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On Pouncare Type Inequalities for Functions from Sobolev-Morrey Type Spaces with Dominant Mixed Derivatives

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Abstract. In this paper, are proved help of method integral representation are proved Pouncare type inequalities for functions from Sobolev-Morrey spaces with dominant mixed derivatives.

Key Words and Phrases: Sobolev-Morrey spaces with dominant mixed derivatives, integral representation, flexible φ-horn.

2010 Mathematics Subject Classifications: Primary 26A33, 46E30, 42B35

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1. Introduction

Let $G \subset R^n$, $1 \le p < \infty$; $\varphi(t) = (\varphi_1(t_1), \varphi_2(t_2), ..., \varphi_n(t_n))$, $\varphi_j(t_j) > 0\,(t_j > 0)$ is Lebesgue measurable functions, $\lim\limits_{t_j \to +0} \varphi_j(t_j) = 0$, $\lim\limits_{t_j \to +\infty} \varphi_j(t_j) \le K \le \infty$, we denote the set of vector-functions $\varphi(t)$ by A. Let $e_n = \{1, 2, ..., n\}$, $e \subseteq e_n$; and $l = (l_1, l_2, ..., l_n)$, $l_j > 0$ are integers $(j \in e_n)$; and $l^e = (l_1^e, ..., l_n^e)$, where $l_j^e = l_j$ for $j \in e$; $l_j^e = 0$ for $j \in e_n \backslash e = e'$;

For any $x \in R^n$ put

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t_j), j \in e_n \right\},$$

and

$$\int\limits_{a^e}^{b^e} f(x)dx = \left(\prod_{j \in e} \int\limits_{a_j}^{b_j} dx_j \right) f(x),$$

i.e. integration is carried and only with respect to the variables x_j whose indices belong to e.

Definition 1. *The Sobolev-Morrey spaces with dominant mixed derivatives $S^l_{p,\varphi,\beta}W(G)$ of locally summable functions f on G having the generalized derivatives $D^{l^e} f (e \subseteq e_n)$ on G with the finite norm*

$$\|f\|_{S^l_{p,\varphi,\beta}W(G)} = \sum_{e \subseteq e_n} \left\| D^{l^e} f \right\|_{p,\varphi,\beta;G}, \tag{1}$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{\substack{x\in G,\\ t>0}} \left(|\varphi([t]_1)|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right), \tag{2}$$

$$|(\varphi([t]_1))|^{-\beta} = \prod_{j\in e_n} \left(\varphi_j\left([t_j]_1\right)\right)^{-\beta_j}, \ \ \beta_j \in [0,1],\ [t_j]_1 = \min\{1, t_j\},\ j \in e_n.$$

Note that the space $S^l_{p,\varphi,\beta}W(G)$ in the case $\beta_j = 0\,(j \in e_n)$ coincides with Sobolev space with dominant mixed derivatives $S^l_pW(G)$ which were introduced and investigated by S.M. Nikolskii and were further considered in the papers [2]-[5].

For any $t_j > 0$ $(j \in e_n)$, there exists a positive constant $C > 0$ such that $|\varphi([t]_1)| \le C$, then the embeddings $L_{p,\varphi,\beta}(G) \to L_p(G)$, $S^l_{p,\varphi,\beta}W9G) \to S^l_pW(G)$, hold i.e.

$$\|f\|_{p,G} \le C\,\|f\|_{p,\varphi,\beta,G},\ \ \|f\|_{S^l_pW(G)} \le C\,\|f\|_{S^l_{p,\varphi,\beta}W(G)}. \tag{3}$$

In this paper, with method of integral representation, we estimate the norms of functions from Sobolev-Morrey spaces with dominant mixed derivatives $S^l_{p,\varphi,\beta}W(G)$ reduced by polynomials, determined in n-dimensional domains satisfying the flexible φ-horn condition.

In order words, we prove the inequalities type Pouncare as

$$\|D^\nu\left(f - P_{l-1}(f,x)\right)\|_{q,\infty} \le C\,|\varphi(1)|\,\|f\|_{S^l_{p,\varphi,\beta}\omega(G)},$$

where

$$\|f\|_{S^l_{p,\varphi,\beta}\omega(G)} = \sum_{\emptyset\ne e\subseteq e_n} \left\|D^{l^e} f\right\|_{p,\varphi,\beta,G},$$

$$\varphi(1) = \max_{e\subseteq e_n} \prod_{j\in e_n} (\varphi_j(1))^{s_e},\ \ s_{e,j} = \begin{cases} \mu_j, & j\in e \\ -\nu_j - (1-\beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right), & j \in e'. \end{cases}$$

Such problems in different spaces were studied in papers [1,9,10].

Let $M_e(\cdot, y, z) \in C_0^\infty(R^n)$, $e \subseteq e_n$, and by such that

$$S(M_e) = \sup pL_e \subset I_{\varphi(T)} = \left\{x : |x_j| < \frac{1}{2}\varphi_j(T_j),\ j \in e_n\right\}.$$

Assume that for any $0 < T_j \le 1$ $(j \in e_n)$

$$V = \underset{0<t_j\le T_j}{U} \left\{ y : \frac{y}{\varphi\left(t^e + T^{e'}\right)} \in S(M_e) \right\},$$

where $\left(t^e + T^{e'}\right) = t_j,\ j\in e;\ \left(t^e + T^{e'}\right) = T_j,\ j \in e'$.

It is clear that $V \subset I_{\varphi(T)}$, and suppose $U + V \subset G$, where U is an open set, contained in the domain G. Let

$$G_{\varphi(T)}(U) = \left(U + I_{\varphi(T)}(x)\right) \cap G = Z,$$

and let $U+V\subset G_{\varphi(T)}(U)$.

Assuming that $\varphi_j(t)$ $(j\in e_n)$ are also differentiable on $[0,T_j]$, $j\in e_n$, and is obtained that in for $f\in S_p^lW(G)$ determined in n-dimensional domains, satisfying the condition of flexible φ-horn, it holds the following integral representation $(\forall x\in U\subset G)$ [8]

$$D^v f(x)=\sum_{e\le e_n}(-1)^{|v|+|l^e|}\prod_{j\in e'}(\varphi_j(T_j))^{-1-v_j}\times$$

$$\times\int\limits_{O^e}^{T^e}\int\limits_{R^n}M_e^{(v)}\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)},\ \frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)},\ \rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\times$$

$$\times\prod_{j\in e}(\varphi_j(t_j))^{l_j-v_j-2}\prod_{j\in e}\varphi_j'(t_j)dt^e dy. \tag{4}$$

2. Main results

Theorem 1. *Let $G\subset R^n$ satisfy the condition of flexible φ-horn [5], $1\le p\le q\le\infty$, $v=(v_1,v_2,...,v_n)$, $v_j\ge 0$ be entire $(\in e_n)$, $f\in S^l_{p,\varphi,\beta}W(G)$, and let*

$$\mu_j=l_j-v_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)>0,\ j\in e_n. \tag{5}$$

Then exists polynomials $P_{l-1}(f,x)$

$$\left\|D^v\left(f-P_{l-1}(f,x)\right)\right\|_{q,G}\le C_1\varphi(1)\sum_{\emptyset\ne e\subseteq e_n}\left\|D^{l^e}f\right\|_{p,\varphi,\beta,G}, \tag{6}$$

where $\varphi(1)=\max\limits_{e\subseteq e_n}\prod_{j\in e_n}(\varphi_j(1))^{S_{e,j}}$,

$$S_{e,j}=\begin{cases}\mu_j,\ j\in e\\ -v_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right),\ j\in e',\end{cases}$$

and C_1 is constant independent of f.

Proof. At first note that in the conditions of our theorem there exists generalized derivatives $D^v f$ in G. Indeed, if $\mu_j>0$ $(j\in e_n)$, $p\le q$, it follows that for $f\in S^l_{p,\varphi,\beta}\to f\in S_p^lW(G)$. Then there exists generalized $D^v f\in L_p(G)$, it holds the following integral representation (**??**). Note that in case

$$\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)=$$

$$=-x\rho\varphi\left(t^e+T^{e'}\right),\,0<t_j\le T_j=1\ \ (j\in e_n)$$

is valid:

$$D^v f(x) = P_{l-1}(f,x) + \sum_{\emptyset\neq e\subseteq e_n} (-1)^{|v|+|l^e|} \int\limits_{0^e}^{1^e} \int\limits_{R^n} M_e^{(v)} \times$$

$$\times \left(\frac{y}{\varphi\left(t^e+1^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e+1^{e'}\right),x\right)}{\varphi\left(t^e+1^{e'}\right)},\ \rho'\left(\varphi\left(t^e+1^{e'}\right),x\right)\right) \times$$

$$\times D^{l^e} f(x+y) \prod_{j\in e} (\varphi_j(t_j))^{l_j-v_j-2} \prod_{j\in e} \varphi_j'(t_j) dt^e dy.$$

The support of this identity is contained in the flexible φ-horn $x+V\subset G$. Hence, by the Minkowski inequality, we have

$$\|D^v\left(f-P_{l-1}(f,\cdot)\right)\|_{q,U} \le \sum_{\emptyset\neq e\subseteq e_n} \left\| K_e\left(\cdot, t^e+1^{e'}\right)\right\|_{q,U} \tag{7}$$

here

$$K_e\left(x,t^e+1^{e'}\right) = \int\limits_{0^e}^{1^e}\int\limits_{R^n} M_e^{(v)} \times$$

$$\times \left(\frac{y}{\varphi\left(t^e+1^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e+1^{e'}\right),x\right)}{\varphi\left(t^e+1^{e'}\right)},\ \rho'\left(\varphi\left(t^e+1^{e'}\right),x\right)\right) \times$$

$$\times D^{l^e} f(x+y) \prod_{j\in e} (\varphi_j(t_j))^{l_j-v_j-2} \prod_{j\in e} \varphi_j'(t_j) dt^e dy. \tag{8}$$

Applying generalized Minkowski inequality (7) for $K_e(x,t)$ defined by equality (8) we get

$$\left\|K_e(\cdot,t^e+1^{e'})\right\|_{q,U} \le C_1 \int\limits_{0^e}^{1^e} \prod_{j\in e} (\varphi_j(t_j))^{l_j-v_j-2} \times$$

$$\times \prod_{j\in e} \varphi_j'(t_j) \left\| R_e\left(\cdot, t^e+1^{e'}\right)\right\| dt^e dy. \tag{9}$$

where

$$R_e\left(x,t^e+1^{e'}\right) = \int\limits_{R^n} M_e^{(v)} \left(\frac{y}{\varphi\left(t^e+1^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e+1^{e'}\right),x\right)}{\varphi\left(t^e+1^{e'}\right)}, \right. \times$$

$$\left. \times \rho'\left(\varphi\left(t^e+1^{e'}\right),x\right) D^{l^e} f(x+y) dy. \right.$$

Help of inequality ($q\le r\le\infty$) we get

$$\left\|R_e\left(\cdot,t^e+1^{e'}\right)\right\|_{q,U} \le \left\|R_e\left(\cdot,t^e+1^{e'}\right)\right\|_{q,U} (mesU)^{\frac{1}{q}-\frac{1}{r}}, \tag{10}$$

let $1 \le p \le r \le \infty,\ s \le r \left(\frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{r}\right)$,

$$\left|M_e^{(v)} D^{l^e} f\right| = \left(\left|M_e^{(v)}\right|^p \left|D^{l^e} f\right|^s\right)^{\frac{1}{p}} \left(\left|D^{l^e} f\right|^p X\right)^{\frac{1}{p}-\frac{1}{r}} \left(\left|M_e^{(v)}\right|^s\right)^{\frac{1}{s}-\frac{1}{r}}$$

and apply Hölder inequality for $\left(\frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{s} - \frac{1}{r}\right) = 1\right)$, we have

$$\left|R_e\left(x, t^e + 1^{e'}\right)\right| \le \left(\int_{R^n} \left|D^{l^e} f(x+y)\right|^p M_e^{(v)} \left(\frac{y}{\varphi\left(t^e + 1^{e'}\right)},\right.\right.$$

$$\left.\left.\frac{\rho\left(\varphi\left(t^e + 1^{e'}, x\right)\right)}{\varphi\left(t^e + 1^{e'}\right)}, \rho'\left(\varphi\left(t^e + 1^{e'}\right), x\right)\right) dy^s\right)^{\frac{1}{r}} \times$$

$$\times \left(\int_{R^n} \left|D^{l^e} f(x+y)\right|^p \chi\left(\frac{y}{\varphi\left(t^e + 1^{e'}\right)}\right) dy\right)^{\frac{1}{p}-\frac{1}{r}} \times$$

$$\times \int_{R^n} \left|M_e^{(v)}\left(\frac{y}{\varphi\left(t^e + 1^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e + 1^{e'}, x\right)\right)}{\varphi\left(t^e + 1^{e'}\right)}, \rho'\left(\varphi\left(t^e + 1^{e'}\right), x\right)\right|^s dy\right)^{\frac{1}{s}-\frac{1}{r}}.$$

Further, we will assume that there exists a function $\tilde{M}_e(x)$ such that $|M_e(x, y, z)| \le$ $\le C\left|\tilde{M}_e(x)\right|$, for all $(y, z) \in R^{2n}$.

