Caspian Journal of Applied Mathematics, Ecology and Economics
V. 8, No 1, 2020, July
ISSN 1560-4055

Inverse Boundary Value Problem for Two-Dimensional Pseudo Parabolic Equation of Third Order with Additional Integral Condition

A.I. Ismailov

Abstract. Inverse boundary value problem for two-dimensional pseudo parabolic equation of third order with additional integral condition is considered. We first reduce our problem to some equivalent (in some sense) one. Using the Fourier method, the equivalent problem, in turn, is reduced to the system of integral equations. Then, using contraction mapping method, we prove the existence and uniqueness for the solution of the system of integral equations, which is also a unique solution of the equivalent problem. Finally, using equivalence, we prove the existence and uniqueness for the classical solution of the original problem.
Key Words and Phrases: inverse boundary value problem, two-dimensional pseudo parabolic equation of third order, Fourier method, classical solution.
2010 Mathematics Subject Classifications: 65N06, 31A25, 35A09

1. Introduction

By the inverse problem for partial differential equations, we mean a problem that requires to find, along with a solution itself, the right-hand side and (or) some coefficient(s) of the equation. Inverse problems arise in many fields of human activities, such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc. which makes them one of the most important problems in today's mathematics. If an inverse problem requires to find not only the solution itself, but also the right-hand side of the equation, then such an inverse problem is linear. And if it requires to find both the solution and at least one of the coefficients, then such an inverse problem is nonlinear. Many mathematicians have studied various inverse problems for some types of partial differential equations, such as A.N.Tikhonov [1], M.M.Lavrentiev [2,3], V.K.Ivanov [4] and their students. More details about these problems can be found in the monograph by A.M.Denisov [5].

Inverse problems for one-dimensional pseudo parabolic equations of third order have been considered in [6–8].

In this work, using Fourier method and contraction mapping principle, we prove the existence and uniqueness of the solution of the nonlocal inverse boundary value problem for a third order two-dimensional pseudo parabolic equation.

2. Problem statement and its reduction to the equivalent problem

Let $D_T = Q_{xy} \times \{0 \le t \le T\}$, where $Q_{xy} = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$. Also, let $\alpha(t) > 0$, $\beta(t) > 0$, $f(x,y,t)$,$\varphi(x,y)$,$h(t)$ be the given functions defined for $x \in [0,1]$, $y \in [0,1]$, $t \in [0,T]$. Consider the following inverse boundary value problem: find a pair $\{u(x,t)\,,\ p(t)\}$ of functions $u(x,t)$, $p(t)$ which satisfy the equation

$$u_t(x,y,t) - \alpha(t)(u_{txx}(x,y,t) + u_{tyy}(x,y,t)) - \beta(t)(u_{xx}(x,y,t) + u_{yy}(x,y,t)) =$$

$$= p(t)u(x,y,t) + f(x,y,t)\,, \tag{1}$$

nonlocal initial condition

$$u(x,y,0) + \delta u(x,y,T) = \varphi(x,y) \ \ (0 \le x \le 1,\ 0 \le x \le 1)\ , \tag{2}$$

boundary conditions

$$u_x(0,y,t) = u(1,y,t) = 0 \ \ (0 \le y \le 1,\, 0 \le t \le T)\ , \tag{3}$$

$$u(x,0,t) = u_y(x,1,t) = 0 \ (0 \le y \le 1, \ \ 0 \le t \le T)\ , \tag{4}$$

and the additional condition

$$\int_0^1 \int_0^1 u(x,y,t)dxdy = h(t) \ (0 \le t \le T), \tag{5}$$

where $\delta \ge 0$ is a given number.

Denote

$$\tilde{C}^{2,2,1}(D_T) = \ \{u(x,y,t) : u(x,y,t) \in C^{2,2,1}(D_T),\, u_{txx}(x,y,t), u_{tyy}(x,y,t) \in C(D_T)\}\,.$$

Definition 1. *By the classical solution of the inverse boundary value problem* (1)-(5)*, we mean a pair* $\{u(x,y,t), p(t)\,\}$ *of functions* $u(x,t)$*,* $p(t)$ *such that* $u(x,y,t) \in \tilde{C}^{2,2,1}(D_T)$*,* $p(t) \in C[0,T]$ *and the relations* (1)-(5) *are satisfied in the usual sense.*

The following theorem is true.

Theorem 1. *Let* $0 < \alpha(t)$*,* $0 < \beta(t) \in C[0,T]$*,* $\varphi(x,y) \in C(Q_{xy})$*,* $f(x,y,t) \in C(D_T)$*,* $h(t) \in C^1[0,T]$ $h(t) \ne 0$ $(0 \le t \le T)$*,* $\delta \ge 0$*, and the coherence condition*

$$\int_0^1 \int_0^1 \varphi(x,y)dxdy = h(0) + \delta h(T)$$

be satisfied. Then the problem of finding the classical solution of the problem (1)-(5) *is equivalent to the one of determining the functions* $u(x,y,t) \in \tilde{C}^{2,2,1}(D_T)$*,*$p(t) \in C[0,T]$ *from the relations* (1)-(4)*,*

$$h'(t) - \alpha(t)\left(\int_0^1 u_{tx}(1,y,t)dy - \int_0^1 u_{ty}(x,0,t)dx\right) -$$

$$-\beta(t)\left(\int_0^1 u_x(1,y,t)dy-\int_0^1 u_y(x,0,t)dx\right)=$$

$$=p(t)h(t)+\int_0^1\int_0^1 f(x,y,t)dxdy \quad (0\le t\le T). \tag{6}$$

Proof. Let $\{u(x,y,t),p(t))\}$ be a classical solution of the problem (1)- (5). On integrating the equation (1) with respect to x and y from 0 to 1, we have:

$$\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)dxdy-$$

$$-\alpha(t)\left(\int_0^1 u_{tx}(1,y,t)-u_{tx}(0,y,t)dy+\int_0^1 u_{ty}(x,1,t)-u_{ty}(x,0,t)dx\right)-$$

$$-\beta(t)\left(\int_0^1 u_x(1,y,t)-u_x(0,y,t)dy+\int_0^1 u_y(x,1,t)-u_y(x,0,t)dx\right)=$$

$$=p(t)\int_0^1\int_0^1 u(x,y,t)dxdy+\int_0^1\int_0^1 f(x,y,t)dxdy\,(\,0\le t\le T).$$

From the last relation, by (3),(4)we obtain

$$\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)dxdy-\alpha(t)\left(\int_0^1 u_{tx}(1,y,t)dy-\int_0^1 u_{ty}(x,0,t)dx\right)-$$

$$-\beta(t)\left(\int_0^1 u_{tx}(1,y,t)dy-\int_0^1 u_{ty}(x,0,t)dx\right)=$$

$$=p(t)\int_0^1\int_0^1 u(x,y,t)dxdy+\int_0^1\int_0^1 f(x,y,t)dxdy\,(\,0\le t\le T). \tag{7}$$

Now, taking $h(t)\in C^1[0,T]$ and differentiating (5), we have

$$\int_0^1\int_0^1 u_t(x,y,t)dxdy=h'(t)\ (0\le t\le T) \tag{8}$$

By (5) and (8), it follows from (7) that the relation (6) is valid.

Now let's assume that $\{u(x,t),p(t)\}$ is a soluton of the problem (1)-(4), (5). Then from (6) and (7) we obtain

$$\frac{d}{dt}\left(\int_0^1\int_0^1 u(x,y,t)dxdy-h(t)\right)=p(t)\left(\int_0^1\int_0^1 u(x,y,t)dxdy-h(t)\right)\quad (\,0\le t\le T\,). \tag{9}$$

By (2) and the coherence condition $\int_0^1\int_0^1\varphi(x,y)dxdy=h(0)+\delta h(T)$, we have

$$\int_0^1\int_0^1 u(x,y,0)dxdy-h(0)+\delta\left(\int_0^1\int_0^1 u(x,y,T)dxdy-h(T)\right)=$$

$$= \int_0^1 \int_0^1 u(x,y,0) + \delta u(x,y,T)dxdy - (h(0) \ + \delta h(T)) =$$

$$= \int_0^1 \int_0^1 \varphi(x,y)dxdy - (h(0) \ + \delta h(T)) = 0. \tag{10}$$

The differential equation (9) has the following general solution:

$$\int_0^1 \int_0^1 u(x,y,t)dxdy - h(t) = Ce^{\int_0^t p(\tau)d\tau} \ , \tag{11}$$

where C is an arbitrary constant. Let's require that the solutions (9) satisfy the conditions (10). Then it is easy to obtain

$$C\left(1 + \delta e^{\int_0^t p(\tau)d\tau}\right) = 0 \ .$$

By $\delta \geq 0$, from the last relation we obtain $C = 0$. Substituting $C = 0$ in (11), we conclude that

$$\int_0^1 \int_0^1 u(x,y,t)dxdy - h(t) \ = 0,$$

i.e. the condition (5) holds. The theorem is proved.

3. The proof of the existence and uniqueness of the classical solution of the inverse boundary value problem

We will search for the first component $u(x,y,t)$ of the solution $\{u(x,y,t), p(t)\}$ of the problem (1)-(4), (6) in the following form:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x \sin\gamma_n y \ , \tag{12}$$

where

$$\lambda_k = \frac{\pi}{2}(2k-1) \ (k = 1,2,...), \quad \gamma_n = \frac{\pi}{2}(2n-1) \ (n = 1,2,...)\,,$$

$$u_{k,n}(t) = 4\int_0^1 \int_0^1 u(x,y,t)\cos\lambda_k x \sin\gamma_n y dxdy \ \ (k = 1,2,...; \ n = 1,2,...).$$

Using the method of separation of variables to define the sought coefficients $u_{k,n}(t)$ $(k = 1,2,...; n = 1,2,...)$ of the function $u(x,t)$, from (1), (2) we obtain

$$\left(1 + \mu_{k,n}^2\alpha(t)\right)u'_{k,n}(t) + \mu_{k,n}^2\beta(t)u_{k,n}(t) =$$

$$= F_{k,n}(t;u,p) \ (k = 1,2,...,n = 1,2,...; \ 0 \leq t \ \leq T)\,, \tag{13}$$

$$u_{k,n}(0) + \delta u_{k,n}(T) = \varphi_{k,n}(k = 1,2,...; n = 1,2,...)\,, \tag{14}$$

where

$$\mu_{k,n}^2 = \lambda_k^2 + \gamma_n^2 \ (k = 1, 2, ...; n = 1, 2, ...),$$

$$F_{k,n}(t; u, p) = f_{k,n}(t) + p(t)u_{k,n}(t) \, (k = 1, 2, ...; n = 1, 2, ...) \,,$$

$$f_{k,n}(t) = 4 \int_0^1 \int_0^1 f(x, y, t) \cos \lambda_k x \sin \gamma_n y dx dy \ \ (k = 1, 2, ...; \ n = 1, 2, ...),$$

$$\varphi_{k,n} = 4 \int_0^1 \int_0^1 \varphi(x, y) \cos \lambda_k x \sin \gamma_n y dx dy \ \ (k = 1, 2, ...; \ n = 1, 2, ...).$$

Solving the problem (13), (14), we find

$$u_{k,n}(t) = \frac{\varphi_{k,n} e^{-\int_0^t \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}}}{1 + \delta e^{-\int_0^T \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}}} + \int_0^t \frac{F_{k,n}(\tau; u, p)}{1 + \mu_{k,n}^2 \alpha(\tau)} e^{-\int_\tau^t \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}} d\tau -$$

$$- \frac{\delta e^{-\int_0^T \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}}}{1 + \delta e^{-\int_0^T \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}}} \int_0^T \frac{F_{k,n}(\tau; u, p)}{1 + \mu_{k,n}^2 \alpha(\tau)} e^{-\int_\tau^t \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}} d\tau \ \ (k = 1, 2, ...; n = 1, 2, ...). \quad (15)$$

Substituting the expressions $u_{k,n}(t)$ $(k = 1, 2, \ldots; n = 1, 2, ...)$ in (12), we have

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{\varphi_{k,n} e^{-\int_0^t \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}}}{1 + \delta e^{-\int_0^T \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}}} + \int_0^t \frac{F_{k,n}(\tau; u, p)}{1 + \mu_{k,n}^2 \alpha(\tau)} e^{-\int_\tau^t \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}} d\tau - \right.$$

$$\left. - \frac{\delta e^{-\int_0^T \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}}}{1 + \delta e^{-\int_0^T \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}}} \int_0^T \frac{F_{k,n}(\tau; u, p)}{1 + \mu_{k,n}^2 \alpha(\tau)} e^{-\int_\tau^t \frac{\mu_{k,n}^2 \beta(s) ds}{1+\mu_{k,n}^2 \alpha(s)}} d\tau \right\} \cos \lambda_k x \sin \gamma_n y. \quad (16)$$

Now, from (6), by (12), we obtain

$$h'(t) + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{k+1} \left(\frac{\gamma_n}{\lambda_k} - \frac{\lambda_k}{\gamma_n} \right) \left(\alpha(t) u'_{k,n}(t) + \beta(t) u_{k,n}(t) \right) =$$

$$= p(t)h(t) + \int_0^1 \int_0^1 f(x, y, t) dx dy \qquad (0 \le t \le T) \,. \quad (17)$$

Further, from (13) we have

$$\mu_{k,n}^2 \left(\alpha(t) u'_{k,n}(t) + \beta(t) u_{k,n}(t) \right) = \ F_{k,n}(t; u, p) \ \ - u'_{k,n}(t) = \frac{\mu_{k,n}^2 \beta(t)}{1 + \mu_{k,n}^2 \alpha(t)} u_{k,n}(t) +$$

$$+\frac{\mu_{k,n}^2\alpha(t)}{1+\mu_{k,n}^2\alpha(t)}F_{k,n}(t;u,p)(k=1,2,...,n=1,2,...;\ 0\le t\ \le T)\,,$$

or

$$\alpha(t)u'_{k,n}(t)+\beta(t)u_{k,n}(t)\ =\frac{\beta(t)}{1+\mu_{k,n}^2\alpha(t)}u_{k,n}(t)+$$

$$+\frac{\alpha(t)}{1+\mu_{k,n}^2\alpha(t)}F_{k,n}(t;u,p)(k=1,2,...,n=1,2,...;\ 0\le t\ \le T). \tag{18}$$

From (17), taking into account (18), we obtain

$$p(t)=[h(t)]^{-1}\left\{h'\left(t\right)-\int_0^1\int_0^1 f(x,y,t)dxdy+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+1}\left(\frac{\gamma_n}{\lambda_k}-\frac{\lambda_k}{\gamma_n}\right)\times\right.$$

$$\left.\times\left(\frac{\beta(t)}{1+\mu_{k,n}^2\alpha(t)}u_{k,n}(t)\ +\ \frac{\alpha(t)}{1+\mu_{k,n}^2\alpha(t)}F_{k,n}(t;u,p)\right)\right\} \tag{19}$$

To obtain the equation for the second component $p(t)$ of the solution $\{u(x,t),p(t)\}$ of the problem (1)-(4), (5), we substitute the expression (15) in (19) to get

$$p(t)=[h(t)]^{-1}\left\{h'\left(t\right)-\int_0^1\int_0^1 f(x,y,t)dxdy+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+1}\left(\frac{\gamma_n}{\lambda_k}-\frac{\lambda_k}{\gamma_n}\right)\times\right.$$

$$\times\left(\frac{\beta(t)}{1+\mu_{k,n}^2\alpha(t)}\left[\frac{\varphi_{k,n}e^{-\int_0^t\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}{1+\delta e^{-\int_0^T\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}+\int_0^t\frac{F_{k,n}(\tau;u,p)}{1+\mu_{k,n}^2\alpha(\tau)}e^{-\int_\tau^t\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}d\tau-\right.\right.$$

$$\left.-\frac{\delta e^{-\int_0^T\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}{1+\delta e^{-\int_0^T\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}}\int_0^T\frac{F_{k,n}(\tau;u,p)}{1+\mu_{k,n}^2\alpha(\tau)}e^{-\int_\tau^t\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}}d\tau\right]+$$

$$\left.\left.+\frac{\alpha(t)}{1+\mu_{k,n}^2\alpha(t)}F_{k,n}(t;u,p)\right)\right\}. \tag{20}$$

Thus, the solution of the problem (1)-(4), (6) is reduced to the solution of the system (16), (20) with respect to the unknown functions $u(x,y,t)$ and $p(t)$.

To treat the uniqueness of the solution of (1)-(4), (6), we will significantly use the following lemma.

Lemma 1. *If* $\{u(x,y,t),p(t)\}$ *is any solution of the problem* (1)-(4), (6)*, then the functions*

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\ \ (k=1,2,...;\ n=1,2,...)$$

satisfy the system (15) *on* $[0,T]$.

Proof. Let $\{u(x,y,t), p(t)\}$ be any solution of the problem (1)-(4), (6). Then, multiplying both sides of the equation (1) by the function $4\cos\lambda_k x \sin\gamma_n y$ $(k=1,2,...;\ n=1,2,...)$, integrating the obtained equality with respect to x and y from 0 to 1 and using the relations

$$4\int_0^1\int_0^1 u_t(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy =$$

$$=\frac{d}{dt}\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=u'_{k,n}(t)(k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{xx}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy =$$

$$=-\lambda_k^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=-\lambda_k^2 u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{yy}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy =$$

$$-\gamma_n^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right)=-\gamma_k^2 u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{txx}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy = -\lambda_k^2 u'_{k,n}(t)\,(k=1,2,...;\ n=1,2,...),$$

$$4\int_0^1\int_0^1 u_{tyy}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy = -\gamma_n^2 u'_{k,n}(t)\,(k=1,2,...;\ n=1,2,...),$$

we get the validity of the equation (13).

Similarly, from (2) it follows that the condition (14) holds.

Thus, $u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...)$ are the solutions of the problem (13), (14). Hence it directly follows that the functions $u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...)$ satisfy the system (15) on $[0,T]$. The lemma is proved.

It is clear that if $u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy$ $(k=1,2,...;\ n=1,2,...)$ are the solutions of the system (15), then the pair $\{u(x,y,t),p(t)\}$ of the functions $u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y$ and $p(t)$ is a solution of the system (16), (20).

Lemma 1 has the following corollary.

Corollary 1. *Let the system* (16), (20) *have a unique solution. Then the problem* (1)-(4), (6) *cannot have more than one solution, i.e. if the problem* (1)-(4),(6) *has a solution, then it is unique.*

1. Denote by $B^3_{2,T}$ [9] the totality of all functions $u(x,y,t)$ of the form

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y$$

in D_T, where each of the functions $u_{k,n}(t)\,(k = 1,2,...;\ n = 1,.2,..)$ is continuously differentiable on $[0,T]$ and

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\,\|u_{k,n}(t)\|_{C[0,T]}\right)^2\ \right\}^{\frac{1}{2}} < +\infty.$$

Define the norm on this set as follows:

$$\|u(x,y,t)\|_{B_{2,T}^3} = \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\,\|u_{k,n}(t)\|_{C[0,T]}\right)^2\ \right\}^{\frac{1}{2}}.$$

2. Denote by E_T^3 the space consisting of topological product

$$B_{2,T}^3 \times C[0,T]\,.$$

The norm of the element $z = \{u,p\}$ is defined by the formula

$$\|z\|_{E_T^3} = \|u(x,y,t)\|_{B_{2,T}^3} + \|p(t)\|_{C[0,T]}\,.$$

It is known that $B_{2,T}^3$ and $E_T^{3;}$ are Banach spaces.

Now let's consider in the space E_T^3 the operator

$$\Phi(u,p) = \{\Phi_1(u,p), \Phi_2(u,p)\},$$

where

$$\Phi_1(u,p) = \tilde{u}(x,y,t) \equiv \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\tilde{u}_{k,n}(t)\cos\lambda_k x\sin\gamma_n y\ \ ,$$

$$\Phi_2(u,p) = \tilde{p}(t),,$$

and $\tilde{u}_{k,n}(t)$ $(k = 1,2,...;\ n = 1,2,...)$ and $\tilde{p}(t)$ are equal to the right-hand sides of (15) and (20), respectively.

