Caspian Journal of Applied Mathematics, Ecology and Economics
V. 9, No 2, 2021, December
ISSN 1560-4055

Basicity of Linear Phase Exponential System in Grand-Sobolev Spaces

Seadet A.Nurieva

Abstract. We define a separable ${MW_{p)}}^1(a,b)$ subspace in grand-Sobolev spaces. Then we show that this subspace is isomorphic to the direct sum of some subspace of grand-Lebesgue space and complex plane and so the system $1 \cup \left\{e^{i(n+\alpha signn)t}\right\}_{n\epsilon Z}$ forms a basis for the space ${MW_{p)}}^1(-\pi,\pi)$, where $\alpha \in C$ is a complex parameter.

Key Words and Phrases: basicity, grand-Lebesgue space, grand-Sobolev space.

2010 Mathematics Subject Classifications: 33B10, 46E30, 54D70

Lately in mathematics, there has been an upsurge of interest in non-standard spaces (see [17, 18, 19, 20, 21, 22]). The study of differential equations in non-standard Sobolev spaces requires the knowledge of basicity properties of trigonometric systems in corresponding non-standard function spaces. Basicity properties of some trigonometric systems in such spaces have been treated in [23, 24, 25, 26, 27, 28, 29].

$$\left\{e^{i(n+\alpha signn)t}\right\}_{n\epsilon Z}, \tag{1}$$

$$1 \cup \left\{e^{i(n+\alpha signn)t}\right\}_{n\neq 0}. \tag{2}$$

The study of basicity properties of the systems (1) and (2) in Lebesgue function space probably dates back to Paley-Wiener [6] and N. Levinson [7]. Riesz basicity of (1)-type systems was studied in L_2 by M.I.Kadets [8], and in L_p by A.M.Sedletski [9] and E.I.Moiseyev [10, 11]. This field was further developed by B.T. Bilalov [12, 13, 14, 15].

Grand-Lebesgue spaces $L^{p)}$ have been introduced in [17] in the study of Jacobian in an open set. These are the functional Banach spaces, and they have wide applications in the theory of partial differential equations, theory of interpolation, etc. The study of some problems of harmonic analysis in these spaces is of special interest.

As these spaces are not separable, basis and approximation-related problems remained unsolved in them. In [25], some $M^{p)}$ subspace was constructed, interesting from the point of view of the theory of differential equations. In [26, 27], basicity properties of the systems (1) and (2) have been studied in this subspace.

Grand-Sobolev spaces have been studied in many works, including [17]. In this work, we explore the basicity of one exponential system for a subspace $MW_{p)}{}^{1}(-\pi,\pi)$ of grand-Sobolev space.

So, let $1<p<\infty$. A space $L^{p)}(a,b)$ of measurable functions satisfying the condition

$$\|f\|_{p)}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{b-a}\int_a^b|f|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}<\infty \tag{3}$$

in the interval $(a,b)\subset$R is called a grand-Lebesgue space.

Denote by $\tilde{M}^{p)}(a,b)$ the set of all functions satisfying the condition $\left\|\hat{f}(\cdot+\delta)-\hat{f}(\delta)\right\|_{p)}\to 0$ as $\delta\to 0$ and belonging to $L^{p)}(a,b)$, where

$$\hat{f}(t)=\begin{cases} f(t)\,,t\in(a,b)\,,\\ 0,\ t\notin(a,b)\ .\end{cases}$$

It is clear that the set $\tilde{M}^{p)}(a,b)$ is a manifold in $L^{p)}(a,b)$. Denote by $M^{p)}(a,b)$ the closure of $\tilde{M}^{p)}(a,b)$ with respect to the norm (3).

Denote by $W_{p)}{}^{1}(a,b)$ the space of functions which belong to $L^{p)}(a,b)$ together with their derivatives equipped with the norm

$$\|f\|_{W_{p)}}=\|f\|_{p)}+\|f'\|_{p)}. \tag{4}$$

We will call this space a grand-Sobolev space:

$$W_{p)}{}^{1}(a,b)=\left\{f\setminus f,f'\in L^{p)}(a,b),\|f\|_{p)}+\|f'\|_{p)}<\infty\right\}.$$

It is easy to prove that this is a Banach space. As is known, $L^{p)}(a,b)$ is not separable. Therefore, $W_{p)}{}^{1}(a,b)$ is also not a separable space. Denote by $\tilde{M}W_{p)}{}^{1}(a,b)$ the set of all functions which satisfy the condition $\left\|\hat{f}'(\cdot+\delta)-\hat{f}'(\delta)\right\|_{p)}\to 0$ as $\delta\to 0$ and belong to $W_{p)}{}^{1}(a,b)$, where

$$\hat{f}(t)=\begin{cases} f(t)\,,t\in(a,b)\,,\\ 0,\ t\notin(a,b)\,.\end{cases}$$

It is clear that the set $\tilde{M}W_{p)}{}^{1}(a,b)$ is a manifold in $W_{p)}{}^{1}(a,b)$. Denote by $MW_{p)}{}^{1}(a,b)$ the closure of $\tilde{M}W_{p)}{}^{1}(a,b)$ with respect to the norm (4).

The following lemma is true.

Lemma 1. *The operator* $A(f,\lambda)=\lambda+\int_a^t f(\tau)\,d\tau$ *creates an isomorphism between the spaces* $M^{p)}(a,b)\oplus\mathbf{C}$ *and* $MW_{p)}{}^{1}(a,b)$*, where* $\mathbf{C}$ *is a complex plane,* $1<p<\infty$.

