{[']} = f_n, n \geq 0, \tag{3}$$

Here, dropping out "n":

$$\begin{gathered} y_1^{['']} - y_0^{['']} = f_0, \\ y_2^{['']} - y_1^{['']} = f_1, \\ \vdots \\ y_n^{['']} - y_{n-1}^{['']} = f_{n-1}, \end{gathered}$$

Adding these expressions, we'll get :

$$y_n^{['']} - y_0^{['']} = \sum_{k=0}^{n-1} f_k,$$

or

$$y_n^{['']} = y_0^{['']} + \sum_{k=0}^{n-1} f_k, n \geq 1, \tag{4}$$

Here, using the designation

$$g_n = g_n\left(y_0^{['']}, f_k\right) = y_0^{['']} + \sum_{k=0}^{n-1} f_k, n \geq 1, \tag{5}$$

the equation will change to

$$y_n^{[']} = g_n, \ n \geq 1, \tag{6}$$

So, the given three-order equation (1)is brought to the two-order equation (6). Using the definition of the discrete multiplicative derivative in the last equation, we'll get:

$$\frac{y_{n+1}^{(\prime)}}{y_n^{(I)}} = g_n, \quad n \geq 1, \tag{7}$$

changing "n":

$$\frac{y_2^{[\prime]}}{y_1^{[\prime]}} = g_1,$$

$$\frac{y_3^{[\prime]}}{y_2^{[\prime]}} = g_2,$$

$$\vdots$$

$$\frac{y_{n-1}^{[\prime]}}{y_{n-2}^{[\prime]}} = g_{n-2},$$

$$\frac{y_n^{[\prime]}}{y_{n-1}^{[\prime]}} = g_{n-1},$$

Multiplying these expressions, we get

$$\frac{y_n^{[\prime]}}{y_1^{[\prime]}} = \prod_{s=1}^{n-1} g_s,$$

or

$$y_n^{[\prime]} = y_1^{[\prime]} \prod_{s=1}^{n-1} g_s, n \geq 2, \tag{8}$$

Here, just like in (3), using the designation

$$h_n = h_n\left(y_1^{[\prime]} g_s\right) = y_1^{[\prime]} \prod_{s=1}^{n-1} g_s, n \geq 2, \tag{9}$$

the equation (7) will change to

$$y_n^{[\prime]} = h_n, n \geq 2, \tag{10}$$

Applying the discrete additive derivative on this equation once more, we'll get

$$\frac{y_{n+1}}{y_n} = h_n, n \geq 2,$$

changing "n":

$$\frac{y_3}{y_2} = h_2,$$

$$\frac{y_4}{y_3} = h_3,$$

$$\vdots$$

$$\frac{y_{n-1}}{y_{n-2}} = h_{n-2},$$

$$\frac{y_n}{y_{n-1}} = h_{n-1},$$

Multiplying them, we get

$$\frac{y_n}{y_2} = \prod_{m=2}^{n-1} h_m,$$

or

$$y_n = y_2 \cdot \prod_{m=2}^{n-1} h_m, \quad n \geq 3, \tag{11}$$

So, we achieve:

Theorem 1. *If f_n $n \geq 0$ is the given valid elemental sequence, the equation* (1) *will have its solving and it is like* (10)*, so $h_n s$ are like in* (9) *and $g_n s$ are like in* (4)*, y_0'', y_1^I and y_2 are optional constants.*

4. Cauchy problem

If the initial conditions

$$y_k = \alpha_k, k = \overline{0,2}, \tag{12}$$

are added to the given third-order equation (1), then, because of

$$y_0^{[\prime\prime]} = \frac{y_0 y_2}{y_1^2} = \frac{\alpha_0 \alpha_2}{\alpha_1^2}, y_1^{[\prime]} = \frac{y_2}{y_1} = \frac{\alpha_2}{\alpha_1}, \tag{13}$$

(1), (10) solving of the Koshi example is defined from (10) as

$$y_n = \alpha_2 \cdot \prod_{m=2}^{n-1} h_m, \quad n \geq 3, \tag{14}$$

So, h_n-s and g_s-s are defined from (4) and (8), taking into consideration (12).

Theorem 2. *Under the terms of Theorem 1, if α_k, $k = \overline{0,2}$, the Koshi example has the only solving and this is given with the help of* (13)*, so $h_n s$ are defined with the help of* (8) *and g_s-s – with the help of* (8) *in the* (4)*.*

5. Boundary problem

Now, taking the $\overline{0, N-3}$ numbers of n in (1), let's see the border conditions of the equation:

$$y_0'' = \alpha, y_1' = \beta, y_N = \gamma, \tag{15}$$

Taking into consideration (14) in (4) and (8):

$$g_n = \alpha + \sum_{k=0}^{n-1} f_k, n \geq 1, \tag{16}$$

$$h_n = \beta \cdot \prod_{s=1}^{n-1} g_s, n \geq 2, \tag{17}$$

Designations of (15) and (16) define g_ns and h_n-s as equal, that is, there's no discretion.

Finally, taking into consideration the general solving of (10) in the third of (14) border conditions, we get

$$\gamma = y_N = y_2 \prod_{m=2}^{N-1} h_m,$$

and

$$y_2 = \frac{\gamma}{\prod_{m=2}^{N-1} h_m}, \tag{18}$$

The general solving of the border example is possible from (**??**) general solving with the help of (16)

$$y_n = \frac{\gamma}{\prod_{m=2}^{N-1} h_m} \cdot \prod_{m=2}^{n-1} h_m = \frac{\gamma}{\prod_{m=n}^{N-1} h_m}, \tag{19}$$

So, we get:

Theorem 3. *Under the terms of Theorem 1, if the given α, β and γ are the true given numbers, there's the only solving of the border example* (1) *and* (14), *and this solving is like* (18). *So, h_ns are given with the help of* (16) *and g_ns – with the help of* (15).

6. Results

Here, the third order presents the Cauchy and boundary problems for the equation with discrete nonlinear differences, and the analytical expressions for the solution of these problems are obtained.

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Asymptotic Behavior of the Distribution Function of the Ahlfors-Beurling Transform

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Abstract. In the present paper, we study the asymptotic behavior of the distribution function of the Ahlfors-Beurling transform of a Lebesgue integrable function as $\lambda \to +\infty$ and as $\lambda \to 0+$.

Key Words and Phrases: Ahlfors–Beurling transform, distribution function, asymptotic behavior.

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1. Introduction

The Ahlfors-Beurling transform of a function $f \in L_p(C)$, $1 \le p < \infty$ is defined as the following singular integral:

$$(Bf)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\{w \in C:\, |z-w| > \varepsilon\}} \frac{f(w)}{(z-w)^2} dm(w).$$

The Ahlfors–Beurling transform is one of the important operators in complex analysis. It is the “Hilbert transform” on complex plane. It has been shown in [1,3,6,9,11,15,17] that this transform plays an essential role in applications to the theory of quasiconformal mappings and to the Beltrami equation with discontinuous coefficients.

From the theory of singular integrals (see [13]) it is known that the Ahlfors–Beurling transform is a bounded operator in the space $L_p(C)$, $1 < p < \infty$, that is, if $f \in L_p(C)$, then $B(f) \in L_p(C)$ and

$$\|Bf\|_{L_p} \le C_p \|f\|_{L_p}. \tag{1}$$

In the case $f \in L_1(C)$ only the weak inequality holds,

$$m\{z \in C: \ |(Bf)(z)| > \lambda\} \le \frac{C_1}{\lambda} \|f\|_{L_1}, \tag{2}$$

where m stands for the Lebesgue measure, C_p, C_1 are constants independent of f and $m\{z \in C: \ |(Bf)(z)| > \lambda\}$ - the distribution function of the Ahlfors-Beurling transform of the function f.

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In [2,4,5,7,8,10,12,14,16] the boundedness of the operator B in other function spaces (in the spaces of Sobolev, Besov, Campanato, Morrey, etc.) was studied.

In the present paper, we study the asymptotic behavior of the distribution function of the Ahlfors-Beurling transform of a Lebesgue integrable function as $\lambda \to +\infty$ and as $\lambda \to 0+$.

2. Asymptotic behavior of the distribution function of the Ahlfors-Beurling transform as $\lambda \to +\infty$

In this section we studying the asymptotic behavior of the distribution function of the Ahlfors-Beurling transform as $\lambda \to +\infty$.

Theorem 1. *Let $f \in L_1(C)$. Then the equation*

$$\lim_{\lambda \to +\infty} \lambda m \{z \in C : \ |(Bf)(z)| > \lambda\} = 0 \tag{3}$$

holds.

Proof. Since $f \in L_1(C)$, then for every $\varepsilon > 0$ there exists $n \in N$ and $R > 0$ such that

$$\|f - [f]_R^n\|_{L_1} \le \frac{\varepsilon}{4C_1}, \tag{4}$$

where $[f]_R^n(z) = [f]^n \chi(B(0;R))(z)$, $[f(z)]^n = f(z)$ for $|f(z)| \le n$, $[f(z)]^n = 0$ for $|f(z)| > n$, $\chi(B(0;R))(z)$- characteristic function of the circle $B(0;R) = \{z \in C : \ |z| < R\}$. It follows from (1) and (4) that for every $\lambda > 0$ the inequality

$$m\left\{z \in C : \ |B(f - [f]_R^n)(z)| > \frac{\lambda}{2}\right\} \le \frac{2C_1}{\lambda} \|f - [f]_R^n\|_{L_1} \le \frac{\varepsilon}{2\lambda} \tag{5}$$

holds. Since the function $[f]_R^n(z)$ is bounded, then we get that $[f]_R^n \in L_p(C)$ for each $p \ge 1$. It follows that $B[f]_R^n \in L_p(C)$ for each $p > 1$. Denote

$$F_1(z) = B[f]_R^n(z) \cdot \chi(B(0;\, 2R)), F_2(z) = B[f]_R^n(z) \cdot \chi(C \backslash B(0;\, 2R)).$$

Then

$$B[f]_R^n(z) = F_1(z) + F_2(z),$$

The function $F_1(z)$ is concentrated on the closed circle $\overline{B(0;\, 2R)}$, and the function $F_2(z)$ is concentrated on the set $C \backslash B(0;\, 2R)$. For every $p > 1$ from the inclusion $B[f]_R^n \in L_p(C)$ it follows that $F_1(z) \in L_p(C)$. Since the function $F_1(z)$ is concentrated on the bounded set, then we have that $F_1(z) \in L_1(C)$. Then for sufficiently large values of $\lambda > 0$

$$\frac{\lambda}{2} m\{z \in C : \ |F_1(z)| > \lambda/2\} \le \int_{\{z \in C: \ |F_1(z)| > \lambda/2\}} |F_1(z)|\, dm(z) < \frac{\varepsilon}{4}. \tag{6}$$

On the other hand, for any $z \in C\backslash B\,(0;\,2r)$ we have

$$|B\,([f]_R^n)\,(z)| = \frac{1}{\pi}\int_{B(0;\,R)} \frac{|[f]_R^n\,(w)|}{|z-w|^2} dm\,(w) \le$$

$$\le \frac{1}{\pi R^2}\int_{B(0;\,R)} |[f]_R^n\,(w)|\,dm\,(w) = \frac{1}{\pi R^2}\,\|[f]_R^n\|_{L_1} \le \frac{1}{\pi R^2}\,\|f\|_{L_1}\,.$$

This shows that the function $F_2\,(z)$ is bounded. Then it follows from (6) that for sufficiently large values of $\lambda > 0$

$$m\,\{z \in C:\ |B\,[f]_R^n\,(z)| > \lambda/2\} \le m\,\{z \in C:\ |F_1\,(z)| > \lambda/2\} < \frac{\varepsilon}{2\lambda}. \tag{7}$$

It follows from (5) and (7) that for sufficiently large values of $\lambda > 0$

$$m\,\{z \in C:\ |(Bf)\,(z)| > \lambda/2\} \le$$

$$\le m\,\{z \in C:\ |B\,[f]_R^n\,(z)| > \lambda/2\} + m\left\{z \in C:\ |B\,(f-[f]_R^n)\,(z)| > \frac{\lambda}{2}\right\} < \frac{\varepsilon}{2\lambda} + \frac{\varepsilon}{2\lambda} = \frac{\varepsilon}{\lambda}.$$

This shows that the equation (3) holds. Theorem 1 is proved. ◀

3. Asymptotic behavior of the distribution function of the Ahlfors-Beurling transform as $\lambda \to 0+$

In this section we studying the asymptotic behavior of the distribution function of the Ahlfors-Beurling transform as $\lambda \to 0+$.