It is not difficult to see that

$$1+\mu_{k,n}^2\alpha(t) > \mu_{k,n}^2\alpha(t), \quad \frac{\mu_{k,n}^2\beta(t)}{1+\mu_{k,n}^2\alpha(t)} < \frac{\beta(t)}{\alpha(t)}, \quad \frac{\mu_{k,n}^2\alpha(t)}{1+\mu_{k,n}^2\alpha(t)} < 1,$$

$$1+\delta e^{-\int_0^T\frac{\mu_{k,n}^2\beta(s)ds}{1+\mu_{k,n}^2\alpha(s)}} \geq 1\,,\ \mu_{k,n}^3 \leq (\lambda_k^2+\gamma_k^2)(\lambda_k+\gamma_n) = \lambda_k^3+\lambda_k^2\,\gamma_n+\gamma_k^2\lambda_k+\gamma_k^3.$$

From these relations we obtain

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\,\|\tilde{u}_{k,n}(t)\|_{C[0,T]}\right)^2\ \right\}^{\frac{1}{2}} \leq 3\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\ \ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}} +$$

$$+3\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\ \left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+3\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\ \left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+3\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\ \left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+3(1+\delta)\left\|\frac{1}{\alpha(t)}\right\|_{C[0,T]}\times$$

$$\times\left[\sqrt{T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\right)+\right.$$

$$\left.+T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\ \left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right],\qquad(21)$$

$$\left\|\tilde{p}(t)\right\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h'(t)-\int_0^1\int_0^1 f(x,y,t)dxdy\right\|_{C[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\frac{\beta(t)}{\alpha(t)}\right\|_{C[0,T]}\left(\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\right.\right.$$

$$+\left\|\frac{1}{\alpha(t)}\right\|_{C[0,T]}\left(\sqrt{T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left|f_{k,n}(\tau)\right|^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_k^3\ \left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right)\right]+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\left|f_{k,n}(t)\right|\right)^2d\tau\right)^{\frac{1}{2}}+$$

$$\left.+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n\left|f_{k,n}(t)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\left\|p(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_k^3\ \left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right\}.\qquad(22)$$

Assume that the data of the problem (1)-(4), (6) satisfy the following conditions:

$$1.\varphi(x,y),\varphi_x(x,y),\varphi_{xx}(x,y),\varphi_y(x,y),\varphi_{xy}(x,y),\varphi_{yy}(x,y)\in C(\bar{Q}_{xy}),$$

$$\varphi_{xxy}(x,y),\varphi_{xyy}(x,y),\varphi_{xxx}(x,y),\varphi_{yyy}(x,y)\in L_2(Q_{xy}),$$
$$\varphi_x(0,y)=\varphi(1,y)=\varphi_{xx}(1,y)=0\,(0\le y\le 1),$$
$$\varphi(x,0)=\varphi_y(x,1)=\varphi_{yy}(x,0)=0\,(0\le x\ \le 1).$$

$$2.f(x,y,t)\in C(D_T),\ f_x(x,y,t),f_y(x,y,t)\in L_2(D_T),$$
$$f\ (1,y,t)=f\ (x,0,t)=0\,(0\le x,y\le 1,\ \ 0\le t\le T).$$

$$3.\delta \geq 0, \quad 0 < \alpha(t) \in C[0,T], 0 < \beta(t) \in C[0,T], \quad h(t) \in C^1[0,T] \ ,$$
$$h(t) \neq 0 \ \ (0 \leq t \leq T).$$

Then from (21)- (22) we obtain

$$\|u(x,y,t)\|_{B^3_{2,T}} = \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_{k,0}(t)\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\mu_k^3 \|u_{k,n}(t)\|_{C[0,T]} \right)^2 \ \right\}^{\frac{1}{2}} \leq$$
$$\leq A_1(T) + B_1(T) \|p(t)\|_{C[0,T]} \|u(x,y,t)\|_{B^3_{2,T}} \,, \tag{23}$$
$$\|\tilde{p}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T) \|p(t)\|_{C[0,T]} \|u(x,y,t)\|_{B^3_{2,T}} \,, \tag{24}$$

where

$$A_1(T) = 5 \|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + 3 \|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} + 3 \|\varphi_{xxy}(x,y)\|_{L_2(Q_{xy})} +$$
$$+3 \|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + (1+\delta) \left\| \frac{1}{\alpha(t)} \right\|_{C[0,T]} \sqrt{T} \left(5 \|f_x(x,y,t)\|_{L_2(D_T)} + 3 \|f_y(x,y,t)\|_{L_2(D_T)} \right),$$
$$B_1(T) = 5(1+\delta) \left\| \frac{1}{\alpha(t)} \right\|_{C[0,T]} T,$$
$$A_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \left\| h'(t) - \int_0^1 \int_0^1 f(x,y,t) dx dy \right\|_{C[0,T]} + \right.$$
$$+ \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^{-2} \right)^{\frac{1}{2}} \left[\left\| \frac{\beta(t)}{\alpha(t)} \right\|_{C[0,T]} \left(\|\varphi_x(x,y)\|_{L_2(Q_{xy})} + \|\varphi_y(x,y)\|_{L_2(Q_{xy})} \right. \right.$$
$$\left. + \left\| \frac{1}{\alpha(t)} \right\|_{C[0,T]} \sqrt{T} \|f(x,y,t)\|_{L_2(D_T)} \right) + \left\| \|f_x(x,y,t)\|_{C[0,T]} \right\|_{L_2(Q_{xy})} +$$
$$\left. \left. + \left\| \|f_y(x,y,t)\|_{C[0,T]} \right\|_{L_2(0,1)} \right] \right\},$$
$$B_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^{-2} \right)^{\frac{1}{2}} \times$$
$$\times \left[\left\| \frac{1}{\alpha(t)} \right\|_{C[0,T]} \left\| \frac{\beta(t)}{\alpha(t)} \right\|_{C[0,T]} T + 1 \right].$$

From the inequalities (23)-(24) it follows

$$\|\tilde{u}(x,y,t)\|_{B^3_{2,T}} + \|\tilde{p}(t)\|_{C[0,T]} \leq$$

$$\leq A(T) + B(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{25}$$

where

$$A(T) = \sum_{i=1}^{2} A_i(T), \;\; B(T) = \sum_{i=1}^{2} B_i(T), \, .$$

So we can prove the following theorem.

Theorem 2. *Let the conditions 1-4 be satisfied and*

$$(A(T)+2)^2 B(T) < 1. \tag{26}$$

Then the problem (1)-(4), (6) *has a unique solution in the ball* $K = K_R(\|z\|_{E^3_T} \leq R = A(T)+2)$ *of the space* E^3_T.

Proof. Consider in the space E^3_T the equation

$$z = \$z, \tag{27}$$

where $z = \{u,p\}$, and the components $\Phi_i(u,p)(i=1,2)$ of the operator $\$(u,p)$ are defined by the right-hand sides of the equations (16), (20), respectively. Consider the operator $\$(u,p)$ in the ball $K = K_R(\|z\|_{E^3_T} \leq R = A(T)+2)$ of E^3_T.

Similar to (25), we obtain the following estimates for every $z, z_1, z_2 \in K_R$:

$$\|\Phi z\|_{E^3_T} \leq A(T) + B(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,y,t)\|_{B^3_{2,T}}\,, \tag{28}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^2_T} \leq B(T) R \left(\|p_1(t)-p_2(t)\|_{C[0,T]} + \|u_1(x,y,t)-u_2(x,y,t)\|_{B^3_{2,T}} \right). \tag{29}$$

Then from the estimates (28) and (29), by (26), it follows that the operator \$ acts in the ball $K = K_R$ and is a contraction operator. Therefore, the operator \$ has a unique fixed point $\{u,p\}$ in the ball $K = K_R$, which is a unique solution of the equation (27), i.e. a unique solution of the system (16), (20) in the ball $K = K_R$.

As an element of the space $B^3_{2,T}$, the function $u(x,y,t)$ is continuous and has continuous derivatives $u_x(x,y,t)$, $u_{xx}(x,y,t)$, $u_y(x,y,t)$, $u_{xy}(x,y,t)$, $u_{yy}(x,y,t)$ in D_T.

Now it is not difficult to see from (13) that

$$\left\{ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left(\mu_{k,n} \left\| u'_{k,n}(t) \right\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}} \leq \sqrt{2} \left\| \frac{1}{\alpha(t)} \right\|_{C[0,T]} \Big[\|u(x,y,t)\|_{B^3_{2,T}} +$$

$$+ \left\| \|f(x,y,t) + p(t)u(x,y,t)\|_{C[0,T]} \right\|_{L_2(Q_{xy})} \Big].$$

Hence, it is clear that $u_t(x,y,t)$, $u_{txx}(x,y,t)$, $u_{tyy}(x,y,t)$ are continuous in D_T.

It is not difficult to verify that the equation (1) and the conditions (2)-(4), (6) are satisfied in the usual sense. Thus, the solution of the problem (1)–(4), (6) is a pair of functions $\{u(x,t), p(t)\}$. By the corollary of Lemma 1, this solution is unique in the ball $K = K_R$. The theorem is proved.

Using Theorems 1 and 2, we obtain the unique solvability of the problem (1)–(5).

Theorem 3. *Let all the conditions of Theorem 2 be satisfied and the coherence conditions*

$$\int_0^1 \int_0^1 \varphi(x,y)dxdy = h(0) + \delta h(T)$$

hold. Then the problem (1)-(5) *has a unique classical solution in the ball* $K = K_R(\|z\|_{E_T^3} \leq R = A(T)+2)$ *of the space* E_T^3.

References

[1] A.I. Tikhonov, *On stability of inverse problems*, Dokl. AN SSSR., **39(5)**, 1943, 195-198 (*in Russian*)

[2] M.M. Lavrentiev, *On one inverse problem for wave equation*, Dokl. AN SSSR., **157(3)**, 1964, 520-521 (*in Russian*)

[3] M.M. Lavrentiev, V.G. Romanov , S.T. Shishatski, *Noncorrect problems of mathematical physics and analysis*, M.: Nauka, 1980, 288 p. (*in Russian*)

[4] V.K. Ivanov, V.V. Vasin, V.P. Tanina, *Theory of linear noncorrect problems and its applications*, M.: Nauka, 1978, 206 p. (*in Russian*)

[5] A.M. Denisov, *Introduction to the theory of inverse problems*, M.: MGU, 1994, 206 p. (*in Russian*)

[6] Y.T.Mehraliyev, G.K. Shafiyeva, *On an inverse boundary value problem for a pseudo parabolic third order equation with integral condition of the first kind*, Journal of Mathematical Sciences, 2015, **204(3)**, 343-350

[7] Y.T.Mehraliyev, G.K.Shafiyeva, *Determination of an unknown coefficient in the third order pseudoparabolic equation with non-self-adjoint boundary conditions*, Journal of Applied Mathematics, 2014

[8] Y.T.Mehraliyev, G.K.Shafiyeva, *Inverse boundary value problem for the pseudoparabolic equation of the third order with periodic and integral conditions*, Applied Mathematical Sciences 8(23), 1145-1155.

[9] K.I. Khudaverdiyev , A.A. Veliyev, *Study of one-dimensional mixed problem for one class of pseudo hyperbolic equations of third order with nonlinear operator right-hand side*, Baku, Casioglu, 2010, 168 p. (*in Russian*)

A.I. Ismailov
Baku State University
E-mail: ismayilov.arif58@mail.ru

Received 28 April 2020
Accepted 12 June 2020

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 8, No 1, 2020, July
ISSN 1560-4055

On the Solvability of One Inverse Boundary Value Problem for the Linearized Benny–Luc Equation with Non-self-adjoint Boundary Conditions

B.K. Velieva

Abstract. An inverse problem is investigated for the linearized Benny-Luc equation with non-self-adjoint boundary conditions. First, the original problem is reduced to an equivalent problem (in a certain sense), for which the existence and uniqueness theorem is proved. Further, on the basis of these facts, the existence and uniqueness of the classical solution to the original problem are proved.

Key Words and Phrases: inverse boundary value problem, Benny–Luc equation, existence, uniqueness of classical solution.

2010 Mathematics Subject Classifications: 35R30

1. Introduction

Many problems in mathematical physics and continuum mechanics are boundary value problems that reduce to the integration of a differential equation or a system of partial differential equations for given boundary and initial conditions. Many problems in gas dynamics, the theory of elasticity, the theory of plates and shells are reduced to the consideration of high-order partial differential equations [1]. Differential equations of the fourth order are of great interest from the point of view of applications (see, for example, [2, 3]). Partial differential equations of Benny – Luc type have applications in mathematical physics (see [3]).

Problems in which, together with the solution of a particular differential equation, it is also required to determine the coefficient (coefficients) of the equation itself, or the right side of the equation, in mathematics and in mathematical modeling are called inverse problems. The theory of inverse problems for differential equations is a dynamically developing branch of modern science. Recently, inverse problems have arisen in various fields of human activity, such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc., which puts them in a number of urgent problems of modern mathematics. Various inverse problems for certain types of partial differential equations have been studied in many works. Let us first of all note here the works of A.N. Tikhonov

[4], M.M. Lavrent'ev [5, 6], V.K. Ivanov [7] and their students. More details can be found in the monograph by A.M. Denisov [8].

The theory of inverse boundary value problems for fourth-order equations is still understudied. The works [9–12] are devoted to inverse boundary value problems for the Benny – Luc equation.

The aim of this work is to prove the existence and uniqueness of solutions to the inverse boundary value problem for the Benny-Luc equation with non-self-adjoint boundary conditions.

2. Statement of the problem and its reduction to an equivalent problem

Let $D_T = \{(x,t): \ 0 \le x \le 1, \ \ 0 \le t \le T\}$. Consider the following inverse boundary value problem in a rectangle D_T: find a pair $\{u(x,t)\,,\ a(t)\}$ of functions $u(x,t)$, $a(t)$ satisfying the equation [3]

$$u_{tt}(x,t) - u_{xx}(x,t) + \alpha u_{xxxx}(x,t) - \beta u_{xxtt}(x,t) = \ a(t)u(x,t) + f(x,t) \ \ (x,t) \in D_T, \quad (1)$$

with initial conditions

$$u(x,0) = \varphi(x), \ \ u_t(x,0) = \psi(x) \ \ \ (0 \le x \le 1), \quad (2)$$

with non-self-adjoint boundary conditions

$$u(1,t) = 0, u_x(0,t) = u_x(1,t) \ \ , \ \ u_{xx}(1,t) \ = 0, \ u_{xxx}(0,t) = u_{xxx}(1,t) \ \ (0 \le t \le T) \quad (3)$$

and with the additional condition

$$u(0,t) = h(t) \quad\ (0 \le t \le T), \quad (4)$$

where $\alpha > 0$, $\beta > 0$ - are fixed numbers, $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h(t)$ - are given functions.

Denote

$$\tilde{C}^{4,2}(D_T) = \left\{u(x,t): \ u(x,t) \in C^2(D_T)\,, u_{ttx}(x,t), \right.$$
$$\left. u_{ttxx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t) \in C(D_T)\right\}.$$

Definition 1. *By the classical solution of the inverse boundary value problem* (1) - (4) *we mean a pair* $\{u(x,t)\,,\ a(t)\}$ *of functions* $u(x,t) \in \tilde{C}^{4,2}(D_T)$, $a(t) \in C[0,T]$, *satisfying equation* (1) *and conditions* (2) - (4) *in the usual sense.*

Similarly to [13], the following theorem is proved.

Theorem 1. *Let* $\varphi(x), \ \ \psi(x) \in C[0,1], \ \ h(t) \in C^2[0,T]\ , \ h(t) \ne 0 \ \ (0 \le t \le T), \ \ f(x,t) \in C(D_T)$ *and the conditions of consistency are hold*

$$\varphi(0) = h(0)\ , \ \ \ \psi(0) = h'0). \quad (5)$$

Then the problem of finding a classical solution to problem (1) - (4) *is equivalent to the problem of determining the functions* $u(x,t) \in \tilde{C}^{4,2}(D_T)$ *and* $a(t) \in C[0,T]$ *from relations* (1) - (3) *and the condition*

$$h''(t) - u_{xx}(0,t) + \alpha u_{xxxx}(0,t) - \beta u_{xxtt}(0,t) = \ a(t)h(t) + f(0,t) \ \ (\,0 \le t \le T\,)\,. \quad (6)$$

3. Solvability of the inverse boundary value problem

It is known that [14], function sequences

$$X_0(x) = 2(1-x),\ \ X_{2k-1}(x) = 4(1-x)\cos\lambda_k x,\ \ X_{2k}(x) = 4\sin\lambda_k x\ (k = 1,2,...),\quad (7)$$

$$Y_0(x) = 1,\ \ Y_{2k-1}(x) = \cos\lambda_k x,\ \ Y_{2k}(x) = x\sin\lambda_k x\ (k = 1,2,...)\quad (8)$$

form a biorthogonal system, and system (7) forms a Riesz basis for $L_2(0,1)$, where $\lambda_k = 2k\pi\,(k = 1,2,...)$. Then an arbitrary function $\vartheta(x) \in L_2(0,1)$ is expanded into a biorthogonal series:

$$\vartheta(x) = \vartheta_0\,X_0(x) + \sum_{k=1}^{\infty}\vartheta_{2k-1}\ X_{2k-1}(x)\ +\ \sum_{k=1}^{\infty}\vartheta_{2k}\ X_{2k}(x)\,,$$

where

$$\vartheta_0\ = \int_0^1 \vartheta_0\,Y_0\,(x)dx\,,\ \ \vartheta_{2k-1}\ = \int_0^1 \vartheta_{2k-1}\,Y_{2k-1}\,(x)dx\,,\ \vartheta_{2k-1}\ = \int_0^1 \vartheta_{2k-1}\,Y_{2k-1}\,(x)dx\,.$$

It is known that [15],

$$\vartheta(x) \in C^{2i-1}[0,1],\quad \vartheta^{(2i)}(x) \in L_2(0,1)\,,$$

$$\vartheta^{(2s)}(1) = 0,\ \ \vartheta^{(2s+1)}(0) = \vartheta^{(2s)}(1)\ \ (s = \overline{0, i-1}\ \ ),$$

then

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i}\vartheta_{2k-1}\ \right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i)}(x)\right\|^2_{L_2(0,1)},$$

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i}\vartheta_{2k}\ \right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i)}(x)x + 2i\vartheta^{(2i-1)}(x)\right\|^2_{L_2(0,1)}.\quad (9)$$

Under assumptions

$$\vartheta(x) \in C^{2i}[0,1],\quad \vartheta^{(2i+1)}(x) \in L_2(0,1)\,,$$

$$\vartheta^{(2s)}(1) = 0,\ \ \vartheta^{(2s-1)}(0) = \vartheta^{(2s-1)}(1)\ \ (i \ge 1,\ s = \overline{0, i}\ \ ),$$

the validity of estimates [15]:

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i+1}\vartheta_{2k-1}\ \right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i+1)}(x)\right\|^2_{L_2(0,1)},$$

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i+1}\vartheta_{2k}\ \right)^2 \le \frac{1}{2}\left\|\vartheta^{(2i+1)}(x)x + (2i+1)\vartheta^{(2i)}(x)\right\|^2_{L_2(0,1)}.\quad (10)$$

is established. In order to study problem (1) - (3), (6), consider the following space.

Denote by $B^5_{2,T}$ [15] the collection of all functions $u\,(x,t)$ of the form

$$u\,(x,t)=\sum_{k=0}^{\infty}u_k\,(t)\,X_k\,(x)\,,$$

considered on D_T, for which all functions $u_k\,(t)\in C[0,T]$ and

$$J_T(u)\equiv\|u_0(t)\|_{C[0,T]}+$$

$$+\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},$$

where the function $X_k\,(x)\ \ (k=0,1,2,...)$ are defined by (7).

The norm in this set is defined as follows: $\|u(x,t)\|_{B^5_{2,T}}=J_{"}(u)$.

Let E^5_T denote the space of vector functions $\{u(x,t),a(t)\}$ such that $u\,(x,t)\in B^5_{2,T}$, $a\,(t)\in C[0,T]$. Equip this space with a norm

$$\|z\|_{E^5_T}=\|u(x,t)\|_{B^5_{2,T}}+\|a(t)\|_{C[0,T]}\,.$$

It is clear that $B^5_{2,T}$ and E^5_T are Banach spaces.