Proof. Let $f\in M^{p)}(a,b)$. Then

$$\left\|\lambda+\int_a^t f(\tau)\,d\tau\right\|_{W_{p)}}=\left\|\lambda+\int_a^t f(\tau)\,d\tau\right\|_{p)}+\|f\|_{p)}\le\|\lambda\|_{p)}+$$
$$+\left\|\int_a^t f(\tau)\,d\tau\right\|_{p)}+\|f\|_{p)}.$$

Obviously, $\|\lambda\|_{p)}\le K_1|\lambda|$, $\left\|\int_a^t f(\tau)\,d\tau\right\|_{p)}\le K_2\|f\|_{L^1}\le K_3\|f\|_{L^{p-\varepsilon}}\le K_4\|f\|_{p)}$, because $L^p\subset L^1$, $L^p\subset L^{p)}\subset L^{p-\varepsilon}$ (K_1,K_2,K_3,K_4 are constants). Thus, $\|A(f,\lambda)\|_{W_{p)}}\le K(|\lambda|+\|f\|_{p)})$, i.e. A is a bounded operator. For $v=\lambda+\int_a^t f(\tau)\,d\tau$ we have $v^{'}=f(t)$. Then $v\in MW_{p)}{}^1(a,b)$.

Let's show that $kerA=\{0\}$. Assume $A(u,\ \lambda)=0$, i.e. $\lambda+\int_a^t f(\tau)\,d\tau=0$. Differentiating both sides, we get $f(t)=0$ a.e. Consequently, $\lambda=0$. Let $\tilde v=\left(v^{'},v(a)\right)$ for $\forall v\in MW_{p)}{}^1(a,b)$. Then $\tilde v\in M^{p)}(a,b)\oplus\mathbf{C}$ and $A(\tilde v)=v$. This means $R_A=MW_{p)}{}^1(a,b)$, where R_A is a range of the operator A. By Banach inverse operator theorem, the inverse of the operator A exists and is continuous. The lemma is proved.

We will significantly use the following theorem.

Theorem 1. *([26]) Let $-2Re\alpha+\frac{1}{p}\notin Z$,$1<p<\infty$. Then the system* (1) *forms a basis for the space $M^{p)}(-\pi,\pi)$,$1<p<\infty$, if and only if $d=\left[-2Re\alpha+\frac{1}{p}\right]=0$ ($[\alpha]$ denotes the integer part of α). The defect of the system* (1) *is $d=\left[-2Re\alpha+\frac{1}{p}\right]$. When $d<0$, the system* (1) *is not complete, but minimal in $M^{p)}(-\pi,\pi)$. When $d>0$, the system* (1) *is complete , but not minimal in $M^{p)}(-\pi,\pi)$.*

So the following theorem is true.

Theorem 2. *Let $-2Re\alpha+\frac{1}{p}\notin Z$, $1<p<\infty$. Then the system*

$$1\cup\left\{e^{i(n+\alpha signn)t}\right\}_{n\epsilon Z} \tag{5}$$

forms a basis for the space $MW_{p)}{}^1(-\pi,\pi)$, $1<p<\infty$, if and only if $\left[-2Re\alpha+\frac{1}{p}\right]=0$.

Proof. Let $\left[-2Re\alpha+\frac{1}{p}\right]=0$. Let's first prove that the system $\hat u_{-1}=\binom{0}{1}$, $\hat u_0=\binom{i\alpha e^{i\alpha t}}{e^{-i\pi\alpha}}$, $\hat u_n^{\pm}=\binom{i(n+\alpha signn)e^{i(n+\alpha signn)t}}{e^{-i\pi(n+\alpha signn)}}$, $n=\pm1,\pm2,\ldots$, forms a basis for the space $M^{p)}(-\pi,\pi)\oplus\mathbf{C}$. To do so, it suffices to show that $\forall\hat u=\binom{u}{\lambda}\in M^{p)}(-\pi,\pi)\oplus\mathbf{C}$ the expansion

$$\hat u=\mathrm{c}_{-1}\hat u_{-1}+\mathrm{c}_0\hat u_0+\sum_{n\neq0}c_n^{\pm}\hat u_n{}^{\pm} \tag{6}$$

exists and is unique. This expansion is equivalent to two following expansions:

$$u\left(t\right)=\mathrm{c}_0 i\alpha e^{i\alpha t}+\sum_{n\neq 0} c_n^{\pm} i(n+\alpha signn)e^{i(n+\alpha signn)t}, \tag{7}$$

$$\lambda=-\pi \mathrm{c}_{-1}+\mathrm{c}_0 e^{-i\pi\alpha}+\sum_{n\neq 0} c_n^{\pm} e^{-i\pi(n+\alpha signn)}. \tag{8}$$

By Theorem 1 ([26]), the expansion (7) exists and is unique. As $\forall\varepsilon\in(0,p-1)$, $L^{p)}\subset L^{p-\varepsilon}$ and $\left[-2Re\alpha+\frac{1}{p}\right]=0$, by [16], Hausdorff-Young inequality is true for the system (1) in grand-Lebesgue space $L^{p)}$, too. That is, if $1<p\leq 2$, then

$$\left(|c_0|^q+\sum_{n\neq 0}\left|c_n^{\pm}n\right|^q\right)^{1/q}\leq M\|u\|_{p-\varepsilon}\leq M\|u\|_{p)},$$

where $p-\varepsilon$ and q are mutually conjugate numbers: $\frac{1}{p-\varepsilon}+\frac{1}{q}=1$.

