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Loan Portfolio Risk Value and Expected Loss Analysis

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Abstract. In this article, I am going to about the risk value of the loan portfolio, which is one of the most important interest-sensitive assets of banks, and the analysis of expected losses. Thus, we are going to calculate the value at risk of the bank's credit portfolio (CrVAR) and meanwhile I am also going to talk about the Expected Loss (EL) and its components, the Probability of Default (PD) and the calculation of the Loss Given Default (LGD).

Key Words and Phrases: credit risk, portfolio at risk, default equivalent risk, credit value at risk, expected loss, unexpected loss, probability of default.

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1. Portfolio at Risk

The risk portfolio (Portfolio at Risk - PAR) is equal to the ratio of the amount of overdue credit to the total loan amount (portfolio). This coefficient groups the credits according to the days of delay (30, 60, 90, 120, 180, 270, 360, etc.). Delay groups should cover monthly for up to one year and annual for more than a year. The current state of the loan portfolio and the direction of the trend are determined by the PAR coefficient relative to the different days of delay. If there is a sharp deterioration in the coefficients of delay (for example, an increase of 1% per month), then the credits should be analyzed in more detail by statistical methods to identify the reasons for the delay and take the indispensable measures.

Portfolio at Risk (PAR) – example

Overdue days	7 – 30	31 – 90	91 – 180	181 –270	271 –360	> 360
Overdue Portfolio	5,723,673	2,196,865	2,692,723	4,281,974	894,405	20,813,491
Total loan Portfolio	187,766,157					
Portfolio at Risk (PAR)	**3.0%**	**1.2%**	**1.4%**	**2.3%**	**0.5%**	**11.1%**
Total PAR	**19.5%**					
Total PAR (AZN)	**36,603,130**					

2. Default Equivalent Risk

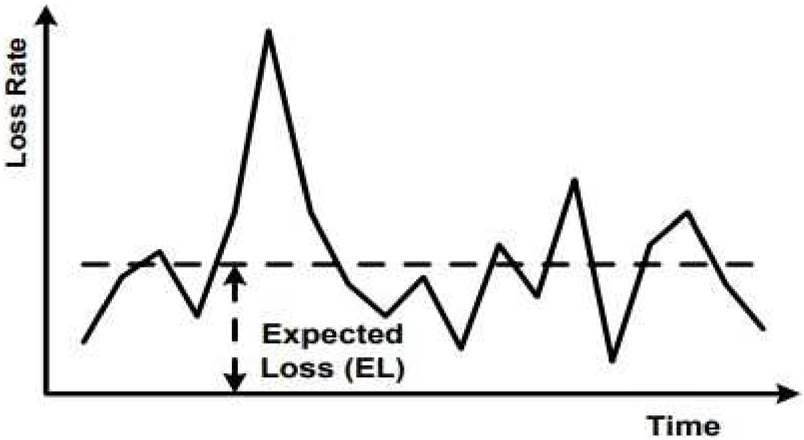
Default Equivalent Risk (DER), potential default of the total credit portfolio is forecasted by giving the probability of default for each group of delays. The possibility of bankruptcy is determined based on empirical data of the bank. According as the delay period increases, this possibility also increases and is defined as 100% for overdue credit groups with a delay period of more than one year. It is possible to use the migration matrix or the transition matrix to calculate the probability of default.

Default equivalent risk (DER) – example

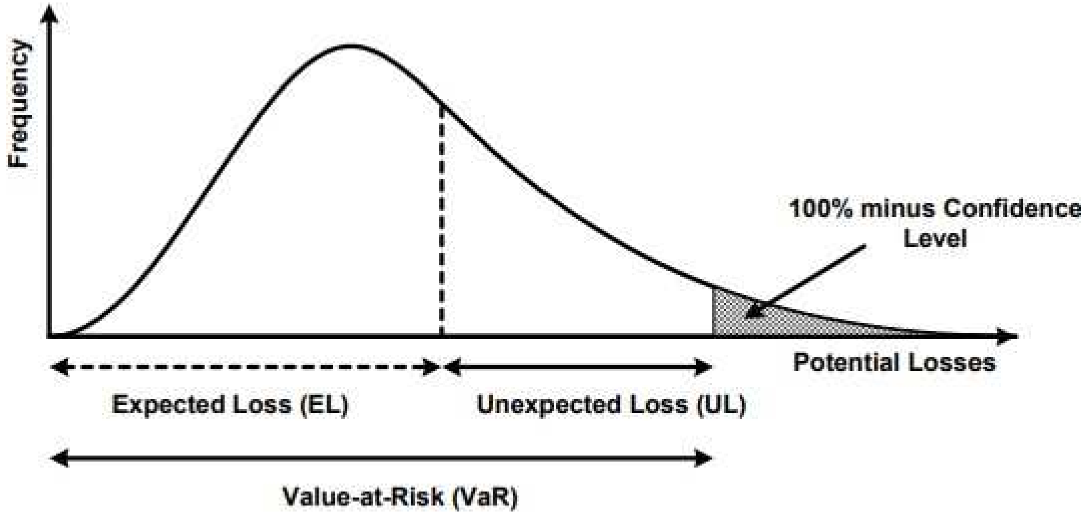
Overdue days	7 – 30	31 – 90	91 – 180	181 –270	271 –360	> 360
Probability of default (PD)	**1%**	**5%**	**20%**	**50%**	**80%**	**100%**
Default equivalent risk (DER)	**0.03%**	**0.1%**	**0.3%**	**1.1%**	**0.4%**	**11.1%**
Total DER	**13.0%**					
Total DER (AZN)	**24,375,626**					

3. Expected Loss

Expected Loss (EL) is the average credit loss we expect from a portfolio in a certain period. Although the losses that a bank will incur in a certain period are not known in advance, banks are able to forecast the expected loss at an acceptable level. These losses are called expected losses and are the averages of losses below the area shown by the dashed lines in the following diagram:



Losses above the dashed line in the picture do not occur every year, but in case of loss, it might be potentially very large. Banks know that Unexpected Loss (UL) whether will happen now or then, but they are not aware of the time and severity of losses in advance. Capital is required in order to compensate such losses. The worst case we can imagine is that banks lose their entire loan portfolio in a year. Although this is a risky occurrence, it is impossible and economically inefficient to keep capital against it. Banks want to reduce risk capital to minimum, as reducing risk capital, increases the amount of money that can be directed in lucrative investments. On the other hand, the smaller a bank's capital, the more likely it is that it will not be able to meet its debt liabilities. That is to say, failure to cover losses in a certain given year with profits and available capital will bankrupt the bank. Thus, banks and regulators must carefully balance the risks and earnings of reserve capital. There are a number of approaches to determining how much capital a bank will have. The IRB (internal ratings based) approach adopted for Basel II targeted to bankrupt the bank. With statistical methods, it is possible to estimate the amount of damage with a predetermined probability. We can show a diagram of expected and unexpected losses and value at risk as follows:



As we noted, the expected loss is equal to the amount of the loss over a determined period. That is to say, it is the amount of loss expected during a certain period. This model is used to calculate the expected loss of the credit portfolio and the amount of reserves to be created for the portfolio. In the expected loss model, the loss is a function of three risk parameters, the probability of default (PD), the equivalent at default (EAD), and the damage that can be caused by default (LGD).