Theorem 2. *Let $f \in L_1\,(C)$. Then the equation*

$$\lim_{\lambda\to 0+} \lambda m\,\{z \in C:\ |(Bf)\,(z)| > \lambda\} = \left|\int_C f\,(z)\,dm\,(z)\right| \tag{8}$$

holds.

At first we prove the auxiliary lemma.

Lemma 1. *If $f \in L_1\,(C)$ and $\int_C f\,(z)\,dm\,(z) = 0$, then the equation*

$$m\,\{z \in C:\ |(Bf)\,(z)| > \lambda\} = o\,(1/\lambda)\,, \lambda \to 0+ \tag{9}$$

holds.

Proof of Lemma 1. At first assume that the function f is concentrated on some circle $B\,(0;\,R) \subset C$. In this case, from the equality

$$(Bf)\,(z) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\{w\in B(0;R)\,:\,|z-w|>\varepsilon\}} \frac{f\,(w)}{(z-w)^2} dm\,(w) =$$

$$= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\{w\in B(0;\,R)\,:\,|z-w|>\varepsilon\}}\frac{f(w)}{(z-w)^2}dm(w)+\frac{1}{\pi}\int_{B(0;\,R)}\frac{f(w)}{(z-z_0)^2}dm(w)=$$

$$=\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\{w\in B(0;\,R)\,:\,|z-w|>\varepsilon\}}(z_0-w)\times$$

$$\times\left[\frac{1}{(z-w)^2(z-z_0)}+\frac{1}{(z-w)(z-z_0)^2}\right]f(w)\,dm(w)\,,z\neq z_0,$$

where $z_0\in C$, we get that

$$|(Bf)(z)|\leq\frac{16}{\pi|z|^3}\int_{B(0;\,R)}|z_0-w|\,|f(w)|\,dm(w)=\frac{k_0}{|z|^3},$$

for values of $|z|>R_0$, where

$$k_0=\frac{16}{\pi}\int_{B(0;\,R)}|z_0-w|\,|f(w)|\,dm(w)\,,R_0=2\max\{R,\ |z_0|\}\,.$$

Then it follows that

$$m\{z\in C:\ |(Bf)(z)|>\lambda\}\leq m\{z\in C:\ |z|\leq R_0\}+m\left\{z\in C:\ \frac{k_0}{|z|^3}>\lambda\right\}=$$

$$=m\{z\in C:\ |z|\leq R_0\}+m\left\{z\in C:\ |z|<\sqrt[3]{\frac{k_0}{\lambda}}\right\}=\pi R_0^2+\pi\left(\frac{k_0}{\lambda}\right)^{2/3},$$

whence it follows asymptotic equality (9).

Now let's consider the general case. From the condition $\int_C f(z)\,dm(z)=0$ it follows that for any $\varepsilon>0$ there exist the functions f_1 and f_2 satisfying the condition: $f=f_1+f_2$, the function f_1 is concentrated on some circle $B(0;\,R)\subset C$ and $\int_C f_1(z)\,dm(z)=0$, the function f_2 satisfies the inequality $\|f_2\|_{L_1}<\frac{\varepsilon}{4C_1}$, where C_1 is a constant in estimation (1). Since the function f_1 is concentrated on the circle $B(0;\,R)\subset C$ and $\int_C f_1(z)\,dm(z)=0$, then for the function f_1 equality (9) is satisfied, and therefore there exists $\lambda(\varepsilon)>0$ such that for $0<\lambda<\lambda(\varepsilon)$ the inequality

$$\lambda m\left\{z\in C:\ |(Bf_1)(z)|>\frac{\lambda}{2}\right\}<\frac{\varepsilon}{2} \tag{10}$$

holds. On the other hand, from the inequality (1) it follows that

$$\lambda m\left\{z\in C:\ |(Bf_2)(z)|>\frac{\lambda}{2}\right\}\leq 2C_1\|f_2\|_{L_2}<\frac{\varepsilon}{2} \tag{11}$$

for any $\lambda>0$. From inequalities (10), (11) and the inclusion

$$\{z\in C:\ |(Bf)(z)|>\lambda\}\subset\left\{z\in C:\ |(Bf_1)(z)|>\frac{\lambda}{2}\right\}\bigcup\left\{z\in C:\ |(Bf_2)(z)|>\frac{\lambda}{2}\right\}$$

we get

$$\lambda m\left\{z\in C:\ \left|\left(Bf\right)(z)\right|>\lambda\right\}<\varepsilon$$

for $0<\lambda<\lambda(\varepsilon)$. This shows that equality (9) was satisfied for all functions $f\in L_1(C)$, satisfying the condition $\int_C f(z)\,dm(z)=0$. This completes the Proof of the Lemma 1 ◄

Proof of Theorem 2. In the case $\int_C f(z)\,dm(z)=0$ the assertion of the Theorem follows from Lemma 1. Let's consider the case $\int_C f(z)\,dm(z)=\eta\neq 0$. Denote by $f_1(z)=\frac{\eta}{\pi}\chi(B(0;1))(z)$, where $\chi(B(0;1))$ is a characteristic function on the unit circle $B(0;1)$ and $f_2(z)=f(z)-f_1(z)$. Then $\int_C f_2(z)\,dm(z)=0$, and from Lemma 1

$$m\left\{z\in C:\ \left|\left(Bf_2\right)(z)\right|>\lambda\right\}=o\left(\frac{1}{\lambda}\right),\lambda\to 0+. \tag{12}$$

Since for any $|z|>2$

$$\left|\left(Bf_1\right)(z)\right|=\frac{|\eta|}{\pi^2}\left|\int_{B(0;1)}\frac{dm(w)}{(z-w)^2}\right|\leq\frac{|\eta|}{\pi}\cdot\frac{1}{(|z|-1)^2},$$

$$\left|\left(Bf_1\right)(z)\right|=\frac{|\eta|}{\pi^2}\left|\int_{B(0;1)}\frac{dm(w)}{(z-w)^2}\right|=\frac{|\eta|}{\pi^2}\left|\int_{B(0;1)}\frac{dm(w)}{(|z|-w)^2}\right|\geq$$

$$\geq\frac{|\eta|}{\pi^2}Re\left(\int_{B(0;1)}\frac{dm(w)}{(|z|-w)^2}\right)\geq\frac{|\eta|}{\pi}\cdot\frac{(|z|-1)^2}{(|z|+1)^4},$$

then for any $0<\lambda<\frac{|\eta|}{49\pi}$

$$m\left\{z\in C:\ \left|\left(Bf_1\right)(z)\right|>\lambda\right\}\leq m\left\{z\in C:\ |z|\leq 2\right\}+m\left\{z\in C:\ \frac{|\eta|}{\pi}\cdot\frac{1}{(|z|-1)^2}>\lambda\right\}=$$

$$=4\pi+m\left\{z\in C:\ |z|<1+\sqrt{\frac{|\eta|}{\pi\lambda}}\right\}=4\pi+\pi\left(1+\sqrt{\frac{|\eta|}{\pi\lambda}}\right)^2, \tag{13}$$

$$m\left\{z\in C:\ \left|\left(Bf_1\right)(z)\right|>\lambda\right\}\geq m\left\{|z|\geq 2:\ \frac{|\eta|}{\pi}\cdot\frac{(|z|-1)^2}{(|z|+1)^4}>\lambda\right\}=$$

$$=m\left\{|z|\geq 2:\ \frac{(|z|+1)^2}{|z|-1}<\sqrt{\frac{|\eta|}{\pi\lambda}}\right\}=m\left\{|z|\geq 2:\ |z|+3+\frac{4}{|z|-1}<\sqrt{\frac{|\eta|}{\pi\lambda}}\right\}\geq$$

$$\geq m\left\{|z|\geq 2:\ |z|+7<\sqrt{\frac{|\eta|}{\pi\lambda}}\right\}\geq\pi\left(\sqrt{\frac{|\eta|}{\pi\lambda}}-7\right)^2-4\pi. \tag{14}$$

It follows from (13) and (14) that

$$\lim_{\lambda\to 0+}\lambda m\left\{z\in C:\ \left|\left(Bf_1\right)(z)\right|>\lambda\right\}=|\eta|. \tag{15}$$

For any $0 < \varepsilon < 1$, by the inclusions

$$\{z \in C: \ |(Bf_1)(z)| > (1+\varepsilon)\lambda\} \setminus \{z \in C: \\ |(Bf_2)(z)| > \varepsilon\lambda\} \subset \{z \in C: \ |(Bf)(z)| > \lambda\} \subset \\ \subset \{z \in C: \ |(Bf_2)(z)| > \varepsilon\lambda\} \bigcup \{z \in C: \ |(Bf_1)(z)| > (1-\varepsilon)\lambda\}$$

and equalities (12), (15) we have

$$\frac{|\eta|}{1+\varepsilon} \le \liminf_{\lambda \to 0+} \lambda \cdot m\{z \in C: \ |(Bf)(z)| > \lambda\} \le$$

$$\le \limsup_{\lambda \to 0+} \lambda \cdot m\{z \in C: \ |(Bf)(z)| > \lambda\} \le \frac{|\eta|}{1-\varepsilon}.$$

This implies the equation (8) and completes the proof of the Theorem 2. ◄

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The Resolvent of the Discrete Dirac Operator

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Abstract. The discrete Dirac operator is considered whose coefficients tend to different limits on $\pm\infty$. An explicit form of the resolvent of this operator is found.

Key Words and Phrases: discrete Dirac operator, resolvent, Yost solution.