Using Hölder's inequality, we obtain

$$|c_0|+\sum_{n\neq 0}\left|c_n^{\pm}\right|=|c_0|+\sum_{n\neq 0}\frac{1}{|n|}\left|c_n^{\pm}n\right|\leq|c_0|+\left(\sum_{n\neq 0}\frac{1}{|n|^p}\right)^{\frac{1}{p}}\left(\sum_{n\neq 0}\left|c_n^{\pm}n\right|^q\right)^{\frac{1}{q}}<\infty.$$

When $2<p$, we can find $\varepsilon>0$ such that $2<p-\ \varepsilon$. Therefore,

$$L^{p)}\subset L^{p-\varepsilon}\subset L^2.$$

Similarly we have

$$|c_0|+\sum_{n\neq 0}\left|c_n^{\pm}\right|=|c_0|+\sum_{n\neq 0}\frac{1}{|n|}\left|c_n^{\pm}n\right|\leq|c_0|+\left(\sum_{n\neq 0}\frac{1}{|n|^2}\right)^{\frac{1}{2}}\left(\sum_{n\neq 0}\left|c_n^{\pm}n\right|^2\right)^{\frac{1}{2}}<\infty.$$

So, the series $\sum_{n\neq 0}|c_n^{\pm}|$ is convergent. Therefore, the expansion (8) also exists and is unique. This implies the existence and uniqueness of the expansion (6), i.e. the system

$$\hat{u}_{-1}\cup\hat{u}_0\cup\left\{\hat{u}_n^{\pm}\right\}\ ,n=\pm1,\pm2,\ldots$$

forms a basis for the space $M^{p)}\left(-\pi,\pi\right)\oplus\mathbf{C}$. As the operator A is an isomorphism, the system

$$\{A\hat{u}_{-1}\}\cup\{A\hat{u}_0\}\cup\left\{A\hat{u}_n^{\pm}\right\},n=\pm1,\pm2,\ldots$$

must form a basis for the space $MW_{p)}{}^1\left(-\pi,\pi\right)$. Simple calculations show that

$$A\hat{u}_{-1}=1,\ \ A\hat{u}_0=e^{i\alpha t},$$
$$A\hat{u}_n^{\pm}=e^{i(n+\alpha signn)t},n=\pm1,\pm2,\ldots$$

That is, the system $1 \cup \left\{e^{i(n+\alpha sign n)t}\right\}_{n \epsilon Z}$ forms a basis for the space $MW_{p)}{}^{1}(-\pi,\pi)$.

Now let $\left[-2Re\alpha+\frac{1}{p}\right]>0$. For certainty, we assume $\left[-2Re\alpha+\frac{1}{p}\right]=1$, i.e. $1< -2Re\alpha+\frac{1}{p}<2$.

Let's rewrite the system (5) as $1 \cup \left\{e^{int}e^{i\alpha t}; e^{-ikt}e^{-i\alpha t}\right\}_{n\geq 0, k\geq 1}$ and multiply every term of it by $e^{-it/2}$. After making some transformations, we obtain:

$$e^{-it/2} \cup \left\{e^{int}e^{i(\alpha-\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha+\frac{1}{2})t}\right\}_{n\geq 0,k\geq 1} \equiv$$

$$\equiv e^{-it/2} \cup \left\{e^{it}e^{i(n-1)t}e^{i(\alpha-\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha+\frac{1}{2})t}\right\}_{n\geq 0,k\geq 1} \equiv$$

$$\equiv e^{-it/2} \cup \left\{e^{int}e^{i(\alpha+\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha+\frac{1}{2})t}\right\}_{n\geq -1,k\geq 1}.$$

Denoting $\alpha^{'}=\alpha+\frac{1}{2}$, we can rewrite the last system as

$$e^{-it/2} \cup \left\{e^{int}e^{i\alpha^{'}t}; e^{-ikt}e^{-i\alpha^{'}t}\right\}_{n\geq -1,k\geq 1}. \tag{9}$$

As $-2Re\alpha^{'}+\frac{1}{p}=-2Re\alpha+\frac{1}{p}-1$, we have $0<-2Re\alpha^{'}+\frac{1}{p}<1$. In this case, due to the fact we have proved above, the system

$$1 \cup \left\{e^{int}e^{i\alpha^{'}t}; e^{-ikt}e^{-i\alpha^{'}t}\right\}_{n\geq 0,k\geq 1}, \tag{10}$$

forms a basis for $MW_{p)}{}^{1}(-\pi,\pi)$. It is clear that if we remove $\{1\}$ from (10) and add the functions $e^{-it/2}$ and $e^{i(\alpha^{'}-1)t}$, we obtain the system (9). It is known from the theory of bases that in this case the system (8) cannot be a basis.

Note that the basicity properties of the systems (9) and (5) are absolutely identical. Because it is easy to verify that the operator of multiplying by $e^{-it/2}$ is an automorphism in $MW_{p)}{}^{1}(-\pi,\pi)$. So, in case $\left[-2Re\alpha+\frac{1}{p}\right]=1$ the system (5) does not form a basis for $MW_{p)}{}^{1}(-\pi,\pi)$. The case of $\left[-2Re\alpha+\frac{1}{p}\right]>1$ can be treated similarly.

Let $\left[-2Re\alpha+\frac{1}{p}\right]<0$. For certainty, assume $\left[-2Re\alpha+\frac{1}{p}\right]=-1$, i.e.$-1<-2Re\alpha+\frac{1}{p}<0$.