Since system (7) forms a Riesz basis in $L_2\,(0,1)$ and system (7) and (8) forms biorthogonal to the system of functions in $L_2\,(0,1)$, then the first component $u\,(x,t)$ of the solution $\{u(x,t),a(t)\}$ of problem (1) - (3), (6) will be sought in the form

$$u\,(x,t)=u_0\,(t)\,X_0\,(x)+\sum_{k=1}^{\infty}u_{2k-1}\,(t)\,X_{2k-1}\,(x)+\sum_{k=1}^{\infty}u_{2k}\,(t)\,X_{2k}\,(x)\,,\tag{11}$$

where

$$u_0(t)=\int_0^1 u(x,t)Y_0(x)dx,$$

$$u_{2k-1}(t)=\int_0^1 u(x,t)Y_{2k-1}(x)dx,\ \ u_{2k}(t)=\int_0^1 u(x,t)Y_{2k}(x)dx\ \ (k=1,2,...)\,,\tag{12}$$

is the solution of the following problem:

$$u_0''\,(t)=F_0\,(t;u,a)\quad(0\le t\le T),\tag{13}$$

$$u_{2k-1}''(t)+\beta_k^2u_{2k-1}(t)=\frac{1}{1+\beta\lambda_k^2}F_{2k-1}\,(t;u,a)\quad(0\le t\le T,\ k=1,2,...),\tag{14}$$

$$u_{2k}''\,(t)+\beta_k^2u_{2k}\,(t)=\frac{1}{1+\beta\lambda_k^2}F_{2k}\,(t;u,a)+$$

$$+\frac{2\lambda_k(1+2\alpha\lambda_k^2)}{1+\beta\lambda_k^2}u_{2k-1}\,(t)+\frac{2\beta\lambda_k}{1+\beta\lambda_k^2}\,u_{2k-1}''\,(t)\ \ (0\le t\le T,\ \ k=1,2,...),\tag{15}$$

$$u_k(0) = \varphi_k, \quad u'_k(0) = \psi_k \quad (k = 0, 1, 2, ...)\,, \tag{16}$$

moreover

$$\beta_k^2 = \frac{\lambda_k^2(1 + \alpha\lambda_k^2)}{1 + \beta\lambda_k^2}, F_k(t; u, a) = a(t)u_k(t) + f_k(t), \quad f_k(t) = \int_0^1 f(x, t)Y_k(x)dx,$$

$$\varphi_k = \int_0^1 \varphi(x)Y_k(x)dx, \quad \psi_k = \int_0^1 \psi(x)Y_k(x)dx \quad (k = 0, 1, ...).$$

Solving problem (13) - (16) we find:

$$u_0(t) = \varphi_0 + \psi_0 t + \int_0^t (t - \tau)F_0(\tau; u, a)d\tau, \tag{17}$$

$$u_{2k-1}(t) = \varphi_{2k-1}\cos\beta_k t + \frac{1}{\beta_k}\psi_{2k-1}\sin\beta_k t + \frac{1}{\beta_k(1 + \beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau; u, a)\sin\beta_k(t - \tau)d\tau, \tag{18}$$

$$u_{2k}(t) = \varphi_{2k}\cos\beta_k t + \frac{1}{\beta_k}\psi_{2k}\sin\beta_k t + \frac{1}{\beta_k(1 + \beta\lambda_k^2)}\int_0^t F_{2k}(\tau; u, a)\sin\beta_k(t - \tau)d\tau +$$

$$+\frac{\lambda_k(1 + 2\alpha\lambda_k^2 + \alpha\beta\lambda_k^4)}{(1 + \beta\lambda_k^2)^3}\left[t\varphi_{2k-1}\sin\beta_k t + \left(\frac{1}{\beta_k}\sin\beta_k t - t\cos\beta_k t\right)\frac{1}{\beta_k}\psi_{2k-1} +\right.$$

$$\left.+\frac{1}{\beta_k(1 + \beta\lambda_k^2)}\int_0^t \left(\int_0^\tau F_{2k-1}(\xi; u, a)\sin\beta_k(t - \xi)\,d\xi\right)\sin\beta_k(t - \tau)\,d\tau\right] +$$

$$+\frac{2\beta\lambda_k}{\beta_k(1 + \beta\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau; u, a)\sin\lambda_k(t - \tau)d\tau. \tag{19}$$

After substituting the expression $u_k(t)$ $(k = 0, 1, ...)$ in (11), to determine the component $u(x, t)$ of the solution to problem (1) - (3), (6), we obtain:

$$u(x, t) = \left(\varphi_0 + \psi_0 t + \int_0^t (t - \tau)F_0(\tau; u, a)d\tau\right)X_0(x) +$$

$$+\left\{\varphi_{2k-1}\cos\beta_k t + \frac{1}{\beta_k}\psi_{2k-1}\sin\beta_k t +\right.$$

$$\left.\frac{1}{\beta_k(1 + \beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau; u, a)\sin\beta_k(t - \tau)d\tau\right\}X_{2k-1}(x) +$$

$$+\sum_{k=1}^{\infty}\left\{\varphi_{2k}\cos\beta_k t + \frac{1}{\beta_k}\psi_{2k}\sin\beta_k t + \frac{1}{\beta_k(1 + \beta\lambda_k^2)}\int_0^t F_{2k}(\tau; u, a)\sin\beta_k(t - \tau)d\tau +\right.$$

$$+\frac{\lambda_k(1 + 2\alpha\lambda_k^2 + \alpha\beta\lambda_k^4)}{(1 + \beta\lambda_k^2)^3}\left[t\varphi_{2k-1}\sin\beta_k t + \left(\frac{1}{\beta_k}\sin\beta_k t - t\cos\beta_k t\right)\frac{1}{\beta_k}\psi_{2k-1} +\right.$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t\left(\int_0^\tau F_{2k-1}\left(\xi;u,a\right)\sin\beta_k\left(t-\xi\right)d\xi\right)\sin\beta_k\left(t-\tau\right)d\tau\Bigg]+$$

$$+\frac{2\beta\lambda_k}{\beta_k(1+\beta\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\Bigg\}X_{2k}(x). \tag{20}$$

Now, from (6), taking into account (11), we have:

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(0,t)+4\sum_{k=1}^{\infty}((\lambda_k^2+\alpha\lambda_k^4)u_{2k-1}(t)+\beta\lambda_k^2u''_{2k-1}(t))\right\}. \tag{21}$$

Further, from (14), taking into account (18), we obtain:

$$(\lambda_k^2+\alpha\lambda_k^4)u_{2k-1}(t)+\beta\lambda_k^2u''_{2k-1}(t)=$$

$$=F_{2k-1}(t;u,a)-u''_{2k-1}(t)=\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a)-\beta_k^2u_{2k-1}(t)=$$

$$=\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a)-\beta_k^2\Bigg(\varphi_{2k-1}\cos\beta_kt+\frac{1}{\beta_k}\psi_{2k-1}\sin\beta_kt+$$

$$+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t-\tau)d\tau\Bigg). \tag{22}$$

In order to obtain an equation for the second component $a(t)$ of the solution $\{u(x,t),a(t)\}$ to problem (1) - (3), (6), we substitute expression (22) into (21):

$$a(t)=[h(t)]^{-1}\Bigg\{h''(t)-f(0,t)+4\sum_{k=1}^{\infty}\Bigg[\frac{\beta\lambda_k^2}{1+\beta\lambda_k^2}F_{2k-1}(t;u,a)-$$

$$-\beta_k^2\left(\varphi_{2k-1}\cos\beta_kt+\frac{1}{\beta_k}\psi_{2k-1}\sin\beta_kt+\frac{1}{\beta_k(1+\beta\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,a)\sin\beta_k(t-\tau)d\tau\right)\Bigg]\Bigg\}. \tag{23}$$

Thus, the solution of problem (1) - (3), (6) is reduced to the solution of system (20), (23) with respect to unknown functions $u(x,t)$ and $a(t)$.

To study the question of uniqueness of the solution of problem (1) - (3), (6), the following lemma plays an important role.

Lemma 1. *If* $\{u(x,t),a(t)\}$ *is any solution to problem* (1) - (3), (6)*, then the functions* $u_k(t)$ $(k=0,1,2,\ldots)$ *defined by relation* (12) *satisfy the counting system* (17), (18) *and* (19) *on* $[0,T]$.

Obviously, if $u_k(t)=\int_0^1 u(x,t)Y_k(x)dx \quad (k=0,1,...)$ is a solution to system (17), (18) and (19), then a pair $\{u(x,t),a(t)\}$ of functions $u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$ and $a(t)$ is a solution to system (20), (23).

Lemma 1 has the following

Corollary 1. *Let system* (20), (23) *have a unique solution. Then problem* (1) - (3), (6) *cannot have more than one solution, i.e. if problem* (1) - (3), (6) *has a solution, then it is unique.*

Now consider the following operator in space E_T^5

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\} \ ,$$

where

$$\Phi_1(u,a) = \tilde{u}(x,t) = \sum_{k=0}^{\infty} \tilde{u}_k(t) X_k(x), \Phi_2(u,a) = \tilde{a}(t),$$

and $\tilde{u}_0(t),\ \tilde{u}_{2k-1}(t),\ \tilde{u}_{2k}(t)$ and $\tilde{a}(t)$ equal corresponding to the right side (17), (18) , (19) and (23).

It is easy to see that

$$1+\beta\lambda_k^2 > \beta\lambda_k^2, \quad \frac{1}{1+\beta\lambda_k^2} < \frac{1}{\beta\lambda_k^2},$$

$$\sqrt{\frac{\alpha}{1+\beta}}\,\lambda_k \le \beta_k \le \sqrt{\frac{1+\alpha}{\beta}}\,\lambda_k, \ \sqrt{\frac{\beta}{1+\alpha}}\,\frac{1}{\lambda_k} \le \frac{1}{\beta_k} \le \sqrt{\frac{1+\beta}{\alpha}}\,\frac{1}{\lambda_k},$$

Taking these relations into account, we find:

$$\|\tilde{u}_0\,(t)\|_{C[0,T]} \le |\varphi_0| + T\,|\psi_0| + T\sqrt{T}\left(\int_0^T |f_0\,(\tau)|^2\,d\tau\right)^{\frac{1}{2}} + T^2\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]}\,, \tag{24}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le$$

$$\le 2\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}} + 2\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k-1}\,|)^2\right) + \frac{2}{\beta}\sqrt{\frac{1+\beta}{\alpha}}$$

$$\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\,\,|)^2 d\tau\right)^{\frac{1}{2}} + T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \tag{25}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le$$

$$\le 3\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k}|)^2\right)^{\frac{1}{2}} + 3\,\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k}\,|)^2\right)^{\frac{1}{2}} + \frac{3}{\beta}\sqrt{\frac{1+\beta}{\alpha}}$$

$$\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k}\,(\tau)\,\,|)^2d\tau\right)^{\frac{1}{2}}+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]+$$

$$+\frac{3(1+2\alpha+\alpha\beta)}{\beta^3}\left[T\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}}+\left(\sqrt{\frac{1+\beta}{\alpha}}+T\right)\sqrt{\frac{1+\beta}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\psi_{2k-1}|)^2\right)^{\frac{1}{2}}+\right.$$

$$+\ \frac{1}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left(T\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\,\,|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+T^2\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right)\right]+$$

$$+\ \frac{6}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\,\,|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.+\ T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \tag{26}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}+\right.$$

$$+4\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\{\frac{1+\alpha}{\beta}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}}+\sqrt{\frac{1+\beta}{\alpha}}\,\left(\sum_{k=1}^{\infty}(\lambda_k^4\,|\,\psi_{2k-1}\,|)^2\right)+\right.\right.$$

$$+\frac{1}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|\,f_{2k-1}\,(\tau)\,\,|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+\ T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right]+$$

$$\left.\left.+\ \left(\sum_{k=1}^{\infty}(\lambda_k^2\left\|f_{2k}\,(t)\right\|_{C[0,T]}\,|\right)^{\frac{1}{2}}+\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right\}\right\}\ . \tag{27}$$

Suppose that the data of problem (1) - (3), (6) satisfy the following conditions:
1.$\alpha>0,\beta>0,h(t)\in C^2[0,T]\,,\ h(t)\neq 0\ (0\leq t\leq T)$.
2.$\varphi(x)\in C^4[0,1],\ \varphi^{(5)}(x)\in L_2(0,1),\ \varphi(1)=0,\varphi'(0)=\varphi'(1),$

$\varphi''(1)=0, \varphi'''(0)=\varphi'''(1),\ \varphi^{(4)}(1)=0.$
3.$\psi(x)\in C^3[0,1],\ \psi^{(4)}(x)\in L_2(0,1),\ \psi(1)=0,\ \psi'(0)=\psi'(1),\ \psi''(1)=0, \psi'''(0)=\psi'''(1).$
4.$f(x,t), f_x(x,t)\in C(D_T),\ f_{xx}(x,t)\in L_2(D_T),\ f(1,t)=0,\ \ f_x(0,t)=f_x(1,t)\ \ (0\le t\le T).$

Then from (24)- (27) we find:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}}=A_1(T)+B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{28}$$

$$\|\tilde{a}(t)\|_{C[0,T]}=A_2(T)+B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{29}$$

where

$$A_1(T)=\|\varphi(x)\|_{L_2(0,1)}+T\,\|\psi(x)\|_{L_2(0,1)}+T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)}+\sqrt{2}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+$$
$$+\sqrt{\frac{2(1+\beta)}{\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+\sqrt{\frac{2(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}+\frac{3}{\sqrt{2}}\left\|\varphi^{(5)}(x)+4\varphi^{(3)}(x)\right\|_{L_2(0,1)}+$$
$$+\frac{3}{\sqrt{2}}\sqrt{\frac{1+\beta}{\alpha}}\left\|\psi^{(4)}(x)+3\psi^{(3)}(x)\right\|_{L_2(0,1)}+\frac{3}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\,\|f_{xx}(x,t)+2f_x(x,t)\|_{L_2(D_T)}+$$
$$+\frac{3(1+2\alpha+\alpha\beta)}{\beta^3}\left(\frac{T}{\sqrt{2}}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\left(\sqrt{\frac{1+\beta}{\alpha}}+T\right)\sqrt{\frac{1+\beta}{2\alpha}}\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+\right.$$
$$\left.+\frac{T}{\beta}\sqrt{\frac{T(1+\beta)}{2\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right)+\frac{6}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\,,$$

$$B_1(T)=T^2+\frac{11T}{\beta}\sqrt{\frac{1+\beta}{\alpha}}\left(1+\frac{3(1+2\alpha+\alpha\beta)}{\beta^3}T\right),$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}+\right.$$
$$+2\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\{\frac{1+\alpha}{\beta}\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\sqrt{\frac{1+\beta}{\alpha}}\,\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}+\right.\right.$$
$$\left.\left.+\frac{1}{\beta}\sqrt{\frac{T(1+\beta)}{\alpha}}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right]+\left\|\|f_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right\},$$

$$B_2(T)=2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\frac{1+\alpha}{\beta^2}\sqrt{\frac{1+\beta}{\alpha}}T+1\right).$$

From inequalities (27), (28) we conclude:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}}+\|\tilde{a}(t)\|_{C[0,T]}\le A(T)+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{30}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).$$

So, the following theorem is proved.

Theorem 2. *Let conditions 1-4 be satisfied and*

$$B(T)(A(T) + 2)^2 < 1. \tag{31}$$

Then problem (1)-(3), *(6) has a unique solution in the ball* $K = K_R(||\,z\,||_{E_T^5} \leq R = A(T) + 2)$ *from* E_T^5.

Proof. In the space E_T^5, consider the equation

$$z = \$z, \tag{32}$$

where $z = \{u, a\}$, the components $\$_i(u, a)\,(i = 1, 2)$ of the operator $\$(u, a)$ are defined by the right-hand sides of equations (20), (23), respectively.

Consider an operator $\$(u, a)$ in a ball $K = K_R$ of E_T^5. Similarly, from (30) we obtain that for any z, z_1, $z_2 \in K_R$ the following estimates are valid:

$$\|\$z\|_{E_T^5} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^5} \leq A(T) + B(T)(A(T) + 2)^2, \tag{33}$$

$$\|\$z_1 - \$z_2\|_{E_T^5} \leq B(T)R\left(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^5}\right). \tag{34}$$

Then, taking into account (31), it follows from estimates (33), (34) that the operator \$ acts in the ball $K = K_R$ and is contracting. Therefore, in the ball $K = K_R$, the operator \$ has a unique fixed point $\{u, a\}$, which is a solution to equation (32), that is, is the only solution in the ball $K = K_R$ to system (20), (23).

A function $u(x,t)$ as an element of space $B_{2,T}^5$, has continuous derivatives $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$ in D_T.

Similarly to [10], one can show that $u_t(x,t)$, $u_{tt}(x,t)$, $u_{tt}(x,t)$, $u_{ttx}(x,t)$, $u_{ttxx}(x,t)$ are continuous in D_T.

It is easy to check that equation (2) and conditions (2), (3) and (6) are satisfied in the usual sense. Hence, $\{u(x,t), a(t)\}$ is a solution to problem (1) - (3), (6), and by virtue of the corollary to Lemma 1, it is unique. ◄

Using Theorem 1, we prove the following

Theorem 3. *Let all conditions of Theorem 2 be satisfied and the conditions of consistency*

$$\varphi(0) = h(0)\ , \quad \psi(0) = h'0).$$

Then problem (1) - (4) *has a unique classical solution in the ball* $K = K_R(||\,z\,||_{E_T^5} \leq R = A(T) + 2)$ *from* E_T^5.

References

[1] Algazin S.D., Kiiko I.A. *Flutter of Plates and Shells*, Nauka, Moscow, 2006 [in Russian].

[2] Shabrov S.A. *About the Estimates of the Function Influence of a Mathematical Model Fourth Order*, Proceedings of Voronezh State University, Series: Physics. Mathematics, **2**, 2015, 168– 179.

[3] Benney D.J., Luke J.C. *On the interactions of permanent waves of finite amplitude*, Journal of Mathematical Physics, **43**, 1964, 309–313.

[4] Tikhonov A.N., *On the stability of inverse problems*, Reports of the Academy of Sciences of the USSR,**39(4)**, 1943, 195-198 (in Russian).

[5] Lavrentiev M.M. *On an inverse problem for the wave equation*, Doklady Akademii Nauk SSSR, **157(3)**, 1964, 520-521.

[6] Lavrent'ev M.M., Romanov V.G., Shishatsky S.T. *Ill-posed problems of mathematical physics and analysis*, Moscow, Nauka, 1980, 288 p.

[7] Ivanov V.K., Vasin V.V., Tanina V.P. *The theory of linear ill-posed problems and its applications*, Moscow, Nauka, 1978, 206 p.

[8] Denisov A.M. *Introduction to the theory of inverse problems*, Moscow, Moscow State University, 1994, 206 p.

[9] Yuldashev T.K. *On a nonlocal inverse problem for a nonlinear integro-differential Benney – Luke equation with a degenerate kernel*, Vestnik TVGU, Series: Applied Mathematics, **3**, 2018, 19–41.

[10] Megraliev Ya.T., Velieva B.K. *Inverse Boundary Value Problem for the Linearized Benny-Luc Equation with Nonlocal Conditions*, Vestn. Udmurtsk. un-that. Mat. Fur. Computer, **29(2)**, 2019, 166–182.

[11] Mehraliyev Ya., Valiyeva B. *On one nonlocal inverse boundary problem for the Benney –Luke equation with integral conditions*, Journal of Physics: Conference Series , **1203**, 2019/4.

[12] Mehraliyev Ya.T., Valiyeva B.K., Ramazanova A.T. *An inverse boundary value problem for a linearized Benny–Luc equation with nonlocal boundary conditions*, Cogent Mathematics & Statistics, **6(1)**, 2019, https://doi.org/10.1080/25742558.2019.1634316.

[13] Megraliev Ya.T. *An inverse boundary value problem for a fourth-order partial differential equation with an integral condition*, Vestn. South-Lv. un-that. Ser. Mat. Fur. Fiz., **5**, 2011, 51–56.

[14] Kaliev I.A., Sabitova M.M. *Problems of Determining the Temperature and Density of Heat Sources from the Initial and Final Temperatures*, Sib. zhurn. industrial mathematics, **12(1:37)**, 2009, 89–97.

[15] Megraliev Ya.T. *On an inverse boundary value problem for a second-order elliptic equation with an integral condition of the first kind*, Tr. IMM UB RAN, **19(1)**, 2013, 226–235.

Bahar K. Velieva
Ganja State University, AZ-2000, Ganja, Sh.I.Khatai Ave, 187
E-mail: bahar.veliyeva.91@inbox.ru

Received 20 April 2020
Accepted 14 June 2020

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 8, No 1, 2020, July
ISSN 1560-4055

Empirical Analysis of Balance of Payments Dynamics

E.G. Orudzhev, N.S. Ayyubova

Abstract. The article deals with the construction of econometric model, which characterizes the dynamics of the balance of payments, formed on the basis of economic and mathematical tools, and serves to predict the preventive signals of the balance of payments crisis. Over time, the unsteady character created by the denomination of the Azerbaijani manat was corrected, and a multidimensional model of linear regression was created for the dynamics of balance of payments development.

Key Words and Phrases: balance of payments, current account, export, import, investment, manat rate, regression, adequacy, t statistics, F-criterion

2010 Mathematics Subject Classifications: C1, C12, C5

1. Introduction

At the beginning of the transformation process in a number of transition countries, such as Azerbaijan, liberalization of the currency market and foreign economic relations increased the level of openness of the economy and its integration into the world economy. At present, the cooperation of the Azerbaijani economy with the world commodity and financial markets has reached a broad and multidimensional level. Therefore, the study of issues related to the regulation of foreign economic relations is particularly relevant.

The balance of payments is central to the macroeconomic regulation of the foreign economic relations sector. The structure of the balance of payments is in principle determined by economic indicators, such as foreign trade, receipts from direct and portfolio foreign investments, in general, the prospects for economic growth. Therefore, along with the problems of ensuring economic growth, high inflation and unemployment, the issues of maintaining stability and stability of the balance of payments are among the priorities of our state's economic policy.

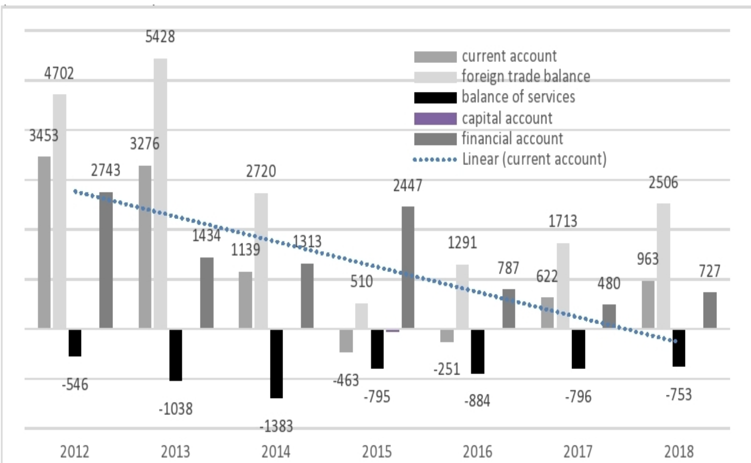
Recent processes such as global economic instability, volatility of oil prices in world markets and surges influenced by various external and endogenous factors, devaluation of national currencies lead to a balance of payments crisis. Monitoring of changes in the structure and dynamics of the balance of payments in order to develop a set of preventive and anti-crisis measures is one of the most difficult issues of monetary policy in state regulation of the economy.

The study of the causes and consequences of balance of payments crises, their early detection and, in general, the search for ways out of the crisis, the development of effective balance of payments methods are the most pressing economic policy issues in any market economy. These issues are particularly important in countries such as Azerbaijan, whose economies are heavily dependent on energy, foreign trade and other external factors [6, 8, 9]. (see: figure 1).

Both at the macro level and the micro level, modern economic theory inevitably turns to mathematical and statistical models and methods, and this approach has now become an important element of economic theory. Applying mathematics to economics first of all reveals more important dependencies of economic variables and objects and creates opportunities for formal description, because studying complex objects requires a high level of abstraction [1]. Precise input information and deduction methods allow obtaining results and making forecasts adequate to the object under study. Mathematical and statistical methods inductively allow obtaining the newest knowledge about the object under study about parameters and forms of dependent variables on the basis of possible observations.