2010 Mathematics Subject Classifications: 34A55, 34B24

1. Introduction and main result

We consider the system of difference equations

$$\begin{cases} a_{1,n}y_{2,n+1} + a_{2,n}y_{2,n} = \lambda y_{1,n}, \\ a_{1,n-1}y_{1,n-1} + a_{2,n}y_{1,n} = \lambda y_{2,n}, \;\; n = 0, \pm 1, \pm 2, ..., \end{cases} \tag{1}$$

where $a_{j,n},, j = 1, 2$, are real coefficients and satisfy the conditions

$$(-1)^{j-1}\, a_{j,n} > 0, n = 0, \pm 1, \pm 2, ..., \; a_{j,n} \to 0, n \to +\infty, j = 1, 2, \tag{2}$$

$$\sum_{n<0} |n| \left| (-1)^{j-1}\, a_{j,n} - 1 \right| < \infty, j = 1, 2. \tag{3}$$

Note that the system of difference equations (1) is a discrete analogue of the one-dimensional Dirac system. In this regard, the operator will be called the discrete Dirac operator. Various questions of the spectral theory of the Dirac operator were studied in [1, 2, 3]. We note that the direct and inverse problems of spectral analysis for the system (1) in various statements and in different classes were considered in [4, 5, 6, 7, 8, 9].

Let $\ell_2\left((-\infty,\infty), C\right)$ denote the Hilbert space of all complex vector sequences $y = \begin{pmatrix} y_{1,n} \\ y_{2,n} \end{pmatrix}_{n=-\infty}^{\infty}$ with the norm $\|y\| = \sum_{n=-\infty}^{\infty} \left\{ |y_{1,n}|^2 + |y_{2,n}|^2 \right\}$. We also define the operator L generated in $\ell_2\left((-\infty,\infty), C\right)$ by (1). By virtue of (2), (3), the operator L is bounded and self-adjoint.

It is known that in studying various problems of the spectral theory of linear operators, of particular interest are formulas for the expansion in eigenfunctions. In the present paper, an explicit form of the operator L resolvent is found. Similar questions for the

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one-dimensional Dirac system, the Schrödinger equation, and its difference analogue were investigated in the works [2, 5, 6, 7, 8, 9].

We denote the operator defined in $\ell_2([0,\infty), C)$ by system of equations (1) for $n \geq 0$ and the boundary condition $y_{1,0} = 0$ by L_0. It follows from the condition (2) that L_0 is a completely continuous self-adjoint operator. Since the eigenvalues of the operator L_0 are simple and L_0 is completely continuous, its spectrum consists of simple eigenvalues$\lambda_n = \pm\mu_n, n = 1, 2, ...,$ where $\lambda_n \to 0$ as $n \to \infty$, and the point $\lambda = 0$. The latter is either a simple eigenvalue of the operator L_0 or the only point of its continuous spectrum. It is known (see, for example, [10], Ch. 7, § 4) that the eigenvectors of a completely continuous self-adjoint operator form an orthogonal basis in the corresponding space. Consequently, the spectral function of the operatorL_0, which we denote by$\rho(\lambda)$, is a step function concentrated at the points $\lambda_n, n = 1, 2, ...$ For the sake of simplicity, in what follows we assume that the spectrum of the operator L_0 lies in the interval $(-\ 2,\ 2)$. Denote by $P_{j,n}(\lambda)$, $Q_{j,n}(\lambda)$ the solutions of the system of equations (1), defined by the initial conditions $P_{1,0}(\lambda) = Q_{2,1}(\lambda) = 0$, $P_{2,1}(\lambda) = 1$, $Q_{1,1}(\lambda) = a_{2,1}^{-1}$.
Consider the spectral function

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \alpha_n^{-1},$$

where

$$\alpha_n = \sum_{k=1}^{\infty} \left\{ P_{1,k}^2(\lambda_n) + P_{2,k}^2(\lambda_n) \right\},\ \sum_{n=1}^{\infty} \alpha_n^{-1} = 1.$$

Following [9], we introduce the Weyl function $m(\lambda) = \langle R_\lambda \delta, \delta \rangle$ of the operator L_0, where R_λ is the resolvent of the operator L_0 and $\delta = \begin{pmatrix} 0,0,0,... \\ 1,0,0,.. \end{pmatrix} \in \ell_2([0,\infty), C)$.
The Weyl function is related to the spectral function (see [11, 12]) by the equality

$$m(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho(t)}{t-\lambda},$$

which implies that

$$m(\lambda) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n(\lambda_n - \lambda)}. \tag{4}$$

We also introduce the Weyl solution

$$f_{j,n}^{+}(\lambda) = Q_{j,n}(\lambda) + m(\lambda) P_{j,n}(\lambda) \tag{5}$$

of the system of equations (1). By (4), the Weyl solution is analytic on the whole complex λ-plane except for the simple poles$\lambda_k, k = 1, 2, ...$ (The point $\lambda = 0$ is a nonisolated singularity of the Weyl solution). In addition, it is known (see, for instance, [11, 12]) that for $n > 0$ the equality $f_{j,n}^{+}(\lambda) = (R_\lambda \delta)_n$ is valid. Consequently, for every $N > -\infty$ the Weyl solution belongs to $\ell_2([N,\infty), C)$ with respect to the variable n.

We denote by Γ the complex $\lambda-$plane with a cut along the segment $[-2,2]$. In the plane we consider the function

$$z = z(\lambda) = -\frac{\lambda^2 - 2}{2} + \frac{\lambda}{2}\sqrt{\lambda^2 - 4},$$

choosing a regular branch of the radical such that $\sqrt{\lambda^2-4} > 0$ with $\lambda > 2$. It is known that the system of equation (1) has solution $\left\{f_{j,n}^{-}(\lambda)\right\}$, $j=1,2$, representable in the form [9]

$$\left.f_{j,n}^{-}(\lambda) = \alpha_j^{-}(n)\left(\frac{z^{-1}-1}{\lambda}\right)^{2-j} z^{-n}\left(1 + \sum_{m\le 1} K_j^{-}(n,m)\, z^{-m}\right),\ n = 0, \pm 1, \pm 2, ...,\right\} \tag{6}$$

and the quantities $\alpha_1^{\pm}(n)$, $\alpha_2^{\pm}(n)$, $K_1^{\pm}(n,m)$, $K_2^{\pm}(n,m)$ satisfy the relations

$$\left.\begin{array}{l} \alpha_j^{-}(n) = 1 + o(1), \quad n \to -\infty,\ j = 1,2, \\ K_j^{-}(n,m) = O\left(\sigma^{-}\left(n + \left[\frac{m}{2}\right] + 1\right)\right), n + m \to -\infty \end{array}\right\} \tag{7}$$

where $\sigma^{-}(n) = \sum_{m\le n}\{|a_{1,m}-1| + |a_{2,m}+1|\}$, by $[x]$ denote the integer part x . According to (6), (7) for each functions $\left\{f_{j,n}^{-}(\lambda)\right\}$, $j=1,2$, are regular in the plane Γ and continuous up to its boundary $\partial\Gamma$.

Let $u_{j,n}$ and $v_{j,n}$ be two solutions of the system of equations (1). We call them the Wronskian quantity $\{u_{j,n}, v_{j,n}\} = a_{1,n-1}(u_{1,n-1}v_{2,n} - u_{2,n}v_{1,n-1})$. Put $w(\lambda) = \left\{f_{j,n}^{+}(\lambda),\right.$ $\left.f_{j,n}^{-}(\lambda)\right\}$. Let us state the main result of this paper.

Theorem 1. *The functions*

$$R_{nm}(\lambda) = \left(\begin{array}{cc} R_{nm}^{11} & R_{nm}^{12} \\ R_{nm}^{21} & R_{nm}^{22} \end{array}\right), R_{nm}^{ij} = -w^{-1}(\lambda)\left\{\begin{array}{l} f_{i,n}^{+}(\lambda)\, f_{j,m}^{-}(\lambda), m \le n, \\ f_{j,m}^{+}(\lambda)\, f_{i,n}^{-}(\lambda), m > n, \end{array}\right. \tag{8}$$

are elements of the operator L resolvent matrix and satisfy the equations

$$\begin{array}{l} a_{1,n}R_{n+1,m}^{22} + a_{2,n}R_{nm}^{22} - \lambda R_{nm}^{12} = 0, \\ a_{1,n}R_{n+1,m}^{21} + a_{2,n}R_{nm}^{21} - \lambda R_{nm}^{11} = \delta_{nm}, \\ a_{1,n-1}R_{n-1,m}^{11} + a_{2,n}R_{nm}^{11} - \lambda R_{nm}^{21} = 0, \\ a_{1,n-1}R_{n-1,m}^{12} + a_{2,n}R_{nm}^{12} - \lambda R_{nm}^{22} = \delta_{nm}, \end{array} \tag{9}$$

where δ_{nm} is the Kronecker symbol.

Proof. Let $h = \{h_{1,n}, h_{2,n}\} \in \ell^2((-\infty,\infty); C)$ be an arbitrary finite sequence. In order to construct the resolvent of the operator L, we need to solve the equation

$$Ly = \lambda y + h.$$

We rewrite the last equation in the form

$$\begin{cases} a_{1,n}y_{2,n+1} + a_{2,n}y_{2,n} = \lambda y_{1,n} + h_{1,n}, \\ a_{1,n-1}y_{1,n-1} + a_{2,n}y_{1,n} = \lambda y_{2,n} + h_{2,n}. \end{cases} \tag{10}$$

We are looking for a solution to the system of equations in the form

$$y_{j,n} = C_n f_{j,n}^{+}(\lambda) + D_n f_{j,n}^{-}(\lambda)\; j = 1, 2, \tag{11}$$

where C_n and D_n are the quantities to be determined. Substituting representation (11) into the system of equations (10) after simple transformations, we obtain

$$\begin{cases} a_{1,n-1}(C_{n-1} - C_n) f_{1,n-1}^{+}(\lambda) + a_{1,n-1}(D_{n-1} - D_n) f_{1,n-1}^{-}(\lambda) = h_{2,n}, \\ a_{1,n-1}(C_{n-1} - C_n) f_{2,n}^{+}(\lambda) + a_{1,n-1}(D_{n-1} - D_n) f_{2,n}^{-}(\lambda) = -h_{1,n-1}. \end{cases}$$

Solving the last system of equations with respect to $C_{n-1} - C_n$ and $D_{n-1} - D_n$, we find that

$$C_{n-1} - C_n = w^{-1}(\lambda)\left[f_{1,n-1}^{-}(\lambda) h_{1,n-1} + f_{2,n}^{-}(\lambda) h_{2,n}\right], \tag{12}$$

$$D_{n-1} - D_n = w^{-1}(\lambda)\left[f_{1,n-1}^{+}(\lambda) h_{1,n-1} + f_{2,n}^{+}(\lambda) h_{2,n}\right]. \tag{13}$$

Note that to fulfil the conditions $y \in \ell^2((-\infty,\infty); C)$ you need to take $C_{-\infty} = 0$, $D_{\infty} = 0$. Adding then equalities (12) for $n = n, n-1, n-2, ...,$ and equalities (13) for $n = n+1, , n+2, n+3, ...,$ we have

$$\begin{aligned} C_n &= -w^{-1}(\lambda)\textstyle\sum_{k=-\infty}^{n-1}\left[f_{1,k}^{-}(\lambda) h_{1,k} + f_{2,k+1}^{-}(\lambda) h_{2,k+1}\right], \\ D_n &= -w^{-1}(\lambda)\textstyle\sum_{k=n}^{\infty}\left[f_{1,k}^{+}(\lambda) h_{1,k} + f_{2,k+1}^{+}(\lambda) h_{2,k+1}\right]. \end{aligned}$$

Substituting the last equalities into representation (11), we obtain

$$\begin{aligned} y_{j,n} = -w^{-1}(\lambda)&\left[\textstyle\sum_{k=-\infty}^{n-1} f_{j,n}^{+}(\lambda) f_{1,k}^{-}(\lambda) h_{1,k} + \sum_{k=n}^{\infty} f_{j,n}^{-}(\lambda) f_{1,k}^{+}(\lambda) h_{1,k}\right] - \\ -w^{-1}(\lambda)&\left[\textstyle\sum_{k=-\infty}^{n-1} f_{j,n}^{+}(\lambda) f_{2,k}^{-}(\lambda) h_{2,k} + \sum_{k=n}^{\infty} f_{j,n}^{-}(\lambda) f_{2,k}^{+}(\lambda) h_{2,k}\right]. \end{aligned}$$

On the other hand, by the definition of the resolvent, we have

$$y_{j,n} = \sum_{k=-\infty}^{\infty}\left[R_{nk}^{j1} h_{1,k} + R_{nk}^{j2} h_{2,k}\right]. \tag{14}$$

Comparison of the last equalities leads us to formulas (8). Using (8), it is directly verified that equations (9) are valid, and it follows from (9) that the vector $y = \{y_{1,n}, y_{2,n}\}_{-\infty}^{\infty}$, defined by formula (14) is a solution to the system of equations (10). Thus, the theorem is proved. ◀

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One Remark on the Eigenvalues of the Schrodinger Operator with Growing Potential

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Abstract. The Schrodinger operator $L = -\frac{d^2}{dx^2} + |x|$ on the whole axis is considered. The spectrum of the operator is investigated. An asymptotic formula for eigenvalues is obtained.