Let's rewrite the system (5) as $1 \cup \left\{e^{int}e^{i\alpha t}; e^{-ikt}e^{-i\alpha t}\right\}_{n\geq 0, k\geq 1}$ and multiply every term of it by $e^{it/2}$. Once again, after making some transformations, we obtain:

$$e^{it/2} \cup \left\{e^{int}e^{i(\alpha+\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha-\frac{1}{2})t}\right\}_{n\geq 0,k\geq 1} \equiv$$

$$\equiv e^{it/2} \cup \left\{e^{-it}e^{i(n+1)t}e^{i(\alpha+\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha-\frac{1}{2})t}\right\}_{n\geq 0,k\geq 1} \equiv$$

$$\equiv e^{it/2} \cup \left\{e^{int}e^{i(\alpha-\frac{1}{2})t}; e^{-ikt}e^{-i(\alpha-\frac{1}{2})t}\right\}_{n\geq 1,k\geq 1}.$$

Denoting $\alpha'' = \alpha - \frac{1}{2}$, we can rewrite the last system as

$$e^{it/2} \cup \left\{ e^{int} e^{i\alpha'' t}; e^{-ikt} e^{-i\alpha'' t} \right\}_{n\geq 1, k\geq 1} . \tag{11}$$

As $-2Re\alpha'' + \frac{1}{p} = -2Re\alpha + \frac{1}{p} + 1$, we have $0 < -2Re\alpha'' + \frac{1}{p} < 1$. In this case, due to the fact we have proved above, the system

$$1 \cup \left\{ e^{int} e^{i\alpha'' t}; e^{-ikt} e^{-i\alpha'' t} \right\}_{n\geq 0, k\geq 1} \tag{12}$$

forms a basis for $MW_{p)}{}^{1}(-\pi,\pi)$. It is clear that if we remove $\{1\}$ and $e^{i\alpha'' t}$ from (12) and add the function $e^{it/2}$, we obtain the system (11). It is known from the theory of bases that in this case the system (11) cannot be a basis.

Note that the basicity properties of the systems (11) and (5) are absolutely identical. Because it is easy to verify that the operator of multiplying by $e^{it/2}$ is an automorphism in $MW_{p)}{}^{1}(-\pi,\pi)$. So, in case $\left[-2Re\alpha + \frac{1}{p}\right] = -1$ the system (5) does not form a basis for $MW_{p)}{}^{1}(-\pi,\pi)$. The case of $\left[-2Re\alpha + \frac{1}{p}\right] < -1$ can be treated similarly. Thus, if the condition $\left[-2Re\alpha + \frac{1}{p}\right] = 0$ is not satisfied, then the system (5) cannot form a basis.

The theorem is proved.

Acknowledgement

The author would like to express her deepest gratitude to Associate Professor V.F. Salmanov from the Azerbaijan State Oil and Industry University for his guidance and valuable comments.

References

[1] S.M. Ponomarev, On an eigenvalue problem, Dokl. Akad. Nauk SSSR, 249:5 (1979), 1068–1070

[2] S.M. Ponomarev, On the theory of boundary value problems for equations of mixed type in three-dimensional domains, Dokl. Akad. Nauk SSSR, 246:6 (1979), 1303–1306

[3] E.I. Moiseev, Some boundary value problems for equations of mixed type, Differ. Uravn., 28:1 (1992), 110–121

[4] E.I. Moiseev, Solution of the Frankl' problem in a special domain, Differ. Uravn., 28:4 (1992), 721–723

[5] E.I. Moiseev, N.O. Taranov, Solution of a Gellerstedt problem for Lavrentiev Bitsadze equation, Differ. Uravn., 45(4) (2009), 543–548. (in Russian)

[6] R. Paley, N. Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloq. Publ., I D19"19"(Amer. Math. Soc., Providence, RI, 1934)

[7] N. Levinson, Gap and Density Theorems, Bull. Amer. Math. Soc. 47:7(1941) 543-547

[8] M.I. Kadets, The exact value of the Paley–Wiener constant, Dokl. Akad. Nauk SSSR, 155:6 (1964), 1253–1254

[9] A.M. Sedletskii, Biorthogonal expansions of functions in series of exponents on intervals of the real axis. Uspekhi Mat. Nauk, 37:5 (227) (1982), 51–95

[10] E.I. Moiseev, The basis property for systems of sines and cosines, Dokl. Akad. Nauk SSSR, 275:4 (1984), 794–798

[11] E.I. Moiseev, On the basis property of a system of sines. Differ. Uravn., 23:1 (1987), 177–179

[12] B.T. Bilalov, Basis Properties of Some Systems of Exponents in L_p, Dokl. RAS, 2003, 392:5 (2003)

[13] B.T. Bilalov, Basis Properties of Some Systems of Exponents, Cosines and Sines, Sibirskiy matem. jurnal, 45:2 (2004), 264-273

[14] B.T. Bilalov, A system of exponential functions with shift and the Kostyuchenko problem, Sibirsk. Mat. Zh., 50:2 (2009), 279–288

[15] B.T. Bilalov, On solution of the Kostyuchenko problem, Sibirsk. Mat. Zh., 53:3 (2012), 509–526

[16] M.H. Karakash, The Hausdorff-Young and Paley type for one system of sines, Proc. of IMM of NAS of Azerbaijan, 23:31 (2005), 59–64

[17] L. Donofrio, C. Sbordone, R. Schiattarella, Grand Sobolev spaces and their application in geometric function theory and PDEs. J. Fixed Point Theory Appl., 13 (2013), 309-340

[18] D.V. Cruz-Vrible, A. Fiorenza, Variable Lebesgue spaces, Springer-Verlag, Basel, 2013.