Relevance of the problem, theoretical and practical significance of methodological and analytical approaches [2, 3, 4, 5], as well as balance of payments modeling, econometric analysis of its interrelation with the main forming factors, analysis of changes in the structure of the balance of payments, taking into account the interest and demand, we have defined the creation of an econometric model that provides the balance of payments as an object of research, forecasting of crisis prevention signals as a goal and includes economic and mathematical tools for crisis prevention.

Figure 1. Dynamics of key indicators of Azerbaijan's balance of payments in 2012-2018 (years) (million US dollars)



The Balance of Payments is a statistical report that systematically reflects the final results of the state's foreign economic operations with other countries. Information on the balance of payments and the position of the state with regard to international investments inevitably play an important role in the formation of domestic and foreign economic policy. This information also includes balance of payments analysis, identification of reasons for contraindications, assessment of regulatory measures, assessment of the role and interdependence of foreign trade and foreign direct investment, external debt, economic growth, income distribution, and the current and financial balance of payments, is very important and invaluable for analyzing the relationship between foreign operations and exchange rates reflected in the accounts.

Any modern economic research is based on a combination of theory and practice, joint application of statistical indicators with an economic model. Thus, while theoretical models are used to describe and explain observed processes, statistical indicators are used for empirical construction and justification of models. For both public policy and any economic entity, the ability to predict the situation means reducing losses and damages, increasing profits and, in general, getting as close to the desired result as possible.

Our research is devoted to building an econometric model based on the adequacy of the balance of payments, which provides the projected values of the balance of payments based on the main factors that determine its dynamics.

The regression methodology includes the analysis of stationary and, most importantly, non-stationary series in order to predict the dynamics. LSM, GLS, VAR, taking into account seasonality, the inclusion of models based on multicollinearity, etc. methods are very popular in this area [4, 5]. The novelty and urgency of our research is connected with application of the analysis of these methods to non-stationary time series.

In our initial research on modeling the dynamics of the balance of payments [7] regression analysis was carried out for the purpose of econometric analysis of dependence of the current account of the balance of payments on general and foreign investments, export and import, exchange rate of Azerbaijani manat. In the research, since the current account of BP's balance of payments depends on a variable, foreign investment FI, EX-exports, IMP-import, the exchange rate of manat against the US dollar, the total investment in GI for 1995-2017, respectively, the explanatory variables.

Table 1. Results of regression analysis (1995-2017)

Variable	Coefficient	Std. Error	t-Statistic	Prob.
Fİ	-1.041597	0.214275	-4.861035	0.0001
EXP	0.855962	0.071390	11.98994	0.0000
İMP	0.040745	0.054360	0.749535	0.4638
ČM	0.127631	0.220403	0.579079	0.5701
GI	-0.074593	0.144352	-0.516739	0.6120
C	-978.5116	1126.953	-0.868281	0.3973
Dependent Variable	*BP*			
Method	Least Squares			
Included observations	23 after adjustments			
Sample (adjusted)	1 23			
R-squared	0.986318		Mean dependent var	4332.196
Adjusted R-squared	0.982293		S.D. dependent var	6943.397
S.E. of regression	923.9354		Akaike info criterion	16.71462
Sum squared resid	14512163		Schwarz criterion	17.01084
Log likelihood	-186.2181		Hannan-Quinn criter	16.78912
F-statistic	245.0926		Durbin-Watson stat	2.928609
Prob(F-statistic)	0.000000			

Source: author's work

Based on the results of regression analysis with the parameters involved in the research, the number of observations: 23; R^2- Determination factor: 0,98; *F statistic* - Fisher Criterion: 245,1; severity level: prob.- 0,00; DW-Durbin Watson's statistic's: 2,92. The results are quite satisfactory. Explanatory variables included in the model according to the determination coefficient explain the result variable by 98%. Criterion F received a fairly reliable estimate with a high probability. However, the result obtained for the DW criterion cannot be considered satisfactory. $n = 23$ and $k = 5$ (number of explanatory factors included in the model) critical boundaries for DW criterion with indicators are $D_L = 0,90$ and $D_U = 1,92$ Since the calculated DW criterion value for the model is higher than 2, the 4-DW value = 1.08 is compared with the critical value. $D_L < 1,08 < D_U$. Alternatively, $4 - D_U < DW < 4 - D_L$ we get a similar result: $2,08 < 2,92 < 3,1$ DW falls into an area of uncertainty and it is impossible to decide if autocorrelation exists.

Units of measurement of independent variables FI, EXP, IMP, GI included in the model for regression analysis are expressed in US dollars. As it is known, in 2006 the denomination of Azerbaijani manat was conducted in the ratio of 1:5000. One of the explanatory factors is the exchange rate of KM manat in the national currency, which created a serious problem for the stability of the considered time series. Thus, large amplitude jumps occurred in time, which, in turn, formed a non-stationary sequence. Unsteady time series lose their importance for econometric studies and are not suitable for forecasting because the model is inadequate. For this purpose, the time series research period was shortened, the number of observations was reduced to 12 and covered 2006-2017 years. Continuing the regression analysis in a new chronological order, we obtained results in Table 2 below.

Table 2. Results of regression analysis (2006-2017)

Variable	Coefficient	Std. Error	t-Statistic	Prob.
Fİ	-0.043786	0.851259	-0.051436	0.9606
EXP	0.940172	0.130751	7.190547	0.0004
İMP	0.075568	0.097305	0.776614	0.4669
CM	-3669.255	3126.497	-1.173600	0.2850
GI	-0.505909	0.394494	-1.282425	0.2470
C	-365.0545	3458.028	-0.105567	0.9194

Dependent Variable	*BP*		
Method	Least Squares		
Included observations	12		
Sample (adjusted)	1 12		
R-squared	0.977251	Mean dependent var	9106.875
Adjusted R-squared	0.958293	S.D. dependent var	6616.478
S.E. of regression	1351.229	Akaike info criterion	17.56227
Sum squared resid	10954912	Schwarz criterion	17.80472
Log likelihood	-99.37361	Hannan-Quinn criter	17.47250
F-statistic	51.54956	Durbin-Watson stat	2.922200
Prob(F-statistic)	0.000075		

Source: author's work

Number of observations from the 2nd regression analysis:12; R^2-0,97; F *statistic* - 51,5 severity level: prob.=0,000075; DW-2,92. Thus, the model we have created for the dynamics of the current account of the balance of payments at this stage of the research looks as follows:

$$BP=-0{,}043786F\dot{I}+0{,}940172EXP+0{,}075568\dot{I}MP-3669{,}255CM-0{,}505909G\dot{I}$$

According to the latest results, there are no significant changes in model quality, i.e. the model quality does not increase or decrease significantly, and DW statistics still fall into the zone of uncertainty. This does not tell us if there is autocorrelation in time, so we cannot be sure that the model is adequate. In such cases, steps such as extending the time sequence and editing explanatory factors in the model can be used to improve the quality of the model. In the next stages of our study, in addition to these steps, appropriate econometric tests will be applied to verify the adequacy of the model in more detail.

Current account deficits generated by the trade balance can be financed by capital inflows in the following forms: foreign loans from other countries, the International Monetary Fund, the World Bank; assets sold to foreign investors; direct investments that bring foreign currency into the country in order to create new production facilities; foreign exchange reserves.

The application of these measures contributes to the reduction of the country's foreign assets. However, if the government increases its external debt, which significantly exceeds the current account deficit, then the country faces a balance-of-payments external debt crisis. Proper regulation of these financial processes is very important for the balance of payments and the dynamic development of the country's economy as a whole.

2. Results

1. The time series of macroeconomic indicators were systematized for econometric modeling of the current account of the balance of payments;

2. The model of multidimensional linear regression was created for the current account of the balance of payments, assuming that foreign investments, exports, imports, exchange rate of manat against the US dollar, total investments - macroeconomic indicators are independent variables;

3. Corrected model for the period 2006-2017. was proposed by eliminating factors that caused serious problems with the stationarity of time series, and an analysis was performed to verify the adequacy of the model.

The urgency of the problem, the need to build an econometric model that reflects the dynamics of the current account in the form of trends and indicators of full adequacy for the forecast evaluation of the balance of payments, as well as the results of our study make it necessary to continue the study.

We believe that the results of the research will make it possible to identify real balance of payments trends and balance of payments dynamics that may be useful for the CIS and Eastern European countries based on the analysis of interdependence of the balance of payments with macroeconomic indicators may be important in shaping regulatory measures.

The results of the research can also be used as teaching material in lectures on economic theory, economic mathematics, econometrics, statistics.

References

[1] Yu.P. Lukashin, *Adaptive methods for short-term forecasting of time series*, Finance and Statistics, Moscow, 2003, 416 p.

[2] D.A. Dickey, W.A. Fuller, *Distribution of Estimators for Autoregressive Time Series with a Unit Root*, Journal of the American Statistical Association, **74**, 1979, 427-431.

[3] R.F. Engle, C.W.J. Granger, *Cointegration and Error Correction: Representation, Estimation and Testing*, Econometrica, **55**, 1987, 251-276.

[4] S. Johansen, *Determination of Cointegration Rank in the Presence of a Linear Trend*, Oxford Bulletin of Economics and Statistics, **54**, 1992, 383-397.

[5] S. Johansen, K. Juselius, *Identification of the Long-run and the Short-run Structure: An Application to the ISLM Model*, Journal of Econometrics, **63**, 1994, 7-36.

[6] E.K. Orudzhev, N.S. Ayyubova, *The cluster factor of sustainable development of the economy of Azerbaijan*, Materials of the XXIII international conference: "Actual problems in modern science and ways to solve them" Russia, Moscow, Eurasian Union of Scientists (ESU) # 12, **21**, 2015, 57-59.

[7] E.G. Orudzhev, N.S. Ayyubova, *The empirical analyses of the factors influencing on the Azerbaijan payment balance*, Actual problems of Economics, **7(181)**, 2016, 400-411.

[8] E. Orudzhev, N. Ayubova, *Some Systematic and Statistical Aspects of Decision of Central Bank of Azerbaijan on Devaluation of Manat (dated 21.02.2015)*, Caspian Journal of Applied Mathematics, Ecology and Economics, **3(1)**, 2015, 59-67.

[9] E. Orudzhev, A. Isazadeh, *Comparative analysis of relationship of the course of the Azerbaijani manat and its major macroeconomic determinants*, Actual Problems of Economics. Scientific Economic Journal, **3(201)**, 2018, 94-104.

Elshar G. Orudzhev
Baku State University, AZ 1148, Baku, Azerbaijan
E-mail: elsharorucov63@mail.ru

Natavan S. Ayyubova
Baku State University, AZ 1148, Baku, Azerbaijan
E-mail: neyyubova@mail.ru

Received: 05 May 2020
Accepted: 15 May 2020

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 8, No 1, 2020, July
ISSN 1560-4055

Investigation of Propagation of Nonlinear Waves in a Structure Consisting of Cylindrical Net System

J.H. Agalarov, M.A. Rustamova, T.J. Hasanova

Abstract. Cylindrical net movement at the smooth cylinder have been obtained on the base of the general net motion theory. On the next basis system of the vectors: in direct of cylinder axes; in tangential (rotated) to a cross-section of the cylinder: perpendicular (rotated) to the cylinder axes. The case of the relative symmetrical filaments position is taken. In this research work the strain impact to the net is considered. The task comes to the hyperbolic system of equations at corresponding conditions. As far as the larger significance of parameters corresponds larger speed of wave spreading, that leads to jumping on the front. To solve the task at the front, there are using the law of mass preservation and law of a motion quantity changing to find out the jump spread speed as a function of incline of filament from cylinder axis and speed of the impact.

Key Words and Phrases: wave front, spread speed, law of a motion, cylindrical base, net motion, tension, angular acceleration

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1. Introduction

On the basis of Kh.A. Rakhmatulin's equations on the motion of a filament, the equations of motion of the net were obtained [1, 2]. On the dynamics of the netthere are solved a number of flat and spatial problems in a rectangular Cartesian coordinate system [3, 4, 5, 6, 7]. Here we consider the problem of the motion of a net on a cylindrical base. In addition to the theoretical interest, the problem is of practical importance, for example, the dynamics of flexible drill pipes.

2. General equations of net motion

The equation of motion of the net taking into account the reaction of the supporting body and the geometric relationships will have the form unlike [2].

$$\frac{\partial}{\partial S_1}\left(\sigma_1\overline{\tau_1}\right)+\frac{\partial}{\partial S_2}\left(\sigma_2\overline{\tau_2}\right)=\rho\frac{\partial^2\overline{r}}{\partial t^2}+p\overline{n}$$

$$\left(1+e_1\right)\overline{\tau_1}=\frac{\partial\overline{r}}{\partial S_1};\qquad\left(1+e_2\right)\overline{\tau_2}=\frac{\partial\overline{r}}{\partial S_2}. \tag{1}$$

Here $\overline{r}$ – is the radius of the particle of the net particle, P is the reaction force of the cylinder l_1, l_2 – are the relative elongations corresponding to the properties of the filaments, S_1, S_2 – Lagrangian coordinates of the particles of filaments, σ_1, σ_2 – are conditional stresses, defined as the sum of the tension of individual threads of one family (intersecting a section of a filament of another family), referred to the original length of the element in question.

Such a distribution of mass and effort is permissible with a sufficiently dense net, ρ – is the mass of the net per unit area in the initial state, $\overline{\tau}_1$,$\overline{\tau}_2$ - are the unit vectors tangent to the filaments, $\overline{n}$ - is the normal to the surface of the cylindrical base.

3. Coordinate system

A basis of a cylindrical system is taken: a unit vector $\overline{i}$ - parallel to the axis of the cylinder, $\overline{j}$ – the unit vector of the tangent(rotating) to the cross section of the cylinder, $\overline{k}$ – unit vector perpendicular(rotating) to the previous ones.

Then

$$\overline{\tau_1} = \cos\gamma_i\overline{i} \ + sin\gamma_1\overline{j};\ \overline{\tau_2} = cos\gamma_2\overline{i} + \sin\gamma_2\overline{j}, \tag{2}$$

where $\gamma_{1,2}$- the filament angles formed with the axis of the cylinder.

Derivatives

$$\frac{\partial\overline{\tau_1}}{\partial S_1} = \cos\gamma_i\frac{\partial\overline{i}}{\partial S_1} + \overline{i}\frac{\partial(cos\gamma_1)}{\partial S_1} + sin\gamma_i\frac{\partial\overline{j}}{\partial S_1} + \overline{j}\frac{\partial(sin\gamma_1)}{\partial S_1}$$

$$\frac{\partial\overline{\tau_2}}{\partial S_2} = cos\gamma_2\frac{\partial\overline{i}}{\partial S_2} + \overline{i}\frac{\partial(cos\gamma_2)}{\partial S_1} + sin\gamma_2\frac{\partial\overline{j}}{\partial S_2} + \overline{j}\frac{\partial(sin\gamma_2)}{\partial S_2}$$

Or considering

$$\frac{\partial\overline{i}}{\partial S_1} = \frac{\partial\overline{i}}{\partial S_2} = 0;\ \frac{\partial\overline{j}}{\partial S_1} = \frac{sin\gamma_1}{r}\overline{k}\frac{\partial\overline{j}}{\partial S_2} = -\frac{sin\gamma_2}{r}\overline{k}$$

We get

$$\frac{\partial\overline{\tau_1}}{\partial S_1} = \frac{\partial(cos\gamma_1)}{\partial S_1}\overline{i} + \frac{sin\gamma_1^2}{r}\overline{k} + \frac{\partial(sin\gamma_1)}{\partial S_1}\overline{j}$$

$$\frac{\partial\overline{\tau_2}}{\partial S_2} = \frac{\partial(cos\gamma_2)}{\partial S_2}\overline{i} - \frac{sin\gamma_2^2}{r}\overline{k} + \frac{\partial(sin\gamma_2)}{\partial S_2}\overline{j} \tag{3}$$

Also taking into account $\overline{r} = x\overline{i} + r\overline{k}$, we have

$$\frac{\partial\overline{r}}{\partial t} = \frac{\partial x}{\partial t}\overline{i} + r\omega\overline{j}$$

$$\frac{\partial^2\overline{r}}{\partial t^2} = \frac{\partial^2 x}{\partial t^2}\overline{i} + r\varepsilon\overline{j} + r\omega^2\overline{k} \tag{4}$$

ω – angular velocity, ε – angular acceleration

4. Equations of motion of a cylindrical net

Substituting (3) and (4) into (1) we obtain

$$\frac{\partial}{\partial S_1}(\sigma_1 \cos\gamma_1) + \frac{\partial}{\partial S_2}(\sigma_2 \cos\gamma_2) = \rho\frac{\partial^2 x}{\partial t^2} \tag{5}$$

$$\frac{\partial}{\partial S_1}(\sigma_1 sin\gamma_1) + \frac{\partial}{\partial S_2}(\sigma_2 sin\gamma_2) = r\varepsilon$$

$$\frac{\sigma_1}{r} sin\gamma_1^2 - \frac{\sigma_2}{r} sin\gamma_2^2 = p + \rho r\omega^2$$

Next, the symmetrical arrangement of the right and left fibers is considered. Then equations (5), taking $\sigma_1 = \sigma_2 = \sigma, \quad \gamma_1 = -\gamma_2 = \gamma, \quad \omega = 0, \ \ \varepsilon = 0,$
will take the form

$$2\frac{\partial}{\partial S}(\sigma, \cos\gamma) = \rho\frac{\partial^2 x}{\partial t^2} \tag{6}$$

$$2\sigma sin\gamma = p$$

5. Geometric relations

We define the derivative of the radius vector $\overline{r}$ in S. Denoting $\overline{r} = x\bar{i} + r\overline{k},$

$$\frac{\partial\overline{r}}{\partial S} = \frac{\partial x}{\partial S}\bar{i} + r\frac{\partial\overline{k}}{\partial S} = \frac{\partial x}{\partial S}\bar{i} + \frac{\partial y}{\partial S}\bar{j}$$

y –circular coordinate, where according to (1) and (3)

$$\frac{\partial x}{\partial S} = (1+e)\, cos\gamma \tag{7}$$

$$\frac{\partial y}{\partial S} = (1+e)\, sin\gamma \tag{8}$$

Since the net does not rotate, then $y = const$, and

$$\frac{\partial\left[(1+e)\ \ sin\gamma\right]}{\partial t} = 0$$

or

$$(1+e)\, sin\gamma = sin\gamma_0 \tag{9}$$

6. Stretching blow on the cylindrical net

Let the infinite unloaded net (Fig. 1) be driven from one end with a constant velocity v.

Since waves with greater deformation propagate faster than waves with less deformation, the wave front will undergo a jump (8). Assuming that the motion is self-similar, we have.

$$\xi = \frac{S}{bt}; \quad x = btf(\xi);$$

$$\frac{\partial \xi}{\partial t} = -\frac{S}{bt^2}; \quad \frac{\partial x}{\partial t} = bf(\xi) - btf'(\xi)\frac{S}{bt^2} = bf(\xi) - \frac{S}{t}f'(\xi)$$

$$\frac{\partial^2 x}{\partial t^2} = \frac{b}{t}\xi^2 f''(\xi) \tag{10}$$

Substituting (10) into (5), we obtain

$$2(\sigma cos\gamma)' = \rho\xi^2 f'' \tag{11}$$

$$(1+e)\,cos\gamma = f' \tag{12}$$

Substituting (9) into (2) with $\sigma = Ee$, we get $sin\gamma_0 ctg\gamma = f'$;

$$\left[cos\gamma\left(\frac{sin\gamma_0}{sin\gamma} - 1\right)\right]' E = \rho\xi^2 f''. \tag{13}$$

In (13), eliminating f, obtaining

$$\left[cos\gamma\left(\frac{sin\gamma_0}{sin\gamma} - 1\right)\right]' E = \rho\xi^2 sin\gamma_0 ctg'\gamma$$

or

$$sin\gamma_0 ctg'\gamma - cos'\gamma = \frac{\xi^2}{a^2} sin\gamma_0 ctg'\gamma$$

or

$$-sin\gamma_0 csc^2\gamma \bullet \gamma' + sin\gamma \bullet \gamma' = -\frac{\xi^2}{a^2} sin\gamma_0 csc^2\gamma \bullet \gamma'. \tag{14}$$

The last equation has two solutions:

1. $\gamma' = 0$– constant parameter area
2. $\xi^2 = a^2\left(1 - \frac{sin^3\gamma}{sin\gamma_0}\right)$ – region of a self-similar (homogeneous) solution

Let us consider the first case

7. Stretching blow on the cylindrical net (Solution)

Let a semi-infinite unloaded cylindrical net be driven from the end with a constant velocity v. Since waves with greater deformation propagate with greater velocity, the wave front will undergo a jump.

Consider the motion of the net in the vicinity of the wave front [Fig. 1]:

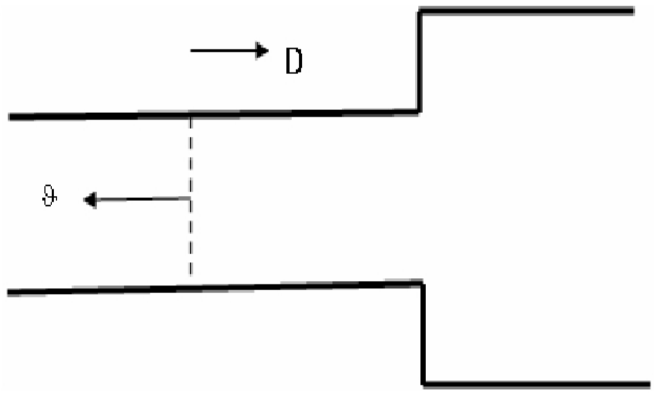

Fig. 1. Motion of the net in the vicinity of the wave front

In time dt, the front propagates to the distance Ddt. For a deformed net, there will be $(v + D)\,dt$. Denoting the values of density ρ_0 for an undeformed net, the law of conservation of mass will have the form [Fig. 2] and ρ for a deformed net

$$\rho\,(D + v) = \rho_0 D. \tag{15}$$

The change in momentum $\rho_0 Dvdt$ will be equal to the momentum of the force

$$\rho_0 Dv + 2\sigma cos\gamma = 0. \tag{16}$$

We connect the net densities with the deformation of the net element [Fig. 2].