PD, EAD and LGD can each be calculated at the level of either individual or sector borrowers. The expected loss is calculated by the following formula:

$$EL = PD * EAD * LGD$$

If we show the expected loss as a percentage, the formula will be as follows:

$$EL_{\%} = PD * LGD$$

Each parameter must be calculated regardless of economic factors, and in this case, the economic effects are analyzed in a more dynamic way.
EAD is the rest of credit or default value. PD is the probability of default occurring during in a determined period. PD can be calculated based on historical data for each product type. LGD is a loss from this credit in case of by the borrower. Another name of LGD is Loss Rate (LR). LGD is calculated by the following formula:

$$LGD = 1 - RR$$

Here, RR (recovery rate) is the coefficient of credit recovery, or in other words, the rate of credit collateral. RR is calculated as follows:

$$RR = Value\ \ of\ \ collateral * k/Credit\ \ amount$$

The coefficient k can be taken as 80% for precious metals, 60% for real estate, 40% for automobiles and 20% for moveable estate / equipment. As a simple example, we can calculate LGD as follows: the residual amount of the credit at risk is 100,000 AZN and the market value of the collateral is 60,000 AZN. In this case, 40,000 AZN the credit remains unguaranteed and the LGD coefficient is 40,000 / 100,000 or 40%.

Expected loss calculation (sample)

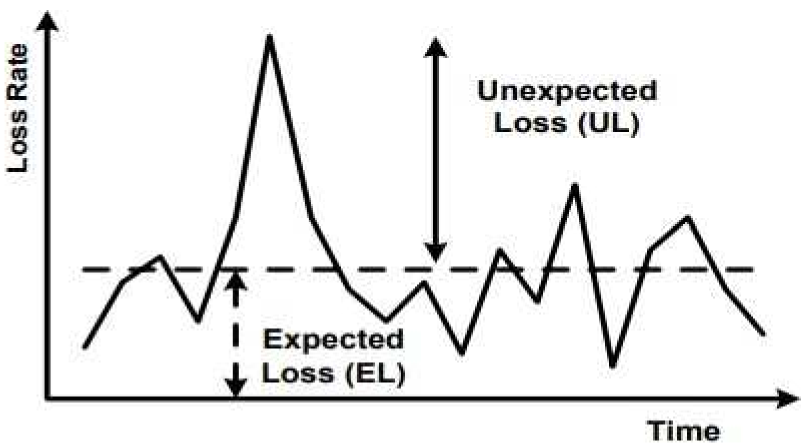
Imagine that the bank has issued a credit of 2,000,000 AZN, and the current balance of the credit is 1,700,000 AZN. 1,700,000 AZN is the default value (EAD). According to the bank's internal rating model, the customer has a 5% probability of default (PD) over the next year. If a default occurs, the estimated instantaneous loss ratio (LGD) is 30%. Now let us calculate the expected loss:

$$EL = PD * EAD * LGD = 5\% * 1700000 * 30\% = 25500$$

Therefore, with a probability of 5% default on a credit with a balance of 1,700,000 AZN, our loss at the time of default will be 25,500 AZN.

4. Unexpected Loss

Unexpected Loss (UL) is the standard error for default losses within a year. Unexpected loss can also be expressed as the volatility of the expected loss. In other words, the unexpected loss is the average of the loss over the average loss. In the following picture above the area shown by the dashed lines is the average of the losses.



Unexpected loss is calculated by the following formula:

$$[UL = EAD * [(PD * \boldsymbol{\sigma}_{LGD}^2) + (LGD^2 * \boldsymbol{\sigma}_{PD}^2)]^{1/2}$$

Here,

$$\boldsymbol{\sigma}^2_{LGD} = LGD * (1 - LGD)/4$$

and

$$\boldsymbol{\sigma}^2_{PD} = PD * (1 - PD)$$

σ^2_{LGD} is a variation of the moment of loss at default and σ^2_{PD} is a variation of the probability of default. Unexpected loss is calculated as the standard deviation from the average at a certain confidence level and is called Credit VaR (Value at Risk). In accordance with, σ^2_{LGD} and σ^2_{PD} are both standard deviations of the (LGD) and default probability (PD).

Unexpected loss calculation (sample)

According to the example above, the default probability (PD) and loss rate (LGD) have standard deviations of 6% and 20%. Now let us calculate the unexpected loss:

$$UL = EAD * [(PD * \sigma^2{}_{LGD}) + (LGD^2 * \sigma^2{}_{PD})]^{1/2} =$$
$$= 1700000 * (5\% * 20\%^2 + 30\% * 6\%^2)^{1/2} = 94346$$

Therefore, a credit with a balance of 1,700,000 AZN will have a probability of default of 5%, a loss ratio of 30% and with a standard deviation of 6% and 20%, we will have a default loss 94,346 AZN.

5. Calculation of the probability of default

By giving the probability of default (PD) for each delay group, it is possible to forecast the potential default volume of the total portfolio. Probability of bankruptcy (PD) is determined regarding migration matrices based on the bank's empirical data. As the delay period increases, this probability also increases, and the probability of default reaches 100% for credit groups with a delay period of more than 360 days.

To calculate the probability of default, you should first calculate the Roll rate coefficient. Roll-rate is the percentage of loans in any delay interval to another delay interval, in other words, its the coefficient. That is, according to the working and delay days of credits from the intervals of 0, 1 - 30, 31 - 60, 61 - 90, ..., 330 - 360, respectively, 1 - 30, 31 - 60, 61 - 90, ..., 330 - the transition to intervals of 360, over 360, is the percentage of migration. To roll-rate analysis, a Migration Matrix of Deferred Loans is prepared and the transition cells from delay intervals to subsequent delay intervals are considered.

First of all, let's form Migration matrices. The migration matrix, also known as the transition matrix, is an analytical report showing the transition of a bank's credit portfolio from one delay interval to another delay interval according to the days of delay. This report is one of the main risk metrics and tools of the loan portfolio. The migration matrix shows the status of credits available during one period for subsequent periods.