[19] R.E. Castilo, H. Rafeiro, An Introductory Course in Lebesgue Spaces. Cham, Switzerland: Springer International Publication, 2016

[20] V. Kokilashvili, A. Meshki, H. Rafeiro, S. Samko, Integral Operators in Non-Standart Function Spaces. Volume 1: Variable Exponent Lebesgue and Amalgam Spaces,Springer, 2016

[21] V. Kokilashvili, A. Meshki, H. Rafeiro, S. Samko, Integral Operators in Non-Standart Function Spaces. Volume 2: Variable Exponent Hölder, Morrey-Campanato and Grand Spaces, and 2016

[22] D.R. Adams, Morrey spaces, Switzherland, Springer, 2016

[23] B.T. Bilalov, Z.G. Guseynov, Basicity of a system of exponents with a piecewise linear phase in variable spaces. Mediterr. J. Math, 9:3 (2012), 487-498

[24] B.T. Bilalov, The basis property of perturbed system of exponentials in Morrey-type spaces, Siberian Math. J., 60:2 (2019), 249-271

[25] M.I. Ismailov, On the Solvability of Riemann Problems in Grand Hardy Classes, Mathematical Notes, 108 (2020), 523–537

[26] M.I. Ismailov, V.Q. Alili, On basicity of the system of exponents and trigonometric systems in grand-Lebesgue spaces. International scientific conference "Modern problems of mathematics and mechanics", dedicated to the 80th anniversary of Academician V. A. Sadovnichy, May 13-15, 2019, p. 193-194

[27] T. Hagverdi, On Stability of Bases Consisting of Perturbed Exponential Systems in Grand Lebesgue Spaces, Journal of Contemporary Applied Mathematics V. 11, No 2, 2021, December, pp. 81-92

[28] V.F. Salmanov, T.Z. Qarayev, On basicity of exponential systems in Sobolev-Morrey spaces, Scientific Annals of Al. I. Cursa university of Iasi, LXIV, f1 (2018), 47-52

[29] V.F. Salmanov, S.A. Nurieva, On basicity of trigonometric systems in Sobolev-Morrey spaces, Caspian Journal of Applied Mathematics, Ecology and Economics, 8:2 (2020), 10-16

Seadet A. Nurieva
Azerbaijan Tourism and Management University, Baku, Azerbaijan
E-mail: sada.nuriyeva@inbox.ru

Received 25 May 2021
Accepted 7 July 2021

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 9, No 2, 2021, December
ISSN 1560-4055

Completeness of the Perturbed Trigonometric System in Generalized Weighted Lebesgue Spaces

S.I. Jafarova

Abstract. A double exponential system with complex-valued complex coefficients is considered in generalized weighted Lebesgue spaces. Completeness of this system in $L_{p(\cdot);\rho}$ spaces is studied.

Key Words and Phrases: exponential system, basicity, variable exponent, generalized Lebesgue space

2010 Mathematics Subject Classifications: 30B60; 42C15; 46A35

1. Introduction

In the context of applications to some problems of mechanics and mathematical physics, since recently there arose great interest in the study of different problems in generalized Lebesgue spaces $L_{p(\cdot)}$ of variable summability rate $p(\cdot)$. Some fundamental results of classical harmonic analysis have been extended to the case of $L_{p(\cdot)}$ (for more details see [9-12]). Note that the use of Fourier method in solving some problems for partial differential equations in generalized Sobolev classes requires the study of approximative properties of perturbed exponential systems in generalized Lebesgue spaces. Some approximation problems in these spaces have been studied by I.I. Sharapudinov (see, e.g., [11]).

In this work, we consider the completeness of a double exponential system with complex-valued complex coefficients in the spaces $L_{p(\cdot);\rho}$. The completeness is reduced to trivial solvability of the corresponding homogeneous Riemann problem in the classes $H^{+}_{q(\cdot);\rho} \times_{-1} H^{-}_{q(\cdot);\rho}$, where $q(t)$ is a conjugate function of $p(t)$. Note that when considering the basicity of such systems in $L_{p(\cdot);\rho}$, unlike in the case of completeness, the solvability of corresponding Riemann problem is studied in the classes $H^{+}_{p(\cdot);\rho} \times_{-1} H^{-}_{p(\cdot);\rho}$. That's why we treat the completeness separately. The scheme we use is not new. We just follow the works [2;4].

2. Needful Information

Let $\omega \equiv \{z : |z| < 1\}$ be a unit ball in the complex plane and $\Gamma = \partial\omega$ be a unit circumference. Let $p : [-\pi, \pi] \to [1, +\infty)$ be some Lebesgue measurable function. The class of all Lebesgue measurable functions on $[-\pi, \pi]$ is denoted by L_0. Denote

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Let

$$\mathrm{L} \equiv \{f \in \mathrm{L}_0 : I_p(f) < +\infty\}.$$

For $p^+ = \sup\limits_{[-\pi,\pi]} vrai\, p(t) < +\infty$, L becomes a linear space with the usual linear operations of addition of functions and multiplication by a number. Equipped with the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

L becomes a Banach space which we denote by $L_{p(\cdot)}$. Let

$$WL \stackrel{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow \\ \Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\Big\}.$$

Throughout this work, $q(\cdot)$ denotes a conjugate function of $p(\cdot)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Denote$p^- = \inf\limits_{[-\pi,\pi]} vrai\, p(t)$.