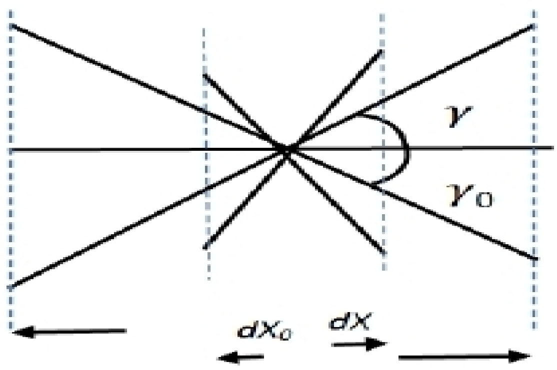

Fig. 2. Connection the deformation of the net with a density

If the mass of the net element dM, the deformation e, the slope angles of the branches in the initial and deformed state γ_0 and γ, then

$$\rho_0 = \frac{dM}{cos\gamma_0 dS} \quad ? \quad \rho = \frac{dM}{(1+e)\,cos\gamma dS}$$

or

$$\rho_0 = \frac{(1+e)\,cos\gamma}{cos\gamma_0}\rho \tag{17}$$

Substituting (17) into (15), we obtain

$$v = -\left[\frac{(1+e)\,cos\gamma}{cos\gamma_0} - 1\right] D \tag{18}$$

Substituting (18) into (16), we obtain

$$D^2 = \frac{\sigma cos\gamma_0}{\rho_0\,[(1+e)\,cos\gamma - cos\gamma_0]} \tag{19}$$

Formulas (9), (18) and (19) allow to determine the shock wave velocity D, strain (tension) and turning angle of the net branches at a given impact speed.

It should be noted that with increasing impact velocity$v\gamma \to 0$, we have

$$D^2 = \frac{\sigma w_1 \gamma_0}{\rho_0\,(1+e-cos\gamma_0)} \tag{20}$$

Setting $\sigma = Ee$, defining from (9)

$$1+e = \frac{sin\gamma_0}{sin\gamma}; \qquad \sigma = E\left(\frac{sin\gamma_0}{sin\gamma} - 1\right)$$

and substituting in (19) we obtain

$$D^2 = \frac{E\left(\frac{sin\gamma_0}{sin\gamma} - 1\right) cos\gamma_0}{\rho_0\,(sin\gamma_0 ctg\gamma - cos\gamma_0)} \tag{21}$$

or

$$D^2 = a^2 \frac{(sin\gamma_0 - sin\gamma)\,cos\gamma_0}{sin\gamma_0 cos\gamma - cos\gamma_0 sin\gamma}$$

or

$$D = a\sqrt{(sin\gamma_0 - sin\gamma)\,/sin\,(\gamma_0 - \gamma)}$$

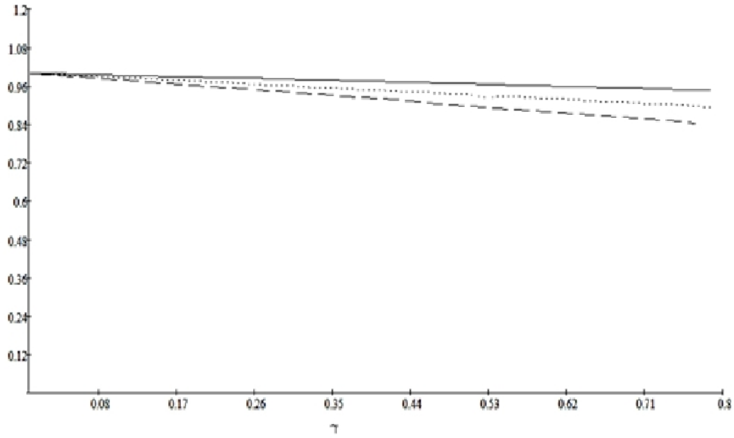

Fig. 3. $\gamma_0 = \frac{\pi}{4}; \gamma_0 = \frac{\pi}{6}; \gamma_0 = \frac{\pi}{12}$

Three variants of shock wave velocity distribution are calculated depending on the impact speed at the initial values of the angle of inclination of the branches of the net to the axis: $\frac{\pi}{4}; \frac{\pi}{6}; \frac{\pi}{12}$.

As can be seen from Graph 3, with increasing impact speed (decrease in γ), the shock wave velocity increases (up to 15%)

References

[1] H.A. Rakhmatulin, *About impact on a flexible thread*, PMM, **X(3)**, 1947. (in Russian)

[2] J.H. Agalarov, *The investigation of the net motion subjected of an impact*, Izv. AN Azerb., Ser. Phys-Tech. and Math. Sciences, Journal of Mathematics and mechanics, **6**, 1982.

[3] J.H. Agalarov, A.N. Efendiev, *The propagation of nonlinear waves in the structure of the net system*, Rakenteiden mekaniika seura RY Finish association for structural mechanics, **21(2)**, 1988, 3-10.

[4] A.I. Seyfullayev, M.A. Guliyeva, *To the solution of the equilibrium problem of the net*, Proceedings of the Institute of Mathematics and Mechanics of NAS of Azerbaijan, **XIII**, 2000, 144-147.

[5] D.G. Agalarov, A.I. Seyfullayev, M.A. Kuliyeva, *Numerical solution of one plane problem of net equilibrium*, Mechanics engineering, **1**, 2001, 4-5.

[6] M.A. Gulieva, *Tension of a rectangular net fastened from two adjacent sides*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **XVI (XXIV)**, 2002, 156-160.

[7] J.H. Agalarov, M.A. Guliyeva, *Waves of strong breaks in nets*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **XVII (XXV)**, 2002, 135-137.

[8] G.I. Barenblat, *On the propagation of instantaneous perturbations in a medium with a nonlinear stress-strain relation*, PMM, **VVII(4)**, 1953.

Jafar H. Agalarov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ 11441, Baku, Azerbaijan
E-mail: agjafar@rambler.ru

Mehseti A. Rustamova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ 11441, Baku, Azerbaijan
E-mail: mehsetir@gmail.com

Tukezban J. Hasanova
Azerbaijan University of Architecture and Construction, AZ 1073, Baku, Azerbaijan
E-mail: atika2014@rambler.ru

Received: 09 April 2020
Accepted: 27 May 2020

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 8, No 1, 2020, July
ISSN 1560-4055

On Riesz-Thorin Type Theorems in the Besov-Morrey Spaces

Nilufer R. Rustamova*

Abstract. In this paper is studied some differential properties of functions belonging to the intersection of Besov-Morrey spaces $B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)$ $(\mu = 1, 2, \ldots, N)$

Key Words and Phrases: Besov-Morrey spaces, integral representation, generalized Hölder condition.
2010 Mathematics Subject Classifications: 26A33, 46E30, 42B35

1. Introduction

In this paper we study differential and differential-difference properties of functions from intersection Besov-Morrey spaces

$$B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi) \tag{1}$$

was introduced in paper [13]. Note that the paper [13] was proved embedding theorems, but in this paper we prove interpolation type theorem in Besov-Morrey space $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$. Such type theorems were first proved in [2] and later in [1, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15].

Let $G \subset R^n$, $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$, $\varphi_j(t) > 0, \varphi'_j(t) > 0$; $j = 1, 2, \ldots, n$ $(t > 0)$ is continuously differentiable functions. Assume that $\lim\limits_{t\to+0} \varphi_j(t) = 0$ and $\lim\limits_{t\to+\infty} \varphi_j(t) = K_j$, $0 < K_j \le \infty, (j = 1, \ldots, n)$. We denote the set of such vector-functions φ by A. We assume that $|\varphi([t]_1)|^{-\beta} = \prod\limits_{j=1}^{n} (\varphi_j([t]_1))^{-\beta_j}$, $\beta_j \in [0, 1]$ $(j = 1, 2, \ldots, n)$ and $[t]_1 = \min\{1, t\}$.

For any $x \in R^n$ we put

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) =$$
$$= G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), (j = 1, 2, ..., n) \right\},$$

Let $l \in (0, \infty)^n$, $m_i \in N$, $k_i \in N_0$, $1 \le p < \infty$, $1 \le \theta \le \infty$. The space $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$ is defined [13] as a linear normed space of functions f, on G, with the finite norm $(m_i > l_i - k_i > 0$ $(i = 1, ..., n))$:

$$\|f\|_{B^l_{p,\theta,\varphi,\beta}(G_\varphi)} = \|f\|_{p,\varphi,\beta;G} +$$

*Corresponding author.

$$+\sum_{i=0}^{n}\left\{\int_{0}^{t_0}\left[\frac{\left\|\Delta_i^{m_i}(\varphi_i(t),G_{\varphi(t)})D_i^{k_i}f\right\|_{p,\varphi,\beta}}{(\varphi_i(t))^{(l_i-k_i)}}\right]^{\theta}\frac{d\varphi_i(t)}{\varphi_i(t)}\right\}^{\frac{1}{\theta}}, \tag{2}$$

where $t_0>0$ is a fixed number and

$$\|f\|_{p,\varphi,\beta;G}=\|f\|_{L_{p,\varphi,\beta}(G)}=\sup_{\substack{x\in G,\\ t>0}}\left(|\varphi([t]_1)|^{-\beta}\|f\|_{p,G_{\varphi(t)}(x)}\right). \tag{3}$$

Let $\lambda_\mu\ge 0$ $(\mu=1,\ldots,N)$, $\sum\limits_{\mu=1}^{N}\lambda_\mu=1$, $\frac{1}{p}=\sum\limits_{\mu=1}^{N}\frac{\lambda_\mu}{p_\mu}$, $\frac{1}{\theta}=\sum\limits_{\mu=1}^{N}\frac{\lambda_\mu}{\theta_\mu}$, $\frac{1}{r}=\sum\limits_{\mu=1}^{N}\frac{\lambda_\mu}{r_\mu}$, $l=\sum\limits_{\mu=1}^{N}\lambda_\mu l^\mu$ and let $\Omega(\cdot,y)$, $M_i(\cdot,y,z)\in C_0^\infty(R^n)$, be such that

$$S(M_i)\subset I_{\varphi(T)}=\left\{y:|y_j|<\frac{1}{2}\varphi_j(T),\ j=1,2,...,n\right\},\ 0<T\le 1.$$

We put

$$V=\bigcup_{0<t\le T}\left\{y:\left(\frac{y}{\varphi(t)}\right)\in S(M_i)\right\}.$$

noting $V\subset I_{\varphi(t)}$ and $U\subset G$, we assume that $U+V\subset G$.

Lemma 1. *Let $1\le p_\mu\le q_\mu\le r_\mu\le\infty$, $0<\eta,t\le T\le 1$, $\nu=(\nu_1,\ldots,\nu_n)$, $\nu_j\ge 0$ be integer $(j=1,2,\ldots,n)$;$\Delta_i^{m_i}(\varphi_i(t))f\in L_{p_\mu,\varphi,\beta}(G)$ and let*

$$B(x)=\prod_{j=1}^{n}\int_{R^n}\int_{R^n}f(x+y+z)\Omega^{(\nu)}\left(\frac{y}{\varphi(T)},\frac{\rho(\varphi(T),x)}{2\varphi(T)}\right)$$

$$\times\Omega\left(\frac{z}{\varphi(T)},\frac{\rho(\varphi(T),x)}{2\varphi(T)}\right)dydz, \tag{4}$$

$$B_\eta^i(x)=\int_0^\eta L_i(x,t)\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \tag{5}$$

$$B_{\eta T}^i(x)=\int_\eta^T L_i(x,t)\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \tag{6}$$

$$Q_T^i=\int_0^T\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}\frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}}dt<\infty \tag{7}$$

$$L_i(x,t)=\int_{R^n}\int_{-\infty}^{+\infty}M_i\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)}\right)\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t),x)}{2\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi_i(t),x)\right)\times$$

$$\times\Delta_i^{m_i}\left(\varphi_i\left(\delta\right)u\right)f\left(x+y+ue_i\right)dudy \tag{8}$$

Then for any $\bar{x}\in U$ *the following inequalities are true*

$$\sup_{\overline{x}\in U}\|B\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_1\prod_{\mu=1}^{N}\{\|f\|_{p_\mu,\varphi,\beta;G}\}^{\lambda_\mu}\times$$

$$\times\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j=1}^{n}\left(\psi_j[\xi]_1\right)^{\beta_j\frac{p}{q}}, \tag{9}$$

$$\sup_{\overline{x}\in U}\left\|B_\eta^i\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_2\prod_{\mu=1}^{N}\left\{\left\|(\varphi_i(t))^{-l_i^\mu}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p_\mu,\varphi,\beta;G}\right\}^{\lambda_\mu}$$

$$\times Q_\eta^i\prod_{j=1}^{n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p}{q}}, \tag{10}$$

$$\sup_{\overline{x}\in U}\left\|B_{\eta T}^i\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_3\prod_{\mu=1}^{N}\left\{\left\|(\varphi_i(t))^{-l_i^\mu}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p_\mu,\varphi,\beta;G}\right\}^{\lambda_\mu}$$

$$\times Q_{\eta T}^i\prod_{j=1}^{n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p}{q}}, \tag{11}$$

where $U_{\psi(\xi)}\left(\overline{x}\right)=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\psi_j\left(\xi\right),j=1,2,...,n\right\}$ $\psi\in A$, C_1 *and* C_2 *-the constants independent of* φ, ξ, η *and* T.

Proof. Apply the generalized Minkowski inequality for $\bar{x}\in U$ we obtain

$$\left\|B_\eta^i\right\|_{q,U_{\psi(\xi)}(\overline{x})}\leq\int_0^\eta\|L_i(\cdot,t)\|_{q,U_{\psi(\xi)}(\overline{x})}\prod_{\mu=1}^{N}(\varphi_j(t))^{-2-\nu_j}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \tag{12}$$

estimate the norm $\|L_i(\cdot,t)\|_{p,U_{\psi(\xi)}(\overline{x})}$. Applying the Holder inequality with exponents

$$\alpha_\mu=\frac{q_\mu}{\lambda_\mu q},\mu=1,2,\ldots,N;\left(\sum_{\mu=1}^{N}\frac{1}{\alpha_\mu}=q,\ \sum_{\mu=1}^{N}\frac{\lambda_\mu}{q_\mu}=1\right)$$

for $|L_i(x,t)|$ we obtain

$$\|L_i(\cdot,t)\|_{q,U_{\psi(\xi)}(\bar{x})}\leq C_1\prod_{\mu=1}^{N}\left\{\|L_i(\cdot,t)\|_{q_\mu,U_{\psi(\xi)}(\bar{x})}\right\}^{\lambda_\mu}. \tag{13}$$

By virtue of the Holder inequality, for $(q_\mu\leq r_\mu)$ $(\mu=1,2,\ldots,N)$ we have

$$\|L_i\left(\cdot,t\right)\|_{p_\mu,U_{\psi(\xi)}(\bar{x})}\leq\prod_{\mu=1}^{N}(\psi_j(\xi))^{\left(\frac{1}{p_\mu}-\frac{1}{r_\mu}\right)}\|L_i\left(\cdot,t\right)\|_{r_\mu,U_{\psi(\xi)}(\bar{x})}. \tag{14}$$

Now estimate the norm $\|L_i(\cdot,t)\|_{r_\mu,U_{\psi(\xi)}(\overline{x})}$.Let χ be a characteristic function of the set $S(M_i)$. Again applying the Holder inequality for representing the function in the function in the form (8) in the case $1 \le p_\mu \le r_\mu \le \infty, s_\mu \le r_\mu$, $\frac{1}{s_\mu} = 1 - \frac{1}{p_\mu} + \frac{1}{r_\mu}$ $(\mu = 1,2,\ldots,N)$, we get

$$\|L_i(\cdot,t)\|_{r_\mu,U_{\psi(\xi)}(\bar{x})}$$

$$\le C \sup_{x\in U_{\psi(\xi)}} \left(\int_{R^n} \left| \int_{-\infty}^{\infty} \Delta^{m^i}(\varphi_i(\delta)u)\, f(x+y+ue_i)\, du \right|^{p_\mu} \chi\left(\frac{y}{\varphi(t)}\right) dy \right)^{\frac{1}{p_\mu}-\frac{1}{r_\mu}} \times$$

$$\times \sup_{x\in V} \left(\int_{U_{\psi(\xi)}} \left| \int_{-\infty}^{\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho(\varphi_i(t),x)}{2\varphi_i(t)}, \frac{1}{2}\rho'(\varphi_i(t),x) \right) \right.\right.$$

$$\left.\left. \times \Delta^{m^i}(\varphi_i(\delta)u)\, f(x+y+ue_i)\, du \right|^{p_\mu} X\left(\frac{y}{\varphi(t)}\right) dy \right)^{\frac{1}{p_\mu}-\frac{1}{r_\mu}}$$

$$\times \sup_{y\in V} \left(\int_{U_{\psi(\xi)}(\bar{x})} \left| \int_{-\infty}^{\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho(t,x)}{2\varphi_i(t)}, \frac{1}{2}\rho'(\varphi_i(t),x) \right) \right.\right.$$

$$\left.\left. \times \Delta^{m^i}(\varphi_i(\delta)u)\, f(x+y+ue_i)\, du \right|^{p_\mu} \chi\left(\frac{y}{\varphi(t)}\right) dy \right)^{\frac{1}{r_\mu}}$$

$$\times \left(\int_{R^n} \left| \widetilde{M}_i\left(\frac{y}{\varphi(t)}\right) \right|^{s_\mu} dy \right)^{\frac{1}{s_\mu}}. \tag{15}$$

It is assumed that $|M_i(x,y)| \le C|\widetilde{M}_i(x)|$, $\widetilde{M}_i \in C_0^\infty(R^n)$. For any $x \in U$ we have

$$\int_{R^n} \left| \int_{-\infty}^{\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho(\varphi_i(t),x)}{2\varphi_i(t)}, \frac{1}{2}\rho'(\varphi_i(t),x) \right) \right.$$

$$\left. \times \Delta^{m^i}(\varphi_i(\delta)u)\, f(x+y+ue_i)\, du \right|^{p_\mu} X\left(\frac{y}{\varphi(t)}\right) dy$$

$$\le \int_{(U+V)_{\varphi(t)}(\bar{x})} \left| \int_{-\infty}^{\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho(\varphi_i(t),x)}{2\varphi_i(t)}, \frac{1}{2}\rho'(\varphi_i(t),x) \right) \right.$$

$$\left. \times \Delta^{m^i}(\varphi_i(\delta)u)\, f\left(x+y+u^{e^i}\right) du \right|^{p_\mu} dy \le$$

$$\le \int_{G_{\varphi(t)}(\bar{x})} \left| \int_{-\infty}^{\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho(\varphi_i(t),x)}{2\varphi_i(t)}, \frac{1}{2}\rho'(\varphi_i(t),x) \right) \right.$$

$$\times\Delta^{m^i}\left(\varphi_i(\delta)u\right)f\left(y+u^{e^i}\right)du\Big|^{p_\mu}dy\le$$

$$\le(\varphi_i(t))^{l_i^\mu p_\mu}\|(\varphi_i(t))^{-l_i^\mu}\Delta_i^{m_i}(\varphi_i(t),G_{\varphi(t)})f\|^{p_\mu}_{p_\mu,G_{\varphi(t)}(x)}\le$$

$$\le(\varphi_i(t))^{l_i^\mu p_\mu}\|(\varphi_i(t))^{-l_i^\mu}\Delta_i^{m_i}(\varphi_i(t),G_{\varphi(t)})f\|^{p_\mu}_{p_\mu,\varphi,\beta}\prod_{j=1}^{n}(\varphi_j(t))^{\beta_j p_\mu}. \tag{16}$$

for $y\in V\quad\left((U+V)_{\psi(\xi)}\subset G_{\varphi(t)}\right)$

$$\int\limits_{U_{\psi_\xi(\bar{x})}}\left|\int\limits_{-\infty}^{\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho'\left(\varphi_i(t),x\right)\right)\right.$$

$$\times\Delta^{m^i}\left(\varphi_i(\delta)u\right)f\left(x+y+ue_i\right)du\Big|^{p_\mu}X\left(\frac{y}{\varphi(t)}\right)dy$$

$$\le\int\limits_{(U+V)_{\varphi(t)}(\bar{x}+y)}\left|\int\limits_{-\infty}^{\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho'\left(\varphi_i(t),x\right)\right)\right.$$

$$\times\Delta^{m^i}\left(\varphi_i(\delta)u\right)f\left(x+u^{e^i}\right)du\Big|^{p_\mu}dy\le$$

$$\le(\varphi_i(t))^{l_i^\mu p_\mu}\|(\varphi_i(t))^{-l_i^\mu}\Delta_i^{m_i}(\varphi_i(t),(U+V)_{\psi(\xi)})f\|^{p_\mu}_{p_\mu,(U+V)_{\psi(\xi)}}\le$$

$$\le(\varphi_i(t))^{l_i^\mu p_\mu}\|(\varphi_i(t))^{-l_i^\mu}\Delta_i^{m_i}(\varphi_i(t),G_{\varphi(t)})f\|^{p_\mu}_{p_\mu,\varphi,\beta}\prod_{j=1}^{n}(\psi_j([\xi]_1))^{\beta_j p_\mu}. \tag{17}$$

$$\int\limits_{R^n}\left|\widetilde{M_i}\left(\frac{y}{\varphi(t)}\right)\right|^{s_\mu}dy=\left\|\widetilde{M_i}\right\|^{s_\mu}_{s_\mu}\prod_{j=1}^{n}\varphi_j(t). \tag{18}$$

From inequalities (15)-(18), we have

$$\|L_i\left(\cdot,t\right)\|_{r_\mu,U_{\psi(\xi)}(\bar{x})}\le C_1\left\|\widetilde{M_i}\right\|_{s_\mu}\|(\varphi_i(t))^{-l_i^\mu}\Delta_i^{m_i}(\varphi_i(t),G_{\varphi(t)})f\|_{p_\mu,\varphi,\beta}\times$$

$$\times(\varphi_i(t))^{l_i^\mu}\prod_{j=1}^{n}(\varphi_j(t))^{\frac{1}{s_\mu}+\beta_j p_\mu\left(\frac{1}{p_\mu}-\frac{1}{r_\mu}\right)}\prod_{j=1}^{n}(\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}} \tag{19}$$

and by inequality (14) we have

$$\|L_i\left(\cdot,t\right)\|_{r_\mu,U_{\psi(\xi)}(\bar{x})}\le C_2\left\|\widetilde{M_i}\right\|_{s_\mu}\|(\varphi_i(t))^{-l_i^\mu}\Delta_i^{m_i}(\varphi_i(t),G_{\varphi(t)})f\|_{p_\mu,\varphi,\beta}\times$$

$$\times(\varphi_i(t))^{l_i^\mu}\prod_{j=1}^{n}(\varphi_j(t))^{\frac{1}{s_\mu}+\beta_j p_\mu\left(\frac{1}{p_\mu}-\frac{1}{r_\mu}\right)}\prod_{j=1}^{n}(\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}}\prod_{j=1}^{n}(\psi_j([\xi]_1))^{\frac{1}{q_\mu}-\frac{1}{r_\mu}} \tag{20}$$

From inequalities (12),(13) for $r_\mu = p_\mu$ and for any $\bar{x} \in U$ reduce to the estimation

$$\sup_{\bar{x}\in U} \|B^i_\eta\|_{q,U_{\psi(\xi)}(\bar{x})} \leq$$

$$\leq C_3 \prod_{\mu=1}^{N} \left\{ \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta} \right\}^{\lambda_\mu} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}}$$

In a similar way can prove inequality (9) and (11).