Let χ be a characteristic function of the set $S(M_e)$. Then, we have

$$\left\|R_e\left(\cdot, t^e + 1^{e'}\right)\right\|_{r, U_\varphi(t^e + 1^e)} \le$$

$$\le \sup_{x \in U} \left(\int_{R^n} \left|D^{l^e} f(x+y)\right|^p \chi\left(\frac{y}{\varphi\left(t^e + 1^{e'}\right)}\right) dy\right)^{\frac{1}{p}-\frac{1}{r}} \times$$

$$\times \sup_{y \in V} \left(\int_U \left|D^{l^e} f(x+y)\right|^p dx\right)^{\frac{1}{r}} \times$$

$$\times \left(\int_{R^n} \left|\tilde{M}_e^{(v)}\left(\frac{y}{\varphi\left(t^e + 1^{e'}\right)}\right)\right|^s dy\right)^{\frac{1}{s}}. \tag{11}$$

For any $x \in U$ we have

$$\int_{R^n} \left|D^{l^e} f(x+y)\right|^p \chi\left(\frac{y}{\varphi\left(t^e + 1^{e'}\right)}\right) dy \le \int_{(U+V)_{\varphi(t^e + 1^e)}} \left|D^{l^e} f(y)\right|^p dy \le$$

$$\leq \int\limits_{Z} \left|D^{l^e} f(y)\right|^p dy \leq \left\|D^{l^e} f\right\|_{p,\varphi,\beta,z} \prod_{j\in e_n} (\varphi_j(1))^{\beta_j p}, \tag{12}$$

for $y \in V$

$$\int\limits_{U_{\varphi\left(t^e+1^{e'}\right)}} \left|D^{l^e} f(x+y)\right|^p dx \leq$$

$$\leq \int\limits_{Z} \left|D^{l^e} f(x)\right|^p dx \leq \left\|D^{l^e} f\right\|_{p,\varphi,\beta,z} \prod_{j\in e_n} \varphi_j(1)^{\beta_j p}, \tag{13}$$

and

$$\int\limits_{R^n} \left|\tilde{M}_e\left(\frac{y}{\varphi\left(t^e+1^{e'}\right)}\right)\right|^s dy = \left\|\tilde{M}_e\right\|_s^s \prod_{j\in e_n} \varphi_j(1). \tag{14}$$

$$\left\|R_e\left(\cdot, t^e+1^{e'}\right)\right\|_{r,U_{\varphi(t^e+1^e)}} \leq C_2 \left\|M_e^{(v)}\right\|_s \prod_{j\in e_n} (\varphi_j(1))^{\frac{1}{s}+\beta_j p\left(\frac{1}{p}-\frac{1}{r}\right)} \times$$

$$\times \left\|D^{l^e} f\right\|_{p,\varphi,\beta,z}. \tag{15}$$

By means inequalities (9), (10) and (15) for $r = q$ we have

$$\left\|K_e\left(\cdot, t^e+1^{e'}\right)\right\|_{q,U} \leq C\varphi(1) \left\|D^{l^e} f\right\|_{p,\varphi,\beta,z}. \tag{16}$$

Substituting the inequality (**??**) in (**??**), we obtain the inequality (**??**).

This completes the proof of Theorem 2.1.

Theorem 2. *Let all the conditions of Theorem 2.1 be fulfilled. Furthermore, let* $l^1 \in N$ *and if*

$$\mu_j^1 = l_j - v_j - (1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right) = l_j^1 > 0,\ j \in e_n.$$

Then

$$\left\|D^v\left(f - P_{l-1}(f,x)\right)\right\|_{S_q^{l^1}W(G)} \leq C_2 \varphi_1(1) \sum_{\emptyset\neq e\subseteq e_n} \left\|D^{l^e} f\right\|_{p,\varphi,\beta;G}, \tag{17}$$

where $\varphi_1(1) = \max\limits_{e\subseteq e_n} \prod_{j\in e_n} (\varphi_1(1))^{S^1_{e,j}}$.

$$s^1_{e,j} = \begin{cases} \mu_j^1, & j \in e \\ -v_j - (1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right) - l_j^1, & j \in e'. \end{cases}$$

C_2 *is constant independent an function on* f*. The theorem is proved analogously to Theorem 2.1.*

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An Application of Generalized Distribution Series on Certain Classes of Univalent Functions

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Abstract. The purpose of the present paper is to obtain some sufficient conditions for generalized distribution series belonging to the classes $\mathcal{S}^*(\alpha, \beta, \gamma)$, $\mathcal{K}(\alpha, \beta, \gamma)$ and inclusion relation of these subclasses by $\Re^\tau(A, B)$. Finally, we obtain some necessary and sufficient conditions of an integral operator associated with the generalized distribution series.

Key Words and Phrases: generalized distribution, Analytic, univalent functions, convex function and starlike functions.

2010 Mathematics Subject Classifications: 30C45

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in C \text{ and } |z| < 1\}$. As usual, by $\mathcal{S}$ we shall represent the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and further, we denote $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{2}$$

The convolution (or Hadamard product) of two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

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A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U}).$$

This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \leq \alpha < 1)$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{U}).$$

This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions.

It is an established fact that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

We recall the class $\mathcal{S}^*(\alpha, \beta, \gamma)$ defined and studied by Kulkarni [10].

Let $\mathcal{S}^*(\alpha, \beta, \gamma)$ the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left|\frac{f'(z) - 1}{2\gamma(f'(z) - \alpha) - (f'(z) - 1)}\right| < \beta, \quad (z \in \mathbb{U}). \tag{3}$$

where $0 < \beta \leqq 1, 0 \leqq \alpha < \frac{1}{2\gamma}$ and $\frac{1}{2} \leqq \gamma \leqq 1$

Recently, some conditions of hypergeometric functions on the class $\mathcal{S}^*(\alpha, \beta, \gamma)$ have been studied by Joshi *et al.* [9].

Now, we define a new class $\mathcal{K}(\alpha, \beta, \gamma)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left|\frac{zf''(z) + f'(z) - 1}{2\gamma(zf''(z) + f'(z) - \alpha) - (zf''(z) + f'(z) - 1)}\right| < \beta, \quad (z \in \mathbb{U}). \tag{4}$$

where $0 < \beta \leqq 1, 0 \leqq \alpha < \frac{1}{2\gamma}$ and $\frac{1}{2} \leqq \gamma \leqq 1$.

By using (3) and (4) we note that

$$f(z) \in \mathcal{K}(\alpha, \beta, \gamma) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha, \beta, \gamma).$$

A function $f \in \mathcal{A}$ is said to be in the class $f \in \Re^\tau(A, B)$ $(\tau \in \mathbb{C}\backslash\{0\}, -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left|\frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]}\right| < 1, \quad (z \in \mathbb{U}).$$

The class $\Re^\tau(A, B)$ was introduced earlier by Dixit and Pal [7].

The applications of hypergeometric functions ([9], [20]), confluent hypergeometric functions [5], generalized hypergeometric functions [8], Wright function [17], Fox-Wright function [6], generalized Bessel functions ([4], [15]) are interesting topics of research in Geometric Function Theory. In 2014, Porwal [13] (see also [2], [11]) introduced Poisson distribution series and obtain necessary and sufficient conditions for certain classes of univalent functions and co-relates probability density function with Geometric Function Theory. After the appearance of this paper several researchers introduced hypergeometric distribution series [1], confluent hypergeometric distribution series [16], Binomial distribution series [12], Mittag-Leffler type Poisson distribution series [3] and obtain some interesting properties of various classes of univalent functions. Recently Porwal [14] introduced generalized distribution series and obtain some necessary and sufficient conditions belonging to the certain classes of univalent functions. Now, we recall the definition of generalized distribution. Let the series $\sum_{n=0}^{\infty} t_n$, where $t_n \geq 0, \forall n \in N$ is convergent and its sum is denoted by S, i.e.

$$S = \sum_{n=0}^{\infty} t_n. \tag{5}$$

Now, we introduce the generalized discrete probability distribution whose probability mass function is

$$p(n) = \frac{t_n}{S}, \quad n = 0, 1, 2, \ldots \tag{6}$$

Obviously $p(n)$ is a probability mass function because $p(n) \geq 0$ and $\sum_n p_n = 1$.

Now, we introduce the series

$$\phi(x) = \sum_{n=0}^{\infty} t_n x^n. \tag{7}$$

From (5) it is easy to see that the series given by (7) is convergent for $|x| < 1$ and for $x = 1$ it is also convergent.

Now, we introduce a power series whose coefficients are probabilities of the generalized distribution

$$K_\phi(z) = z + \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n. \tag{8}$$

Further, we define the function

$$TK_\phi(z) = z - \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n. \tag{9}$$

Next, we introduce the convolution operator $TK_\phi(f, z)$ for functions f of the form (2) as follows

$$TK_\phi(f, z) = K_\phi(z) * f(z) = z - \sum_{n=2}^{\infty} |a_n| \frac{t_{n-1}}{S} z^n. \tag{10}$$

In the present paper, motivated with the above mentioned work, we obtain necessary and sufficient conditions for generalized distribution series belonging to the classes $\mathcal{K}(\alpha, \beta, \gamma)$, $\mathcal{S}^*(\alpha, \beta, \gamma)$ and inclusion relation of these subclasses by $\Re^\tau(A, B)$.

2. Main Results

To establish our main results we shall require the following lemmas.

Lemma 1. *([9]) A function $f \in \mathcal{A}$ and of the form* (1) *belongs to the class $\mathcal{S}^*(\alpha,\beta,\gamma)$ if*

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]|a_n| \leq 2\beta\gamma(1-\alpha). \tag{11}$$

Our next lemma is a direct consequences of definition (4).

Lemma 2. *A function $f \in \mathcal{A}$ and of the form* (1) *belongs to the class $\mathcal{K}(\alpha,\beta,\gamma)$ if*

$$\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]|a_n| \leq 2\beta\gamma(1-\alpha).$$

Lemma 3. *[7] A function $f \in \Re^{\tau}(A,B)$ is of form* (1), *then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{12}$$

The bound given in (12) *is sharp.*

Theorem 1. *If $f \in \mathcal{A}$ is of the form* (1) *and the inequality*

$$(1+\beta(1-2\gamma))\left[\phi''(1)+3\phi'(1)+(\phi(1)-\phi(0))\right] \leq 2\beta\gamma(1-\alpha)S \tag{13}$$

is satisfied, then $K_\phi(z)$ is of the form (8) *is in the class $\mathcal{K}(\alpha,\beta,\gamma)$.*

Proof. To prove that $K_\phi(z) \in \mathcal{K}(\alpha,\beta,\gamma)$from Lemma 2 it suffices to prove that

$$\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]\frac{t_{n-1}}{S} \leq 2\beta\gamma(1-\alpha).$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]\frac{t_{n-1}}{S} \\
&= \frac{1+\beta(1-2\gamma)}{S}\left[\sum_{n=2}^{\infty} n^2\right] t_{n-1} \\
&= \frac{1+\beta(1-2\gamma)}{S}\sum_{n=2}^{\infty}\left[(n-1)(n-2)+3(n-1)+1\right] t_{n-1} \\
&= \frac{1+\beta(1-2\gamma)}{S}\sum_{n=1}^{\infty}\left[n(n-1)t_n+3nt_n+t_n\right]
\end{aligned}$$

$$= \frac{1+\beta(1-2\gamma)}{S}\left[\phi''(1)+3\phi'(1)+(\phi(1)-\phi(0))\right]$$
$$\leq 2\beta\gamma(1-\alpha).$$

This completes the proof of Theorem 1. ◄

Theorem 2. *If $f \in \mathcal{A}$ is of the form* (1) *and the inequality*

$$[1+\beta(1-2\gamma)]\left[\phi'(1)+(\phi(1)-\phi(0))\right] \leq 2\beta\gamma(1-\alpha)S. \tag{14}$$

is satisfied, then $K_\phi(z)$ is of the form (8) *is in the class $\mathcal{S}^*(\alpha,\beta,\gamma)$.*

Proof. To prove that $K_\phi(z) \in \mathcal{S}^*(\alpha,\beta,\gamma)$ from Lemma 1 it suffices to prove that

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]\frac{t_{n-1}}{S} \leq 2\beta\gamma(1-\alpha).$$

Now

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]\frac{t_{n-1}}{S}$$
$$= \frac{[1+\beta(1-2\gamma)]}{S}\left[\sum_{n=2}^{\infty}(n-1)+1\right]t_{n-1}$$
$$= \frac{[1+\beta(1-2\gamma)]}{S}\sum_{n=1}^{\infty}[nt_n+t_n]$$
$$= \frac{[1+\beta(1-2\gamma)]}{S}\left[\phi'(1)+(\phi(1)-\phi(0))\right]$$
$$\leq 2\beta\gamma(1-\alpha).$$

Thus the proof of Theorem 2 is established. ◄

Remark 1. *The conditions* (13) *and* (14) *are also necessary for the distribution series $TK_\phi(z)$ defined by* (9).