The following generalized Hölder inequality is true:

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where

$$c\left(p^-; p^+\right) = 1 + \frac{1}{p^-} - \frac{1}{p^+}.$$

We will significantly use the following easy-to-prove:

Statement 1. *Suppose*

$$p \in WL,\ p(t) > 0, \forall t \in [-\pi, \pi]; \{\alpha_i\}_0^m \subset R.$$

The weight function

$$\rho(t) = |t|^{\alpha_0} \prod_{i=1}^{m} |t - \tau_i|^{\alpha_i} \tag{1}$$

belongs to the space $L_{p(\cdot)}$ *if the following inequalities are true:*

$$\alpha_i > -\frac{1}{p(\tau_i)}, \forall i = \overline{0, m};$$

where $-\pi = \tau_1 < \tau_2 < ... < \tau_m = \pi,\ \tau_0 = 0,\ \tau_i \ne 0, \forall i = \overline{1, m}.$

To obtain our main results, we will also use the following important fact:

Property B. *If $p(t): 1 < p^- \leq p^+ < +\infty$, then the class $C_0^\infty(-\pi, \pi)$ (class of finite, infinitely differentiable functions on $(-\pi, \pi)$) is everywhere dense in $L_{p(\cdot)}$.*

Define the weighted class $h_{p(\cdot),\rho}$ of functions which are harmonic inside the unit circle ω with the variable summability rate $p(\cdot)$, where the weight function $\rho(\cdot)$ is defined by (1).

Denote

$$h_{p(\cdot),\rho} \equiv \left\{ u : \Delta u = 0 \ in \ \omega \ and \ \|u\|_{p(\cdot),\rho} = \sup_{0<r<1} \left\| u\left(re^{it}\right) \right\|_{p(\cdot),\rho} < +\infty \right\}.$$

We will need the following

Lemma 1. *Let $p \in WL, p^- \geq 1$, and the weight $\rho(\cdot)$ satisfy the condition*

$$-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)}, \ k = \overline{0, m}. \tag{2}$$

If $f \in L_{p(\cdot),\rho}$, then $\exists p_0 \geq 1 : f \in L_{p_0}$.

The following lemma is also true:

Lemma 2. *Let $p \in WL, p^- \geq 1$, and the weight $\rho(\cdot)$ satisfy the condition* (2). *If $u \in h_{p(\cdot),\rho}$, then $\exists p_0 \in [1, +\infty] : u \in h_{p_0}$.*

Using these lemmas, one can prove the following theorem:

Theorem 1. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $u \in h_{p(\cdot),\rho}$, then $\exists f \in L_{p(\cdot),\rho}$:*

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(t) dt, \tag{3}$$

where

$P_r(\alpha) = \frac{1-r^2}{1+r^2-2r\cos\alpha}$ *is a Poisson kernel.*

On the contrary, if $f \in L_{p(\cdot),\rho}$, then the function u defined by (3) *belongs to the class $h_{p(\cdot),\rho}$.*

Similarly we define the weighted Hardy classes $H^\pm_{p(\cdot),\rho}$. By $H^+_{p_0}$ we denote the usual Hardy class, where $p_0 \in [1, +\infty)$ is some number. Let

$$H^\pm_{p(\cdot),\rho} \equiv \left\{ f \in H_1^+ : f^+ \in L_{p(\cdot),\rho}(\partial\omega) \right\},$$

where f^+ are nontangential boundary values of $f(\cdot)$ on $\partial\omega$.

It is absolutely clear that $f(\cdot)$ belongs to the space $H^+_{p(\cdot),\rho}$ only when Ref and Imf belong to the space $h_{p(\cdot),\rho}$. Therefore, many properties of the functions from $h_{p(\cdot),\rho}$ are transferred to the functions from $H^+_{p(\cdot),\rho}$. Taking into account the relationship between the Poisson kernel $P_r(\alpha)$ and the Cauchy kernel $K_z(t) = \frac{e^{it}}{e^{it}-z}$, it is easy to derive from Theorem 2.1 the validity of the following:

Theorem 2. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $F \in H^+_{p(\cdot),\rho}$, then $F^+ \in L_{p(\cdot),\rho}$:*

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{F^+(t)dt}{1 - ze^{-it}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_z(t)F^+(t)dt. \tag{4}$$

On the contrary, if $F^+ \in L_{p(\cdot),\rho}$, then the function F defined by (4) *belongs to the class $H^+_{p(\cdot),\rho}$, where $F^+(\cdot)$ are nontangential boundary values of $F(\cdot)$ on $\partial\omega$.*

Following the classics, we define the weighted Hardy class ${}_mH^-_{p(\cdot),\rho}$ of analytic functions on $C\backslash\bar{\omega}$ of order $k \leq m$ at infinity. Let $f(z)$ be an analytic function on $C\backslash\bar{\omega}$ of finite order $k \leq m$ at infinity, i.e.

$$f(z) = f_1(z) + f_2(z),$$

where $f_1(z)$ is a polynomial of degree $k \leq m$, $f_2(z)$ is the principal part of Laurent decomposition of the function $f(z)$ at infinity. If the function $\varphi(z) \equiv \overline{f_2\left(\frac{1}{\bar{z}}\right)}$ belongs to the class $H^+_{p(\cdot),\rho}$, then we will say that the function $f(z)$ belongs to the class ${}_mH^-_{p(\cdot),\rho}$.

Absolutely similar to the classical case, one can prove the following:

Theorem 3. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $f \in H^+_{p(\cdot),\rho}$, then*

$$\left\|f(re^{it}) - f^+(e^{it})\right\|_{p(\cdot),\rho} \to 0,\ r \to 1-0,$$

$$\left\|f\left(re^{it}\right)\right\|_{p(\cdot),\rho} \to \left\|f^+\left(e^{it}\right)\right\|_{p(\cdot),\rho},\ r \to 1-0,$$

where f^+ are nontangential boundary values of f on $\partial\omega$.