Key Words and Phrases: Schrodinger operator, Airy equation, Airy functions, eigenvalues.

2010 Mathematics Subject Classifications: 34A55

1. Introduction and main result

The spectral properties of the Airy operator $L_D y = -y'' + xy$, $y(0) = 0$ or $L_N y = -y'' + xy$, $y'(0) = 0$ were studied in the works of quite a few authors (see [1, 2, 3, 4, 5, 6, 7, 8, 9] also the literature there). The interest is also the corresponding Schrodinger operator on the whole axis.

In the space $L_2(-\infty, \infty)$ we consider the operator$L$, generated by the differential expression

$$l(y) = -y'' + |x|\, y$$

with the domain

$$D(L) = \left\{ y \in L_2(-\infty, \infty) : y \in W^2_{2,loc},\ l(y) \in L_2(-\infty, \infty) \right\}.$$

Note that the operator L is densely defined, because its domain contains infinitely differentiable functions compactly supported on $(-\infty, \infty)$, the set of these functions is well known to be dense in $L_2(-\infty, \infty)$, since its domain of definition contains infinitely differentiable functions with compact support on the interval, the set of which is dense in. Moreover, L is a self-adjoint operator. Obviously, the spectrum of the operator is discrete and consists of eigenvalues $\lambda_n, n = 1, 2, ...,$ where $\lambda_n \to \infty$ for $n \to \infty$.

We will be interested the asymptotic behavior of eigenvalues λ_n.

First, consider the equation

$$-y'' + xy = \lambda y,\ -\infty < x < \infty, \quad \lambda \in C. \tag{1}$$

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It is well known [10] that this equation has two linearly independent solutions in the form $Ai\,(x-\lambda)\,,\;Bi\,(x-\lambda)$, where $Ai\,(z)\,,\;Bi\,(z)$ are the Airy functions of the first and second kind, respectively.

We note some properties of these functions. As is known (see [8, 10]), both functions are entire functions of order 3/2 and type 2/3. The function $Ai\,(z)$ admits the following asymptotic representations as $|z|\to\infty$

$$\begin{gathered} \mathrm{A}i\,(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\; |\arg z|<\pi, \\ Ai'\,(z) \sim -\pi^{-\frac{1}{2}} z^{\frac{1}{4}} e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\; |\arg z|<\pi, \end{gathered}$$

$$\begin{gathered} \mathrm{A}i\,(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \sin\left(\zeta+\frac{\pi}{4}\right)\left[1+O\left(\zeta^{-1}\right)\right],\; |\arg z|<\frac{2\pi}{3}, \\ Ai'\,(-z) \sim -\pi^{-\frac{1}{2}} z^{\frac{1}{4}} \cos\left(\zeta+\frac{\pi}{4}\right)\left[1+O\left(\zeta^{-1}\right)\right],\; |\arg z|<\frac{2\pi}{3}, \end{gathered}$$

where $\zeta=\frac{2}{3}z^{\frac{3}{2}}$. In the sector $|z|<\frac{\pi}{3}$ the function $Bi\,(z)$ has an asymptotic representation

$$Bi\,(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{\zeta}\left[1+O\left(\zeta^{-1}\right)\right].$$

Thus, the functions $Bi\,(z)$ grow exponentially as $|z|\to\infty$ along any ray in this sector. For the Wronskian of functions $Ai\,(z)\,,Bi\,(z)$ the equality

$$\{Ai\,(z)\,,Bi\,(z)\} = Ai\,(z)\,Bi'\,(z) - Ai'\,(z)\,Bi\,(z) = \pi^{-1} \tag{2}$$

is valid.

We now consider the equation

$$-y''+|x|\,y=\lambda y,\;\; -\infty<x<\infty,\quad \lambda\in C. \tag{3}$$

According to the general theory (see [11]), equation (3) has two linearly independent solutions $\psi_\pm\,(x,\lambda)$, which for each $\lambda, Im\lambda>0$ satisfy the conditions $\psi_\pm\,(x,\lambda)\in L_2\,(0,\pm\infty)$. Since equation (3) does not change when x replaced by $-x$, the function $\psi_\pm\,(-x,\lambda)$ is also its solution. Therefore, we can assume that $\psi_-\,(x,\lambda)=\psi_+\,(-x,\lambda)$.

On the other hand, since $Ai\,(x-\lambda)\in L_2\,(0,\infty)$, the functions $\psi_+\,(x,\lambda)\,,\;Ai\,(x-\lambda)$ coincide up to a factor. Based on these considerations, for $x\geq 0$ we set $\psi_+\,(x,\lambda)=Ai\,(x-\lambda)$. Further, when $x\leq 0$ looking at the solution $\psi_+\,(x,\lambda)$ in the form

$$\psi_+\,(x,\lambda)=\alpha Ai\,(-x-\lambda)+\beta Bi\,(-x-\lambda)\,,$$

since the functions $Ai\,(-x-\lambda)\,,\,Bi\,(-x-\lambda)$ form the fundamental system of solutions of equation (1) for $x\leq 0$. Taking into account that the solution $\psi_+\,(x,\lambda)$ and its derivative $\psi_+'\,(x,\lambda)$ are continuous at a point $x=0$, to determine the coefficients α,β we obtain the following system of equations

$$\begin{cases} Ai\,(-\lambda)\,\alpha+Bi\,(-\lambda)\,\beta=Ai\,(-\lambda) \\ Ai'\,(-\lambda)\,\alpha+Bi'\,(-\lambda)\,\beta=-Ai'\,(-\lambda) \end{cases}$$

Solving the last system with respect to the coefficients α, β and taking into account equality (2), we obtain

$$\alpha = -\pi \left(Ai\left(-\lambda\right) Bi\left(-\lambda\right)\right)', \\ \beta = -2\pi Ai\left(-\lambda\right) Ai'\left(-\lambda\right).$$

So, we have proved the following theorem.

Theorem 1. *Equation* (3) *has special solutions* $\psi_{\pm}(x,\lambda)$*, which can be represented in the form*

$$\psi_{+}(x,\lambda) = \begin{cases} Ai\left(x-\lambda\right), x \geq 0, \\ -\pi\left(Ai\left(-\lambda\right) Bi\left(-\lambda\right)\right)' Ai\left(-x-\lambda\right) - 2\pi Ai\left(-\lambda\right) Ai'\left(-\lambda\right) Bi\left(-x-\lambda\right), x < 0 \end{cases}$$

$$\psi_{-}(x,\lambda) = \begin{cases} -\pi\left(Ai\left(-\lambda\right) Bi\left(-\lambda\right)\right)' Ai\left(x-\lambda\right) - 2\pi Ai\left(-\lambda\right) Ai'\left(-\lambda\right) Bi\left(x-\lambda\right), x \geq 0, \\ Ai\left(-x-\lambda\right), x < 0 \end{cases}$$

We return now to the study of the spectrum of the operator L. From the fact that $\psi_{\pm}(x,\lambda) \in L_2(0,\pm\infty)$, if $\lambda = \lambda_n$ is an eigenvalue, then the solutions $\psi_{+}(x,\lambda_n)$ and $\psi_{-}(x,\lambda_n)$ are linearly dependent. In fact, since

$$\psi_{+}(x,\lambda_n) = \begin{cases} Ai\left(x-\lambda_n\right), x \geq 0, \\ (-1)^{n-1} Ai\left(-x-\lambda_n\right), x < 0, \end{cases}$$

$$\psi_{-}(x,\lambda_n) = \begin{cases} (-1)^{n-1} Ai\left(x-\lambda_n\right), x \geq 0, \\ Ai\left(-x-\lambda_n\right), x < 0, \end{cases}$$

then following equality holds

$$\psi_{+}(x,\lambda_n) = (-1)^{n-1}\psi_{-}(x,\lambda_n).$$

From these arguments it follows that the eigenvalues of the operator coincide with the zeros of the function

$$\Delta(\lambda) = \{\psi_{+}(x,\lambda), \psi_{-}(x,\lambda)\}.$$

Taking advantage of the fact that the Wronskian of the two solutions does not depend on x, we obtain

$$\Delta(\lambda) = \left.\{\psi_{+}(x,\lambda), \psi_{-}(x,\lambda)\}\right|_{x=0} = -2Ai\left(-\lambda\right) Ai'\left(-\lambda\right). \tag{4}$$

From the last formula and the known properties of the zeros of functions $Ai(\lambda)$, $Ai'(\lambda)$ (see [10]) it follows that the eigenvalues $\lambda_n, n = 1, 2, ...$ of the operator L are located only on the positive semi-axis and holds the following asymptotic equality

$$\lambda_n = \left(\frac{3\pi\left(2n-1\right)}{8}\right)^{\frac{2}{3}}\left(1 + O\left(n^{-2}\right)\right), n \to \infty.$$

Let us prove that the eigenvalues of the operator L are simple. We introduce normalization numbers $\alpha_n, n = 1, 2, ...$, setting

$$\alpha_n = \sqrt{\int_{-\infty}^{\infty} |\psi_{\pm}(x, \lambda_n)|^2 \, dx}. \tag{5}$$

Let us agree with dots to denote differentiation with respect to λ, and strokes with respect to x:

$$u' = \frac{\partial}{\partial x} u, \ \dot{u} = \frac{\partial}{\partial \lambda} u.$$

Since $\psi_{\pm}(x, \lambda)$ decreases exponentially for $x \to \pm\infty$, from the standard (see, e.g., [12]) identity

$$f^2 = \left\{ \dot{f}, f \right\}'$$

and (5) it follows that

$$\begin{aligned}
&(\alpha_n)^2 = \textstyle\int_{-\infty}^{\infty} \psi_+^2 (x, \lambda_n) \, dx = \int_0^{\infty} \psi_+^2 (x, \lambda_n) \, dx + \int_{-\infty}^{0} \psi_-^2 (x, \lambda_n) \, dx = \\
&= \left\{ \dot{\psi}_+ (x, \lambda_n), \psi_+ (x, \lambda_n) \right\}\Big|_0^{\infty} + \left\{ \dot{\psi}_- (x, \lambda_n), \psi_- (x, \lambda_n) \right\}\Big|_{-\infty}^{0} = \\
&= - \left\{ \dot{\psi}_+ (x, \lambda_n), \psi_+ (x, \lambda_n) \right\}\Big|_{x=0} + \left\{ \dot{\psi}_- (x, \lambda_n), \psi_- (x, \lambda_n) \right\}\Big|_{x=0} = \\
&= - (-1)^{n-1} \left\{ \dot{\psi}_+ (x, \lambda_n), \psi_- (x, \lambda_n) \right\}\Big|_{x=0} - \\
&(-1)^{n-1} \left\{ \psi_+ (x, \lambda_n), \dot{\psi}_- (x, \lambda_n) \right\}\Big|_{x=0} = - (-1)^{n-1} \dot{\Delta} (\lambda_n).
\end{aligned}$$

Therefore, $\dot{\Delta}(\lambda_n) \neq 0$, i.e. the eigenvalues of the operator L are simple.