Theorem 3. *If $f \in R^\tau(A,B)$ is of the form* (2) *and the operator $TK_\phi(f,z)$ defined by* (10) *is in the class $\mathcal{K}(\alpha,\beta,\gamma)$, if and only if*

$$\frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S}\left[\phi'(1)+(\phi(1)-\phi(0))\right] \leq 2\beta\gamma(1-\alpha). \tag{15}$$

Proof. To prove $TK_\phi(f,z) \in \mathcal{K}(\alpha,\beta,\gamma)$, from Lemma 2, it suffices to prove that

$$P_1 = \sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]|a_n| \leq 2\beta\gamma(1-\alpha).$$

Since $f \in R^{\tau}(A,B)$ then by using Lemma 3 we have

$$|a_n| \le \frac{(A-B)|\tau|}{n}.$$

Hence

$$\begin{aligned}
P_1 \le & \frac{(A-B)|\tau|}{S} \sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]t_{n-1} \\
= & \frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S} \left[\sum_{n=2}^{\infty}(n-1)+1\right] t_{n-1} \\
= & \frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S} \sum_{n=1}^{\infty} [nt_n + t_n] \\
= & \frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S} \left[\phi'(1) + (\phi(1)-\phi(0))\right] \\
\le & 2\beta\gamma(1-\alpha).
\end{aligned}$$

Thus the proof of Theorem 3 is established. ◄

Theorem 4. *If $f \in R^{\tau}(A,B)$ is of the form* (2) *and the operator $TK_{\phi}(f,z)$ defined by* (10) *is in the class $\mathcal{S}^*(\alpha,\beta,\gamma)$, if and only if*

$$\frac{(A-B)[1+\beta(1-2\gamma)]|\tau|}{S} [\phi(1)-\phi(0)] \le 2\beta\gamma(1-\alpha). \tag{16}$$

Proof. The proof of above theorem is similar to that of Theorem 3. Therefore we omit the details involved. ◄

3. An Integral Operator

In this section, we introduce an integral operator $TG_{\phi}(z)$ as follows

$$TG_{\phi}(z) = \int_0^z \frac{TK_{\phi(t)}}{t} dt, \tag{17}$$

and obtain a necessary and sufficient condition for $TG_{\phi}(z)$ belonging to the classes $\mathcal{S}^*(\alpha,\beta,\gamma)$ and $\mathcal{K}(\alpha,\beta,\gamma)$.

Theorem 5. *If $TK_{\phi}(z)$ defined by* (10)*, then $TG_{\phi}(z)$ defined by* (17) *is in the class $\mathcal{K}(\alpha,\beta,\gamma)$, if and only if* (14) *satisfies.*

Proof. Since

$$TG_{\phi}(z) = z - \sum_{n=2}^{\infty} \frac{t_{n-1}}{nS} z^n$$

by Lemma 2, we have to prove that

$$\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]\frac{t_{n-1}}{nS} \leq 2\beta\gamma(1-\alpha).$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^2[1+\beta(1-2\gamma)]\frac{t_{n-1}}{nS} \\
=&\frac{1}{S}\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]t_{n-1} \\
&= \frac{[1+\beta(1-2\gamma)]}{S}\sum_{n=1}^{\infty}[(n+1)-1]\,t_n \\
&= \frac{[1+\beta(1-2\gamma)]}{S}\sum_{n=1}^{\infty}[n+1]\,t_n \\
&= \frac{[1+\beta(1-2\gamma)]}{S}\left[\phi^{'}(1)+[\phi(1)-\phi(0)]\right] \\
&\leq 2\beta\gamma(1-\alpha).
\end{aligned}$$

This completes the proof of Theorem 5. ◄

Theorem 6. *If $TK_\phi(z)$ defined by* (10)*, then $TG_\phi(z)$ defined by* (17) *is in the class $\mathcal{S}^*(\alpha,\beta,\gamma)$, if and only if*

$$\frac{[1+\beta(1-2\gamma)]}{S}\left[\phi(1)-\phi(0)\right] \leq 2\beta\gamma(1-\alpha).$$

satisfies.

Proof. The proof of above theorem is much akin to that of Theorem 5. Therefore we omit the details involved. ◄

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On Basicity of Trigonometric Systems in Sobolev-Morrey Spaces

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Abstract. This work is devoted to the study of basicity of the system $1 \cup t \cup \{\sin nt\}_{n\geq 1}$ and $\{\cos nt\}_{n\geq 0}$ in one subspace of Sobolev-Morrey space.

Key Words and Phrases: Sobolov-Morrey spaces, basicity.

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1. Introduction

The concept of Morrey space was introduced by Morrey in 1938. Since then, various problems related to this space have been intensively studied. Playing an important role in the qualitative theory of elliptic differential equations (see, for example, [5, 13]), this space also provides a large class of examples of mild solutions to the Navier–Stokes system [12]. In the context of fluid dynamics, Morrey spaces have been used to model flow when vorticity is a singular measure supported on certain sets in R^3 [7]. There are sufficiently wide investigations related to fundamental problems in these spaces in view of differential equations, potential theory, maximal and singular operator theory, approximation theory and others (see, for example, [6] and the references above). More details about Morrey spaces can be found in [15, 19].

In recent years there has been a growing interest in the study of various subjects related to Morrey-type spaces. For example, some problems in harmonic analysis and approximation theory have been treated in [8-11,17].

The basis properties of trigonometric systems in classical spaces are well studied [1, 2, 14]. Study of the problems of the approximation theory in spaces such as Morrey has recently started and it remains much to learn. Basicity of exponential systems in Morrey type spaces is studied in [3,4,18]. Basicity of exponential system [5] in Sobolev-Morrey spaces studied in [16]. In this paper we study the problem of basicity of trigonometric systems in Sobolev-Morrey spaces. In the future, our goal is to follow the scheme of work [16].

2. Morrey-Lebesgue space

Let us give a definition for above-mentioned spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$.

By the Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0 \le \alpha \le 1$, $p \ge 1$, we mean a normed space of all functions $f(\xi)$ measurable on Γ equipped with a finite norm $\|\cdot\|_{L^{p,\alpha}(\Gamma)}$:

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \left(\sup_B \left| B \bigcap \Gamma \right|^{\alpha-1} \int_{B \bigcap \Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty,$$

where the sup is taken over all disks B centered on Γ. $L^{p,\alpha}(\Gamma)$ is a Banach space and $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$.

The embedding $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 \le \alpha_1 \le \alpha_2 \le 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_p(\Gamma)$, $\forall \alpha \in [0,1]$, $\forall p \ge 1$. The case of $\Gamma = [0,\pi]$ will be denoted by $L^{p,\alpha}$.

Denote by $\tilde{L}^{p,\alpha}$ the linear subspace of $L^{p,\alpha}$ consisting of functions whose shifts are continuous in $L^{p,\alpha}$, i.e.

$$\tilde{L}^{p,\alpha} = \{ f \in L_{p,\alpha} : \|f(\cdot + \delta) - f(\cdot)\| \to 0, \delta \to 0 \}.$$

The closure of $\tilde{L}^{p,\alpha}$ in $L^{p,\alpha}$ will be denoted by $ML^{p,\alpha}$, i.e. $ML^{p,\alpha} = \overline{\tilde{L}^{p,\alpha}}$.

3. Morrey-Sobolev space

Let $0 \le \alpha \le 1$, $p \ge 1$. By $W^1_{p,\alpha}$ we denote the space of functions which belong, together with their derivatives of first order, to the space $L^{p,\alpha}(\Gamma)$ equipped with the norm

$$\|f\|_{W^1_{p,\alpha}} = \|f\|_{L^{p,\alpha}} + \|f'\|_{L^{p,\alpha}}. \tag{1}$$

Denote by $\tilde{W}^1_{p,\alpha}$ the linear subspace of $W^1_{p,\alpha}$ consisting of functions whose first order derivatives are continuous with respect to the shift operator. By $MW^1_{p,\alpha}$ we denote the closure of this space with respect to the norm (1).

By $\mathscr{L}_{p,\alpha}$ we denote the direct sum of $ML^{p,a}$ and C (C is the complex plane)

$$\mathscr{L}_{p,\alpha} = ML^{p,\alpha} \oplus C.$$

Let us define the norm in $\mathscr{L}_{p,\alpha}$ in the following way:

$$\|\hat{u}\|_{\mathscr{L}_{p,\alpha}} = \|u\|_{L^{p,\alpha}} + |\lambda|, \forall \hat{u} = (u;\lambda) \in \mathscr{L}_{p,\alpha}.$$

The following lemma is true.

Lemma 1. *The operator* $(A\hat{u})(t) = \lambda + \int_0^t u(\tau)\, d\tau$ *is an isomorphism from* $\mathscr{L}_{p,\alpha}$ *onto* $MW^1_{p,\alpha}$.

Proof. At first let us show that $v(t) = (A\hat{u})\, t \in W^1_{p,\alpha}$. Indeed, since $L^{p,\alpha} \subset L_p \subset L_1$, then

$$\|v(t)\|_{L^{p,\alpha}} = \left\|\lambda + \int_0^t u(\tau)\, d\tau\right\|_{L^{p,\alpha}} \leq \|\lambda\|_{L_{p,\alpha}} + \left\|\int_0^t u(\tau)\, d\tau\right\|_{L^{p,\alpha}} \leq$$

$$\leq (\pi)^{\frac{\alpha}{p}} |\lambda| + \sup_{I\subset(0,\pi)} \left\{\frac{1}{|I|^{1-\alpha}} \int_I \left|\int_0^t u(\tau)\, d\tau\right|^p dt\right\}^{1/p} \leq$$

$$\leq (\pi)^{\frac{\alpha}{p}} |\lambda| + \sup_{I\subset(0,\pi)} \left\{\frac{1}{|I|^{1-\alpha}} \int_I \left(\int_0^\pi |u(\tau)|\, d\tau\right)^p dt\right\}^{1/p} =$$

$$= (\pi)^{\frac{\alpha}{p}} |\lambda| + (\pi)^{\frac{\alpha}{p}} \|u\|_{L_1(0,\pi)} < +\infty. \tag{2}$$

Also, since $v'(t) = u(t) \in L^{p,\alpha}$, we have $v(t) \in W^1_{p,\alpha}$.