In order to form a migration matrix, first of all we are preparing the distribution on the credit portfolio to a certain date (must be at least 30 days before the current date) according to 0, 1-30, 31-60, 61-90, ..., 301-330 and 331-360 days of delay. It is possible to enter both the number and amount of credits in the portfolio separately for these intervals. In this way, we prepare two separate reports on the number and amount. We enter the distribution of these delay days vertically in the first column of the matrix. Then, 30 days after the date that we mentioned above, we prepare the distribution of the credit portfolio, which includes these loans, on days of delay of 1-30, 31-60, 61-90, ..., 301-330, 331-360 and more than 360 days. We enter the distribution of these delay days horizontally in the first row of the matrix. In the end we will get a matrix in the following form.

overdue days	total loans	Closed loans	due loans	1–30	31–60	61–90	91 –120	121 -150	151 -180	181 –210	211-240	241–270	271 -300	301-330	331 –360	>361
Due loans	17,816	123	17,459	234	0	0	0	0	0	0	0	0	0	0	0	0
1 – 30	1,234	12	345	855	22	0	0	0	0	0	0	0	0	0	0	0
31 – 60	53	5	14	23	4	7	0	0	0	0	0	0	0	0	0	0
61 – 90	36	2	3	5	14	6	6	0	0	0	0	0	0	0	0	0
91 – 120	31	1	2	2	1	4	5	16	0	0	0	0	0	0	0	0
121 –150	19	0	0	0	1	2	2	5	9	0	0	0	0	0	0	0
151 –180	21	0	0	0	0	0	0	0	0	21	0	0	0	0	0	0
181 –210	20	0	0	0	0	0	2	2	1	0	15	0	0	0	0	0
211 –240	28	0	0	0	0	0	0	0	1	0	4	23	0	0	0	0
241 –270	29	0	0	0	0	0	0	0	0	1	0	2	26	0	0	0
271 –300	35	2	4	0	5	0	2	0	0	2	0	5	5	10	0	0
301 –330	27	0	1	0	0	0	0	0	0	0	0	0	0	1	25	0
331 –360	16	1	0	0	0	0	0	0	0	0	0	0	0	0	0	15

Thus we obtain the migration matrix. Now let us show this migration matrix as a percentage.

Overdue days	total loans	closed loans	due loans	1–30	31–60	61–90	91–120	121–150	151–180	181–210	21 –240	24 –270	27 –300	301–330	331–360	>361
Dueloans	17,816	0.7%	98.0%	1.3%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
1 – 30	1,234	1.0%	28.0%	69.3%	1.8%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
31 – 60	53	9.4%	26.4%	43.4%	7.5%	13.%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
61 – 90	36	5.6%	8.3%	139%	38.9%	16.7%	16.7%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
91 – 120	31	3.2%	6.5%	6.5%	3.2%	12.%	16.1%	51.6%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
121 –150	19	0.0%	0.0%	0.0%	5.3%	10.5%	10.5%	26.3%	47.4%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
151 –180	21	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	100.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
181 –210	20	0.0%	0.0%	0.0%	0.0%	0.0%	10.0%	10.0%	5.0%	0.0%	75.0%	0.0%	0.0%	0.0%	0.0%	0.0%
211 –240	28	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	3.6%	0.0%	14.3%	82.1%	0.0%	0.0%	0.0%	0.0%
241 –270	29	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	3.4%	0.0%	6.9%	89.7%	0.0%	0.0%	0.0%
271 –300	35	5.7%	11.4%	0.0%	14.3%	0.0%	5.7%	0.0%	0.0%	5.7%	0.0%	14.3%	14.3%	28.6%	0.0%	0.0%
301 –330	27	0.0%	3.7%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	3.7%	92.6%	0.0%
331 –360	16	6.3%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	93.8%

As can be seen from the matrix, this form of analysis shows how much or what percentage of credits of the bank's credit portfolio on overdue days were closed for certain periods, remained in the current status and migrated to the periods after 30 days. As mentioned above, Roll-rate is the percentage of loans in any delay interval to another delay interval, in other words, the coefficient. In other words, Roll-rate ratios are the ratios on the diagonal of the migration matrix. If we look at the example above, our Roll-rate report will be like this:

overdue days	1-30	31–60	61–90	91–120	121–150	151–180	181–210	211–240	241–270	271–300	301–330	331–360	>361
due loans	1.3%												
1 – 30		1.8%											
31 – 60			13.2%										
61 – 90				16.7%									
91 – 120					51.6%								
121 – 150						47.4%							
151 – 180							100%						
181 – 210								75%					
211 – 240									82.1%				
241 – 270										89.7%			
271 – 300											28.6%		
301 – 330												92.6%	
331 – 360													93.8%

As can be seen from the matrix, the Roll-rate shows the diagonal of the Migration Matrix. According to the report, 1.3% of non-overdue credits are in the range of 1 to 30 days, 1.8% of credits with a delay of 1 to 30 days are in the range of 31 to 60 days, and 13.2% of credits with a delay of 31 to 60 days are 61 - 90-day delay interval, 16.7% of credits with a delay of 61-90 days - 91-120 days, etc. passed. Our roll-rate report will be summarized as follows:

Overdue days	**Roll rate**
Due loans	1.3%
1 – 30	1.8%
31 – 60	13.2%
61 – 90	16.7%
91 – 120	51.6%
121 – 150	47.4%
151 – 180	100%
181 – 210	75%
211 – 240	82.1%
241 – 270	89.7%
271 – 300	28.6%
301 – 330	92.6%
331 – 360	93.8%

Now let us calculate the probability of default. In order to calculate the probability of default on a delay interval, it is necessary to multiply that delay interval by the roll-rate ratios of all subsequent delay intervals. According to this method, we can calculate the probability of default (PD) as shown in the following chart:

Overdue days	**Roll rate**	**Probability of default (PD)**
Due loans	1.30%	0.00%
1 – 30	1.80%	0.00%
31 – 60	13.20%	0.07%
61 – 90	16.70%	0.56%
91 – 120	51.60%	3.36%
121 – 150	47.40%	6.50%
151 – 180	100%	13.72%
181 – 210	75%	13.72%
211 – 240	82.10%	18.29%
241 – 270	89.70%	22.28%
271 – 300	28.60%	24.84%
301 – 330	92.60%	86.86%
331 – 360	93.80%	93.80%

As a result, we calculated the probability of default of overdue credits for each delay interval. Now let us calculate the expected losses for each delay interval using the default probabilities we calculated and the expected loss formula:

Overdue days	**PD**	**EAD**	**LGD**	**EL = PD x EAD x LGD**
Due loans	0.00%	17,816	40%	0
1 – 30	0.00%	1,234	40%	0
31 – 60	0.07%	53	40%	0
61 – 90	0.56%	36	40%	0
91 – 120	3.36%	31	40%	0
121 – 150	6.50%	19	40%	0
151 – 180	13.72%	21	40%	1
181 – 210	13.72%	20	40%	1
211 – 240	18.29%	28	40%	2
241 – 270	22.28%	29	40%	3
271 – 300	24.84%	35	40%	3
301 – 330	86.86%	27	40%	9
331 – 360	93.80%	16	40%	6

Finally, we calculated the expected losses of overdue credits (numerically) for each delay interval with a 40% loss coefficient (LGD) and default probabilities (PDs) for each delay interval.