Thus, the following theorem holds.

Theorem 2. *The spectrum of the operator L consists of a sequence of simple real eigenvalues $\lambda_n, n \geq 1$, located on the positive semi-axis and*

$$\lambda_n = \left(\frac{3\pi (2n-1)}{8} \right)^{\frac{2}{3}} \left(1 + O\left(n^{-2}\right)\right), n \to \infty.$$

Remark 1. *In the space $L_2(0, \infty)$ we consider the operators $L_D y = -y'' + |x| \, y$, $y(0) = 0$ and $L_N y = -y'' + |x| \, y$, $y'(0) = 0$. Formula (4) shows that the spectrum of the operator consists of the union of the spectra of the operators L_D and L_N.*

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On an Inverse Boundary-value Problem for the Equation of Motion of a Homogeneous Beam

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Abstract. In this work, a classical solution of an inverse boundary-value problem for the equation of motion of a homogeneous beam with periodic boundary conditions is studied. Firstly, the original problem is reduced to an equivalent (in a defined sense) problem, for which the existence and uniqueness theorem of the solution is proved. Further, using the unique solvability of the equivalent problem, the classical solvability of the original problem is showed.

Key Words and Phrases: Inverse boundary-value problem, classical solution, Fourier's method, homogeneous beam.
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1. Introduction

Recently, there are many cases in which the needs in a practice lead to the problems of determining the coefficients of a differential equation (ordinary or in partial derivatives) from some known functional of its solution. Such problems are called inverse problems of mathematical physics. The applied importance of inverse problems is so great (they arise in various fields of human activity, such as seismology, mineral exploration, biology, medicine, desalination of seawater, movement of liquid in a porous medium, etc.) which puts them a series of the most actual problems of modern mathematics. The presence in the inverse problems of additional unknown functions requires that in the complement to the boundary conditions that are natural for a particular class of differential equations, impose some additional conditions - overdetermination conditions. The basics of the theory and practice of investigating inverse problems of mathematical physics were established and developed in the fundamental works of the outstanding scientists A.N.Tikhonov [1], M.M.Lavrent'ev [2], V.K.Ivanov [3], and their followers.

Inverse problems associated with equations of various types, have been studied by many papers and monographs, in particular, [4]-[13]. But the problem statement and the proof techniques used in this paper are different from those presented in these works. The technique used in this paper is based on the passing from the original inverse problem to the new equivalent one, the study of the solvability of the equivalent problem, and then in the reverse transition to the original problem.

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Moreover, the vibrations and wave movements of an elastic beam on an elastic base were investigated by Yu.A. Mitropolsky [14], J.M.Thompson [15], B.S.Bardin [16], V.Z. Vlasov [17], D.V.Kostin [18], T.P.Goy [19], Ya.T.Mehraliyev [20], and et al. The simplest nonlinear model of the motion of a homogeneous beam is described by the equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + k\frac{\partial^2 w}{\partial x^2} + \alpha w + w^3 = 0,$$

where ω is beam deflection (after the displacements of the points of the midline of the elastic beam located along the x-axis). Note that a similar equation also arises in the theory of crystals, in which ω is parameter of order [21].

2. Statement of the problem

This paper is concerned with the following inverse problem of finding a pair $\{u(x,t), p(t)\}$ in the domain $D_T = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ for the following system

$$u_{tt}(x,t)+u_{xxxx}(x,t)+\beta u_{xx}(x,t)+\alpha u(x,t)+u^3(x,t) = p(t)g(x,t)+f(x,t),\ (x,t) \in D_T, \quad (1)$$

$$u(x,0) + \delta u(x,T) = \varphi(x),\ u_t(x,0) + \delta u_t(x,T) = \psi(x), \quad 0 \le x \le 1, \quad (2)$$

$$u(0,t) = u(1,t), \quad u_x(0,t) = u_x(1,t),$$
$$u_{xx}(0,t) = u_{xx}(1,t), \quad u_{xxx}(0,t) = u_{xxx}(1,t), \quad 0 \le t \le T, \quad (3)$$

$$u(x_0,t) = h(t), \quad 0 \le t \le T, \quad (4)$$

where $x_0 \in (0,1)$ is fixed number, $\alpha > 0,\ \beta > 0$, and δ are given numbers, and $\beta < 4\alpha,\ g(x,t),\ f(x,t),\ \varphi(x),\ \psi(x),\ h(t)$ are known functions.

We introduce the set of functions

$$\tilde{C}^{2,4}(D_T) = \{u(x,t) : u(x,t) \in C^2(D_T), u_{xxxx}(x,t) \in C(D_T)\}.$$

Definition 1. *The pair $\{u(x,t), p(t)\}$ defined on D_T is said to be a classical solution of the problem (1)-(4), if the functions $u(x,t) \in \tilde{C}^{2,4}(D_T)$ and $p(t) \in C[0,T]$ satisfies Eq. (1), condition (2) on $[0,1]$, and the statements (3)-(4) on the interval $[0,T]$.*

It's easy to prove that

Lemma 1. *Suppose that $f(x,t),\ g(x,t) \in C(D_T),\ g(0,t) \ne 0,\ 0 \le t \le T,\ \varphi(x),\ \psi(x) \in C[0,1],\ h(t) \in C^2[0,T],\ \delta \ne \pm 1$, and the condition*

$$\varphi(x_0) = h(0) + \delta h(T),\ \psi(x_0) = h'(0) + \delta h'(T)$$

holds.Then the problem of finding a classical solution of (1)-(4) is equivalent to the problem of determining the functions $u(x,t) \in \tilde{C}^{2,4}(D_T)$ and $p(t) \in C[0,T]$ from the (1)-(3), and satisfying the condition

$$h''(t) + u_{xxxx}(x_0,t) + \beta u_{xx}(x_0,t) + \alpha h(t) + u^3(x_0,t)$$
$$= p(t)g(x_0,t) + f(x_0,t),\ 0 \le t \le T. \quad (5)$$

3. Classical solvability of inverse boundary-value problem

It is known that [22] the system

$$1, \cos\lambda_1 x, \sin\lambda_1 x, ..., \cos\lambda_k x, \sin\lambda_k x, ... \tag{6}$$

are a bases in $L_2(0,1)$, for $\lambda_k = 2k\pi$ $(k = 0, 1, ...)$.

Since the system (6) forms a basis in $L_2(0,1)$, then it is obvious that the first component of the solution $\{u(x,t), p(t)\}$ has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x,\ \ \lambda_k = 2k\pi,\ \ k = 0, 1, ..., \tag{7}$$

where

$$u_{10}(t) = \int_0^1 u(x,t)dx,\ \ u_{1k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx,\ \ k = 0, 1, ...,$$

$$u_{2k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx,\ \ k = 0, 1, ...$$

Then applying the formal scheme of the Fourier method, from (1) and (2) we have

$$u''_{10}(t) + \alpha u_{10}(t) = F_{10}(t; u, p),\ \ 0 \le t \le T, \tag{8}$$

$$u''_{ik}(t) + (\lambda_k^4 - \beta\lambda_k^2 + \alpha)u_{ik}(t) = F_{ik}(t; u, p),\ \ 0 \le t \le T;\ \ i = 1, 2;\ \ k = 1, 2, ..., \tag{9}$$

$$u_{10}(0) + \delta u_{10}(T) = \varphi_{10},\ \ u'_{10}(0) + \delta u'_{10}(T) = \psi_{10}, \tag{10}$$

$$u_{ik}(0) + \delta u_{ik}(T) = \varphi_{ik},\ \ u'_{ik}(0) + \delta u'_{ik}(T) = \psi_{ik},\ \ i = 1, 2;\ \ k = 1, 2, ..., \tag{11}$$

where

$$F_{1k}(t; u, p) = p(t)g_{1k}(t) + f_{1k}(t) - G_{1k}(t, u),\ \ k = 0, 1, 2...,$$

$$g_k(t) = m_k\int_0^1 g(x,t)\cos\lambda_k x dx,$$

$$F_{1k}(t; u, p) = f_{1k}(t) + p(t)u_{1k}(t),\ \ k = 0, 1, 2...,$$

$$f_{10}(t) = \int_0^1 f(x,t)dx,\ \ f_{1k}(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx,\ \ k = 1, 2, ...,$$

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$$G_{10}(t,u)=\int_0^1 u^3(x,t)dx,\ G_{1k}(t,u)=2\int_0^1 u^3(x,t)\cos\lambda_k x dx,\ k=1,2,...,$$

$$\varphi_{10}=\int_0^1 \varphi(x)dx,\ \psi_{10}=\int_0^1 \psi(x)dx,$$

$$\varphi_{1k}=2\int_0^1 \varphi(x)\cos\lambda_k x dx,\ \psi_{1k}=2\int_0^1 \psi(x)\cos\lambda_k x dx,\ k=1,2,...,$$

$$F_{2k}(t;u,p)=p(t)g_{2k}(t)+f_{2k}(t)-G_{2k}(t,u),\ k=0,1,2,...,$$

$$f_{2k}(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx,\ g_{2k}(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx,\ k=1,2,...,$$

$$G_{2k}(t,u)=2\int_0^1 u^3(x,t)\sin\lambda_k x dx,\ k=1,2,...,$$

$$\varphi_{2k}=2\int_0^1 \varphi(x)\sin\lambda_k x dx,\ \psi_{2k}=2\int_0^1 \psi(x)\sin\lambda_k x dx,\ k=1,2,....$$