Now we show that $v(t) \in MW^1_{p,\alpha}$. From $u \in ML^{p,\alpha}$ it follows

$$\|v(\cdot+\delta) - v(\cdot)\|_{W^1_{p,\alpha}} = \|v(\cdot+\delta) - v(\cdot)\|_{L^{p,\alpha}} + \left\|v'(\cdot+\delta) - v'(\cdot)\right\|_{L^{p,\alpha}} =$$

$$= \left\|\int_\cdot^{\cdot+\delta} u(\tau)\, d\tau\right\|_{L^{p,\alpha}} + \|u(\cdot+\delta) - u(\cdot)\|_{L^{p,\alpha}} \to 0, \quad \delta \to 0.$$

Let us show that A is a bounded operator. We have

$$\|A(\hat{u})\|_{W^1_{p,\alpha}} = \left\|\lambda + \int_0^t u(\tau)\, d\tau\right\|_{L^{p,\alpha}} + \|u(\tau)\|_{L^{p,\alpha}}.$$

Taking into account (2)

$$\|A(\hat{u})\|_{W^1_{p,\alpha}} \leq (\pi)^{\frac{\alpha}{p}} |\lambda| + (\pi)^{\frac{\alpha}{p}} \|u\|_{L_1(0,\pi)} + \|u\|_{L^{p,\alpha}}.$$

As the following relation holds

$$\|u\|_{L_1} \leq C_1 \|u\|_{L_p} \leq C_2 \|u\|_{L^{p,\alpha}},$$

we have the validity of the following inequality

$$\|A(\hat{u})\|_{W^1_{p,\alpha}} \leq M\left(|\lambda| + \|u\|_{L^{p,\alpha}}\right) = M \|\hat{u}\|_{\mathscr{L}_{p,\alpha}}, M = const.$$

Let us show that $\ker A = \{0\}$. Let $A\hat{u} = 0$, i.e. $\lambda + \int_0^t u(\tau)\, d\tau = 0$. If we differentiate both sides, we get $u(t) = 0$, a.e. . Thus $\lambda = 0$. We have $\hat{u} = 0$. For $\forall v \in MW^1_{p,\alpha}$ taking $\hat{v} = (v'; v(-\pi))$ we have $\hat{v} \in L_{p,\alpha}$ and $A(\hat{v}) = v$. It means that $R_A = MW^1_{p,\alpha}$, where R_A is a range of the operator A . It follows from Banach's theorem on the inverse operator that the inverse of A is a continuous operator. The lemma is proved.

The following theorem is true.

Theorem 1. *System* $1 \cup t \cup \{\sin nt\}_{n\geq 1}$ *forms a basis for* $MW^1_{p,\alpha}(0,\pi)$.

Proof. It is known that system $\{\cos nt\}_{n\geq 0}$ is a basis in space $ML^{p,\alpha}$ [18].

Let us prove that the system $\{\hat{u}_{-1}\} \cup \{\hat{u}_n\}_{n\geq 0}$ forms a basis for $\mathscr{L}_{p,\alpha}(0,\pi)$, where

$$\hat{u}_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \hat{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{u}_n = \begin{pmatrix} n\cos nt \\ 0 \end{pmatrix}, \ n \geq 1.$$

Let us show that for $\forall \hat{u} \in \mathscr{L}_{p,\alpha}$ there exists the decomposition

$$\hat{u} = c_{-1}\hat{u}_{-1} + \sum_{n=0}^{\infty} c_n \hat{u}_n, \tag{3}$$

and this decomposition is unique. This decomposition is equivalent to the next decomposition

$$u\left(t\right) = c_0 + \sum_{n=1}^{\infty} c_n n \cos nt, \tag{4}$$

and equality $\lambda = c_{-1}$.

Following [18] we obtain that there exists the decomposition (4) and it is unique. Therefore the decomposition (3) also exists and unique. I.e. a system $\{\hat{u}_{-1}\} \cup \{\hat{u}_n\}_{n\geq 0}$ forms a basis for $\mathscr{L}_{p,\alpha}(0,\pi)$.

We can easily calculate that for the operator

$$A\hat{u} = \lambda + \int_0^t u(\tau)d\tau,$$

the following relations are true

$$A(\hat{u}_{-1}) = 1, \ A(\hat{u}_0) = t,$$

$$A(\hat{u}_n) = \sin nt, \quad n \geq 1$$

If A is isomorphism, then a system $1 \cup t \cup \{\sin nt\}_{n\geq 1}$ forms a basis for $MW^1_{p,\alpha}(0,\pi)$.

The theorem 1 is proved.

Theorem 2. $\{\cos nt\}_{n\geq 0}$ *forms a basis for* $MW^1_{p,\alpha}\left(0,\pi\right)$.

Proof. It is known that system $\{\sin nt\}_{n\geq 1}$ is a basis in space $ML^{p,\alpha}$ [18].

Let us prove that the system $\{\hat{u}_0\} \cup \{\hat{u}_n\}_{n\geq 1}$ forms a basis for $\mathscr{L}_{p,\alpha}(0,\pi)$, where

$$\hat{u}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \hat{u}_n = \begin{pmatrix} -n\sin nt \\ 1 \end{pmatrix}, \ n \geq 1.$$

Let us show that for $\forall \hat{u} \in \mathscr{L}_{p,\alpha}$ there exists the decomposition

$$\hat{u} = \sum_{n=0}^{\infty} c_n \hat{u}_n, \tag{5}$$

and this decomposition is unique. This decomposition is equivalent to the next two decompositions

$$u(t) = -\sum_{n=1}^{\infty} c_n n \sin nt, \tag{6}$$

$$\lambda = c_0 + \sum_{n=1}^{\infty} c_n. \tag{7}$$

Following [18] we obtain that there exists the decomposition (6) and it is unique. Let us note that the decomposition (6) belongs to the space $ML^{p,\alpha}$ and since $L^{p,\alpha} \subset L_p$, then Hausdorf-Young inequality holds for the system $\{\sin nt\}_{n\geq 1}$ in Morrey spaces $L^{p,\alpha}$. I.e. if $1 < p \leq 2$ then

$$\left(\sum_{n=1}^{\infty} |nc_n|^q\right)^{1/q} \leq M\|u\|_{L_p}$$

where $1/p + 1/q = 1$.

Applying Hölder's inequality, we obtain

$$\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} \frac{|nc_n|}{n} \leq \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^p \left(\sum_{n=1}^{\infty} |nc_n|^q\right)^{1/q} < +\infty.$$

In the case of $p > 2$ since $L^{p,\alpha} \subset L_p \subset L_2$ then

$$\left(\sum |nc_n|^2\right)^{1/2} \leq M\|u\|_{L_2}$$

and similarly

$$\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} \frac{|nc_n|}{n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{n=1}^{\infty} |nc_n|^2\right)^{1/2} < +\infty$$

So, we show that the series $\sum\limits_{n=0}^{\infty} |c_n|$ is convergent. Therefore in the decomposition (7) the coefficient c_0 is uniquely defined. Thus, we have shown the existence and uniqueness of the decomposition (5) for $\forall \hat{u} \in \mathscr{L}_{p,\alpha}$. I.e a system $\{\hat{u}_n\}_{n\geq 0}$ forms a basis for $\mathscr{L}_{p,\alpha}$. We can easily calculate that for the operator

$$A\hat{u} = \lambda + \int_0^t u(\tau)d\tau,$$

the following relations are true

$$A(\hat{u}_0) = 1,\ A(\hat{u}_n) = \cos nt,\ n \geq 1$$

If A is isomorphism, then a system $\{\cos nt\}_{n\geq 0}$ forms a basis for $MW^1_{p,\alpha}$.

The Theorem 2 is proved.

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On Basicity of Eigenfunctions of a Spectral Problem in spaces $L_p \oplus C$ and L_p

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Abstract. In this paper we study the spectral problem for a discontinuous second order differential operator with a summable potential and with a spectral parameter in transmission conditions, that arises in solving the problem of vibration of a loaded string with fixed ends. Using abstract theorems on the stability of basis properties of multiple systems in a Banach space with respect to certain transformations, as well as theorems on basicity of perturbed systems are proved theorems on the basicity of eigenfunctions of a discontinuous differential operator in spaces $L_p \oplus C$ and L_p.

Key Words and Phrases: spectral problem, eigenfunctions, basicity.
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1. Introduction

Consider the following spectral problem with a point of discontinuity:

$$l(y) = -y'' + q(x)\,y = \lambda y, x \in \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right), \tag{1}$$

$$\left.\begin{array}{c} y(0) = y(1) = 0, \\ y(\frac{1}{3} - 0) = y(\frac{1}{3} + 0), \\ y'(\frac{1}{3} - 0) - y'(\frac{1}{3} + 0) = \lambda m y(\frac{1}{3}), \end{array}\right\} \tag{2}$$

where $q(x)$ is a complex valued, summable function, λ is the spectral parameter, m is non-zero complex number. Such spectral problems arise when the problem of vibrations of a loaded string with fixed ends is solved by applying the Fourier method. The practical significance of such problems is noted in the wellknown monographs (for example [1-3]). In case when the load is placed at the middle of the string, some aspects of this spectral problem have been studied in [4,5]. In [6] was found asymptotical formulas for eigenvalues and eigenfunctions, in [7] was proved completeness and minimality of eigenfunctions of the problem (1),(2) in spaces $L_p \oplus C$ and L_p. In [8-13] the problem (1),(2) was considered in a special case, when $q(x) \equiv 0$, where the theorems on completeness and basicity of eigenfunctions was proved in weight spaces $L_p \oplus C$ and L_p , also in Morrey-Lebesgue type

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spaces. We also note the work [14-17], where using different methods was studied the spectral properties of the above considered problem in the case, when the load is fixed one or both ends of string.
The spectral parameter with discontinuity point and with spectral parameter in boundary conditions was considered in [18-20]. Such problems play an important role in mathematics, mechanics, physics and other fields of natural science, and their applications associated with the discontinuity of the physical properties of material. The study of basis properties of the spectral problems with a point of discontinuity sometimes requires completely new research methods, different from the known ones. In [21,22] new method for exploring basis properties of discontinuous differential operators has been suggested. The present paper is an extension of the method of [6,7,12], and here developing the methods of [21,22] we study the basicity of eigenfunctions and associated functions of the (1),(2) in spaces $L_p \oplus C$ and L_p.

2. Necessary information and preliminary results

For obtaining the main results we need some notions and facts from the theory of basis in a Banach space.

Definition 1. *Let X- be a Banach space. If there exists a sequence of indexes, such that $\{n_k\} \subset N$, $n_k < n_{k+1}, n_0 = 0$, and any vector $x \in X$ is uniquely represented in the form*

$$x = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i u_i.$$

then the system $\{u_n\}_{n\in N} \in X$ is called a basis with parentheses in X .

For $n_k = k$ the system $\{u_n\}_{n\in N}$ forms a usual basis for X.
We need the following easily proved statements.

Statement 1. *Let the system $\{u_n\}_{n\in N}$ forms a basis with parentheses for a Banach space X. If the sequence $\{n_{k+1} - n_k\}_{k\in N}$ is bounded and the condition*

$$\sup_n \|u_n\| \, \|\vartheta_n\| < \infty,$$

holds, where $\{\vartheta_n\}_{n\in N}$- is a biorthogonal system, then the system $\{u_n\}_{n\in N}$ forms a usual basis for X.

Statement 2. *Let the system $\{x_n\}_{n\in N}$ forms a Riesz basis with parentheses for a Hilbert space X. If the system $\{x_n\}_{n\in N}$ uniformly minimal and almost normalized, the sequence $\{n_{k+1} - n_k\}_{k\in N}$ is bounded, then this system forms a usual Riesz basis for X.*

Take the following

Definition 2. *The basis $\{u_n\}_{n\in N}$ of Banach space X is called a p-basis, if for any $x \in X$ the condition*

$$\left(\sum_{n=1}^{\infty} |\langle x, \vartheta_n\rangle|^p\right)^{\frac{1}{p}} \le M \|x\|,$$

holds, where $\{\vartheta_n\}_{n\in N}$- is a biorthogonal system to $\{u_n\}_{n\in N}$.

Definition 3. *The sequences $\{u_n\}_{n\in N}$ and $\{\phi_n\}_{n\in N}$ of Banach space X are called a p-close, if the following condition holds:*

$$\sum_{n=1}^{\infty} \|u_n - \phi_n\|^p < \infty.$$

We will also use the following results from [23] (see also [24]).

Theorem 1. *Let $\{x_n\}_{n\in N}$ forms a q-basis for a Banach space X, and the system $\{y_n\}_{n\in N}$ is p-close to $\{x_n\}_{n\in N}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then the following properties are equivalent:*
i) $\{y_n\}_{n\in N}$- is complete in X;
ii) $\{y_n\}_{n\in N}$- is minimal in X;
iii) $\{y_n\}_{n\in N}$- forms an isomorphic basis to $\{x_n\}_{n\in N}$ for X.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system and $\left\{\hat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ be its biorthogonal system:

$$\hat{u}_n = (u_n; a_{n1}, ..., a_{nm})\,; \hat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm})\,.$$

Let $J = \{n_1, ..., n_m\}$ some set of m natural numbers. Suppose

$$\delta = det\,\|\beta_{n_i j}\|_{\ i,j=\overline{1,m}}.$$

In [25](see also [26]) has been proved the following theorem :

Theorem 2. *Let the system $\{\hat{u}_n\}_{n\in N}$ forms a basis for X_1. In order to the system $\{u_n\}_{n\in N_J}$, where $N_J = N\backslash J$ forms a basis for X it is necessary and sufficient that the condition $\delta \neq 0$ be satisfied. In this case the biorthogonal system to $\{u_n\}_{n\in N_J}$ is defined by*

$$\vartheta_n^* = \frac{1}{\delta}\begin{vmatrix} \vartheta_n & \vartheta_{n1} & \dots & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & \dots & \beta_{n_m 1} \\ \dots & \dots & \dots & \dots \\ \beta_{nm} & \beta_{n_1 m} & \dots & \beta_{n_m m} \end{vmatrix}.$$

In particular, if X is a Hilbert space and the system $\{u_n\}_{n\in N}$- forms a Riesz basis for X_1, then under the condition $\delta \neq 0$, the system $\{u_n\}_{n\in N_J}$ also forms a Riesz basis for X.