6. Credit Value at Risk - CrVaR

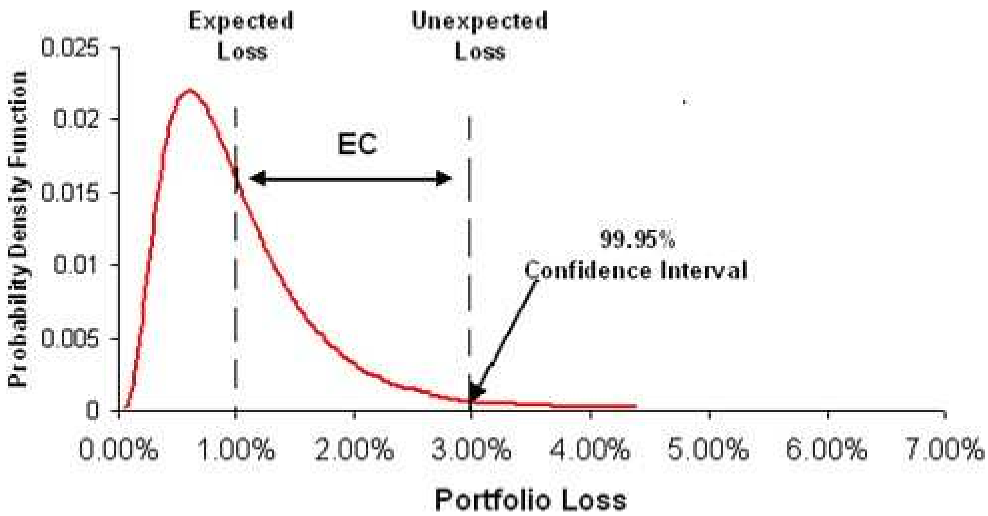
The value of the credit portfolio at risk (Credit Value at Risk - CrVaR) represents the maximum amount of probable loss with a calculated confidence level for a predetermined period.

As can be seen from the definition, Risk Value at Risk involves two factors such as time interval and confidence level. When applying any risk model a minimum of 95% or 99% is taken as the confidence level. The following formula is used to calculate CrVaR:

$$CrVaR = UL + EL$$

7. Economic capital

Economic Capital (EC) is the amount of capital required to cover unforeseen losses (UL). That is, reserves allocated for unexpected losses constitute economic capital. We can show economic capital on the graph as follows:



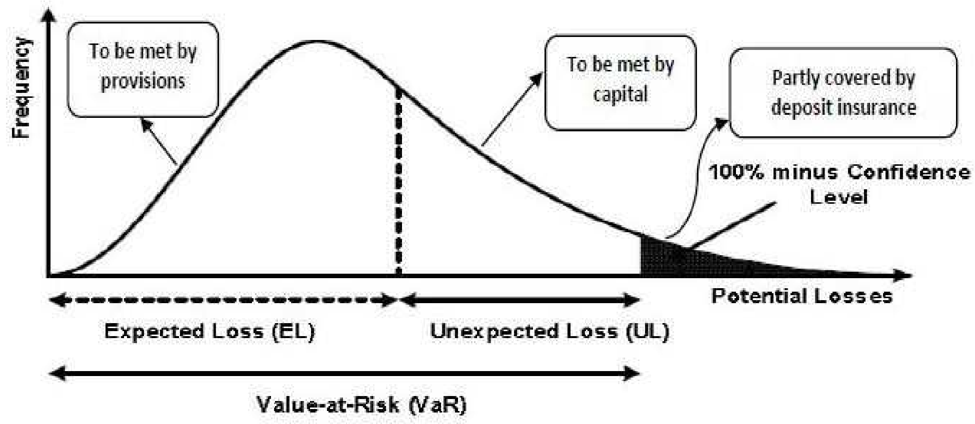
The CrVaR method is used to calculate economic capital. The following formula is used to calculate economic capital:

$$EC = CrVaR - EL$$

or

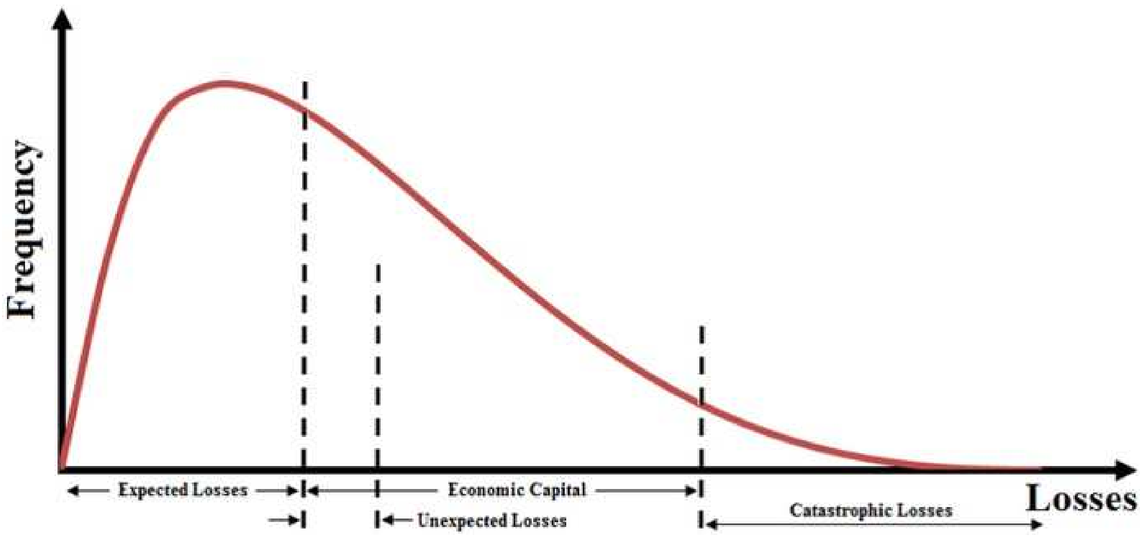
$$EC = UL$$

The Bank creates credit reserves in the face of expected losses and economic capital in the face of unexpected losses. We can show this with the following graph:

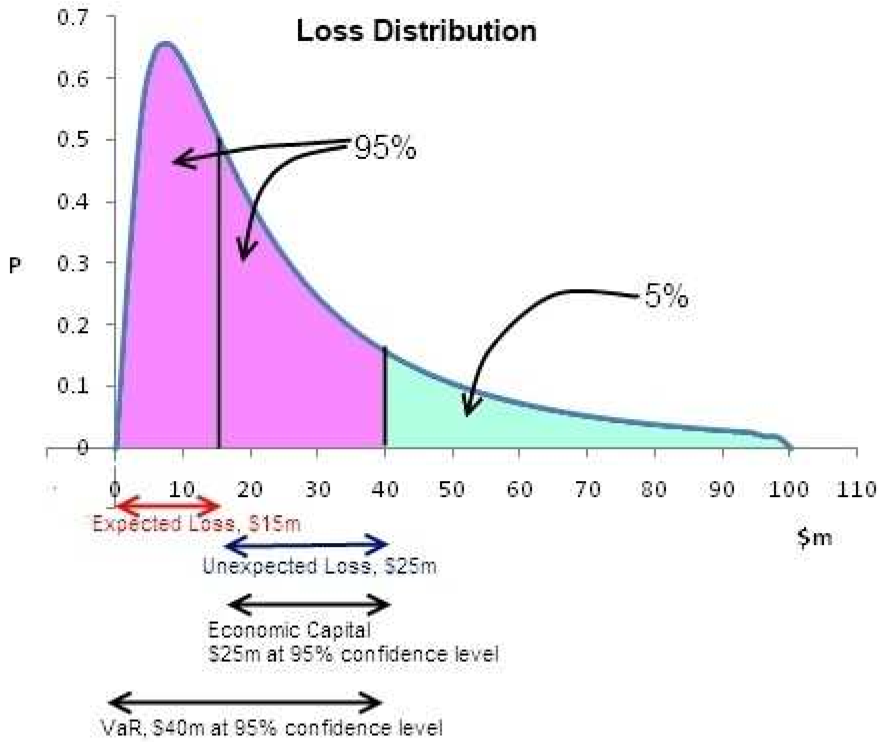


8. Distribution of losses

Loss distribution is the total distribution of unexpected losses, as well as losses that are outside the risk value of the credit portfolio. We can show this in the following diagram:



The distribution of losses includes both credit reserves, economic capital, and exceptional losses. Exceptional losses are losses due to an extraordinary situation that does not occur on a regular basis. In our samples and approaches, this covers 5% and 1%, which are not covered by CrVaR. We can show this with the following diagram:



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Integral Inequalities for Function Spaces with a Finite Collection of Generalized Smoothness

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Abstract. In this paper the function space $\bigcap_{k=0}^{n} \Lambda_{p_k,\theta_k}^{\langle m^k;N^k\rangle}(G,\varphi_k)$ is defined. This function spaces is the generalization of classical Sobolev-Slobodetskii and Nikolskii-Besov spaces. We established sufficient conditions under which the embedding theorems for these spaces are proved. We reduce the analog of integral representations of functions given by S.L. Sobolev for functions form the space $\bigcap_{k=0}^{n} \Lambda_{p_k,\theta_k}^{\langle m^k;N^k\rangle}(G,\varphi_k)$.

Key Words and Phrases: **Key Words and Phrases:** Generalized Hölder space, strong $a\,(h)$-horn condition, integral representation, embedding theorem.

2010 Mathematics Subject Classifications: 46E30, 46E35

1. Introduction

The theory of embedding of spaces of differentiable functions of several variables developed as a new direction of mathematics in the 30s of the 20th century as a result of the works of S.L. Sobolev, which is presented in detail in monograph [5]. This theory studies important connections and relations of differential properties of functions in various metrics. In addition to its independent interest from the point of view of function theory, it also has numerous and effective applications in the theory of partial differential equations. Such applications were given by S.L. Sobolev in [5] (see, also [3]). S.L. Sobolev studied isotropic spaces $W_p^{(l)}(G)$ of functions f defined on a domain $G \subset R^n$ with the norm

$$\|f\|_{W_p^{(l)}(G)} = \sum_{|\alpha|\le l} \|D^\alpha f\|_{L_p(G)} \ ,$$

where l is a non-negative integer and $p \ge 1$. S.L. Sobolev proved embedding theorems for function space $W_p^{(l)}(G)$ in domains of n-dimensional Euclidean spaces. Namely, theorems on the summability in power q of derivatives $D^\beta f$ with respect to a domain G or manifolds of lower dimension belonging to G .

In subsequent years, the theory of embedding developed intensively in various directions and received new interesting and important applications. Among these works, one can note the works of S.M. Nikolskii, O.V. Besov, V.P. Ilin, N. Aronszajn, V.M. Babich, L.N. Slobodetskii, A.S. Jafarov, G. Freud, D. Kralik, V.I. Burenkov, A.J. Jabrailov and others. For more details we refer the readers to [1] and [4].

S.L. Sobolev established embedding theorems using integral representations of functions in terms of their weak derivatives. This method of integral representations was developed in the works of V.P. Ilin and, in particular, was carried over to cases of representation through differences. One of the significant advantages of the method of integral representations is that the representation of a function at a given point is constructed from the values of this function at the points of a bounded cone with vertex at this point. This creates an opportunity to study function spaces of functions defined on an open set of a sufficiently general form.

The remainder of the paper is structured as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. In Section 3 we reduce the class of domains satisfying special horn conditions. Our principal assertions, concerning the embedding of Hölder spaces with generalized smoothness to Lebesgue spaces are formulated and proved in Section 4. We establish sufficient conditions on a domain $G \subset R^n$ for the validity of embedding theorem.

2. Preliminaries

Let R^n be the $n-$ dimensional Euclidean space of points $x = (x_1, ..., x_n)$, and let G be a Lebesgue measurable set of R^n. Suppose $f : G \to R^n$ is a Lebesgue measurable function and let $1 \leq p < \infty$. Throughout this paper we will assume that all sets and functions are Lebesgue measurable.

Definition 1. *The Lebesgue space $L_p(G)$ is the class of all measurable functions f defined on G such that*

$$\|f\|_{L_{p(G)}} = \|f\|_{p,G} = \left(\int_G |f(x)|^p \, dx \right)^{\frac{1}{p}}. \tag{1}$$

In the case $p = \infty$, the space $L_\infty(G)$ will be defined as all measurable functions such that

$$\|f\|_{\infty,G} = vrai \sup_{x \in G} |f(x)|. \tag{2}$$

Let

$$\left. \begin{array}{c} m = (m_1, .., m_n) \\ N = (N_1, .., N_n) \end{array} \right\} \tag{3}$$

be the vectors with integer non-negative components.