Solving problem (8) - (11), we find

$$u_{10}(t)=\frac{1}{\sqrt{\alpha}\rho_0(T)}\{\sqrt{\alpha}(\cos\sqrt{\alpha}t+\delta\cos\sqrt{\alpha}(T-t))\varphi_0$$
$$+(\sin\sqrt{\alpha}t-\delta\sin\sqrt{\alpha}(T-t))\psi_0-\delta\int_0^T F_0(\tau;u,p)(\sin\sqrt{\alpha}(T+t-\tau)$$
$$+\delta\sin\sqrt{\alpha}(t-\tau))d\tau\}+\frac{1}{\sqrt{\alpha}}\int_0^t F_0(\tau;u,p)\sin\sqrt{\alpha}(t-\tau)d\tau, \tag{12}$$

$$u_{ik}(t)=\frac{1}{\beta_k\rho_k(T)}\{\beta_k(\cos\beta_k t+\delta\cos\beta_k(T-t))\varphi_{ik}+(\sin\beta_k t$$
$$\left.-\delta\sin\beta_k(T-t))\psi_{ik}-\delta\int_0^T F_{ik}(\tau;u,p)(\sin\beta_k(T+t-\tau)+\delta\sin\beta_k(t-\tau))d\tau\right\}$$
$$+\frac{1}{\beta_k}\int_0^t F_{ik}(\tau;u,p)\sin\beta_k(t-\tau)d\tau,\ i=1,2;\ k=1,2,\ldots, \tag{13}$$

where

$$\rho_0(T) = 1 + 2\delta\cos\sqrt{\alpha}T + \delta^2,\ \beta_k = \sqrt{\lambda_k^4 - \beta\lambda_k^2 + \alpha},$$

$$\rho_k(T) = 1 + 2\delta\cos\beta_k T + \delta^2,\ k = 1, 2, \ldots.$$

Differentiating twice (13) gives

$$u'_{ik}(t) = \frac{1}{\rho_k(T)}(\beta_k(-\sin\beta_k t + \delta\sin\beta_k(T-t))\varphi_{ik} + (\cos\beta_k t$$

$$\left. +\delta\cos\beta_k(T-t))\psi_{ik} - \delta\int_0^T F_{ik}(\tau;u,p)(\cos\beta_k(T+t-\tau) + \delta\cos\beta_k(t-\tau))d\tau \right\}$$

$$+\int_0^t F_{ik}(\tau;u,p)\cos\beta_k(t-\tau)d\tau,\ i=1,2;\quad k=1,2,\ \ldots, \tag{14}$$

$$u''_{ik}(t) = F_{ik}(t;u,p) - \frac{\beta_k}{\rho_k(T)}\{\beta_k(\cos\beta_k t + \delta\sin\beta_k(T-t))\varphi_{ik}$$

$$+(\sin\beta_k t - \delta\sin\beta_k(T-t))\psi_{ik}$$

$$\left. -\delta\int_0^T F_{ik}(\tau;u)(\sin\beta_k(T+t-\tau) + \delta\sin\beta_k(t-\tau))d\tau \right\}$$

$$-\beta_k\int_0^t F_{ik}(\tau;u)\sin\beta_k(t-\tau)d\tau,\ i=1,2;\ k=1,2,\ldots. \tag{15}$$

In order to determine the first component of the solution of the problem (1)-(3), (5) we substitute of $u_{1k}(t)$ $(k = 0, 1, 2, ...)$ and $u_{2k}(t)$ $(k = 1, 2, ...)$ into (7), we obtain

$$u(x,t) = \frac{1}{\sqrt{\alpha}\rho_0(T)}\{\sqrt{\alpha}(\cos\sqrt{\alpha}t + \delta\cos\sqrt{\alpha}(T-t))\varphi_{10}$$

$$+(\sin\sqrt{\alpha}t - \delta\sin\sqrt{\alpha}(T-t))\psi_{10}$$

$$\left. -\delta\int_0^T F_{10}(\tau;u,p)(\sin\sqrt{\alpha}(T+t-\tau) + \delta\sin\sqrt{\alpha}(t-\tau))d\tau \right\}$$

$$+\frac{1}{\sqrt{\alpha}}\int_0^t F_{10}(\tau;u,p)\sin\sqrt{\alpha}(t-\tau)d\tau$$

$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\beta_k\rho_k(T)}[\beta_k(\cos\beta_k t + \delta\cos\beta_k(T-t))\varphi_{1k} + (\sin\beta_k t\right.$$

$$+\delta\sin\beta_k(T-t))\psi_{1k}-\delta\int\limits_0^T F_{1k}(\tau;u,p)(\sin\beta_k(T+t-\tau)+\delta\sin\beta_k(t-\tau))d\tau\Bigg]$$

$$\left.+\frac{1}{\beta_k}\int\limits_0^t F_{1k}(\tau;u,p)\sin\beta_k(t-\tau)d\tau\right\}\cos\lambda_k x$$

$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\beta_k\rho_k(T)}[\beta_k(\cos\beta_k t+\delta\cos\beta_k(T-t))\varphi_{2k}+(\sin\beta_k t\right.$$

$$+\delta\sin\beta_k(T-t))\psi_{2k}-\delta\int\limits_0^T F_{2k}(\tau;u,p)(\sin\beta_k(T+t-\tau)+\delta\sin\beta_k(t-\tau))d\tau\Bigg]$$

$$\left.+\frac{1}{\beta_k}\int\limits_0^t F_{2k}(\tau;u,p)\sin\beta_k(t-\tau)d\tau\right\}\sin\lambda_k x. \tag{16}$$

Now, from (5), taking into account (6), we have

$$p(t)=[g(0,t)]^{-1}\{h''(t)+\alpha h(t)-f(x_0,t)+u^3(x_0,t)$$

$$+\sum_{k=1}^{\infty}(\lambda_k^4-\beta\lambda_k^2)u_{1k}(t)\cos\lambda_k x_0+\sum_{k=1}^{\infty}(\lambda_k^4-\beta\lambda_k^2)u_{2k}(t)\sin\lambda_k x_0. \tag{17}$$

In this way to obtain the equation for the second component of the solution to the problem (1) - (3), (5) we substitute expression (13) into (17) and get

$$p(t)=[g(0,t)]^{-1}\left\{h''(t)+\alpha h(t)-f(x_0,t)+u^3(x_0,t)+\sum_{k=1}^{\infty}(\lambda_k^4-\beta\lambda_k^2)\right.$$

$$\times\left[\frac{1}{\beta_k\rho_k(T)}[\beta_k(\cos\beta_k t+\delta\cos\beta_k(T-t))\varphi_{1k}+(\sin\beta_k t-\delta\sin\beta_k(T-t))\psi_{1k}\right.$$

$$-\delta\int\limits_0^T F_{1k}(\tau;u,p)(\sin\beta_k(T+t-\tau)+\delta\sin\beta_k(t-\tau))d\tau\Bigg]$$

$$\left.+\frac{1}{\beta_k}\int\limits_0^t F_{1k}(\tau;u,p)\sin\beta_k(t-\tau)d\tau\right]\cos\lambda_k x_0+\sum_{k=1}^{\infty}(\lambda_k^4-\beta\lambda_k^2)$$

$$\times\left[\frac{1}{\beta_k\rho_k(T)}[\beta_k(\cos\beta_k t+\delta\cos\beta_k(T-t))\varphi_{2k}+(\sin\beta_k t-\delta\sin\beta_k(T-t))\psi_{2k}\right.$$

$$\left.-\delta\int_0^T F_{2k}(\tau;u,p)(\sin\beta_k(T+t-\tau)+\delta\sin\beta_k(t-\tau))d\tau\right]$$

$$\left.+\frac{1}{\beta_k}\int_0^t F_{2k}(\tau;u,p)\sin\beta_k(t-\tau)d\tau\right]\sin\lambda_k x_0. \tag{18}$$

Thus, finding the solution of problem (1) - (3), (5) is reduced to the finding solution of system (16), (18) with respect to unknown functions $u(x,t)$ and $p(t)$.

The following lemma plays an important role in studying the uniqueness of the solution to problem (1) - (3), (5):

Lemma 2. *If $\{u(x,t),p(t)\}$ is a solution of (1)-(3), (5), then the functions*

$$u_{10}(t)=\int_0^1 u(x,t)dx,\ \ u_{1k}(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx,\ \ k=1,2,...,$$

$$u_{2k}(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx,\ \ k=1,2,...,$$

satisfy the system (12) and (13) on the interval $[0,T]$.

Remark 1. *It follows from Lemma 2 that in order to prove the uniqueness of a solution to the problem (1) - (3), (5) it is sufficient to prove the uniqueness of a solution to system (13), (15).*

Now, to study problem (1) - (3), (5), we consider the following spaces.

Denote by $B^5_{2,T}$ an aggregate of all the functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_k x,\ \ \lambda_k=2\pi k,$$

considered in D_T, where each of the functions $u_{1k}(t)$ $(k=0,1,2,...)$ and $u_{2k}(t)$ $(k=1,2,...)$ is continuous on $[0,T]$, and

$$J_T(u)\equiv\|u_{10}(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_{1k}(t)\|_{C[0,T]})^2\right)^{\frac12}+\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac12}<+\infty.$$

The norm in this set is defined as follows

$$\|u(x,t)\|_{B^5_{2,T}} = J(u).$$

Now, we denote by E^5_T the space of vector-functions $z(x,t) = \{u(x,t), p(t)\}$, which $u(x,t) \in B^5_{2,T},\ p(t) \in C[0,T]$.

The norm in the set E^5_T will be

$$\|z(x,t)\|_{E^5_T} = \|u(x,t)\|_{B^5_{2,T}} + \|p(t)\|_{C[0,T]}\,.$$

It is known that $B^5_{2,T}$ and E^5_T are the Banach spaces [23].

We now consider the operator

$$\Phi(u,p) = \{\Phi_1(u,p), \Phi_2(u,p)\},$$

in the space E^5_T, where

$$\Phi_1(u,p) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t)\sin\lambda_k x, \quad \Phi_2(u,p) = \tilde{p}(t),$$

where the functions $\tilde{u}_{10}(t),\ \tilde{u}_{ik}(t)\ (i=1,2;\ k=1,2,...)$, and $\tilde{p}(t)$ are equal to the right-hand sides of (12), (13), and (15), respectively.