For $\delta = 0$ the system $\{u_n\}_{n\in N_J}$ is not complete and is not minimal in X.

Let X be a Banach space and the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ is any system in X. Let $a_{ik}^{(n)}, i,k=\overline{1,m}, n\in N$, any complex numbers. Let

$$A_n=\left(a_{ik}^{(n)}\right)_{i,k=\overline{1,m}} \quad and \ \Delta_n=detA_n \ , n\in N.$$

Consider the following system in space X :

$$\hat{u}_{kn}=\sum_{i=1}^{m}a_{ik}^{(n)}u_{in}, k=\overline{1,m}; n\in N. \tag{3}$$

Following theorems have been proved in [12] (also [21,22])

Theorem 3. *If the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ forms basis for X and*

$$\Delta_n\neq 0, \forall n\in N, \tag{4}$$

then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms basis with parentheses for X. If in addition the following conditions

$$\sup_n\left\{\|A_n\|, \left\|A_n^{-1}\right\|\right\}<\infty, \quad \sup_n\left\{\|u_{kn}\|, \|\vartheta_{kn}\|\right\}<\infty, \tag{5}$$

hold, where $\{\vartheta_{kn}\}_{k=\overline{1,m};n\in N}\subset X^*$- is biorthogonal to $\{u_{kn}\}_{k=\overline{1,m};n\in N}$, then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a usual basis for X.

Theorem 4. *If X-is a Hilbert space, and the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ forms a Riesz basis for X, for holding the condition (4) the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a Riesz basis with parentheses for X. If in addition the condition (5) holds, then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a usual Riesz basis for X.*

We need some results from [6]. For their formulation introduce the following functions:

$$q_1(x)=\frac{1}{2}\int_0^x q(t)\,dt, q_2(x)=\frac{1}{2}\int_x^1 q(t)\,dt. \tag{6}$$

Theorem 5. *The eigenvalues of the problem (1),(2) are asymptotically simple and consist of three series: $\lambda_{i,n}=\rho_{i,n}^2, i=1,2,3; n=1,2,...,$ where the numbers $\rho_{i,n}$ hold the following asymptotically formulas:*

$$\left\{\begin{array}{c}\rho_{1,n}=3\pi n+O\left(\frac{1}{n^2}\right), \\ \\ \rho_{2,n}=3\pi n+\frac{\alpha_1}{n}+O\left(\frac{1}{n^2}\right), \\ \\ \rho_{3,n}=3\pi n-\frac{3\pi}{2}+\frac{\alpha_2}{n}+O\left(\frac{1}{n^2}\right),\end{array}\right. \tag{7}$$

here indicated $\alpha_1=\frac{3+2mq_1+2mq_2}{3\pi m}, \alpha_2=-\frac{1+mq_2}{3\pi m}, q_1=q_1\left(\frac{1}{3}\right), q_2=q_2\left(\frac{1}{3}\right)$.

Theorem 6. *The eigenfunctions $y_{i,n}(x)$ of the problem (1),(2) corresponding to the eigenvalues $\lambda_{in} = (\rho_{1,n})^2, i = \overline{1,3}; n \in N$, hold the following asymptotically formulas*

$$y_{1n}(x) = \begin{cases} \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right], \\ \gamma_1 \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \tag{8}$$

$$y_{2,n}(x) = \begin{cases} \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right], \\ \gamma_2 \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \tag{9}$$

$$y_{3,n}(x) = \begin{cases} O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right], \\ \gamma_{3,n} \cos 3\pi \left(n - \frac{1}{2}\right) x + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \tag{10}$$

here indicated $\gamma_1 = (1 + mq_1), \gamma_2 = \frac{mq_1 - mq_2}{3}$, $\gamma_3 = m$.

3. Main results

Now consider a problem on basicity of eigenfunctions of the problem (1),(2) in spaces $L_p(0,1) \oplus C$ and $L_p(0,1)$.

Theorem 7. *The root vector system $\{\hat{y}_{in}\}_{i=\overline{1,3};n\in N}^{\infty}$ of the operator L, which linearized the problem* (1), (2) *forms basis in space $L_p(0,1) \oplus C, 1 < p < \infty$, and for $p = 2$ it forms a Riesz basis.*

Proof. Since the operator L has compact resolvent, the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=\overline{1,3};n\in N}^{\infty}$ of eigenfunctions and associated functions is minimal in $L_p(0,1) \oplus C$. The conjugate system $\{\hat{z}_0\} \cup \{\hat{z}_{in}\}_{i=\overline{1,3};n\in N}^{\infty}$ is the eigenvectors and associated vectors of the conjugating operator L^* and is in the $\hat{z}_{in} = \left(z_{in}(x); \overline{m} z_{in}\left(\frac{1}{3}\right)\right)$ form, where $z_{in}(x), i = \overline{1,3}; n \in N$ are the eigenvectors and associated vectors of the conjugate problem and anologically we obtain the asymptotically formulas:

$$z_{1n}(x) = c_{1n} \cdot \begin{cases} \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right], \\ \bar{\gamma}_1 \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \tag{11}$$

$$z_{2,n}(x) = c_{2n} \cdot \begin{cases} \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right], \\ \bar{\gamma}_2 \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \tag{12}$$

$$z_{3,n}(x) = c_{3n} \begin{cases} O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right], \\ \bar{\gamma}_3 \cos 3\pi \left(n - \frac{1}{2}\right) x + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \tag{13}$$

Here indicated $\gamma_1 = (1 + mq_1)\,, \gamma_2 = \frac{mq_1 - mq_2}{3}$, $\gamma_3 = m$, and c_{1n}, c_{2n}, c_{3n} are the normalized multipliers. We can easily calculate that, the c_{1n}, c_{2n}, c_{3n} normalized multipliers hold

$$c_{1n} = \frac{6}{1 + 2|\gamma_1|^2} + O\left(\frac{1}{n}\right), c_{2n} = \frac{6}{1 + 2|\gamma_2|^2} + O\left(\frac{1}{n}\right), c_{3n} = \frac{3}{|m|^2} + O\left(\frac{1}{n}\right)$$

If we consider these at formulas (11)-(13), we will obtain for $z_{in}(x)$ the following formulas:

$$z_{1n}(x) = \begin{cases} \frac{6}{1+2|\gamma_1|^2} \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right], \\ \frac{6\bar{\gamma}_1}{1+2|\gamma_1|^2} \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \tag{14}$$

$$z_{2,n}(x) = \begin{cases} \frac{6}{1+2|\gamma_2|^2} \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right], \\ \frac{6\bar{\gamma}_2}{1+2|\gamma_2|^2} \sin 3\pi nx + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \tag{15}$$

$$z_{3,n}(x) = \begin{cases} O\left(\frac{1}{n}\right), & x \in \left[0, \frac{1}{3}\right], \\ \frac{3}{\gamma_3} \cos 3\pi \left(n - \frac{1}{2}\right) x + O\left(\frac{1}{n}\right), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \tag{16}$$

Let us introduce the following functional system for the separating the head part of the asymptotically formulas:

$$\begin{cases} u_{1,n}(x) = \begin{cases} sin3\ \pi nx, x \in \left[0, \frac{1}{3}\right], \\ \gamma_1 sin3\pi nx, x \in \left[\frac{1}{3}, 1\right], \end{cases} \\ u_{2,n}(x) = \begin{cases} sin3\ \pi nx, x \in \left[0, \frac{1}{3}\right], \\ \gamma_2 sin3\ \pi nxx \in \left[\frac{1}{3}, 1\right], \end{cases} \\ u_{3,n}(x) = \begin{cases} 0, x \in \left[0, \frac{1}{3}\right], \\ \gamma_3 cos3\ \pi \left(n - \frac{1}{2}\right) x, x \in \left[\frac{1}{3}, 1\right]. \end{cases} \end{cases} \tag{17}$$

Then from the formulas (8)-(10) implies that, the following relations are true:

$$\begin{cases} y_{1,n}(x) = u_{1,n}(x) + O\left(\frac{1}{n}\right), \\ y_{2,n}(x) = u_{2,n}(x) + O\left(\frac{1}{n}\right), \\ y_{3,n}(x) = u_{3,n}(x) + O\left(\frac{1}{n}\right), \end{cases} \tag{18}$$

One can easily seen that the system (17) implies from the following system by the conversion

$$u_{i,n} = \sum_{j=1}^{3} a_{ij} e_{j,n}$$

$$\left\{\begin{array}{c} e_{1,n}(x)=\left\{\begin{array}{c} sin3\ \pi nx, x\in\left[0,\frac{1}{3}\right], \\ 0, x\in\left[\frac{1}{3},1\right], \end{array}\right. \\ e_{2,n}(x)=\left\{\begin{array}{c} 0, x\in\left[0,\frac{1}{3}\right], \\ sin3\ \pi nx, x\in\left[\frac{1}{3},1\right], \end{array}\right. \\ e_{3,n}(x)=\left\{\begin{array}{c} 0, x\in\left[0,\frac{1}{3}\right], \\ cos3\ \pi\left(n-\frac{1}{2}\right)x, x\in\left[\frac{1}{3},1\right], \end{array}\right. \end{array}\right. \quad (19)$$

where the numbers a_{ij} are the elements of the following matrix

$$A=\left(\begin{array}{ccc} 1 & \gamma_1 & 0 \\ 1 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{array}\right) \quad (20)$$

Note that, since $\gamma_3=m\neq 0$, for $\gamma_1\neq\gamma_2$, i.e. $2mq_1+mq_2+3\neq 0$ the determinant will be

$$detA\ =\gamma_2\gamma_3-\gamma_1\gamma_3\neq 0$$

On the other hand the system $\{e_{i,n}\}_{i=\overline{1,3};n\in N}$ forms a basis for $L_p(0,1)$, $1<p<\infty$. Really, according to the decomposition $L_p(0,1)=L_p\left(0,\frac{1}{3}\right)\oplus L_p\left(\frac{1}{3},1\right)$ and since the systems $\{e_{1,n}\}_{n\in N}$, $\{e_{i,n}\}_{i=1,2;n\in N}$ form basis in $L_p\left(0,\frac{1}{3}\right)$, $L_p\left(\frac{1}{3},1\right)$, their combination will form a basis in $L_p(0,1)$. If we take it into consideration and apply Theorem 3, then we obtain that the system $\{u_{i,n}\}_{i=\overline{1,3};n\in N}$ forms basis in $L_p(0,1)$. Consider the system in $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ in $L_p(0,1)\oplus C$, where

$$\hat{u}_0=(0;1)\,,\hat{u}_{i,n}=(u_{i,n};0)\,,i=\overline{1,3};n\in N. \quad (21)$$

It is clear that, the system $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ forms basis in $L_p(0,1)\oplus C$. Let us show that it also forms a q-basis, where $q=p/(p-1)$. One can easily check that the system $\{\hat{\vartheta}_0\}\cup\left\{\hat{\vartheta}_{i,n}\right\}_{i=\overline{1,3};n\in N}$, which biorthogonal to it is in the following form:

$$\hat{\vartheta}_0=(0;1)\,,\hat{\vartheta}_{i,n}=(\vartheta_{i,n};0)\,,i=\overline{1,3};n\in N, \quad (22)$$

where

$$\vartheta_{1n}(x)=\left\{\begin{array}{ll} \frac{6}{1+2|\gamma_1|^2}\sin 3\pi nx, & x\in\left[0,\frac{1}{3}\right], \\ \\ \frac{6\bar{\gamma}_1}{1+2|\gamma_1|^2}\sin 3\pi nx, & x\in\left[\frac{1}{3},1\right], \end{array}\right. \quad (23)$$

$$\vartheta_{2,n}(x)=\left\{\begin{array}{ll} \frac{6}{1+2|\gamma_2|^2}\sin 3\pi nx, & x\in\left[0,\frac{1}{3}\right], \\ \\ \frac{6\bar{\gamma}_2}{1+2|\gamma_2|^2}\sin 3\pi nx, & x\in\left[\frac{1}{3},1\right], \end{array}\right. \quad (24)$$

$$\vartheta_{3,n}(x)=\left\{\begin{array}{ll} 0, & x\in\left[0,\frac{1}{3}\right], \\ \\ \frac{3}{\gamma_3}\cos 3\pi\left(n-\frac{1}{2}\right)x, & x\in\left[\frac{1}{3},1\right], \end{array}\right. . \quad (25)$$