The mixed derivative of order $|m| = m_1 + ...m_n$ is defined by

$$D^m f(x) = Dl_1{}^{m_1}...D_n{}^{m_n} f(x_1, .., x_n) = \frac{\partial^{|m|}}{\partial x_1{}^{m_1}...\partial x_n{}^{m_n}}. \tag{4}$$

We denote by

$$\Delta^N(t) f(x) = \Delta_1{}^{N_1}(t_1) ... \Delta_n{}^{N_n}(t_n) f(x_1, .., x_n) \tag{5}$$

the $|N| = N_1 + ...N_n$-order finite mixed difference of a function $f = f(x)$, corresponding to mixed step of a vector $t = (t_1, ..t_n)$. Here

$$\begin{cases} \Delta_k{}^{N_k}(t_k) f(...x_k...) = \Delta_k{}^1(t_k) \left\{\Delta_k{}^{N_k-1}(t_k) f(...x_k...)\right\} \\ \Delta_k{}^0(t_k) f(..., x_k, ...) = f(..., x_k, ...) \\ \Delta_k{}^1(t_k) f(..., x_k, ...) = f(..., x_k + t_k, ...) - f(..., x_k, ...) \end{cases} \tag{6}$$

Therefore

$$\Delta_k{}^{N_k}(t_k) f(..., x_k, ...) \tag{7}$$

Is the finite difference of a function $f = f(x)$ of order N_k in the direction of variable x_k with step t_k.We observe that in the domain $G \subset E_n$ the expression

$$\Delta^N(t, G) f(x) = \Delta^N(t) f(x) \Delta^N(t, G) f(x) = \Delta^N(t) f(x), \tag{8}$$

is the mixed difference of a function $f = f(x)$. In this case we suppose that the mixed difference is constructed from the vertices of a polyhedron that lies entirely in the domain $G \subset E_n$. Otherwise, we assume that

$$\Delta^N(t, G) f(x) = 0. \tag{9}$$

Let $\varphi = \varphi(t) = (\varphi_1(t_1), ..., \varphi_n(t_k))$ be a vector-function such that $\varphi_j = \varphi_j(t_j) > 0$, if $t_j \neq 0$ and $\varphi_j(t_j) \downarrow 0$ for $t \to 0$ and for all $j = 1, 2, .., n$.

Let $1 \leq \theta < \infty$ and let $\frac{dt}{t} = \prod_{j \in E_n} \frac{dt_j}{t_j}$. We consider the following semi-norm

$$\|f\|_{\Lambda_{p,\theta}^{\langle m,n\rangle}}(G, \varphi) = \left\{ \int_{E_{|E_n|}} \left\| \frac{\Delta^N\left(\frac{t}{N}; G\right) D^m f}{\prod_{j \in E_n} \varphi_j(t_j)} \right\|_{p,G}^{\theta} \frac{dt}{t} \right\}^{\frac{1}{\theta}}. \tag{10}$$

For $\theta = \infty$, we suppose that

$$\|f\|_{\Lambda_{p,\infty}^{\langle m,n\rangle}(G,\varphi)} = vrai \sup_{t \in E_N} \left\| \frac{D^N\left(\frac{t}{N}, G\right)}{\prod_{j \in E_n} \varphi_j(t_j)} \right\|_{P,G}. \tag{11}$$

Here $E_n = \sup p\, N$ is a support of a vector $N = (N_1, ...N_n)$. In other words E_n is a set of nonzero indices of the coordinates of vector N. Thus, $E_n \subset \{1, 2, ...n\} = e_n$.

Let us $\frac{t}{N} = \left(\frac{t_1}{N_1}, ... \frac{t_n}{N_n}\right)$ and we use the convention $\frac{0}{0} = 0$.

Therefore $E_{|E_N|} = \left\{t \in E_N; t_j = 0 \left(j \in {}^{e_n}\!/_{E_n}\right)\right\}$.

Let

$$\left.\begin{matrix} m^k = \left(m_1^k, .. m_n^k\right) \\ N^k = \left(N_1^k, ..., N_n^k\right) \end{matrix}\right\} \qquad (k = 0, 1, ..., n) \tag{12}$$

be the vectors with integer non-negative components. Thus,

$\left.\begin{matrix} m_j^k \geq 0 \\ N_j^k \geq 0 \end{matrix}\right\}$ for all $\left(j = \overline{1, n}\right)$ and $k = \overline{0, n}$.

Suppose that any vector-function from collection of $(n+1)$ vector function $\varphi^k = \varphi^k(t) = \left(\varphi_1^k(t_1, ..., \varphi_n^k(t_n))\right)$ satisfy following conditions:
$\varphi_j = \varphi_j(t_j) > 0$ for $t_j \neq 0$
$\varphi_j(t_j \downarrow 0)$ for $t \to 0$.

Definition 2. *Let $1 \leq p_k \leq \theta_k \leq \infty$, and $k = 0, ..., n$. The space*

$$\bigcap_{k=0}^{n} \Lambda_{p_k,\theta_k}^{\langle m^k;N^k \rangle}(G, \varphi_k) \tag{13}$$

is defined as the closure of sufficiently smooth functions $f = f(x)$ with compact support on R^n by the norm

$$\|f\|_{\bigcap\limits_{k=0}^{n} \Lambda_{p_k\theta_k}^{\langle m^k;N^k \rangle}(G;\varphi^k)} = \sum_{k=0}^{n} \|f\|_{\Lambda_{p_k\theta_k}^{\langle m^k,N^k \rangle}(G,\varphi_k)} < \infty. \tag{14}$$

Remark 1. *We observe that the space given by (2.13) in the case $1 \leq p_k \leq \theta_k \leq \infty$ ($k = 0, n$) is a generalization of the classical Sobolev-Slobodetskii space $W_p^r(G)$. Also, in the case $1 \leq p_k \leq \theta_k \leq \infty$ and $\sup pm^k \subseteq \sup pN_k = E_{N^k}$ the space $\bigcap\limits_{k=0}^{n} \Lambda_{p_k,\theta_k}^{\langle m^k;N^k \rangle}(G, \varphi_k)$ is a generalization of Nikolskii-Besov space $B_{p,\theta}^r(G)$ (see, [2]).*

3. The class of domains $G \subset E_n$

Let

$$a(\upsilon) = (a_1(\upsilon), ..., a_n(\upsilon))\, a(\upsilon) = (a_1(\upsilon), ..., a_n(\upsilon)) \tag{15}$$

be a differentiable vector-function in $[0; h]$ such that

$$\left.\begin{array}{ll} a_j = a_j(\nu) > 0, & \upsilon \in (0; h] \\ \lim\limits_{\upsilon \to 0+} a_j(\upsilon) = 0 & \\ \frac{d}{d\upsilon} a_j(\upsilon) > 0, & \upsilon \in (0; h] \end{array}\right\} \tag{16}$$

for all $j = 1, n$.
Let $\delta = (\delta_1, ..., \delta_n)$ be a vector such that $\delta_j = \pm 1$. We put
$R_\delta(a(h)) = \bigcup\limits_{0<\upsilon\leq h} \left\{y \in E_n; c_j \leq \frac{y_j - \delta_j}{a_j(\upsilon)} \leq A_j^*\right\} \left(j = \overline{1, n}\right)$ for all $\upsilon \in (0; h]$. (**??**)
The set $X + R_\delta(a(h))$ is called $\leq a(h) \geq$-horn with vertices in $x \in R^n$.