The similar fact is true also in ${}_mH^-_{p(\cdot),\rho}$ classes.

Theorem 4. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $f \in {}_mH^-_{p(\cdot),\rho}$, then*

$$\left\|f(re^{it}) - f^-(e^{it})\right\|_{p(\cdot),\rho} \to 0,\ r \to 1+0,$$

$$\left\|f\left(re^{it}\right)\right\|_{p(\cdot),\rho} \to \left\|f^-\left(e^{it}\right)\right\|_{p(\cdot),\rho},\ r \to 1+0,$$

where f^- are nontangential boundary values of $\theta(t) \equiv \arg G\left(e^{it}\right)$ on $\partial\omega$ from outside ω.

The following analog of the classical Smirnov theorem is valid:

Theorem 5. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. If $u \in H^+_1$ and $L_{p(\cdot),\rho}$, then $u \in H^+_{p(\cdot),\rho}$.*

Denote the restrictions of the classes $H^+_{p(\cdot),\rho}, {}_mH^-_{p(\cdot),\rho}$ to $\partial\omega$ by $L^+_{p(\cdot),\rho}$ and ${}_mL^-_{p(\cdot),\rho}$, respectively, i.e.

$$L^+_{p(\cdot),\rho} = H^+_{p(\cdot),\rho}/_{\partial\omega};\ {}_mL^-_{p(\cdot),\rho} = {}_mH^-_{p(\cdot),\rho}/_{\partial\omega}.$$

We will need the following result:

Theorem 6. *Let $p \in WL, p^- > 1$, and the inequalities* (2) *be fulfilled. Then the system $E^{(0)}_+ = \left\{e^{int}\right\}_{n\geq 0}$($E^{(m)}_- = \left\{e^{-int}\right\}_{n\geq m}$) forms a basis for $L^+_{p(\cdot),\rho}({}_mL^-_{p(\cdot),\rho})$, $1 < p < +\infty$.*

We will also need the following easy-to-prove lemma, which is derived immediately from the definition of weighted space $L_{p(\cdot),\rho}$.

Lemma 3. *Let*
$p \in C[-\pi,\pi]$ *and* $p(t) > 0$, $\forall t \in [-\pi,\pi]$.
Then the function $\xi(t) = |t-c|^{\alpha}$ *belongs to* $L_{p(\cdot),\rho}$, *if*
$\alpha > -\frac{1}{p(c)}$, *for* $c \neq \tau_k$, $\forall k = \overline{1,m}$,
and
$\alpha + \alpha_{k_0} > -\frac{1}{p(c)}$, *for* $c = \tau_{k_0}$.

3. Main Assumptions and Riemann Problem Statement

Let's state the Riemann problem in the classes $H^{\pm}_{p(\cdot);\rho}$. Let the complex-valued function $G(t)$ on $[-\pi,\pi]$ satisfy the following conditions:
i) Function $|G(t)|$ *belongs to the space* $L_{r(\cdot)}$ *for some* $r : 0 < r^- \le r^+ < +\infty$, *and* $|G(t)|^{-1} \in L_{\omega(\cdot)}$ *for* $\omega : 0 < \omega^- \le \omega^+ < +\infty$.
ii) Argument $\theta(t) \equiv \arg G(t)$ *has a following decomposition:*

$$\theta(t) = \theta_0(t) + \theta_1(t),$$

where $\theta_0(t)$ *is a continuous function on* $[-\pi,\pi]$ *and* $\theta_1(t)$ *is a function of bounded variation on* $[-\pi,\pi]$.

It is required to find a piecewise analytic function $F^{\pm}(z)$ on the complex plane with a cut $\partial\omega$ which satisfies the following conditions:

a) $F^+(z) \in H^+_{p(\cdot)} : 0 < p^- \le p^+ < +\infty$;

b) $F^-(z) \in {}_mH^-_{\nu(\cdot)}; 0 < \nu^- \le \nu^+ < +\infty$;

c) nontangential boundary values on the unit circumference $\partial\omega$ *satisfy the relation*
$F^+(e^{it}) - G(t)F^-(e^{it}) = g(t)$, *for a.e.* $t \in (-\pi,\pi)$,
where $g \in L_{\rho(\cdot)} : 0 < \rho^- \le \rho^+ < +\infty$ *is some given function.*

Note that in the case of constant summability rate, the theory of such problems has been well studied (see [3]).

Consider the following homogeneous Riemann problem in the classes $H^+_{p(\cdot),\rho} \times {}_mH^-_{p(\cdot),\rho}$:

$$F^+(z) - G(z)F^-(z) = 0,\ z \in \partial\omega. \tag{5}$$

By the solution of the problem (5) we mean a pair of analytic functions

$$(F^+(z); F^-(z)) \in H^+_{p(\cdot),\rho} \times {}_mH^-_{p(\cdot),\rho},$$

whose boundary values satisfy a.e. the equation (5). Introduce the following functions $X_i(z)$ analytic inside (with the sign "+") and outside (with the sign "-") the unit circle:

$$X_1(z) \equiv \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G\left(e^{it}\right)\right| \frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi}\theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

where $\theta(t) \equiv \arg G\left(e^{it}\right)$. Define

$$Z_i(z) \equiv \begin{cases} X_i(z), & |z|<1, \\ [X_i(z)]^{-1}, & |z|>1, \quad i=1,2 \ \ and \ \ Z(z_i) = Z_1(z)\times Z_2(z) \end{cases}$$

Let $\{s_k\}_1^r : -\pi < s_1 < ...s_r < \pi$ be points of discontinuity of the function $\theta(t)$ and

$$\{h_k\}_1^r : \ h_k = \theta(s_k+0) - \theta(s_k-0), k=\overline{1,r},$$

be the corresponding jumps of this function at these points. Denote

$$h_0 = \theta(-\pi) - \theta(\pi); h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi).$$

Let

$$u_0(t) \equiv \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta_0(\tau)\, ctg\frac{t-\tau}{2}d\tau\right\}$$

and

$u(t) = \prod_{k=0}^r \left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{\frac{h_k}{2\pi}}$, where $s_0 = \pi$.