Let $1 < p \leq 2$. Then according to inequality Hausdorf-Young for trigonometric system (see [27], p.153) for each $f \in L_p(0,1)$ the inequality

$$\left(\sum_{i=1}^{3}\sum_{n=1}^{\infty}|< f, e_{i,n} >|^q\right)^{\frac{1}{q}} \leq M\|f\|_{L_p}$$

is fulfilled, where M > 0 is a fixed number which does not depend on f. Taking into consideration that, the system $\{\vartheta_{i,n}\}_{i=\overline{1,3};n\in N}$ implies from the system $\{e_{i,n}\}_{i=\overline{1,3};n\in N}$ by conversion

$$u_{i,n} = \sum_{j=1}^{3} b_{ij}e_{j,n}$$

where b_{ij} are the elements of matrix the $(A^{-1})^*$. We obtain from here that for an arbitrary $\hat{f} \in L_p(0,1) \oplus C$ the following inequality holds:

$$\left(\sum_{i=1}^{3}\sum_{n=1}^{\infty}\left|\left\langle \hat{f}, \hat{\vartheta}_{i,n}\right\rangle\right|^q\right)^{\frac{1}{q}} \leq \mathrm{M}\left\|\hat{f}\right\|_{L_p\oplus C}$$

and implies the system $\{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ is a q-basis in $L_p(0,1) \oplus C$. Let's point

$$\hat{y}_{i,n} = \left(y_{i,n}(x)\,; m y_{i,n}\left(\frac{1}{3}\right)\right), i = \overline{1,3}; n \in N,$$

According the formulas (8)-(10) since $y_{i,n}\left(\frac{1}{3}\right) = O\left(\frac{1}{n}\right)$, from (18) implies that the systems $\{\hat{y}_{i,n}\}_{i=\overline{1,3};n\in N}$ and $\{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ are p-close,

$$\sum_{i=1}^{3}\sum_{n=1}^{\infty}\|\hat{y}_{i,n} - \hat{u}_{i,n}\|^p < \infty.$$

Thus, all the conditions of Theorem 1 are fulfilled and according t is theorem the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=\overline{1,3};n\in N}^{\infty}$ also forms an isomorphic basis to the system $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ in $L_p(0,1) \oplus C$.

Now suppose that $p > 2$, then $1 < q < 2$. Taking into account that in this case the following inclusion is fulfilled:

$$L_p(0,1) \subset L_q(0,1)\,,$$

Then for $\hat{f} \in L_p(0,1) \oplus C$ we obtain:

$$\left(\sum_{i=1}^{3}\sum_{n=1}^{\infty}\left|\left\langle \hat{f}, \hat{\vartheta}_{i,n}\right\rangle\right|^p\right)^{\frac{1}{p}} \leq \mathrm{M}\left\|\hat{f}\right\|_{L_q\oplus C} \leq M\left\|\hat{f}\right\|_{L_p\oplus C},$$

This implies that the system $\{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ forms a p-basis in $L_p(0,1) \oplus C$. Besides, the systems $\{\hat{y}_{i,n}\}_{i=\overline{1,3};n\in N}$ and $\{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ are q-close in $L_p(0,1) \oplus C$:

$$\sum_{i=1}^{3}\sum_{n=1}^{\infty}\|\hat{y}_{i,n} - \hat{u}_{i,n}\|_{L_p\oplus C}^q < \infty.$$

According the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=\overline{1,3};n\in N}^{\infty}$ is minimal in $L_p(0,1) \oplus C$ and again applying the Theorem 1, we obtain that it is an isomorphic basis to $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ in $L_p(0,1) \oplus C$.

In the case $p = 2$ according the Theorem 4 the system $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ forms a Riesz basis in $L_2(0,1) \oplus C$. Besides the systems $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=\overline{1,3};n\in N}^{\infty}$ and $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=\overline{1,3};n\in N}$ are square-close,

$$\sum_{i=1}^{3}\sum_{n=1}^{\infty} \|\hat{y}_{i,n} - \hat{u}_{i,n}\|^2 < \infty$$

and according Theorem 1 the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=\overline{1,3};n\in N}^{\infty}$ forms a Riesz basis in $L_2(0,1) \oplus C$ and this completes the proof of the theorem. The theorem is proved. ◄

Now consider the basicity $\{y_0\} \cup \{y_{i,n}\}_{i=\overline{1,3};\ n\in N}^{\infty}$ of the system of eigenfunctions and associated functions of the problem (1),(2) in $L_p(0,1)$. Applying the Theorem 2 and 6, we obtain the honesty of the following theorem.

Theorem 8. *In order the system $\{y_0\} \cup \{y_{i,n}\}_{i=\overline{1,3};\ n\in N}^{\infty}$ of eigenfunctions and associated functions of the problem (1),(2) forms a basis in $L_p(0,1)$, $1 < p < \infty$, and for $p = 2$ forms a Riesz basis, after eliminate any function $y_{i,n_0}(x)$ it is necessary and sufficient that the corresponding function $z_{i,n_0}(x)$ of the biorthogonal system satisfy the condition $z_{i,n_0}\left(\frac{1}{3}\right) \neq 0$. If $z_{i,n_0}\left(\frac{1}{3}\right) = 0$, then after the eliminating function $y_{1,n_0}(x)$ from the system, obtaining system does not form basis in $L_p(0,1)$, moreover in this case it is not complete and not minimal in this space.*

In Theorems 6 and 7 the parameter m which included in the problem (1), (2), generally speaking is a complex number. But in some particular cases it is possible to refine the root subspaces of the operator L. So, if $m > 0$ and $q(x)$ is a real function, then the operator L , linearized of the problem (1),(2), is a self-adjoint operator in $L_2 \oplus C$, and in this case all the eigenvalues are simple and for each eigenvalue there corresponds only one eigenvector . If $m < 0$ and $q(x)$- is a real function, then the operator L is a J-self-adjoint operator in $L_2 \oplus C$, and in this case applying the results of [28,29], we obtain that all eigenvalues are real and simple, with the exception of, may be either one pair of complex conjugate simple eigenvalues or one non-simple real value. In the case of a complex value m the operator L has an infinite number of complex eigenvalues that are asymptotically simple and, consequently, the operator L can have a finite number of associated vectors. If there are associated vectors, they are determined up to a linear combination with the corresponding eigenvector. Therefore depending on the choice of the coefficients of the linear combination there are associated vectors satisfying the condition $z_{2,n}\left(\frac{1}{3}\right) \neq 0$, and there are also associated vectors not satisfying this condition.

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Asymptotic Behavior of the Distribution Function of the Riesz Transform

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Abstract. It is known that the Riesz transform of a Lebesgue integrable function is not Lebesgue integrable. In the present paper, we study the asymptotic behavior of the distribution function of the Riesz transform of a Lebesgue integrable function as $\lambda \to +\infty$ and as $\lambda \to 0+$.

Key Words and Phrases: Riesz transform, distribution function, asymptotic behavior, summability.

2010 Mathematics Subject Classifications: 44A15, 42B20.

1. Introduction

The j-th Riesz transform of a function $f \in L_p(R^d)$, $1 \leq p < +\infty$ is defined as the following singular integral:

$$R_j(f)(x) = \gamma_{(d)} \lim_{\varepsilon \to 0} \int_{\{y \in R^d : |x-y| > \varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy, \ j = \overline{1,d},$$

where $C_{(d)} = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$.

The Riesz transform is one of the important operators in harmonic analysis. It has been shown in [2, 6, 10, 11] that this transform plays an essential role in applications to the theory of elliptic partial differential equations.

From the theory of singular integrals (see [10]) it is known that the Riesz transform is a bounded operator in the space $L_p(R^d)$, $p > 1$, that is, if $f \in L_p(R^d)$, then $R_j(f) \in L_p(R^d)$ and the inequality

$$\|R_j f\|_{L_p} \leq C^{(p)} \|f\|_{L_p} \tag{1}$$

holds. In the case $f \in L_1(R^d)$ only the weak inequality holds:

$$m\{x \in R^d : \ |(R_j f)(x)| > \lambda\} \leq \frac{C_1}{\lambda} \|f\|_{L_1}, \tag{2}$$

where m stands for the Lebesgue measure, $C^{(p)}$, C_1 are constants independent of f.

In [3, 4, 5, 7, 8, 9, 10] the boundedness of the operator R_j in other function spaces (in the spaces of Sobolev, Besov, Campanato, Morrey, etc.) was studied.

Note that the Riesz transform of a function $f \in L_1(R^d)$ is not Lebesgue integrable. In work [1] using the notion of A-integrating functions, the analogue of Riss equality is proved for the class of functions $f \in L_1(R^d)$. In this paper we study the asymptotic behavior of the distribution function of the Riesz transform of a Lebesgue integrable function as $\lambda \to +\infty$ and as $\lambda \to 0+$ and find a necessary condition and a sufficient condition for the summability of the Riesz transform.

2. Asymptotic behavior of the distribution function of the Riesz transform as $\lambda \to +\infty$

In this section we studying the asymptotic behavior of the distribution function of the Riesz transform as $\lambda \to +\infty$.

Theorem 1. Let $f \in L_1(R^d)$. Then the equation

$$\lim_{\lambda\to+\infty} \lambda m\{x \in R^d : \ |(R_j f)(x)| > \lambda\} = 0 \tag{3}$$

holds.

Proof: Since $f \in L_1(R^d)$, then for every $\varepsilon > 0$ there exists $n \in N$ and $r > 0$ such that

$$\|f - [f]_r^n\|_{L_1} \le \frac{\varepsilon}{4C_1}, \tag{4}$$

where $[f]_r^n(x) = [f]^n \chi(B(0;r))(x)$, $[f(x)]^n = f(x)$ for $|f(x)| \le n$, $[f(x)]^n = 0$ for $|f(x)| > n$, $\chi(B(0;r))(x)$ - characteristic function of the ball $B(0;r) = \{x \in R^d : \ |x| < r\}$. It follows from (2) and (4) that for every $\lambda > 0$ the inequality

$$m\{x \in R^d : \ |R_j(f - [f]_r^n)(x)| > \frac{\lambda}{2}\} \le \frac{2C_1}{\lambda}\|f - [f]_r^n\|_{L_1} \le \frac{\varepsilon}{2\lambda} \tag{5}$$

holds. Since the function $[f]_r^n(x)$ is bounded, then we get that $[f]_r^n \in L_p(R^d)$ for each $p \ge 1$. It follows that $R_j[f]_r^n \in L_p(R^d)$ for each $p > 1$. Denote

$$F_1(x) = R_j([f]_r^n)(x) \cdot \chi(B(0;2r)), F_2(x) = R_j([f]_r^n)(x) \cdot \chi(R^d \backslash B(0;2r)).$$

Then

$$R_j([f]_r^n)(x) = F_1(x) + F_2(x),$$

The function $F_1(x)$ is concentrated on the closed ball $\overline{B(0;2r)}$, and the function $F_2(x)$ is concentrated on the set $R^d \backslash B(0;2r)$. For every $p > 1$ from the inclusion $R_j([f]_r^n) \in L_p(R^d)$ it follows that $F_1(x) \in L_p(R^d)$. Since the function $F_1(x)$ is concentrated on the bounded set, then we have that $F_1(x) \in L_1(R^d)$. Then for sufficiently large values of $\lambda > 0$

$$\frac{\lambda}{4} m\{x \in R^d : \ |F_1(x)| > \frac{\lambda}{4}\} \le \int_{\{x\in R^d : \, |F_1(x)|>\lambda/4\}} |F_1(x)| dx < \frac{\varepsilon}{8}. \tag{6}$$

On the other hand, for any $x \in R^d \backslash B(0;2r)$ we have

$$|R_j([f]_r^n)(x)| \le \gamma_{(d)} \int_{B(0;r)} \frac{|x_j - y_j|}{|x-y|^{d+1}} \cdot |[f]_r^n(y)| dy$$

$$\le \frac{\gamma_{(d)}}{r^d} \int_{B(0;r)} |[f]_r^n(y)| dy = \frac{\gamma_{(d)}}{r^d} \|[f]_r^n\|_{L_1} \le \frac{\gamma_{(d)}}{r^d} \|f\|_{L_1}.$$

This shows that the function $F_2(x)$ is bounded. Then it follows from (6) that for sufficiently large values of $\lambda > 0$

$$m\{x \in R^d : \ |R_j([f]_r^n)(x)| > \frac{\lambda}{2}\} \le m\{x \in R^d : \ |F_1(x)| > \frac{\lambda}{4}\} < \frac{\varepsilon}{2\lambda}. \tag{7}$$

It follows from (5) and (7) that for sufficiently large values of $\lambda > 0$

$$m\{x \in R^d : \ |(R_j f)(x)| > \lambda\}$$

$$\le m\{x \in R^d : |R_j([f]_r^n)(x)| > \frac{\lambda}{2}\} + m\{x \in R^d : |R_j(f - [f]_r^n)(x)| > \frac{\lambda}{2}\} < \frac{\varepsilon}{\lambda}.$$

This shows that the equation (3) holds. Theorem 1 is proved.