Note that at each point $x \in R^n$, you can give2^n-number of "$a(h)$"-horns with vertex in $x \in R^n$.

If a vector $\delta = (\delta_1, ..., \delta_n)$ be fixed, then at each point $x \in R^n$ there is only single $\leq a(h) \geq$-horn (for the same vector function (15) - (3.2) the vertex at this point $x \in R^n$).

A subdomain $\Omega \subset G$ is considered to be a subdomain satisfying the $a(h)$-horn condition, if there is a vector $\delta = (\delta_1, ..., \delta_n)$ with $\delta_j = \pm 1$ for which $X + R_\delta(a(h)) \subset G$ for all $x \in \Omega$.

Definition 3. *A subdomain $G \subset E_n$ is called a domain satisfying "$a(h)$- horn" condition, if there exists a finite collection of subdomains*

$$\Omega_1, \Omega_2, ..., \Omega_m \subset G,$$

with $a(h)$-horn condition such that $\bigcup\limits_{k=1}^{M} \Omega_k = G$.

By $C(a(h))$ we denote the class of domains $G \subset E_n$ satisfying the $a(h)$-horn condition.

Definition 4. *Let $k = \overline{1, M}$ and let $\Omega_{k,\varepsilon} = \{y \in \Omega_k; \rho(y; G \backslash \Omega_k) > \varepsilon\}$ is a set of points $y \in \Omega_k$ spaced from $G \backslash \Omega_k$ at a distance greater than $\varepsilon > 0$. A set $G \in C(a(h))$ is called a domain satisfying strong $a(h)$-horn condition, if in addition to condition* $\bigcup\limits_{k=1}^{M} \Omega_k = C$, *there is also a covering*

$\bigcup\limits_{k=1}^{M} \Omega_{k,\varepsilon} \supseteq G$ *for some* $\varepsilon > 0$.

By $C_\varepsilon(a(h))$ we denote the class of domains $G \subset R^n$ satisfying strong "$a(h)$ -horn".

We observe that the notions of a domain $G \subset R^n$ satisfying the $a(h)$-horn condition and strong $a(h)$-horn conditions are introduced in [3] by O.V. Besov, respectively.

4. Main results

In this section of our paper we state and prove our principal assertions.

Theorem 1. *Let $1 \le p_k \le \theta_k \le \infty$ and let $f \in \bigcap\limits_{k=0}^{n} \Lambda_{p_k,\theta_k}^{\langle m^k; N^k \rangle}(G, \varphi^k)$ $(k = \overline{0, n})$. Suppose that* $\left.\begin{matrix} m^k = (m_1^k, ..., m_n^k) \\ N^k = (N_1^k, ..., N_n^k) \end{matrix}\right\}$ *is the vectors with integer non-negative components such that $\{k\} \subset \sup p\,(m^k + N^k)$ $\quad (k = 1, \overline{n})$.*

Let $\varphi^k(t) = (\varphi_1^k(t_1), ..., \varphi_n^k(t_n))$ be a vector-function satisfying condition $\varphi_j(t) = \varphi_j(t_j) > 0$ for $t_j \ne 0$, and $\varphi_j(t_j) \downarrow 0$ for $t \to 0$. Suppose that a domain $G \subset E_n$ is satisfy "$a(h)$ -horn" condition, i.e. $G \in C(a(h))$ and a vector-function $a(\upsilon) = (a_1(\upsilon), ..., a_n(\upsilon))$ satisfy condition (3.2) for all $\upsilon \in [0, n)$

Let $\upsilon = (\upsilon_1, ..., \upsilon_n)$ be a vector with integer-nonnegative components satisfy matching condition with respect to the vectors $\left.\begin{matrix} m^k = (m_1^k, ..., m_n^k) \\ N^k = (N_1^k, ..., N_n^k) \end{matrix}\right\}$ $m_j^k \ge 0, \quad N_j^k \ge 0$, as the form:

$$\left.\begin{matrix} \nu_j \ge m_j^0 + N_j^0 \quad (j = 1, 0) \\ \nu_j \ge m_j^k + N_j^k \quad (j \ne k) \\ \nu_k < m_k^k + N_k^k \quad (j = k) \end{matrix}\right\}, (k = 1, n)\ .$$

Here $H_k(h) \le const < \infty$, $\qquad 1 \le p_k \le q < \infty \qquad (k = \overline{1, n})$ and

$$H_k = \int_0^h \prod_{j=1}^{n} (a_j(\upsilon))^{m_j^k - \frac{\upsilon}{j} - \frac{1}{p_k} - \frac{1}{q}} \left\{ \prod_{j \in E_{N_k}} \varphi_j^k(a_j(\upsilon)) \right\} \frac{da_k(\upsilon)}{a_k(\upsilon)} \tag{17}$$

for all $k \in e_n = \{1, 2, ..., n\}$.

Then

$$D^{\upsilon} f \in L_q(G), \tag{18}$$

and the integral inequality holds

$$\|D^{\upsilon} f\|_{L_q(G)} \le C \sum_{k=0}^{n} H_k(h) \|f\|_{\Lambda_{p_k,\theta_k}^{\langle m^k, n^k \rangle}(G;\varphi^k)}, \tag{19}$$

where $C > 0$ is a constant independent of function $f = f(x)$ and $h > 0$. Also $H_k(h)$ is defined by (4.1) for $k = \overline{1, n}$, and for $k = 0$

$$H_0(h) = \prod_{j=1}^{n} (a_j(h))^{m_j^o - \upsilon_j - \frac{1}{p_0} + \frac{1}{q}} \prod_{j \in E_{N_0}} \varphi_j^0(a_j(h)).$$

We observe that $E_{N_k} = \sup pN^k \quad \left(k = \overline{0, n}\right)$.
Other formulation of Theorem 3.1 we can give as following form.

Remark 2. *Under the conditions of Theorem 3.1 the following embedding holds:*

$$D^{\nu} : \bigcap_{k=0}^{n} \Lambda_{p_k,\theta_k}^{\langle m^k, N^k \rangle}\left(G; \varphi^k\right) \subset L_q(G) \tag{20}$$

In particular, for $\upsilon = 0$ *we have* $\bigcap_{k=0}^{n} \Lambda_{p_k,\theta_k}^{\langle m^k, N^k \rangle}\left(G; \varphi^k\right) \subset L_q(G)$.