Corollary 1. *For $1 \leq \tau_1 \leq \tau_2 \leq \infty$ the following inequalities:*

$$\sup_{\bar{x}\in U} \|B\|_{q,\psi,\beta_1;U} \leq C^1 \prod_{\mu=1}^{N} \left\{ \|f\|_{p_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu} \tag{21}$$

$$\sup_{\bar{x}\in U} \|B^i_\eta\|_{q,\psi,\beta_1;U} \leq$$

$$\leq C^2 \prod_{\mu=1}^{N} \left\{ \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta} \right\}^{\lambda_\mu} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}} \tag{22}$$

$$\sup_{\bar{x}\in U} \|B^i_{\eta,T}\|_{q,\psi,\beta_1;U} \leq$$

$$\leq C^2 \prod_{\mu=1}^{N} \left\{ \|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f\|_{p_\mu,\varphi,\beta} \right\}^{\lambda_\mu} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{\beta_j p_\mu}{r_\mu}} \tag{23}$$

2. Main results

Prove two theorems on the properties of the functions from the space $\bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)$.

Theorem 1. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn [11], $1 \leq p_\mu \leq q_\mu \leq \infty$, $1 \leq \theta_\mu \leq \infty (\mu = 1, 2, \ldots, N)$; $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \geq 0$ integer $j = 1, 2, ..., n$, $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ and let $f \in \bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)$. Then the following embeddings hold*

$$D^\nu : \bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi) \hookrightarrow L_{q,\psi,\beta^1}(G)$$

i.e. for $f \in \bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)$ there exists a generalized derivative $D^\nu f$ in G and the following inequalities are true

$$\|D^\nu f\|_{q,G} \leq C^1 B(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{\bigcap_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu}, \tag{24}$$

$$\|D^{\nu}f\|_{q,\psi,\beta^1;G}\leq C^2\prod_{\mu=1}^{N}\left\{\|f\|_{\bigcap\limits_{\mu=1}^{N}B^{l^{\mu}}_{p_{\mu},\theta_{\mu},\varphi,\beta}(G_{\varphi})}\right\}^{\lambda_{\mu}},\ p_{\mu}\leq q_{\mu}<\infty, \tag{25}$$

in particular, if

$$Q^{i}_{T,0}=\int\limits_{0}^{T}\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-(1-\beta_j p)\frac{1}{p}}\frac{\varphi_i'(t)}{(\varphi_i(t))^{1-\sum\limits_{\mu=1}^{n}l_i^{\mu}\lambda_{\mu}}}dt<\infty,\left(i=\overline{1,n}\right), \tag{26}$$

then $D^{\nu}f\,(x)$ is continuous on G, and

$$\sup_{x\in G}|D^{\nu}f(x)|\leq C_1B_1^0(t)\prod_{\mu=1}^{N}\left\{\|f\|_{\bigcap\limits_{\mu=1}^{N}B^{l^{\mu}}_{p_{\mu},\theta_{\mu},\varphi,\beta}(G_{\varphi})}\right\}^{\lambda_{\mu}} \tag{27}$$

where $0<T\leq\min\{1,T_0\}$ is a fixed number, C_1, C_2 are the constants independent of f, C_1 are independent also on T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^{\nu}f$. Indeed, from the condition $Q^i_T<\infty$ $\{i=1,2,...,n\}$, it follows that for $f\in\bigcap\limits_{\mu=1}^{N}B^{l^{\mu}}_{p_{\mu},\theta_{\mu},\varphi,\beta}(G_{\varphi})\to B^{l^{\mu}}_{p_{\mu},\theta_{\mu},\varphi,\beta}(G_{\varphi})\to B^{l^{\mu}}_{p_{\mu},\theta_{\mu}}(G_{\varphi})$ there exists a generalized derivative $D^{\nu}f\in L_p(G)$ and for it integral representation with the kernels is valid [13].

$$D^{\nu}f(x)=f^{(\nu)}_{\varphi(T)}(x)+(-1)^{|\nu|}\sum_{i=1}^{n}\int\limits_{0}^{T}\int\limits_{-\infty}^{+\infty}\int\limits_{R^n}K_i^{(\nu)}\left(\frac{u}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)}\right)\times$$

$$\times\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho(\varphi_i(t),x)}{2\varphi_i(t)},\frac{1}{2}\rho'(\varphi_i(t),x)\right)\Delta^{m^i}(\varphi_i(\delta)u)\times$$

$$\times f(x+y+ue_i)\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dtdudy, \tag{28}$$

$$f^{(\nu)}_{\varphi(T)}(x)=\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-2}\int\limits_{R^n}\int\limits_{R^n}\Omega^{(\nu)}\left(\frac{u}{\varphi(T)},\frac{\rho(\varphi(T),x)}{2\varphi(T)}\right)\times$$

$$\times\Omega^{(\nu)}\left(\frac{z}{\varphi(T)},\frac{\rho(\varphi(T),x)}{2\varphi(T)}\right)f(x+y+z)dydz. \tag{29}$$

Based around the Minkowsky inequality, from identities (28) and (29) we get

$$\|D^{\nu}f\|_{q,G}\leq\left\|f^{(\nu)}_{\varphi(T)}\right\|_{q,G}+\sum_{i=1}^{n}\left\|B^i_T\right\|_{q,G}. \tag{30}$$

By means of inequality (9) for $U = G$, $M_i = \Omega$ we get

$$\left\| f_{\varphi(t)}^{(\nu)} \right\|_{q,G} \le \prod_{\mu=1}^{N} \left\{ \|f\|_{p_\mu,\theta_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu} \prod_{j=1}^{n} (\varphi_j(T))^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}}, \quad (31)$$

and by means inequality (10) for $U = G$, $M_i = K_i^{(\nu)}$ $\eta = T$ we get

$$\left\| B_T^i \right\|_{q,G} \le C_2 |Q_T^i| \prod_{\mu=1}^{N} \left\{ \left\| (\varphi_i(t))^{-l_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)} \right) f \right\|_{p,\varphi,\beta} \right\}^{\lambda_\mu}. \quad (32)$$

Substituting (31), (32) for $1 \le \theta_\mu \le \infty$, $p_\mu \le \theta_\mu$ $(\mu = 1, 2, \ldots, N)$, we get inequality (24). By means of inequalities (21) and (22) for $\eta = T$ we get inequality (25) .

Now let conditions $Q_{T,0}^i < \infty$ $(i = 1, 2, \ldots, n)$. Then from identities (28), (29) and by the inequality (24) for $q = \infty, p \le \theta$ we get

$$\left\| D^\nu f(x) - f_{\varphi(T)}^{(\nu)}(x) \right\|_{\infty,G} \le C_1 \sum_{i=1}^{n} Q_{T,0}^i$$

$$\times \prod_{\mu=1}^{N} \left\{ \left(\int_0^{t_0} \left[\frac{\left\| \Delta_i^{m_i}(\varphi_i(t), G_{\varphi(t)}) f \right\|_{p_\mu,\varphi,\beta}}{(\varphi_i(t))^{l_i^\mu}} \right]^{\theta_\mu} \frac{d\varphi_i(t)}{\varphi_i(t)} \right)^{\frac{1}{\theta_\mu}} \right\}^{\lambda_\mu}.$$

As $T \to 0$, the left side of this inequality tends to zero, since $f_{\varphi(T)}^{(\nu)}$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f$ is continuous on G.

Theorem 1 is proved.

Let γ be an n-dimensional vector.

Theorem 2. *Let all the conditions of Theorem 2.1 be satisfied. Then for $Q_T^i < \infty$ $(i = 1, 2, ..., n)$ the generalized derivative $D^\nu f$ satisfies on G the generalized Hölder condition, i.e. the following inequality is valid:*

$$\left\| \Delta(\gamma, G) D^\nu f \right\|_{q,G} \le C \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu} \cdot |H(|\gamma|, \varphi; T)|. \quad (33)$$

In particular, if $Q_{T,0}^i < \infty$, $(i = 1, 2, \ldots, n)$, then

$$\sup_{x \in G} |\Delta(\gamma, G) D^\nu f(x)| \le C \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu} \cdot |H_0(|\gamma|, \varphi, T)|, \quad (34)$$

where C - is a constant independent of f, $|\gamma|, \varphi, T$ and H.

$$H(|\gamma|, \varphi, T) = \max_i \left\{ |\gamma|, Q_{|\gamma|}^i, Q_{|\gamma|,T}^i \right\}$$

$$\left(H_0(|\gamma|, \varphi, T) = \max_i \left\{ |\gamma|, Q_{|\gamma|,0}^i, Q_{|\gamma|,T,0}^i \right\} \right)$$

Proof. According to lemma 8.6 from [3] there exists a domain

$$G_\omega \subset G\left(\omega = \vartheta r\left(x\right),\ \vartheta > 0,\ r\left(x\right) = \rho\left(x, \partial G\right),\ x \in G\right)$$

and assume that $|\gamma| < \omega$,then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identities (28) and (29) with the same kernels are valid. After same transformations, we get

$$\left|\Delta\left(\gamma, G\right) D^\nu f\left(x\right)\right| \le \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-1-\nu_j} =$$

$$= C_1 B\left(x,\gamma\right) + C_2 \sum_{i=1}^{n} \left(B_1\left(x,\gamma\right) + B_2\left(x,\gamma\right)\right), \tag{35}$$

where $0 < T \le \{1, T_0\}$ we also assume that $|\gamma| < t$,consequently $|\gamma| < \min\left(\omega, T\right)$. If $x \in G \setminus G_\omega$, then

$$\Delta\left(\gamma, G\right) D^\nu f\left(x\right) = 0.$$

From (35) we get

$$\left\|\Delta\left(\gamma, G\right) D^\nu f\right\|_{q,G} \le \left\|B\left(\cdot,\gamma\right)\right\|_{q,G_\omega}$$

$$+\sum_{i=1}^{n} \left(\left\|B_1\left(\cdot,\gamma\right)\right\|_{q,G_\omega} + \left\|B_2\left(\cdot,\gamma\right)\right\|_{q,G_\omega}\right), \tag{36}$$

$$B\left(x,\gamma\right) \le \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-\nu_j-2} \int_0^{|\gamma|} d\zeta \int_{R^n}\int_{R^n} \left|f\left(x+\zeta e_\gamma + y\right)\right|$$

$$\times \left|D_j \Omega^{(\nu)}\left(\frac{y}{\varphi\left(T\right)}, \frac{\rho\left(\varphi(t), x\right)}{2\varphi(t)}\right) \Omega^{(\nu)}\left(\frac{z}{\varphi\left(T\right)}, \frac{\rho\left(\varphi(t), x\right)}{2\varphi(t)}\right)\right| dydz.$$

Taking into account $\xi e_\gamma + G_\omega \subset G$, and from inequality (21) for $U = G$, we have

$$\left\|B\left(\cdot,\gamma\right)\right\|_{q,G_\omega} \le C_1 \left|\gamma\right| \left\|f\right\|_{p,\varphi,\beta;G}. \tag{37}$$

By means of inequality (22), for $U = G$, $\eta = |\gamma|$, $M_i = K_i^{(\nu)}$ we get

$$\left\|B_1\left(\cdot,\gamma\right)\right\|_{q,G_\omega} \le C_2 \left|Q^i_{|\gamma|}\right| \left\|\left(\varphi_i(t)\right)^{-l_i} \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G} \tag{38}$$

and by means of inequality (23) for $U = G$, $\eta = |\gamma|$, $M_i = K_i^{(\nu)}$ we get

$$\left\|B_2\left(\cdot,\gamma\right)\right\|_{q,G_\omega} \le C_3 \left|Q^i_{|\gamma|,T}\right| \left\|\left(\varphi_i(t)\right)^{-l_i} \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G}. \tag{39}$$

From inequalities (36)-(39) for cases $p_\mu \le \theta_\mu$ we get the required inequality (33).

Now suppose that, $|\gamma| \ge \min\left(\omega, T\right)$. Then we have

$$\left\|\Delta\left(\gamma, G\right) D^\nu f\right\|_{q,G} \le 2\left\|D^\nu f\right\|_{q,G} \le C\left(\omega T\right) \left\|D^\nu f\right\|_{q,G} \left|h\left(\left|\gamma\right|, \varphi; T\right)\right|.$$

Estimating for $\left\|D^\nu f\right\|_{q,G}$ by means of inequality (24), in this case we get estimation. This completes the proof of Theorem 2.

References

[1] A. Akbulut, A. Eroglu and A.M.Najafov, *Some Embedding Theorems on the Nikolskii-Morrey Type Spaces*, Adv. in Anal., **1**, (2016), no. 1, 18–26.

[2] Bergh J., Lefstrem J., Interpolation spaces, M.Mir, 1980, 264p.

[3] O.V. Besov, V.P. Il'yin and S. M.Nicolskii, *Integral representations of functions and embeddings theorems*, M. Nauka, (1996), 480

[4] V.S. Guliyev and M.N. Omarova, *Multilinear singular and fractional integral operators on generalized weighted Morrey spaces*, Azerb. J. Math., **5**, (2015), no. 1 104–132.

[5] D.I. Hakim, Y. Sawano and T. Shimomura, *Boundedness of Generalized Fractional Integral Operators From the Morrey Space $L_{1,\phi}(X;\mu)$ to the Campanato Space $L_{1,\psi}(X;\mu)$ Over Non-doubling Measure Spaces*, Azerb. J. Math. **6**,(2016),no. 2, 117–127

[6] V.P. Il'yin, *On some properties of the functions of spaces $W^l_{p,a,\chi}(G)$*, Zap. Nauch.Sem. LOMI AN USSR, **2**, 1971, 33–40.

[7] V.Kokilashvili, A. Meskhi and H. Rafeiro, *Sublinear operators in generalized weighted Morrey spaces*, Dokl. Math. **94**, (2016), no.2, 558–560.

[8] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, // Trans. Amer. Math. Soc. **43**, (1938), 126–166.

[9] E. Nakai, *Generalized fractional integrals on generalized Morrey spaces*, Math. Nachr. **287**, (2014),no.2-3, 339–351.

[10] A.M. Najafov, *On Some Properties of Functions in the Sobolev-Morrey-Type Spaces* $W^l_{p,a,\varkappa,\tau}(G)$, Sib. Math. J.,**46**, (2005), no.3, 501-513

[11] A.M.Najafov, *The embedding theorems of space $W^l_{p,\varphi,\beta}(G)$*, Math. Aeterna, **3**, (2013), no. 4, 299 – 308.

[12] A.M.Najafov and A.T. Orujova, *On properties of the generalized Besov-Morrey spaces with domiant mixed derivatives*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **41**, (2015), no. 1, 3–15.

[13] A. M. Najafov, N. R. Rustamova, On some properties of functions from a Besov–Morrey type spaces, Afrika Matematika, volume 29, (2018), 1007–1017

[14] Yu. V. Netrusov, *On some imbedding theorems of Besov-Morrey type spaces* Zap. Nauch.Sem. LOMI AN USSR, **139** (1984), 139–147 (Russian).

[15] Y. Sawano, *Idendification of the image of Morrey spaces by the fractional integral operators*, Proc. A. Razmadze Math. Inst.,(2009), no. 149, 87–93.

Nilufer R. Rustamova
Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, Az-1141, Baku, Azerbaijan
E-mail: niluferustamova@gmail.com

Received 5 May 2020
Accepted 10 June 2020

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 8, No 1, 2020, July
ISSN 1560-4055

Inverse Problem For A Third Order Hyperbolic Equation

Ulvu Alizadeh

Abstract. In this work a nonlinear inverse boundary value problem for a hyperbolic equation of the third order is investigated. Using the Fourier method, the problem is reduced to solving a system of integral equations, and using the contraction mapping method, the existence and uniqueness of a solution to a system of integral equations are proved. The existence and uniqueness of the classical solution to the initial problem are proved.

Key Words and Phrases: hyperbolic equation, inverse problem, integral condition of overdetermination.

2010 Mathematics Subject Classifications: 35L25, 35R30

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1. Introduction

There are many cases when the needs of practice lead to problems of determining the coefficients or the right-hand side of a differential equation according some known data of its solution. Such problems are called inverse problems of mathematical physics. Inverse problems are an actively developing branch of modern mathematics. Inverse problems for partial differential equations of various types were studied in many works [1–5]. In inverse problems, along with the initial and boundary conditions characteristic of a particular direct problem, additional information is given, the need for which is due to the presence of unknown coefficients or the right-hand side of the equation. Additional information, called an overdetermination condition, can be presented in various forms.

In the proposed article, an inverse boundary value problem with additional integral conditions for a third-order hyperbolic equation is studied.

2. Statement of the problem and its reduction to an equivalent problem

Let $D_T = \{(x,t):\ 0 \le x \le 1,\ 0 \le t \le T\}$. Next, let $f(x,t)$, $g(x,t)$, $\omega(x)$,$\varphi_i(x)$, $(i = 1, 2.3)$, $h_i(t)\,(i = 1, 2)$ - be the given functions defined for $x \in [0,1]$, $t \in [0,T]$. Consider the following inverse boundary value problem: It is required to find the triple $\{u(x,t), a(t), b(t)\}$ of the functions $u(x,t), a(t), b(t)$ related by the equation [6]:

$$u_{ttt}(x,t) - u_{txx}(x,t) + u_{tt}(x,t) - \alpha u_{xx}(x,t) = a(t)u(x,t) + b(t)g(x,t) + f(x,t) \quad (1)$$

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when the initial conditions are fulfilled for the function $u(x,t)$

$$u(x,0)=\varphi_0(x),\quad u_t(x,0)=\varphi_1(x),\quad u_{tt}(x,0)=\varphi_2(x)\quad (0\le x\le 1) \tag{2}$$

boundary conditions

$$u_x(0,t)=0,\quad u(1,t)=0\quad (0\le t\le T), \tag{3}$$

and with additional conditions

$$\int_0^1 \omega(x)u(x,t)dx\ = h_1(t)\ \ (0\le t\le T), \tag{4}$$

$$u(0,t)=h_2(t)\quad (0\le t\le T), \tag{5}$$

where $0<\alpha<1$ – is a given number.

Denote

$$\tilde{C}^{(2,3)}\left(D_T\right)=\{u(x,t)\ ,\, u_x(x,t),u_{xx}(x,t),u_t(x,t),u_{tx}(x,t),u_{txx}(x,t),$$

$$u_{tt}(x,t),u_{ttt}(x,t)\in C\left(D_T\right)\}\,.$$

Definition 1. *By the classical solution of the inverse boundary value problem* (1) - (5) *we mean a triple* $\{u(x,t),a(t),b(t)\}$ *of functions* $u\left(x,t\right)\in\tilde{C}^{(2,3)}\left(D_T\right)$, $a\left(t\right)\in C[0,T]$, $b\left(t\right)\in C[0,T]$*, satisfying equation* (1) *and conditions* (2) - (5) *in the usual sense.*

Similarly to [7], the following theorem is proved.