3. Asymptotic behavior of the distribution function of the Riesz transform as $\lambda \to 0+$

In this section we studying the asymptotic behavior of the distribution function of the Riesz transform as $\lambda \to 0+$.

Theorem 2. Let $f \in L_1(R^d)$. Then the equation

$$\lim_{\lambda \to 0+} \lambda m\{x \in R^d : |(R_j f)(x)| > \lambda\} = \gamma_{(d)} \theta_{(d)} \left| \int_{R^d} f(x) dx \right| \tag{8}$$

holds, where $\theta_{(d)} = \frac{2^d}{d(d-1)!!} (\frac{\pi}{2})^{[\frac{d-1}{2}]}$ and $[\frac{d-1}{2}]$ - integer part of a number $\frac{d-1}{2}$.

At first we prove the auxiliary lemma.

Lemma 1. If $f \in L_1(R^d)$ and $\int_{R^d} f(x) dx = 0$, then the equation

$$m\{x \in R^d : |(R_j f)(x)| > \lambda\} = o(1/\lambda), \lambda \to 0+ \tag{9}$$

holds.

Proof of Lemma 1. At first assume that the function f is concentrated on some ball $B(0;r) \subset R^d$. In this case, for values of $|x| > 2r$ from the equality

$$(R_j f)(x) = \gamma_{(d)} \int_{B(0;r)} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) dy$$

$$= \gamma_{(d)} \int_{B(0;r)} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) dy - \gamma_{(d)} \int_{B(0;r)} \frac{x_j}{|x|^{d+1}} f(y) dy$$

$$= \gamma_{(d)} \int_{B(0;r)} \left[\frac{x_j - y_j}{|x-y|^{d+1}} - \frac{x_j}{|x|^{d+1}} \right] f(y)dy,$$

we get that

$$|(R_j f)(x)| \le \gamma_{(d)} \int_{B(0;r)} \left[|x_j| \frac{||x|^{d+1} - |x-y|^{d+1}|}{|x-y|^{d+1} \cdot |x|^{d+1}} + \frac{|y_j|}{|x-y|^{d+1}} \right] |f(y)|dy$$

$$\le \gamma_{(d)} \int_{B(0;r)} \left[|x_j| \cdot |y| \cdot \sum_{k=1}^{d+1} \frac{1}{|x|^k \cdot |x-y|^{d+2-k}} + \frac{|y_j|}{|x-y|^{d+1}} \right] |f(y)|dy$$

$$\le \frac{c_0}{|x|^{d+1}},$$

where $c_0 = \gamma_{(d)} r(d+2)2^{d+1}\|f\|_{L_1}$. Then it follows that

$$m\{x \in R^d : |(R_j f)(x)| > \lambda\} \le m\{x \in R^d : |x| \le 2r\} + m\{x \in R^d : \frac{c_0}{|x|^{d+1}} > \lambda\}$$

$$= \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \cdot (2r)^d + m\left\{ x \in R^d : |x| < \sqrt[d+1]{\frac{c_0}{\lambda}} \right\} = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \cdot \left[(2r)^d + \left(\frac{c_0}{\lambda}\right)^{\frac{d}{d+1}} \right],$$

whence it follows asymptotic equality (9).

Now let's consider the general case. From the condition $\int_{R^d} f(x)dx = 0$ it follows that for any $\varepsilon > 0$ there exist the functions f_1 and f_2 satisfying the condition: $f = f_1 + f_2$, the function f_1 is concentrated on some ball $B(0;r) \subset R^d$ and $\int_{R^d} f_1(x)dx = 0$, the function f_2 satisfies the inequality $\|f_2\|_{L_1} < \frac{\varepsilon}{4C_1}$, where C_1 is a constant in estimation (2). Since the function f_1 is concentrated on the ball $B(0;r) \subset R^d$ and $\int_{R^d} f_1(x)dx = 0$, then for the function f_1 equality (9) is satisfied, and therefore there exists $\lambda(\varepsilon) > 0$ such that for $0 < \lambda < \lambda(\varepsilon)$ the inequality

$$\lambda m\{x \in R^d : |(R_j f_1)(x)| > \frac{\lambda}{2}\} < \frac{\varepsilon}{2} \tag{10}$$

holds. On the other hand, from the inequality (2) it follows that

$$\lambda m\{x \in R^d : |(R_j f_2)(x)| > \frac{\lambda}{2}\} \le 2C_1\|f_2\|_{L_1} < \frac{\varepsilon}{2} \tag{11}$$

for any $\lambda > 0$. From inequalities (10), (11) we get

$$\lambda m\{x \in R^d : |(R_j f)(x)| > \lambda\}$$

$$\le \lambda m\{x \in R^d : |(R_j f_1)(x)| > \frac{\lambda}{2}\} + \lambda m\{x \in R^d : |(R_j f_2)(x)| > \frac{\lambda}{2}\} < \varepsilon$$

for $0 < \lambda < \lambda(\varepsilon)$. This shows that equality (9) was satisfied for all functions $f \in L_1(R^d)$, satisfying the condition $\int_{R^d} f(x)dx = 0$. This completes the Proof of the Lemma 1.

Proof of Theorem 2. In the case $\int_{R^d} f(x)dx = 0$ the assertion of the Theorem follows from Lemma 1. Let's consider the case $\int_{R^d} f(x)dx = \alpha \neq 0$. Denote by $f_1(x) = \alpha\eta_{(d)}\chi(B(0;\,1))(x)$, where $\eta_{(d)} = \frac{\Gamma(\frac{d}{2}+1)}{\pi^{\frac{d}{2}}}$, $\chi(B(0;\,1))$ is a characteristic function on the unit circle $B(0;\,1)$ and $f_2(x) = f(x) - f_1(x)$. Then $\int_{R^d} f_2(x)dx = 0$, and from Lemma 1

$$m\{x \in R^d:\ |(R_j f_2)(x)| > \lambda\} = o\left(\frac{1}{\lambda}\right), \lambda \to 0+. \tag{12}$$

Since for any $x \in \{x \in R^d : x_j > 2\}$

$$|(R_j f_1)(x)| = \eta_{(d)}\gamma_{(d)}|\alpha| \left| \int_{B(0;\,1)} \frac{x_j - y_j}{|x-y|^{d+1}} dy \right|$$

$$\leq \eta_{(d)}\gamma_{(d)}|\alpha| \int_{B(0;\,1)} \frac{|x_j|+1}{||x|-1|^{d+1}} dy = \gamma_{(d)}|\alpha| \frac{|x_j|+1}{||x|-1|^{d+1}}$$

$$\leq \gamma_{(d)}|\alpha| \frac{|x_j|}{||x|-1|^{d+1}} + \gamma_{(d)}|\alpha| \frac{2^d}{|x|^{d+1}},$$

$$|(R_j f_1)(x)| \geq \gamma_{(d)}|\alpha| \frac{|x_j|}{||x|+1|^{d+1}} - \gamma_{(d)}|\alpha| \frac{2^d}{|x|^{d+1}},$$

and for any $\lambda > 0$

$$m\{x \in R^d:\ \frac{|x_j|}{|x|^{d+1}} > \lambda\} = \int_{\{x \in R^d:\ |x_j| > \lambda |x|^{d+1}\}} dx = \frac{\theta_{(d)}}{\lambda},$$

$$m\{x \in R^d:\ \frac{1}{|x|^{d+1}} > \lambda\} = m\left\{x \in R^d:\ |x| < \left(\frac{1}{\lambda}\right)^{\frac{1}{d+1}}\right\} = \frac{1}{\eta_{(d)}} \left(\frac{1}{\lambda}\right)^{\frac{d}{d+1}},$$

then we get that

$$\limsup_{\lambda \to 0+} \lambda m\{x \in R^d:\ |(R_j f_1)(x)| > \lambda\} \leq \gamma_{(d)}\theta_d|\alpha|, \tag{13}$$

$$\liminf_{\lambda \to 0+} \lambda m\{x \in R^d:\ |(R_j f_1)(x)| > \lambda\} \geq \gamma_{(d)}\theta_d|\alpha|. \tag{14}$$

It follows from (13), (14) that

$$\lim_{\lambda \to 0+} \lambda m\{x \in R^d : |(R_j f_1)(x)| > \lambda\} = \gamma_{(d)}\theta_d|\alpha|. \tag{15}$$

For any $0 < \varepsilon < 1$, by the inclusions

$$\{x \in R^d:\ |(R_j f_1)(x)| > (1+\varepsilon)\lambda\} \backslash \{x \in R^d:\ |(R_j f_2)(x)| > \varepsilon\lambda\} \subset$$

$$\subset \{x \in R^d:\ |(R_j f)(x)| > \lambda\} \subset$$

$$\subset \{x \in R^d : \ |(R_j f_2)(x)| > \varepsilon\lambda\} \cup \{x \in R^d : \ |(R_j f_1)(x)| > (1-\varepsilon)\lambda\}$$

and equalities (12), (15) we have

$$\limsup_{\lambda \to 0+} \lambda m\{x \in R^d : \ |(R_j f)(x)| > \lambda\} \le \frac{\gamma_{(d)}\theta_d |\alpha|}{1-\varepsilon},$$

$$\liminf_{\lambda \to 0+} \lambda m\{x \in R^d : \ |(R_j f)(x)| > \lambda\} \ge \frac{\gamma_{(d)}\theta_d |\alpha|}{1+\varepsilon},$$

This implies the equation (8) and completes the proof of the Theorem 2.

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Estimations of Functions from Lizorkin-Triebel Spaces Reduced by Corresponding Polynomials

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Abstract. In this paper the integral inequalities as estimation of the norms of functions reduced by polynomials, are proved.

Key Words and Phrases: integral representation, the space Lizorkin-Triebel, flexible φ-horn, generalized derivatives.