Then we obtain

$$\|\tilde{u}_{10}(t)\|_{C[0,T]} = \frac{\rho(T)}{\sqrt{\alpha}}\left\{\sqrt{\alpha}(1+|\delta|)\,|\varphi_{10}|\right.$$

$$\left. +(1+|\delta|)\,|\psi_{10}| + (1+|\delta|\,(1+|\delta|)\sqrt{T}\left(\int_0^T |F_{10}(\tau,u,p)|^2\right)^{\frac{1}{2}}\right., \tag{19}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{3}\rho(T)(1+|\delta|)\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{3}\rho(T)(1+|\delta|)\varepsilon\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_{ik}|)^2\right)^{\frac{1}{2}} + \sqrt{3T}(1+|\delta|\rho(T)(1+|\delta|)$$

$$\times\varepsilon\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3|F_{ik}(\tau;u,p)|)^2 d\tau\right)^{\frac{1}{2}},\ i=1,2, \tag{20}$$

$$\|\tilde{p}(t)\|_{C[0,T]} \le \left\|[g(0,t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t) + \alpha h(t) + u^3(x_0,t) + f(x_0,t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+\beta)\left[\rho(T)(1+|\delta|)\sum_{i=1}^{2}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}}\right.$$

$$+\rho(T)(1+|\delta|)\varepsilon\sum_{i=1}^{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_{ik}|)^2\right)^{\frac{1}{2}}$$

$$\left.\left.+\sqrt{T}(1+|\delta|\,\rho(T)(1+|\delta|)\varepsilon\sum_{i=1}^{2}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|F_{ik}(\tau;u,p)|)^2d\tau\right)^{\frac{1}{2}}\right]\right\}, \quad (21)$$

where

$$\rho(T)\equiv\sup_k\rho_k^{-1}(T)\le\frac{1}{(1+\delta^2-2\,|\delta|)},\ \ \sup_k\left(\frac{\lambda_k^2}{\sqrt{\lambda_k^4-\beta\lambda_k^2+\alpha}}\right)=\frac{1}{\varepsilon}.$$

Suppose that the data of problem (1) - (3), (5) satisfy the conditions

(A_1) $\varphi(x)\in C^4[0,1],\ \varphi^{(5)}(x)\in L_2(0,1),\ \varphi(0)=\varphi(1),\ \varphi'(0)=\varphi'(1),$
$\varphi''(0)=\varphi''(1),\varphi'''(0)=\varphi'''(1),\ \varphi^{(4)}(0)=\varphi^{(4)}(1);$

(A_2) $\varphi(x)\in C^4[0,1],\ \psi'''(x)\in L_2(0,1),\ \psi(0)=\psi(1),\ \psi'(0)=\psi'(1),\ \psi''(0)=\psi''(1);$

(A_3) $f(x,t),\ f_x(x,t),\ f_{xx}(x,t)\in C(D_T),\ f_{xxx}(x,t)\in L_2(D_T)$, and $f(0,t)=f(1,t),$
$f_x(0,t)=f_x(1,t),\ f_{xx}(0,t)=f_{xx}(1,t),\ 0\le t\le T;$

(A_4) $g(x,t),\ g_x(x,t),\ g_{xx}(x,t)\in C(D_T),\ g_{xxx}(x,t)\in L_2(D_T)$, and $g(0,t)=g(1,t),$
$g_x(0,t)=g_x(1,t),\ g_{xx}(0,t)=g_{xx}(1,t)=0,\ g(0,t)\ne0,\ 0\le t\le T;$

(A_5) $\alpha>0,\ \beta>0,\ \delta\ne\pm1,\ \beta<4\alpha,\ h(t)\in C^2[0,T],\ 0\le t\le T.$

Then from relations (16) - (18), correspondingly we have

$$\|\tilde u_0(t)\|_{C[0,T]}=\frac{\rho(T)}{\sqrt{\alpha}}\{\sqrt{\alpha}(1+|\delta|)\,\|\varphi\|_{L_2(0,1)}+(1+|\delta|)\,\|\psi\|_{L_2(0,1)}$$

$$+(1+|\delta|\,(1+|\delta|)\sqrt{T}\left\|p(t)g(x,t)+f(x,t)+u^3\right\|_{L_2(D_T)}\}, \quad (22)$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le\sqrt{3}\rho(T)(1+|\delta|)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+\sqrt{3}\rho(T)(1+|\delta|)\varepsilon\left\|\psi'''(x)\right\|_{L_2(0,1)}+\sqrt{3T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon$$

$$\times\left\|p(t)g_{xxx}(x,t)+f_{xxx}(x,t)+6u_x^3+18u\cdot u_x\cdot u_{xx}+3u^2\cdot u_{xxx}\right\|_{L_2(D_T)}, \quad (23)$$

$$\|\tilde p(t)\|_{C[0,T]}\le\left\|[g(x_0,t)]^{-1}\right\|_{C[0,T]}\{\left\|h''(t)+\alpha h(t)+u^3(x_0,t)+f(x_0,t)\right\|_{C[0,T]}$$

$$+2(1+\beta)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}[\rho(T)(1+|\delta|)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+\rho(T)(1+|\delta|)\varepsilon\left\|\psi'''(x)\right\|_{L_2(0,1)}+\sqrt{T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon$$

$$\times\left\|p(t)g_{xxx}(x,t)+f_{xxx}(x,t)+6u_x^3+18u\cdot u_x\cdot u_{xx}+3u^2\cdot u_{xxx}\right\|_{L_2(D_T)}]\}. \qquad (24)$$

We denote by

$$A_1(T)=\frac{\rho(T)}{\sqrt{\alpha}}\{\sqrt{\alpha}(1+|\delta|)\left\|\varphi\right\|_{L_2(0,1)}+(1+|\delta|)\left\|\psi\right\|_{L_2(0,1)}$$

$$+(1+|\delta|\,(1+|\delta|)\sqrt{T}\left\|f(x,t)\right\|_{L_2(D_T)}\}$$

$$+2\sqrt{3}\rho(T)(1+|\delta|)\varepsilon\left\|\psi'''(x)\right\|_{L_2(0,1)}+2\sqrt{3}\rho(T)(1+|\delta|)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+6\sqrt{T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T)=6\sqrt{T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon(\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}+1)$$

$$+\frac{\rho(T)(1+|\delta|\,(1+|\delta|))\sqrt{T}}{\sqrt{\alpha}}(\left\|g(x,t)\right\|_{L_2(D_T)}+1),$$

$$A_2(T)=\left\|[g(x_0,t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)+\alpha h(t)+f(x_0,t)\right\|_{C[0,T]}\right.$$

$$+2\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+\beta)[\rho(T)(1+|\delta|)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+\rho(T)(1+|\delta|)\varepsilon\left\|\psi'''(x)\right\|_{L_2(0,1)}$$

$$+\sqrt{T}(1+|\delta|\,\rho(T)(1+|\delta|)\varepsilon\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}],$$

$$B_2(T)=\left\|[g(x_0,t)]^{-1}\right\|_{C[0,T]}\left[2\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+\beta)\right.$$

$$\times\sqrt{T}(1+|\delta|\rho(T)(1+|\delta|)\varepsilon(\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}+1)],$$

and rewrite (22) - (24) as

$$\left\|\tilde{u}(x,t)\right\|_{B_{2,T}^5}\leq A_1(T)+B_1(T)(\left\|p(t)\right\|_{C[0,T]}$$

$$+\left\|u^3\right\|_{L_2(D_T)}+\left\|6u_x^3+18u\cdot u_x\cdot u_{xx}+3u^2\cdot u_{xxx}\right\|_{L_2(D_T)}), \qquad (25)$$

$$\left\|\tilde{p}(t)\right\|_{C[0,T]}\leq A_2(T)+B_2(T)(\left\|u^3(0,t)\right\|_{C[0,T]}+\left\|p(t)\right\|_{C[0,T]}$$

$$+\left\|6u_x^3+18u\cdot u_x\cdot u_{xx}+3u^2\cdot u_{xxx}\right\|_{L_2(D_T)}). \qquad (26)$$

From the inequalities (25), (26), we conclude

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}} + \|\tilde{p}(t)\|_{C[0,T]} \le A(T) + B(T)(\left\|u^3(x_0,t)\right\|_{C[0,T]} + \|p(t)\|_{C[0,T]}$$

$$+\left\|u^3\right\|_{L_2(D_T)} + \left\|6u_x^3 + 18u\cdot u_x\cdot u_{xx} + 3u^2\cdot u_{xxx}\right\|_{L_2(D_T)}), \tag{27}$$

where

$$A(T) = A_1(T) + A_2(T),\ B(T) = B_1(T) + B_2(T).$$

Thus, the following assertion is valid

Theorem 1. *If conditions* $(A_1) - (A_5)$ *and*

$$64B(T)(A(T)+2)^3 < 1. \tag{28}$$

holds, then problem (1)-(3), (5) has a unique solution in the ball $K = K_R(||z||_{E^5_T} \le R = A(T)+2)$ *of the space* E^5_T*.*

Proof. In the space E^5_T, we consider the equation

$$z = \Phi z, \tag{29}$$

where $z = \{u,p\}$, the components $\Phi_i(u,p),\ i = 1,2$, of operator $\Phi(u,p)$, defined by the right sides of equations (16) and (18), respectively.

Now, consider the operator $\Phi(u,p)$ in the ball $K = K_R$ of the space E^5_T. Similarly to (27), we obtain that for any $z = \{u,p\},\ z_1 = \{u_1,p_1\},\ z_2 = \{u_2,p_2\} \in K_R$ the following inequalities hold:

$$\|\Phi z\|_{E^5_T} \le A(T) + 64B(T)R^3, \tag{30}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^5_T} \le 64B(T)R^2\left(\|p_1(t)-p_2(t)\|_{C[0,T]} + \|u_1(x,t)-u_2(x,t)\|_{B^5_{2,T}}\right). \tag{31}$$

Then by (28), from (30) and (31) it follows that the operator Φ acts in the ball $K = K_R$, and satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{u,p\}$, in the ball $K = K_R$, which is a solution of equation (29), i.e. in the ball $K = K_R$ is the unique solution of the systems (16), (18).

Then the function $u(x,t)$, as an element of space $B^5_{2,T}$, is continuous and has continuous derivatives $u_x(x,t), u_{xx}(x,t), u_{xxx}(x,t)$, and $u_{xxxx}(x,t)$ in D_T.

From (9) it is easy to see that

$$\left(\sum_{k=1}^{\infty}(\lambda_k\left\|u''_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le 2(1+\beta+\alpha)(\rho(T)(1+|\delta|)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+\rho(T)(1+|\delta|)\varepsilon\left\|\psi'''(x)\right\|_{L_2(0,1)} + T(1+|\delta|\rho(T)(1+|\delta|)\varepsilon$$

$$\times\left\|p(t)g_{xxx}(x,t) + f_{xxx}(x,t) + 6u_x^3 + 18u\cdot u_x\cdot u_{xx} + 3u^2\cdot u_{xxx}\right\|_{L_2(D_T)})$$

$$+2\left\| \left\| p(t)g_x(x,t)+f_x(x,t)+3u^2\cdot u_x \right\|_{C[0,T]} \right\|_{L_2(0,1)},\ i=1,2.$$

Hence, we conclude that the function $u_{tt}(x,t)$ is continuous in the domain D_T.

Further, it is easy to verify that equation (1), and conditions (2), (3), and (5) are satisfied in the usual sense. Consequently, $\{u(x,t),p(t)\}$ is a solution of (1)-(3), (5), and by Lemma 2 it is unique in the ball $K=K_R$. The proof is complete.

From Lemma 1 and Theorem 1, implies the unique solvability of the original problem (1) - (4).

In summary, we conclude the following result.

Theorem 2. *Suppose that all assumptions of Theorem 1, and*

$$\varphi(x_0)=h(0)+\delta h(T),\ \psi(x_0)=h'(0)+\delta h'(T).$$

hold. Then problem (1)-(4) has a unique classical solution in the ball $K=K_R(||z||_{E_T^5}\leq R=A(T)+2)$.

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A Variational View on Dupuit's formula

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Abstract. In this paper, Dupuit's formula on discharge from the well is studied in dependence of bottom hole zone and layer geometries. The term "conductivity" has been used to propose a new result in this regard. The obtained result useful for deriving of new Dupuit's formulas suitable to a concrete bottom-hole zone and layer constructions. It happens thanking a variational nature of conductivity of layer. Also same approach is considered for porous medium obeying Forthamel's low.

Key Words and Phrases: filtration, porous media, viscous flow, velocity of fluid, liquid, non-Newtonian fluids.