Theorem 1. *Let* $f(x,t),g(x,t)\in C(D_T)$, $\varphi_i(x)\in C[0,1]$ $(i=1,2,3)$,$h_i\left(t\right)\in C^3\left[0,T\right]$ $(i=1,2)$ *and the conditions of agreement are fulfilled:*

$$\int_0^1\omega(x)\varphi_0\left(x\right)dx=h_1\left(0\right),\ \int_0^1\omega(x)\varphi_1\left(x\right)dx=h_1'\left(0\right),\ \int_0^1\omega(x)\varphi_2\left(x\right)dx=h_1''\left(0\right)\ ,$$

$$\varphi_0\left(0\right)=h_2\left(0\right),\ \ \varphi_1\left(0\right)=h_2'\left(0\right)\ ,\ \ \varphi_2\left(0\right)=h_2''\left(0\right)\ .$$

Then the problem of finding a classical solution to problem (1) - (5) is equivalent to the problem of determining functions $u\left(x,t\right)\in\tilde{C}^{(2,3)}\left(D_T\right)$, $a\left(t\right)\in C[0,T]$, $b\left(t\right)\in C[0,T]$ from relations (1) - (3) and

$$a(t)h_1(t)\ +\ b(t)\int_0^1\omega(x)g\left(x,t\right)dx=$$

$$=h_1'''(t)-\int_0^1\omega(x)f\left(x,t\right)dx-\int_0^1\omega(x)u_{txx}(x,t)dx\ +h_1''(t)-\alpha\int_0^1\omega(x)u_{xx}(x,t)dx\ \ , \tag{6}$$

$$a(t)h_2(t)\left(t\right)\ +\ b(t)g(0,t)=h_2'''(t)\ -f(0,t)-\ u_{txx}(0,t)\ +\ h_2''(t)\ -\ \alpha u_{xx}(0,t)\,, \tag{7}$$

moreover

$$h\left(t\right)\equiv h_1(t)\,g\left(0,t\right)-h_2(t)\int_0^1\omega(x)g\left(x,t\right)dx\quad (0\le t\le T).$$

3. Solvability of the problem

The first component $u(x,t)$ of the solution $\{u(x,t), a(t), b(t)\}$ to problem (1) - (3), (6), (7) will be sought in the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \cos \lambda_k x, \quad \lambda_k = \frac{\pi}{2}(2k-1), \tag{8}$$

where

$$u_k(t) = 2 \int_0^1 u(x,t) \cos \lambda_k x dx \; (k = 1, 2, ...).$$

Then, applying the formal Fourier scheme, from (1) and (2) we have:

$$u_k'''(t) + u_k''(t) + \lambda_k^2 u_k'(t) + \alpha \lambda_k^2 u_k(t) = F_k(t; u, a, b) \;\; (k = 1, 2, ...; \;\; 0 \le t \le T), \tag{9}$$

$$u_k(0) = \varphi_{0k}, \;\; u_k'(0) = \varphi_{1k}, \;\; u_k''(0) = \varphi_{2k} \;\; (k = 1, 2, ...), \tag{10}$$

where

$$F_k(t; u, a, b) = f_k(t) + a(t)u_k(t) + b(t)g_k(t), f_k(t) = 2 \int_0^1 f(x,t) \cos \lambda_k x dx,$$

$$g_k(t) = 2 \int_0^1 g(x,t) \cos \lambda_k x dx, \;\; \varphi_{ik} = 2 \int_0^1 \varphi_i(x) \cos \lambda_k x \, dx (i = 0, 1, 2; \;\; k = 1, 2, ...).$$

Solving problem (9), (10), we find:

$$\begin{aligned} u_k(t) = \frac{1}{b_k} \Bigg\{ \Bigg[(\gamma_k^2 + \beta_k^2) e^{\alpha_k t} + e^{\gamma_k t} \Bigg[\alpha_k(\alpha_k - 2\gamma_k) \cos \beta_k t + \\ + \frac{1}{\beta_k} (\gamma_k^3 + \alpha_k \gamma_k^2 - \alpha_k \beta_k^2 - \alpha_k^2 \gamma_k) \sin \beta_k t \Bigg] \Bigg] \varphi_{0k} + \\ + \Bigg[-2\gamma_k e^{\alpha_k t} + e^{\gamma_k t} \Bigg[2\gamma_k \cos \beta_k t + \frac{1}{\beta_k} (\alpha_k^2 + \beta_k^2 - \gamma_k^2) \sin \beta_k t \Bigg] \Bigg] \varphi_{1k} + \\ + \Bigg[e^{\alpha_k t} + e^{\gamma_k t} \Bigg[-\cos \beta_k t + \frac{1}{\beta_k} (\gamma_k - \alpha_k) \sin \beta_k t \Bigg] \Bigg] \varphi_{2k} + \int_0^t F_k(\tau; u, a, b) \times \\ \times \Bigg[e^{\alpha_k (t-\tau)} + e^{\gamma_k (t-\tau)} \Bigg[\frac{\gamma_k - \alpha_k}{\beta_k} \sin \beta_k (t - \tau) - \cos \beta_k (t - \tau) \Bigg] \Bigg] d\tau \Bigg\} \;\; (k = 1, 2, ...), \end{aligned} \tag{11}$$

where

$$\alpha_k = \alpha_{1k} + \beta_{1k} - \frac{1}{3}, \;\; \beta_k = \frac{\sqrt{3}}{2}(\alpha_{1k} - \beta_{1k}), \;\; \gamma_k = -\frac{1}{3} - \frac{1}{2}(\alpha_{1k} + \beta_{1k}),$$

$$b_k = \alpha_k^2 + \beta_k^2 + \gamma_k^2 - 2\alpha_k \gamma_k,$$

moreover

$$\alpha_{1k}=\left\{-\frac{1}{2}\left(\left(\alpha-\frac{1}{3}\right)\lambda_k^2+\frac{2}{27}\right)+\left[\frac{1}{4}\left(\left(\alpha-\frac{1}{3}\right)\lambda_k^2+\frac{2}{27}\right)^2+\frac{1}{27}\left(\lambda_k^2-\frac{1}{3}\right)^3\right]^{1/2}\right\}^{1/3}, \tag{12}$$

$$\beta_{1k}=\left\{-\frac{1}{2}\left(\left(\alpha-\frac{1}{3}\right)\lambda_k^2+\frac{2}{27}\right)-\left[\frac{1}{4}\left(\left(\alpha-\frac{1}{3}\right)\lambda_k^2+\frac{2}{27}\right)^2+\frac{1}{27}\left(\lambda_k^2-\frac{1}{3}\right)^3\right]^{1/2}\right\}^{1/3}. \tag{13}$$

After substituting the expressions$u_k(t)$ $(k=1,2,...)$ in (8), to determine the components $u(x,t)$ of the solution to problem (1) - (3), (6), (7), we obtain:

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\frac{1}{b_k}\left\{\left[(\gamma_k^2+\beta_k^2)e^{\alpha_k t}+e^{\gamma_k t}\left[\alpha_k(\alpha_k-2\gamma_k)\cos\beta_k t+\right.\right.\right.\right.$$

$$\left.\left.+\frac{1}{\beta_k}(\gamma_k^3+\alpha_k\gamma_k^2-\alpha_k\beta_k^2-\alpha_k^2\gamma_k)\sin\beta_k t\right]\right]\varphi_{0k}+$$

$$+\left[-2\gamma_k e^{\alpha_k t}+e^{\gamma_k t}\left[2\gamma_k\cos\beta_k t+\frac{1}{\beta_k}(\alpha_k^2+\beta_k^2-\gamma_k^2)\sin\beta_k t\right]\right]\varphi_{1k}+$$

$$+\left[e^{\alpha_k t}+e^{\gamma_k t}\left[-\cos\beta_k t+\frac{1}{\beta_k}(\gamma_k-\alpha_k)\sin\beta_k t\right]\right]\varphi_{2k}+\int_0^t F_k(\tau;u,a,b)\times$$

$$\left.\left.\times\left[e^{\alpha_k(t-\tau)}+e^{\gamma_k(t-\tau)}\left[\frac{\gamma_k-\alpha_k}{\beta_k}\sin\beta_k(t-\tau)-\cos\beta_k(t-\tau)\right]\right]d\tau\right\}\right\}\cos\lambda_k x. \tag{14}$$

Differentiating (13) we find:

$$u_k'(t)=\frac{1}{b_k}\left\{\left[\alpha_k(\gamma_k^2+\beta_k^2)e^{\alpha_k t}+e^{\gamma_k t}\left[-\alpha_k(\gamma_k^2+\beta_k^2)\cos\beta_k t+\frac{\alpha_k}{\beta_k}(\gamma_k-\alpha_{kk})\times\right.\right.\right.$$

$$\left.\left.\times(\gamma_k^2+\beta_k^2)\sin\beta_k t\right]\right]\varphi_{0k}+\left[-2\alpha_k\gamma_k e^{\alpha_k t}+e^{\gamma_k t}\left[(\alpha_k^2+\beta_k^2+\gamma_k^2)\cos\beta_k t+\right.\right.$$

$$\left.\left.+\frac{\gamma_k}{\beta_k}(\alpha_k^2-\beta_k^2-\gamma_k^2)\sin\beta_k t\right]\right]\varphi_{1k}+\left[\alpha_k e^{\alpha_k t}+e^{\gamma_k t}\left[-\alpha_k\cos\beta_k t+\right.\right.$$

$$\left.\left.+\frac{1}{\beta_k}(\beta_k^2+\gamma_k^2-\alpha_k\gamma_k)\sin\beta_k t\right]\right]\varphi_{2k}+\int_0^t F_k(\tau;u,a,b)\left[\alpha_k e^{\alpha_k(t-\tau)}+e^{\gamma_k(t-\tau)}\times\right.$$

$$\left.\left.\times\left[\left(\frac{\gamma_k}{\beta_k}(\gamma_k-\alpha_k)+\beta_k\right)\sin\beta_k(t-\tau)-\alpha_k\cos\beta_k(t-\tau)\right]\right]d\tau\right\}\quad (k=1,2,...). \tag{15}$$

Now from (6) and (7) taking into account (8), respectively, we have:

$$a(t)h_1(t)+b(t)\int_0^1\omega(x)g(x,t)\,dx=$$

$$= h_1'''(t) + h_1''(t) - \int_0^1 \omega(x) f(x,t)\,dx + \sum_{k=1}^{\infty} \lambda_k^2 (u_k'(t) + \alpha u_k(t)) \int_0^1 \omega(x) \cos \lambda_x x dx \ , \quad (16)$$

$$a(t) h_2(t)\,(t) + b(t) g(0,t) = h_2'''(t) + h_2''(t) - f(0,t) + \sum_{k=1}^{\infty} \lambda_k^2 (u_k'(t) + \alpha u_k(t)) \,. \quad (17)$$

Suppose that

$$h(t) \equiv h_1(t)\, g(0,t) - h_2(t) \int_0^1 \omega(x) g(x,t)\,dx \neq 0 \ \ (0 \le t \le T).$$

Then from (16) and (17) we obtain:

$$a(t) = [h(t)]^{-1} \left\{ \left(h_1'''(t) + h_1''(t) - \int_0^1 \omega(x) f(x,t)\,dx \right) g(0,t) - \right.$$
$$- \left(h_2'''(t) + h_2''(t) - f(0,t) \right) \int_0^1 \omega(x) g(x,t)\,dx +$$
$$\left. + \sum_{k=1}^{\infty} \lambda_k^2 (u_k'(t) + \alpha u_k(t)) \left(g(0,t) \int_0^1 \omega(x) \cos \lambda_k dx - \int_0^1 \omega(x) g(x,t)\,dx \right) \right\} , \quad (18)$$

$$b(t) = [h(t)]^{-1} \left\{ \left(h_2'''(t) + h_2''(t) - f(0,t) \right) h_1(t) - \right.$$
$$\left(h_1'''(t) + h_1''(t) - f(0,t) - \int_0^1 \omega(x) f(x,t)\,dx \right) h_2(t) +$$
$$\left. + \sum_{k=1}^{\infty} \lambda_k^2 (u_k'(t) + \alpha u_k(t)) \left(h_1(t) - h_2(t) \int_0^1 \omega(x) \cos \lambda_k dx \right) \right\} . \quad (19)$$

Further, from (11) and (15), we obtain:

$$u_k'(t) + \alpha u_k(t) = \frac{1}{b_k} \left\{ \left[(\alpha + \alpha_k)(\gamma_k^2 + \beta_k^2) e^{\alpha_k t} + e^{\gamma_k t} \left[\alpha_k (\alpha \alpha_k - 2\alpha \gamma_k - \gamma_k^2 - \beta_k^2) \times \right.\right.\right.$$
$$\left.\left. \times \cos \beta_k t + \frac{\alpha_k}{\beta_k} ((\gamma_k - \alpha_k)(\gamma_k^2 + \beta_k^2) + \alpha(\gamma_k^2 - \beta_k^2 - \alpha_k \gamma_k)) \sin \beta_k t \right]\right] \varphi_{0k} +$$
$$+ \left[-2(\alpha + \alpha_k) \gamma_k e^{\alpha_k t} + e^{\gamma_k t} \left[(2\alpha \gamma_k + \alpha_k^2 + \beta_k^2 + \gamma_k^2) \cos \beta_k t + \right.\right.$$
$$\left.\left. + \frac{1}{\beta_k} (\alpha(\alpha_k^2 + \beta_k^2 - \gamma_k^2) + \gamma_k(\alpha_k^2 - \beta_k^2 - \gamma_k^2)) \sin \beta_k t \right]\right] \varphi_{1k} + \left[(\alpha + \alpha_k) e^{\alpha_k t} + \right.$$
$$\left. + e^{\gamma_k t} \left[-(\alpha + \alpha_k) \cos \beta_k t + \frac{1}{\beta_k} (\alpha \gamma_k - \alpha \alpha_k + \beta_k^2 + \gamma_k^2 - \alpha_k \gamma_k) \sin \beta_k t \right]\right] \varphi_{2k} +$$

$$+\int_0^t F_k(\tau;u,a,b)\left[(\alpha+\alpha_k)e^{\alpha_k(t-\tau)}+e^{\gamma_k(t-\tau)}\left[(-\alpha+\alpha_k)\cos\beta_k(t-\tau)+\right.\right.$$

$$\left.\left.+\left(\frac{\gamma_k+\alpha}{\beta_k}(\gamma_k-\alpha_k)+\beta_k\right)\sin\beta_k(t-\tau)\right]\right]d\tau\Bigg\}\quad (k=1,2,...)\ . \tag{20}$$

In order to obtain an equation for the second and third components of the solution $\{u(x,t),a(t),b(t)\}$ of problem (1) - (3), (6), (7), we substitute expression (20) into (18) and (19) :

$$a(t)=[h(t)]^{-1}\left\{\left(h_1'''(t)+h_1''(t)-\int_0^1\omega(x)f(x,t)\,dx\right)g(0,t)-\right.$$

$$-\left(h_2'''(t)+h_2''(t)-f(0,t)\right)\int_0^1\omega(x)g(x,t)\,dx+$$

$$+\sum_{k=1}^{\infty}\frac{\lambda_k^2}{b_k}\left\{\left[(\alpha+\alpha_k)(\gamma_k^2+\beta_k^2)e^{\alpha_k t}+e^{\gamma_k t}\left[\alpha_k(\alpha\alpha_k-2\alpha\gamma_k-\gamma_k^2-\beta_k^2)\times\right.\right.\right.$$

$$\left.\left.\times\cos\beta_k t+\frac{\alpha_k}{\beta_k}((\gamma_k-\alpha_k)(\gamma_k^2+\beta_k^2)+\alpha(\gamma_k^2-\beta_k^2-\alpha_k\gamma_k))\sin\beta_k t\right]\right]\varphi_{0k}+$$

$$+\left[-2(\alpha+\alpha_k)\gamma_k e^{\alpha_k t}+e^{\gamma_k t}\left[(2\alpha\gamma_k+\alpha_k^2+\beta_k^2+\gamma_k^2)\cos\beta_k t+\right.\right.$$

$$\left.\left.+\frac{1}{\beta_k}(\alpha(\alpha_k^2+\beta_k^2-\gamma_k^2)+\gamma_k(\alpha_k^2-\beta_k^2-\gamma_k^2))\sin\beta_k t\right]\right]\varphi_{1k}+\left[(\alpha+\alpha_k)e^{\alpha_k t}+\right.$$

$$\left.+e^{\gamma_k t}\left[-(\alpha+\alpha_k)\cos\beta_k t+\frac{1}{\beta_k}(\alpha\gamma_k-\alpha\alpha_k+\beta_k^2+\gamma_k^2-\alpha_k\gamma_k)\sin\beta_k t\right]\right]\varphi_{2k}+$$

$$+\int_0^t F_k(\tau;u,a,b)\left[(\alpha+\alpha_k)e^{\alpha_k(t-\tau)}+e^{\gamma_k(t-\tau)}\left[(-\alpha+\alpha_k)\cos\beta_k(t-\tau)+\right.\right.$$

$$\left.\left.+\left(\frac{\gamma_k+\alpha}{\beta_k}(\gamma_k-\alpha_k)+\beta_k\right)\sin\beta_k(t-\tau)\right]\right]d\tau\Bigg\}$$

$$\left(g(0,t)\int_0^1\omega(x)\cos\lambda_k x dx-\int_0^1\omega(x)g(x,t)\,dx\right)\Bigg\}\,, \tag{21}$$

$$b(t)=[h(t)]^{-1}\left\{\left(h_2'''(t)+h_2''(t)-f(0,t)\right)h_1(t)-\right.$$

$$\left(h_1'''(t)+h_1''(t)-f(0,t)-\int_0^1\omega(x)f(x,t)\,dx\right)h_2(t)+$$

$$+\sum_{k=1}^{\infty}\frac{\lambda_k^2}{b_k}\left\{\left[(\alpha+\alpha_k)(\gamma_k^2+\beta_k^2)e^{\alpha_k t}+e^{\gamma_k t}\left[\alpha_k(\alpha\alpha_k-2\alpha\gamma_k-\gamma_k^2-\beta_k^2)\times\right.\right.\right.$$

$$\times \cos\beta_k t\ + \frac{\alpha_k}{\beta_k}((\gamma_k - \alpha_k)(\gamma_k^2 + \beta_k^2) + \alpha(\gamma_k^2 - \beta_k^2 - \alpha_k\gamma_k)) \sin\beta_k t \Big]\Big] \varphi_{0k} +$$

$$+ \Big[-2(\alpha + \alpha_k)\gamma_k e^{\alpha_k t} + e^{\gamma_k t} \Big[(2\alpha\gamma_k + \alpha_k^2 + \beta_k^2 + \gamma_k^2) \cos\beta_k t +$$

$$+ \frac{1}{\beta_k}(\alpha(\alpha_k^2 + \beta_k^2 - \gamma_k^2) + \gamma_k(\alpha_k^2 - \beta_k^2 - \gamma_k^2)) \sin\beta_k t \Big]\Big] \varphi_{1k} + \Big[(\alpha + \alpha_k) e^{\alpha_k t} +$$

$$+ e^{\gamma_k t} \Big[-(\alpha + \alpha_k) \cos\beta_k t + \frac{1}{\beta_k}(\alpha\gamma_k - \alpha\alpha_k + \beta_k^2 + \gamma_k^2 - \alpha_k\gamma_k) \sin\beta_k t \Big]\Big] \varphi_{2k} +$$

$$+ \int_0^t F_k(\tau; u, a, b) \Big[(\alpha + \alpha_k) e^{\alpha_k (t-\tau)} + e^{\gamma_k (t-\tau)} \Big[(-\alpha + \alpha_k) \cos\beta_k (t - \tau) +$$

$$+ \ \left(\frac{\gamma_k + \alpha}{\beta_k}(\gamma_k - \alpha_k) + \beta_k \right) \sin\beta_k (t - \tau) \Big]\Big] d\tau \Big\} \ \left(h_1(t) - h_2(t) \int_0^1 \omega(x) \cos\lambda_k x dx \right) \ \Big\}. \tag{22}$$

Thus, the solution of problem (1) - (3), (6), (7) is reduced to the solution of system (14), (21), (22) with respect to unknown functions $u(x,t)$, $a(t)$ and $b(t)$.

The following lemma is true.

Lemma 1. *If* $\{u(x,t), a(t), b(t)\}$ *–is any classical solution to problem* (1) (1)-(3), (6), (7), *then the functions*

$$u_k(t) = 2\int_0^1 u(x,t) \cos\lambda_k x dx \quad (k = 1, 2, ...)$$

satisfy the system (11).

Corollary 1. *Lemma 1 implies that to prove the uniqueness of the solution to problem* (1) - (3), (6), (7), *it suffices to prove the uniqueness of the solution to system* (14), (21), (22).

Now, in order to study problem (1) - (3), (6), (7), consider the following spaces:

1. Let us denote by $B_{2,T}^3$ [8] the collection of all functions $u(x,t)$ of the form

$$u(x_i t) = \sum_{k=1}^{\infty} u_k(t) \cos\lambda_k x, \lambda_k = \frac{\pi}{2}(2k - 1),$$

considered inD_T for which all functions $u_k(t) \in C[0,T]$ and

$$J_T(u) \equiv \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|\, u_k(t) \,\|_{C[0,T]})^2 \right)^{1/2} < \infty.$$

The norm in this set is defined as follows:

$$\|u(x,t)\|_{B_{2,T}^{3,}} = J_T(u).$$

2. Let us denote by E_T^3 the spaces of the vector of functions $\{u(x,t), a(t), b(t)\}$ such that

$$u(x,t) \in B_{2,T}^3, a(t) \in C[0,T]\,, b(t) \in C[0,T]\ .$$

We equip this space with a norm:

$$\|z\|_{E_T^3} = \|u(x,t)\|_{B_{2,T}^3} + \|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]}\,.$$

It is known that $B_{2,T}^3$ and E_T^3 are Banach spaces.