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1. Introduction

In this paper, with help method of integral representation, we estimate the norms of functions from the Lizorkin-Triebel spaces $F^l_{p,\theta}(G)$, where $l \in (0,\infty)^n$, $p, \theta \in (1,\infty)$, $G \subset R^n$ (introduced and studied from point of view of embedding theory in papers [1]), reduced by polynomials, determined in n-dimensional domains satisfying the flexible φ-horn condition. In other words, we prove inequality type

$$\|D^v (f - P_{l-1}(f,x))\|_{q,G} \le C \left|\tilde{A}(1)\right| \|f\|_{\tilde{F}_{p,\theta}(G_\varphi)}, \tag{1}$$

where

$$\tilde{A}(1) = \max_i A^i(1) = \int_0^1 \prod_{j=1}^n (\varphi_j(t))^{-v_j - \frac{1}{p} + \frac{1}{q}} \frac{\varphi_i^1(t)}{\varphi_i(t)} dt,$$

$$\|f\|_{\tilde{F}_{p,\theta}(G_\varphi)} = \sum_{i=1}^n \left\| \left\{ \int_0^{t_0} \left[\frac{\delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) D_i^{k_i} f(\cdot)}{(\varphi_i(t))^{l_i - k_i}} \right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)} \right\}^{\frac{1}{\theta}} \right\|_p . \tag{2}$$

Note that the normed linear space $F^l_{p,\theta}(G_\varphi)$ of functions $f \in L^{loc}(G)$ with the finite norm, defined in paper [5]

$$\|f\|_{F^l_{p,\theta}(G_\varphi)} = \|f\|_{p,G} +$$

$$+\sum_{i=1}^{n}\left\|\left\{\int_{0}^{t_0}\left[\frac{\delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)D_i^{k_i}f\left(\cdot\right)}{\left(\varphi_i(t)\right)^{l_i-k_i}}\right]^{\theta}\frac{d\varphi_i(t)}{\varphi_i(t)}\right\}^{\frac{1}{\theta}}\right\|_p, \tag{3}$$

where

$$\|f\|_{p,G}=\|f\|_{L_p(G)}=\left(\int_G|f(x)|^p\,dx\right)^{\frac{1}{p}},$$

$$\delta_i^{m_i}\left(\varphi_i(t)\right)f(x)=\int_{-1}^{1}\left|\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right),f(x)\right|dt,$$

$$\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f(x)=\begin{cases}\Delta_i^{m_i}\left(\varphi_i(t)\right)f(x), & \text{for}\quad [x,x+m_i\varphi_i(t)e_i]\subset G_{\varphi(t)},\\ 0, & \text{for}\quad [x,x+m_i\varphi_i(t)e_i]\not\subseteq G_{\varphi(t)},\end{cases}$$

$$\Delta_i^{m_i}\left(\varphi_i(t)\right)f(x)=\sum_{j=0}^{m_i}(-1)^{m_i-j}C_{m_i}^{j}f\left(x+j\varphi_i(t)e_i\right),\quad e_i=\{0,...,0,1,0,...,0\},$$

and let $G\subset R^n$; $l\in(0,\infty)^n$; $m_i\in N$; $k_i\in N_0$; $1<p,\theta<\infty$; $\varphi(t)=(\varphi_1(t),...,\varphi_n(t))$, $\varphi_j(t)>0\,(t>0)$ be continuously-differentiable functions, $\lim\limits_{t\to+0}\varphi_j(t)=0$, $\lim\limits_{t\to+0}\varphi_j(t)=$ $=L_j\leq\infty$, $j=1,2,...,n$. We denote the set of such vector functions by A. For any $x\in R^n$ we assume

$$G_{\varphi(t)}(x)=G\cap I_{\varphi(t)}(x)=G\cap\left\{y:|y_j-x_j|<\frac{1}{2}\varphi_j(t),\,j=1,2,...,n\right\}.$$

In case $\varphi_j(t)=t^{\lambda_j},\lambda_j>0$, $j=1,2,...,n$ the spaces $F_{p,\theta}^{l}(G)$, was introduced and studied view of theory embedding in monograph [1].

Was proved that [6] for $f\in F_{p,\theta}^{l}(G_\varphi)$, $p,\theta\in(1,\infty)$, $l\in(0,\infty)^n$, if

$$A^i(T)=\int_0^T\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-t_j}\frac{\varphi_i^1(t)}{\left(\varphi_i(t)\right)^{1-l_i}}dt<\infty,\ i=1,2,...,n,$$

then there exists $D^vf\in L_p(G)$ and the following identity is valid [5]

$$D^vf(x)=f_{\varphi(T)}^{(v)}(x)+\sum_{i=1}^{n}\int_0^T\int_{R^n}L_i^{(v)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}\right)\times$$

$$\times f_i(x+y,t)\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-1-v_j}\frac{\varphi_i^1(t)}{\varphi_i(t)}dtdy, \tag{4}$$

$$f_{\varphi(T)}^{(v)}(x)=\prod_{j=1}^{n}\left(\varphi_j(T)\right)^{-1-v_j}\int_{R^n}\Omega^{(v)}\left(\frac{y}{\varphi(T)},\frac{\rho\left(\varphi(T),x\right)}{3\varphi(T)}\right)f(x+y)dy, \tag{5}$$

where

$$|f_i(x,t)| \leq \int_{-1}^{1} \left|\delta_i^{m_i}(\varphi_i(t))\, f(x+u\varphi_i(t))\right| du.$$

2. Main results.

Theorem 1. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn $1 < p < q \leq \infty$, $v = (v_1, ..., v_n)$, $v_j \geq 0$ be entire $j = 1, ..., n$; $A^i(1) < \infty$ $(i = 1, ..., n)$ and let $f \in F^l_{p,\theta}(G_\varphi)$. Then*

$$\left\| D^v \left(f - P_{l-1}(f,x)\right)\right\|_{q,G} \leq C\,|A(1)|\, \|f\|_{\tilde{F}^l_{p,\theta}(G_\varphi)},$$

where $A(1) = \max_i A^i(1)$, $A^i(1) = \int_0^1 \prod_{j=1} (\varphi_j(t))^{-v_j-\frac{1}{p}+\frac{1}{q}} \frac{\varphi_i^1(t)}{\varphi_i(t)^{1-l_i}}dt$ and C the constant independent of f.

Proof. Under the conditions of our theorem, there exist generalized derivatives $D^v f \in L_p(G)$ and for almost each point $x \in G$ the integral representation (4) and (5) with the kernels is valid. In (4) and (5) if $\rho(\varphi(t), x) = -x\varphi(t)$, $0 < t \leq T = 1$ we get identity

$$D^v f(x) = P^{(v)}_{l-1} + \sum_{i=1}^{n} \int_0^1 \int_{R^n} L_i^{(v)} \left(\frac{y}{\varphi(t)}, \frac{p(\varphi(t), x)}{\varphi(t)}\right) f_i(x+y,t) \times$$

$$\times \prod_{j=1}^{n} (\varphi_j(t))^{-1-v_j} \frac{\varphi_i^1(t)}{\varphi_i(t)} dt, \tag{6}$$

where

$$|f_i(x,t)| \leq \int_{-1}^{1} \left|\delta_i^{m_i}(\varphi_i(t))\right| f(x+u\varphi_i(t))\, u,$$

the support of this identity (6) is contained in the flexible φ horn

$$x + V(\varphi) = x + \bigcup_{0<t\leq T \leq 1} \left\{ y : \left(\frac{y}{\varphi(t)}\right) \in S(L_i),\, i = 1, ..., n \right\},$$

where $L_i(\cdot, y) \in C_0^\infty(R^n)$ $(i = 1, ..., n)$, and let

$$S(L_i) = \sup p L_i \subset I_{\varphi(1)} = \left\{ x : |x_j| < \frac{1}{2}\varphi_j(1), j = 1, ..., n \right\}.$$

Let U be is open set contained in the domain G; hence forth we always assume that $U + V(\varphi) \subset G$.

Hence, by the Minkowski inequality, we have:

$$\left\|D^v\left(f-p_{l-1}(f,x)\right)\right\|_{q,U} \leq \sum_{i=1}^{n}\left\|F_i\left(\cdot,t\right)\right\|_{q,U}, \tag{7}$$

here

$$F_i(x,t)=\int_0^1\int_{R^n} L_i^{(v)}\left(\frac{y}{\varphi(t)},\frac{p\left(\varphi(t),x\right)}{\varphi(t)}\right) f_i(x+y,t)\times$$
$$\times\prod_{j=1}^{n}\left(\varphi_j(t)^{-1-v_j}\frac{\varphi_i(t)}{\varphi_i(t)}\right)dtdy. \tag{8}$$

Applying generalized Minkowski inequality (8) for $F_i(x,t)$, we get

$$\left\|F_i\left(\cdot,t\right)\right\|_{q,U} \leq \int_0^1 \left\|E_i\left(\cdot,t\right)\right\|_{q,U}\left|\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-1-v_j}\frac{\varphi_i^1(t)}{\left(\varphi_i(t)\right)^{1-l_j}}\right|dt, \tag{9}$$

here

$$E_i(x,t)=\int_{R^n} L_i^{(v)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}\right) f_i(x+y,t)dy. \tag{10}$$

From the Holder inequality ($q \leq r \leq \infty$) we have

$$\left\|E_i\left(\cdot,t\right)\right\|_{q,U} \leq \left\|E_i\left(\cdot,t\right)\right\|_{r,U}(mesU)^{\frac{1}{q}-\frac{1}{r}}. \tag{11}$$

Now estimate the norm $\left\|E_i\left(\cdot,t\right)\right\|_{r,U}$. Let χ be a characteristic function of the set $S(L_i)$. Again applying the Holder inequality for representing the function in the form (10) in the case $1<p<r\leq\infty$, $s\leq r$ as $\left(\frac{1}{s}=1-\frac{1}{p}+\frac{1}{r}\right)$, we get

$$\left\|E_i\left(\cdot,t\right)\right\|_{r,U} \leq \sup_{x\in U}\left(\int_{R^n}\left|f_i(x+y)\right|^p\chi\left(\frac{y}{\varphi(t)}\right)\right)^{\frac{1}{p}-\frac{1}{r}}\times$$
$$\times\sup_{y\in V}\left(\int_U\left|f_i(x+y,t)\right|^p dx\right)^{\frac{1}{r}}\left(\int_U\left|\tilde{L}_i\left(\frac{y}{\varphi(t)}\right)\right|^s dy\right)^{\frac{1}{s}}. \tag{12}$$

It is assumed that $\left|L_i(x,y,z)\right|\leq C\left|\tilde{L}_i(x)\right|$, and $\tilde{L}_i\in C_0^\infty\left(R^n\right)$.

For any $x\in U$ we have

$$\int_{R^n}\left|f_i(x+y,t)\right|^p\chi\left(\frac{y}{\varphi(t)}\right)dy\leq$$

$$\leq \int\limits_{U+V} |f_i(x+y,t)|^p \, dy \leq (\varphi_i(t))^{pl_i} \left\| (\varphi_i(t))^{-l_i} \, \delta_i^{m_i} \left(\varphi_i(t)\right) f \right\|_{p,C}^p . \tag{13}$$

For $y \in V$

$$\int\limits_U |f_i(x+y,t)|^p \, dx \leq \int\limits_{U+V} |f_i(x,t)|^p \, dx \leq (\varphi_i(t))^{l_i} \left\| (\varphi_i(t))^{-l_i} \, \delta_i^{m_i} \left(\varphi_i(t)\right) f \right\|_{p,G} , \tag{14}$$

$$\int\limits_{R^n} \left| \widetilde{L}_i \left(\frac{y}{\varphi(t)} \right) \right|^s dy = \left\| \widetilde{L}_i \right\|_s^s \prod_{j=1}^n \varphi_j(t). \tag{15}$$

From inequalities (12)-(15) it follows that

$$\left\| E_i \left(\cdot, t \right) \right\|_{r,U} \leq C_1 \left\| \tilde{L}_i \right\|_s \prod_{j=1}^n \left(\varphi_j(t) \right)^{\frac{1}{s}} \left(\varphi_i(t) \right)^{l_i} \times$$

$$\times \left\| (\varphi_i(t))^{-l_i} \, \delta_i^{m_i} \left(\varphi_i(t) \right) \right\|_{p,G} , \tag{16}$$

and by the inequality (12) we have

$$\left\| E_i \left(\cdot, t \right) \right\|_{q,U} \leq C_2 \prod_{j=1}^n \left(\varphi_j(t) \right)^{\frac{1}{s}} \left(\varphi_i(t) \right)^{l_i} \times$$

$$\times \left\| (\varphi_i(t))^{-l_i} \, \delta_i^{m_i} \left(\varphi_i(t) \right) f \right\|_{p,G} . \tag{17}$$

From inequalities (7) and (9) for $(r = q)$ we have

$$\left\| D^v \left(f - P_{l-1}(f,x) \right) \right\|_{q,G} \leq C \left| \tilde{A}(1) \right| \, \|f\|_{F_{p,\theta}(G_\varphi)} .$$

This completes the poof of Theorem 2.

The following theorem is proved analogously to Theorem 1.

Theorem 2. *Let all the conditions of Theorem 2.1 be fulfilled. Furthermore, let* $l^1 \in (0,\infty)^n$, $1 < \theta < \theta_1 < \infty$, *if*

$$A^{i,1}(1) = \int\limits_0^1 \prod_{j=1}^n \left(\varphi_j(t) \right)^{-v_j - l_j^1 - \frac{1}{p} + \frac{1}{q}} \frac{\varphi_i^1(t)}{\left(\varphi_i(t) \right)^{1-l_i}} dt < \infty$$

$i = 1, 2, ..., n,$ *then*

$$\left\| D^v \left(f - P_{l-1}(f,x) \right) \right\|_{F^{l^1}_{q,\theta}(G_\varphi)} \leq C \left| A^1(1) \right| \, \|f\|_{F^l_{p,\theta}(G_\varphi)} ,$$

where $A^{(1)}(1) = \max\limits_i A^{i,1}(1)$ *and* C *the constant independent of* f.

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