2010 Mathematics Subject Classifications: 76A02, 76S99, 76M30, 76S05

1. Introduction

The Dupuit formula relating to a debit and a depression in the oil wells is well-known (see e.g. [1, p. 61] or [4, p.40]) . Let $\Delta p = P_k - P_c$ be a debit,- it is difference of pressures on the bottom hole zone and in the end of layer. Then the discharge Q from the well over cylindrical well-bore of radius r_c, height h is found as

$$Q = \frac{2\pi k h \Delta p}{\mu \ln \frac{R_k}{r_c}}, \tag{1}$$

where R_k is the limit radius of layer, k is its permeability, μ is viscosity of fluid (oil).

There are a lot of versions of formula (1) relating to a single hole and multi-hole cases, where different form bottom-hole zones is considered (see e.g. [2]). From those it follows that the coefficient of proportionality of discharge Q on depression Δp significantly depends on geometry of layer both at infinity γ and in the inter-layer surfaces Γ. In the paper, to characterize the impact of those geometries, we have employed the mathematical term "*conductivity*". Using this term we derive a Dupuit formula, which characterizes the coefficient of proportionality in the dependence of discharge via the depression, provided arbitrary bottom-hole zone and layer to be considered. Though this formula contains the abstract mathematical term conductivity, in general, it may be exactly calculated finding a solution of variational problem (6) below. Solution of variational problem allows to find the conductivity-$\mathbb{P}(G)$ in order to be inserted in to (7). For example, in case of formula

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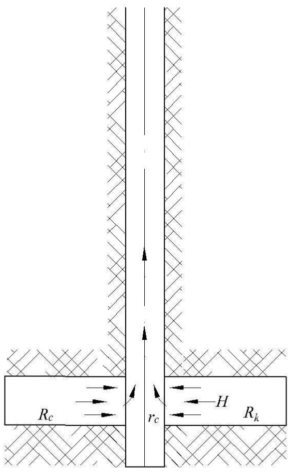

Figure 1: A circular cylinder of radius r_c height h.

(1) bottom-hole zone is a circular cylinder of height H radius r_c and layer limits in the infinity is a sphere of radius R_k (see, Figure 1)

It is a well known fact in potential theory of mathematics (see e.g. [5]) that the Wiener's capacity of a body coincides with capacity of its boundary surface. This property can also be attributed to the conductivity too. Also it is known that the capacity of an 2-dimensional surface is positive. From the Dupuit's type formula founded in the paper it is seen that the discharge increases as the contact surface ∂W of layer Ω with bottom hole zone W increase. This proves that the considered approach is true in the sense that, the discharge Q remains constant if the volume of bottom-hole zone W decreases but the capacity of its surface remains constant. In other words, it follows from formula (7) below that by taking the volume of contact zone W as for as small, but the contact surface ∂W sufficiently "big" we will increase the productivity of well. This result proves the increase of productivity of rocky and stony layers in the hydraulic fracture method exploitations. Though the interior volume in the hydraulic fractures (that stands a bottom-hole zone of well W) is almost zero over interior of the fracture, the total conductivity of the contact surface ∂W may be sufficiently larger (see, Figure 2)

This explains a reason of increase of productivity of rock and stone wells in exploitation by hydraulic fracture technologies (see e.g. [6, 7]).

In this paper, we have considered also a case of porous medium layer. In this case too a proper conductivity is introduced in order to characterize the productivity of wells. In dependence of the geometry of bottom hole zone and layer a Dupuit formula for a porous medium layer obeying Forchamel's low of filtration has been produced.

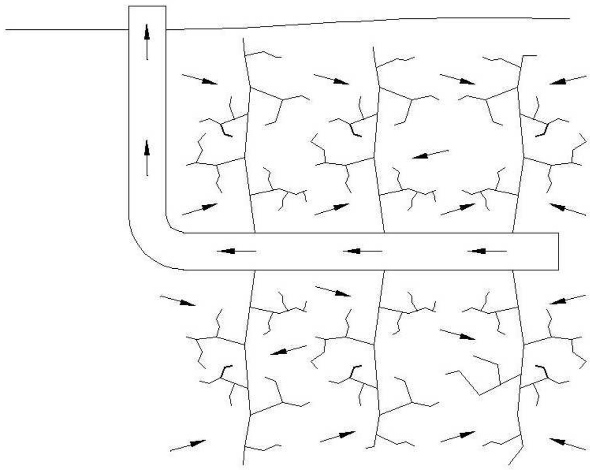

Figure 2: The hydraulic fractured well pipe.

2. A conductivity characterization of Dupuit's formula for Darcy flittering medium

Assume that the considered layer is restricted from upper and bottom by inter-layer surfaces Γ. The fluid filtrating from the medium (layer) through $\Omega\backslash$ and obeying the Darcy low arrives to the bottom-hole zone W (see, Figure 3).

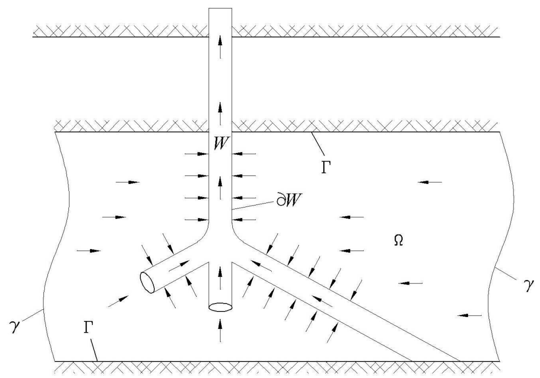

Figure 3: The multy-pipe bottom-hole well.

The filtration equation corresponding to this process in the steady stage is

$$\operatorname{div}\left(\frac{k}{\mu}\nabla\mathcal{P}\right)=0,\quad (x,y,z)\in\Omega\setminus W, \tag{2}$$

where $\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$, μ is viscosity of liquid, k -its permeability; for simplicity, having considered homogeneous and incompressible fluid, we can assume these quantities as constant. Let the bottom-hole pressure be P_c on W, the pressure at the end of medium on γ be P_k.

Denoting the discharge of well as Q we have $Q = \iint\limits_{\partial W} \rho v_n ds,$ where ds is an element of small area of surface $\partial W, Q$ -amount of fluid outing from well at unit time (the productivity of well), ρ is fluid density, $v_n = v \cdot \overline{n}$ is the liquid velocity passing through the bottom-hole zone, $\overline{n}$-unit normal to ∂W ordered out the layer. By the Darcy low [3], $v = -\frac{k}{\mu}$ grad p. For $\rho,\ k,\ \mu$ to be constants we have

$$Q = -\frac{\rho k}{\mu} \iint\limits_{\partial W} \frac{\partial p}{\partial n}\, ds, \tag{3}$$

while the boundary conditions are

$$p|_{\partial W} = P_c, \quad p|_{\gamma} = P_k, \quad \left.\frac{\partial p}{\partial n}\right|_{\Gamma} = 0;$$

Γ-the interlayer surface, γ is a limit surface of layer on infinity.

Introducing the auxiliary function $\mathcal{P} = \frac{P_k - p}{P_k - P_c}$, we have

$$\frac{\partial^2 \mathcal{P}}{\partial x^2} + \frac{\partial^2 \mathcal{P}}{\partial y^2} + \frac{\partial^2 \mathcal{P}}{\partial z^2} = 0. \tag{4}$$

and the conditions

$$\mathcal{P}|_{\partial W} = 1, \quad \mathcal{P}|_{\infty} = 0, \quad \left.\frac{\partial \mathcal{P}}{\partial n}\right|_{\Gamma} = 0.$$

Now, multiply equation (4) by $\mathcal{P}$ and integrate over the domain $G = \Omega \backslash W$. Then since $\mathcal{P}$ equals one on ∂W, it follows from Green's formula that

$$\iint\limits_{\partial W} \frac{\partial \mathcal{P}}{\partial n}\, ds = \iiint\limits_{\Omega \backslash \mathrm{W}} |\nabla \mathcal{P}|^2 dxdydz$$

Using (3) and the notation for $\mathcal{P}$ the left hand side equals $\frac{\mu Q}{\rho k (P_k - P_c)}$. Therefore,

$$\frac{\mu Q}{\rho k (P_k - P_c)} = \iiint\limits_{\Omega \backslash W} |\nabla \mathcal{P}|^2\, dxdydz \tag{5}$$

It is proved in the potential theory in mathematics [5] that, the right hand side is conductivity $\mathbb{P}(G)$ of domain $G = \Omega \setminus W$. Where also was proved that solutions of (4) are minimizers of the functional

$$\mathbb{P}(G) = \inf \iiint\limits_{G} |\nabla \psi|^2\, dxdydz\ , \tag{6}$$

over the class of functions ψ that are greater than one in W, and vanishes at the end of medium (not the interlayer surface!). Observe, the inter-layer surfaces Γ are free from the conditions for a minimizer ψ. From (5) we get $\mathbb{P}\left(\Omega \backslash W\right) = \frac{\mu Q}{\rho k (P_k - P_c)}$ or

$$Q = \frac{(P_k - P_c)\,\rho k}{\mu} \cdot \mathbb{P}(G) \tag{7}$$

The obtained formula (7) is one of the main results of the paper. In applications it can be found (or estimated) as a solution of variation problem (6) by approximate or accurate calculations for the minimizing functions ψ.

Example 1. Let the wellbore be a cylinder of radius r_c, height H. It is not difficult to show that $\mathbb{P}\left(\Omega \backslash W\right) \simeq \frac{H}{\ln \frac{R_k}{r_c}}$. Taking into account the last from (7) it follows

$$Q \simeq \frac{(P_k - P_c)\,\rho k H}{\mu \,\ln \frac{R_k}{r_c}}. \tag{8}$$

This formula is known as Dupuit's formula [4]. To prove it let us calculate the conductivity $\mathbb{P}\left(\Omega \backslash W\right)$ in formula (7). For that, we search for a minimum of variation problem (6) in the class of functions $F = f_z\left(x, y\right)\ cos \frac{\pi z}{l}$, where $f_z\left(x, y\right)$ is a function of variables x, y greater than one on lateral surface of cylinder and is zero on the infinity. Inserting the function f_z in (6), we get (8).

Exampe 2. Let well-head be a sphere of radius r_c with center at zero and the medium is a ball of radius R_k also with center in the origin. This means $\Omega = Q(0, R_k), W = Q(0, r_c)$ and $G = Q(0, R_k) \setminus Q(r_c)$. To calculate $\mathbb{P}(G)$ for this case in order to get the analog of formulas (1) or (8). Take the function

$$\psi(r) = \left(1 - \frac{r_c}{R_k}\right)^{-1} \left(\frac{r_c}{r} - \frac{r_c}{R_k}\right), \quad r_c < r < R_k, \quad r = \sqrt{x^2 + y^2 + z^2}$$

and calculate integral (6) in the right hand side

$$\mathbb{P}(G) = \iiint\limits_{Q(0,R_k)\backslash Q(0,r_c)} |\nabla \psi|^2 dxdydz = \iint\limits_{r=r_c} \frac{\partial \psi}{\partial r} ds = \frac{4\pi r_c R_k}{R_k - r_c}$$

Inserting this into (7) we get the following Dupuit's type formula

$$Q = \frac{(P_k - P_c)\,\rho k}{\mu} \cdot \frac{4\pi r_c R_k}{R_k - r_c}. \tag{9}$$

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