Consider the following operator in space E_T^3

$$\Phi(u,a,b) = \{\Phi_1(u,a,b), \Phi_2(u,a,b), \Phi_3(u,a,b)\},$$

where

$$\Phi_1(u,a,b) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(t) \cos\lambda_k x, , \Phi_2(u,a,b) = \tilde{a}(t),\ \ , \Phi_3(u,a,b) = \tilde{b}(t)$$

where $\tilde{u}_k(t)$ $(k = 1,2,...)$, $\tilde{a}(t)$ and $\tilde{b}(t)$ are equal, respectively, to the right-hand sides (11), (21) and (22).

Accept the notation

$$\alpha_{2k} = -\frac{1}{2}\left(\left(\alpha - \frac{1}{3}\right)\lambda_k^2 + \frac{2}{27}\right) + \left[\frac{1}{4}\left(\left(\alpha - \frac{1}{3}\right)\lambda_k^2 + \frac{2}{27}\right)^2 + \frac{1}{27}\left(\lambda_k^2 - \frac{1}{3}\right)^3\right]^{1/2}, \quad (23)$$

$$\beta_{2k} = \frac{1}{2}\left(\left(\alpha - \frac{1}{3}\right)\lambda_k^2 + \frac{2}{27}\right) + \left[\frac{1}{4}\left(\left(\alpha - \frac{1}{3}\right)\lambda_k^2 + \frac{2}{27}\right)^2 + \frac{1}{27}\left(\lambda_k^2 - \frac{1}{3}\right)^3\right]^{1/2}. \quad (24)$$

Then

$$\alpha_{1k} = \sqrt[3]{\alpha_{2k}}\,, \quad \beta_{1k} = -\sqrt[3]{\beta_{2k}}\,.$$

Hence, taking into account (23) and (24), we obtain:

$$\alpha_{1k} + \beta_{1k} = \left|\sqrt[3]{\alpha_{2k}} - \sqrt[3]{\beta_{2k}}\right| = \left|\frac{\alpha_{2k} - \beta_{2k}}{\sqrt[3]{\alpha_{2k}^2} + \sqrt[3]{\alpha_{2k}\beta_{2k}} + \sqrt[3]{\beta_{2k}^2}}\right| \leq \frac{9\alpha}{2} + \frac{11}{2}.$$

It is easy to see that

$$|\alpha_k| \leq \left|\alpha_{1k} + \beta_{1k} - \frac{1}{3}\right| \leq \frac{9\alpha}{2} + \frac{13}{6} \equiv \varepsilon_1\,, |\gamma_k| = \left|-\frac{1}{3} - \frac{\alpha_{1k} + \beta_{1k}}{2}\right| \leq \frac{9\alpha}{4} + \frac{5}{4} \equiv \varepsilon_2\,,$$

$$\varepsilon_3\lambda_k \equiv \frac{\sqrt{2}}{3}\lambda_k \leq \beta_k \leq \sqrt[3]{\frac{1}{2}\left(\alpha - \frac{1}{27}\right) + \sqrt{\frac{1}{4}\left(\alpha - \frac{1}{27}\right)^2 + \frac{1}{27}}}\ {}_k\lambda_k \equiv \varepsilon_4\lambda_k\,,$$

$$b_k = (\alpha_k - \gamma_k)^2 + \beta_k^2 \geq \beta_k^2 \geq \varepsilon_3^2 \lambda_k^2 \,,$$

Taking these relations into account, we find:

$$\left(\sum_{k=1}^{\infty} (\lambda_k^3 \, \|\tilde{u}_k(t)\|_{C[0,T]})^2 \right)^{1/2} \leq \rho_0(T) \left(\sum_{k=1}^{\infty} (\lambda_k^3 \, | \, \varphi_{0k} \, |)^2 \right)^{1/2} + \rho_1(T) \left(\sum_{k=1}^{\infty} (\lambda_k^3 \, | \, \varphi_{1k} \, |)^2 \right)^{1/2} +$$

$$+\rho_2(T) \left(\sum_{k=1}^{\infty} (\lambda_k^2 \, | \, \varphi_{2k} \, |)^2 \right)^{1/2} + \rho_2(T)\sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 \, | \, f_k(\tau) \, |)^2 d\tau \right)^{1/2} +$$

$$+\rho_2(T) T \, \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \, \|u_k(t)\|_{C[0,T]})^2 \right)^{1/2} ,$$

$$+\rho_2(T)\sqrt{T} \, \|b(t)\|_{C[0,T]} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 \, | \, g_k(\tau) \, |)^2 d\tau \right)^{1/2} , \tag{25}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\| [h(t)]^{-1} \right\|_{C[0,T]} \times$$

$$\times \left\{ \left\| \left(h_1'''(t) + \, h_1''(t) \, - \int_0^1 \omega(x) f\,(x,t)\, dx \right) g(0,t) - \right. \right.$$

$$- \left(h_2'''(t) + \, h_2''(t) \, - f(0,t) \right) \int_0^1 \omega(x) g\,(x,t)\, dx \, \Bigg\|_{C[0,T]} +$$

$$+ \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{1/2} \left(\|g(0,t)\| \;_{C[0,T]} \, \|\omega(x)\| \;_{L_2(0,1)} \; + \left\| \int_0^1 \omega(x) g\,(x,t)\, dx) \right\|_{C[0,T\,]} \right) \times$$

$$\times \left[\rho_3(T) \left(\sum_{k=1}^{\infty} (\lambda_k^3 \, | \, \varphi_{0k} \, |)^2 \right)^{1/2} + \rho_4(T) \left(\sum_{k=1}^{\infty} (\lambda_k^3 \, | \, \varphi_{1k} \, |)^2 \right)^{1/2} - \right.$$

$$+\rho_5(T) \left(\sum_{k=1}^{\infty} (\lambda_k^2 \, |\varphi_{2k}|)^2 \right)^{1/2} + \; \rho_5(T)\sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 \, |f_k(\tau)|)^2 d\tau \right)^{1/2} +$$

$$+\rho_5(T) T \; \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \, \|u_k(t)\|_{C[0,T]})^2 \right)^{1/2} +$$

$$\left. \left. +\rho_5(T)\sqrt{T} \; \|b(t)\| \;_{C[0,T]} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 \, |g_k(\tau)|)^2 d\tau \right)^{1/2} \right] \right\} , \tag{26}$$

$$\left\| \tilde{b}(t) \right\|_{C[0,T]} \leq \left\| [h(t)]^{-1} \right\|_{C[0,T]} \times$$

$$\times\left\{\left\|\left(h_2'''(t)+h_2''(t)-f(0,t)\right)h_1(t)-\right.\right.$$

$$\left.-\left(h_1'''(t)+h_1''(t)-f(0,t)-\int_0^1\omega(x)f(x,t)\,dx\right)h_2(t)\right\|_{C[0,T]}+$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{1/2}\left(\|h_1(t)\|_{C[0,T]}+\|h_2(t)\|_{C[0,T]}\|\omega(x)\|_{L_2(0,1)}\right)\times$$

$$\times\left[\rho_3(T)\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{0k}|)^2\right)^{1/2}+\rho_4(T)\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{1k}|)^2\right)^{1/2}-\right.$$

$$+\rho_5(T)\left(\sum_{k=1}^{\infty}(\lambda_k^2\,|\varphi_{2k}|)^2\right)^{1/2}+\rho_5(T)\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_k(\tau)|)^2d\tau\right)^{1/2}+$$

$$+\rho_5(T)T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{1/2}+$$

$$\left.\left.+\rho_5(T)\sqrt{T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|g_k(\tau)|)^2d\tau\right)^{1/2}\right]\right\}, \qquad (27)$$

where

$$\rho_0(T)=\frac{\sqrt{6}}{\varepsilon_3^2}\left\{(\varepsilon_2^2+\varepsilon_4^2)e^{\varepsilon_1T}+\varepsilon_1e^{\varepsilon_2T}\left[\varepsilon_1+2\varepsilon_2+\frac{1}{\varepsilon_3}(\varepsilon_2^2+\varepsilon_4^2+\varepsilon_1\varepsilon_3)\right]\right\},$$

$$\rho_1(T)=\frac{\sqrt{6}}{\varepsilon_3^2}\left\{2\varepsilon_2e^{\varepsilon_1T}+_1e^{\varepsilon_2T}\left[2\varepsilon_2+\frac{1}{\varepsilon_3}(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_4^2)\right]\right\},$$

$$\rho_2(T)=\frac{\sqrt{6}}{\varepsilon_3^2}\left\{e^{\varepsilon_1T}+e^{\varepsilon_2T}\left[1+\frac{1}{\varepsilon_3}(\varepsilon_1+\varepsilon_2)\right]\right\},$$

$$\rho_3(T)=\frac{1}{\varepsilon_3^2}\left\{(\alpha+\varepsilon_1)(\varepsilon_2^2+\varepsilon_4^2)e^{\varepsilon_1T}+e^{\varepsilon_2T}\left[\varepsilon_1(\alpha\varepsilon_1+2\alpha\varepsilon_2+\right.\right.$$

$$\left.\left.+\varepsilon_2^2+\varepsilon_4^2+\frac{\varepsilon_2}{\varepsilon_3}((\varepsilon_1+\varepsilon_2)(\varepsilon_2^2+\varepsilon_4^2)+\alpha(\varepsilon_2^2+\varepsilon_4^2+\varepsilon_1\varepsilon_3))\right]\right\},$$

$$\rho_4(T)=\frac{1}{\varepsilon_3^2}\left\{2\varepsilon_2(\alpha+\varepsilon_1)e^{\varepsilon_1T}+e^{\varepsilon_2T}\left[\varepsilon_1^2+\varepsilon_2^2+\varepsilon_4^2+\right.\right.$$

$$\left.\left.+2\alpha\varepsilon_2+\frac{1}{\varepsilon_3}(\varepsilon_2(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_4^2)+\alpha(\varepsilon_1^2+\varepsilon_4^2+\varepsilon_1\varepsilon_3))\right]\right\},$$

$$\rho_5(T)=\frac{1}{\varepsilon_3^2}\left\{(\alpha+\varepsilon_1)e^{\varepsilon_1T}+e^{\varepsilon_2T}\left[\alpha+\varepsilon_1+\frac{1}{\varepsilon_3}(\alpha\varepsilon_2+\alpha\varepsilon_1+\varepsilon_2^2+\varepsilon_4^2+\varepsilon_1\varepsilon_2)\right]\right\}.$$

Suppose that the given problem (1) - (3), (6), (7) satisfy the following conditions:

1. $\varphi_i(x) \in C^2[0,1],\ \varphi_i'''(x) \in L_2(0,1)$ and $\varphi_i'(0) = \varphi_i(1) = \varphi_i''(1) = 0\ \ (i = 0,1)$.

2. $\varphi_2(x) \in C^1[0,1],\ \varphi_2''(x) \in L_2(0,1)$ and $\varphi_2'(0) = \varphi_2(1) = 0$.

3. $f(x,t),\ f_x(x,t) \in C(D_T),\ f_{xx}(x,t) \in L_2(D_T)$ and $f_x(0,t) = f(1,t) = 0\ \ (0 \le t \le T)$.

4. $g(x,t),\ g_x(x,t) \in C(D_T),\ g_{xx}(x,t) \in L_2(D_T)$ and $g_x(0,t) = g(1,t) = 0\ \ (0 \le t \le T)$.

5. $h(t) \in C^3[0,T]\ ,h\,(t) \equiv h_1(t)\,g\,(0,t) - h_2(t)\int_0^1 \omega(x)g\,(x,t)\,dx \ne 0\ (0 \le t \le T), \omega(x) \in L_2(0,1)$.

Then from (25) - (27) we have:

$$\left\|\tilde{u}(x,t)\right\|_{B^3_{2,T}} \le A_1(T) + B_1(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}} + C_1(T)\left\|b(t)\right\|_{C[0,T]}, \tag{28}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]} \le A_2(T) + B_2(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}} + C_3(T)\left\|b(t)\right\|_{C[0,T]}, \tag{29}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \le A_3(T) + B_3(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}} + C_3(T)\left\|b(t)\right\|_{C[0,T]}. \tag{30}$$

where

$$A_1(T) = \rho_0(T)\left\|\varphi_0'''(x)\right\|_{L_2(0,1)} + \rho_1(T)\left\|\varphi_1'''(x)\right\|_{L_2(0,1)} +$$

$$+\rho_2(T)\left\|\varphi_2''(x)\right\|_{L_2(0,1)} + \rho_2(T)\sqrt{T}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T) = \rho_2(T)T, C_1(T) = \rho_2(T)\sqrt{T}\left\|g_{xx}(x,t)\right\|_{L_2(D_T)},$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \times$$

$$\times\left\{\left\|\left(h_1'''(t) + h_1''(t) - \int_0^1 \omega(x)f\,(x,t)\,dx\right)g(0,t) -\right.\right.$$

$$\left.-\left(h_2'''(t) + h_2''(t) - f(0,t)\right)\int_0^1 \omega(x)g\,(x,t)\,dx\right\|_{C[0,T]} +$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{1/2}\left(\left\|g(0,t)\right\|_{C[0,T]}\left\|\omega(x)\right\|_{L_2(0,1)} + \left\|\int_0^1 \omega(x)g\,(x,t)\,dx\right\|_{C[0,T]}\right)\times$$

$$\times\left(\rho_3(T)\left\|\varphi_0'''(x)\right\|_{L_2(0,1)} + \rho_4(T)\left\|\varphi_1'''(x)\right\|_{L_2(0,1)} +\right.$$

$$\left.\left.+\rho_5(T)\left\|\varphi_2''(x)\right\|_{L_2(0,1)} + \rho_5(T)\sqrt{T}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}\right)\right\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{1/2} \times$$

$$\times\left(\|g(0,t)\|_{C[0,T]}\|\omega(x)\|_{L_2(0,1)}+\left\|\int_0^1\omega(x)g\left(x,t\right)dx\right)\right\|_{C[0,T]}\right)\rho_5(T)T\ ,$$

$$C_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\times$$

$$\times\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{1/2}\left(\|g(0,t)\|_{C[0,T]}\|\omega(x)\|_{L_2(0,1)}+\right.$$

$$\left.+\left\|\int_0^1\omega(x)g\left(x,t\right)dx\right)\right\|_{C[0,T]}\right)\rho_5(T)\sqrt{T}\left\|g_{xx}(x,t)\right\|_{L_2(D_T)}\ ,$$

$$A_3(T)\ =\ \left\|[h(t)]^{-1}\right\|_{C[0,T]}\times\left\{\left\|\left(h_2'''(t)+\ h_2''\left(t\right)\ -f\left(0,t\right)\right)\ h_1\left(t\right)\ -\right.\right.$$

$$\left.-\left(h_1'''(t)+\ h_1''\left(t\right)\ -f(0,t)-\int_0^1\omega(x)f\left(x,t\right)dx\right)\ h_2\left(t\right)\ \right\|_{C[0,T]}+$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{1/2}\left(\left\|h_1\left(t\right)\right\|_{C[0,T]}\ +\left\|h_2\left(t\right)\right\|_{C[0,T]}\|\omega(x)\|_{L_2(0,1)}\right)\times$$

$$\times\left(\rho_3(T)\left\|\varphi_0'''(x)\right\|_{L_2(0,1)}+\rho_4(T)\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}+\right.$$

$$\left.\left.\rho_5(\,T)\left\|\varphi_2''(x)\right\|_{L_2(0,1)}+\rho_5(T)\sqrt{T}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}\right)\right\}\ ,$$

$$B_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}$$

$$\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{1/2}\left(\left\|h_1\left(t\right)\right\|_{C[0,T]}\ +\left\|h_2\left(t\right)\right\|_{C[0,T]}\|\omega(x)\|_{L_2(0,1)}\right)\rho_5(T)T\ ,$$

$$C_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\times$$

$$\times\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{1/2}\left(\left\|h_1\left(t\right)\right\|_{C[0,T]}\ +\left\|h_2\left(t\right)\right\|_{C[0,T]}\|\omega(x)\|_{L_2(0,1)}\right)$$

$$\rho_5(T)\sqrt{T}\left\|g_{xx}(x,t)\right\|_{L_2(D_T)}\ .$$

From inequalities (28) - (30) we conclude:

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}}+\|\tilde{a}(t)\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq$$

$$\leq\ A(T)+B(T)\left\|a(t)\right\|_{C[0,T]}\left\|u\left(x,t\right)\right\|_{B^{3,}_{2,T}}+C(T)\left\|b(t)\right\|_{C[0,T]}, \tag{31}$$

where

$$A(T)=A_1(T)+A_2(T)\ +A_3(T)$$

$$B(T)=B_1(T)+B_2(T)\ +B_3(T)\,,C(T)=C_1(T)+C_2(T)\ +C_3(T)\,.$$

So, the following theorem is proved.

Theorem 2. *Let conditions 1-5 be satisfied and*

$$(A(T)+2)(B(T)(A(T)+2)+C(T))<1\,. \tag{32}$$

Then the problem (1)-(3), (6),(7) *has a unique solution in the ball* $K=K_R(\|\,z\,\|_{E_T^3}\leq R=A(T)+2)$ *from* E_T^3.

Proof. In space E_T^3 consider the equation

$$z=\Phi z\,, \tag{33}$$

where $z=\{u,\,a,b\}$, the components $\Phi_i(u,a,b)$ $(i=1,2,3)$, of the operator $\Phi(u,a,b)$, are defined by the right-hand sides of equations (14), (21), (22).

Consider the operator $\Phi(u,a,b)$ in the ball $K=K_R$ from E_T^3. Similarly to (31), we obtain that for any $z_1,z_2,z_3\in K_R$ the following estimates are valid:

$$\|\Phi z\|_{E_T^3}\leq A(T)+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3}+C(T)\,\|b(t)\|_{C[0,T]}\,,$$

$$\leq A(T)+(A(T)+2)(B(T)(A(T)+2)+C(T)), \tag{34}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^3}\leq$$

$$\leq B(T)R(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3})+C(T)\,\|b_1(t)-b_2(t)\|_{C[0,T]}\,. \tag{35}$$

Then, by virtue of (32), from (34) and (35), it is clear that the operator $\Phi(u,a,b)$, satisfies the conditions of the contraction mapping principle on the set $K=K_R$. Therefore, the operator $\Phi(u,a,b)$, in the ball $K=K_R$ has a unique fixed point $\{z\}=\{u\,,\,a,b\}$, which is a solution to equation. (33), i.e. is the only solution of systems (14), (21), (22) in the ball $K=K_R$.

The function $u(x,t)$, as an element of space $B_{2,T}^3$, is continuous and has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$ in D_T.

Similarly, [7], it can be shown that $u_t(x,t)$, $u_{txx}(x,t)$ $u_{tt}(x,t)$, $u_{ttt}(x,t)$ are continuous in D_T.

It is easy to check that equation (1), conditions (2), (3), (6) and (7) are satisfied in the usual sense. Then, $\{u(x,t),a(t),b(t)\}$ is a solution of problem (1) - (3), (6), (7). By the corollary of Lemma 1, it is unique in the ball $K=K_R$. Theorem is proved.

Using Theorem 1, the last theorem implies the unique solvability of the initial problem (1) - (4).

Theorem 3. *Let all conditions of Theorem 2 be satisfied and*

$$\int_0^1\omega(x)\varphi_0\,(x)\,dx=h_1\,(0)\,,\;\int_0^1\omega(x)\varphi_1\,(x)\,dx=h_1'\,(0)\,,\;\int_0^1\omega(x)\varphi_2\,(x)\,dx=h_1''\,(0)\;,$$

$$\varphi_0\,(0)=h_2\,(0)\,,\;\;\varphi_1\,(0)=h_2'\,(0)\;,\;\;\varphi_2\,(0)=h_2''\,(0)\;..$$

Then problem (1) - (5) has a unique classical solution in the ball $K=K_R(\|\,z\,\|_{E_T^3}\leq R=A(T)+2)$ *from* E_T^3 .

References

[1] A.N. Tikhonov, *On the stability of inverse problems*, Sov. Math. Dokl, **39(5)**, 1943, 195-198.

[2] M.M. Lavrentiev, V.G. Romanov, V.G. Vasiliev, *Multidimensional Inverse Problems for Differential Equations*, Novosibirsk, 1969.

[3] M.M. Lavrent'ev, V.G. Romanov, S.T. Shishatsky, *Ill-posed problems of mathematical physics and analysis*, Moscow, Nauka, 1980 (in Russian).

[4] V.K. Ivanov, V.V. Vasin, V.P. Tanana, *Theory of linear ill-posed problems and its applications*,Moscow, 1978, 206 p.

[5] V.G. Romanov, *Inverse problems of mathematical physics*, Moscow, 1984, 264 p.

[6] V.V. Varlamov, *On an initial-boundary value problem for a hyperbolic equation of the third order*, Differential Equations, **26(8)**, 1990, 1455-1457.

[7] Ya.T. Megraliev, U.S. Alizadeh, *On the solvability of an inverse boundary value problem for a third order equation describing the propagation of longitudinal waves in a dispersive medium with an integral condition*, Vestnik TVGU, Series: Applied Mathematics, **2**, 2019, 88–106

[8] K.I. Khudaverdiyev, A.A. Veliyev, *Investigation of a one-dimensional mixed problem for a class of pseudohyperbolic equations of third order with non-linear operator right hand side*, Baku, Chashyoghly, 2010, 168 p. (in Russian).

Ulvu Alizadeh
Baku State University
E-mail: ulvi.alizada1918@gmail.com

Received 10 May 2020
Accepted 16 June 2020