Thus, the inclusion (4.4) *is characterized the differential properties of functions from* $\bigcap_{k=0}^{n} \Lambda_{p_k,\theta_k}^{\langle m^k, N^k \rangle}\left(G; \varphi^k\right)$.

Proof. Theorem 3.1 is proved by the method of integral representations of functions $f = f(x)$, developed by S.L. Sobolev in [1]. The method of the proof of Theorem 3.1 is the integral identities given by the equality

$$D^{\nu} f = (-1)^{|\nu + m^0|} C_0 A_0(h) \int_{E_{\left|E_{N^0}\right|}} dz^0 \times \int_{E_n} \left\{ \Delta^{N^0} \left(\frac{Z^0}{N^0} \right) D^{m^0} f(x+y) \right\} M_{\delta,0} dy +$$

$$+ \sum_{k=1}^{n} (-1)^{|\nu + m_k|} C_k \int_0^h A_k(\upsilon) \frac{da_k(\upsilon)}{a_k(\upsilon)} \times$$

$$\times \int_{E_{\left|E_{N^k}\right|}} dz^k \int_{E_n} \left\{ \Delta^{N^k} \left(\frac{z^k}{N^k} D^{m^k} f(x+y) \right) \right\} M_{\delta,k} dy. \tag{21}$$

Here C_k are the constants independent on $f = f(x)$ and $h > 0$, where

$$\left|\nu + m^k\right| = \sum_{j=0}^{n} \left(\nu_j + m_j^k\right),$$

$$A_k(\upsilon)=\prod_{j=1}^{n}(a_j(\upsilon))^{m\frac{k}{j}-\nu_j-1}\prod_{j\in E_{N^k}}(a_j(\upsilon))^{-1}\qquad\left(k=\overline{0,n}\right).$$

In (4.5), the kernels $M_{\delta,0}$ and $M_{\delta,k}$

$$M_{\delta,0}=M_{\delta,0}\left(\frac{y}{a(h)};\frac{z^0}{a(h)}\right),$$

$$M_{\delta,k}=M_{\delta,k}\left(\frac{y}{a(h)};\frac{z^k}{a(h)}\right),(k=\overline{1,n})$$

are sufficiently smooth functions with compact support on R^n, respectively. Here

$$\frac{y}{a(\upsilon)}=\left(\frac{y_1}{a_1(\upsilon)},...,\frac{y_n}{a_n(\upsilon)}\right),$$

$$\frac{z^k}{a(\upsilon)}=\left(\frac{z_{k,1}}{a_1(\upsilon)},...,\frac{z_{k,n}}{a_n(\upsilon)}\right),k=(1,n)$$

while the supports of these kernels satisfy condition:

$$\sup pM_{\delta,k}\left(y;z^k\right)\subset\left\{\begin{array}{l}0<y_j-\delta_j\le 1\qquad\left(j=\overline{1,n}\right)\\ \left(y;z^k\right)\in E_n\times E_{\left|E_{n^k}\right|}:0<z_{k,j}\delta_j\le 1\end{array}\right\},(j\in E_{N_k}).$$

Also, a vector $\delta=(\delta_1,...,\delta_n)$, with $\delta_j=\pm1$ $\left(j=\overline{1,n}\right)$ be fixed.

We observe that in (4.6)

$$\int_{E_{\left|E_{N^k}\right|}}(...)\,dz^k=\underbrace{\int_{E_1}....\int_{E_1}}_{E_{N^k}}(....)\prod_{j\in E_{N^k}}dz_{k,j},$$

where by $|E_{N_k}|$ the number of elements of the set $E_{N_k}=\sup pN^k$.
Moreover, the construction of auxiliary functions given by equality

$$f_{\nu,\delta^i}(x)=(-1)^{\left|m^0+\nu\right|}C_0A_0(h)\int_{\left|E^{N^0}\right|}dz^0\times$$

$$\times\int_{E_n}\left\{\Delta^{N^0}\left(\tfrac{z^0}{N^0};\Omega_i+R_{\delta^i}\right)D^{m^0}f(x+y)\right\}M_{\delta^i0}dy+$$

$$+\sum_{k=1}^{n}(-1)^{\left|m^k+\nu\right|}C_k\int_0^hA_k(\upsilon)\frac{da_k(\upsilon)}{a_k(\upsilon)}\int dz^k\times$$

$$\int_{E_n}\left\{\Delta^{N^k}\left(\frac{z^k}{N^k};\Omega_i+R_{\delta^i}\right)D^{m^k}f(x+y)\right\}M_{\delta^i,k}dy=J_{0,\delta^i}(f)+\sum_{k=1}^{n}J_{0,\delta^i}(f)\qquad(i=1,n)\tag{22}$$

and proof of inequality

$$\|D^{\nu}f\|_{q,G} \leq C\sum_{i=1}^{M}\|D^{\nu}f\|_{q,\Omega_i+R_{\delta_i}} \leq C\sum_{i=1}^{M}\|f_{\nu,\delta_i}\|_{q_1 E_k} \leq C\sum_{i=1}^{M}\sum_{k=0}^{n}\left\|J_{k,\delta^i}\right\|_{q_1 E_n} \tag{23}$$

shows that estimates of integral expressions $\|D^{\nu}f\|_{q,G}$ reduce to estimates of integral operators J_{k,δ^i} $\left(k=\overline{0,n}\right)$ in Lebesgue space $L_q(G)$.
Then, using the Hölder inequality and Young inequality for convolution, we have (see, [2])

$$\|D^{\nu}f\|_{q,G} \leq C\sum_{i=1}^{M}\sum_{k=0}^{n}\left\|J_{k,\delta^i}(f)\right\|_{q_1 E_n} \leq C\sum_{k=0}^{n}Q_k(h)\left(\sum_{i=1}^{M}\|f\|\,\Lambda_{p_k,\theta_k}^{\langle m_k,N_k\rangle}\left(\Omega_i+R_{\delta_i},\varphi_k\right) \leq\right.$$

$$\leq C\sum_{k=0}^{n}Q_k(h)\,\|f\|_{\Lambda_{p_k,\theta_k}^{\langle m_k,N_k\rangle}(G;\varphi_k)}.$$

Here

$$Q_k(h)=\int_0^h\prod_{j=1}^{n}\left((a_j(\upsilon))^{m\frac{k}{j}-\nu_j-1}\right)\left\{\prod_{j\in E_{N^k}}\varphi_j^k(a_j(\upsilon))\right\}\frac{da_k(\upsilon)}{a_k(\upsilon)}.$$

This complete the proof of Theorem 3.1.

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