As is known, (see [3]), the boundary values $\left|Z_2^-(\tau)\right|$ are defined by the formula

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\,[u(t)]^{-1}\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}},$$

i.e.

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\prod_{k=0}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{-\frac{h_k}{2\pi}}.$$

It follows directly from Sokhotskii-Plemelj formula that

$$\sup_{(-\pi,\pi)} vrai \left\{\left|Z_1^-\left(e^{it}\right)\right|^{\pm 1}\right\} < +\infty.$$

Thus, for $\left|Z^-\left(e^{it}\right)\right|^{-1}$ we have the representation

$$\left|Z^-\left(e^{it}\right)\right|^{-1} = \left|Z_1^-\left(e^{it}\right)\right|^{-1}\left|u_0(t)\right|^{-1}\prod_{k=0}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{\frac{h_k}{2\pi}}. \quad (6)$$

Represent the product $\left|Z^-\rho\right|^{-1}$ as follows:

$$\left|Z^-\rho\right|^{-1} \approx \left|Z_1^-\right|^{-1}\left|u_0\right|^{-1}\prod_{k=0}^{l}\left|t-t_k\right|^{\beta_k},$$

where

$\{t_k\}_{k=0}^{l} \equiv \{\tau_k\}_{k=1}^{m} \bigcup \{s_k\}_{k=0}^{r}$, and β_k's are defined by

$$\beta_k = -\sum_{i=1}^{m} \alpha_i \chi_{\{t_k\}}(\tau_i) + \frac{1}{2\pi} \sum_{i=0}^{r} h_i \chi_{\{t_k\}}(s_i), \ \ k = \overline{0,l}\,. \tag{7}$$

By virtue of Lemma 2.3, we obtain that if the inequalities

$$\beta_k > -\frac{1}{q(t_k)}\ , \ \ k = \overline{0,r}, \tag{8}$$

are true, then the product $|Z^- \rho|^{-1}$ belongs to the space $L_{q(\cdot)}$, i.e. $|Z^-|^{-1} \in L_{q(\cdot),\rho^{-1}}$. So, if the inequalities (8) are true, then the function $\Phi(z) = \frac{F(z)}{Z(z)}$ belongs to the classes $H_1^{\pm}$. Then, according to [3], $\Phi(z)$ is a polynomial $P_{m_0}(z)$ of degree $m_0 \leq m$. Thus,

$$F^-(z) = P_{m_0}(z) Z^-(z).$$

Let's find out under which conditions the function $F^-(z)$ belongs to the space $H^-_{p(\cdot),\rho}$. We have

$$\left|Z^- \rho\right| \approx |Z_1|\,|u_0| \prod_{k=0}^{l} |t - t_k|^{-\beta_k}\,.$$

Consequently, if the inequalities

$$\beta_k < \frac{1}{p(t_k)}\ , \ k = \overline{0,r},$$

are true, then it is clear that $F^-(\tau) \in L_{p(\cdot),\rho}$, and hence $F^- \in {}_mH^-_{p(\cdot),\rho}$. So, if the inequalities

$$-\frac{1}{q(t_k)} < \beta_k < \frac{1}{p\,(t_k)}, \ \ k = \overline{0,r}, \tag{9}$$

are true, then the general solution of homogeneous problem

$$F_0^+(\tau) = G_1(\tau)\, F_0^-(\tau)\,, \quad \ \tau \in \partial\omega,$$

in the classes $H^+_{p(\cdot),\,\rho} \times_m H^-_{p(\cdot),\,\rho}$ can be represented as follows:

$$F_0(z) = P_{m_0}(z)\, Z(z)\,,$$

where $P_{m_0}(z)$ is an arbitrary polynomial of degree $m_0 \leq m$.So the following theorem is valid:

Theorem 7. *Let the* $\{\beta_k\}_1^r$*'s be defined by* (7) *and the inequalities* (9) *be true. If*

$$-\frac{1}{p\,(\tau_k)} < \alpha_k < \frac{1}{q\,(\tau_k)}, \quad \ k = \overline{1,m},$$

then the general solution of the homogeneous Riemann problem (5) *in the classes* $H^{+}_{p(\cdot),\,\rho}\times_m H^{-}_{p(\cdot),\,\rho}$ *can be represented as*

$$F(z)=P_{m_0}(z)\,Z(z)\,,$$

where $Z(\cdot)$ *is a canonical solution of homogeneous problem, and* $P_{m_0}(\cdot)$ *is a polynomial of degree* $m_0\le m$.

This theorem has the following direct:

Corollary 1. *Let all the conditions of Theorem 3.1 be satisfied. Then the homogeneous Riemann problem* (5) *is trivially solvable in the Hardy classes* $H^{+}_{p(\cdot),\rho}\times{}_{-1}H^{-}_{p(\cdot),\rho}$.

4. Reducing The Completeness of Exponential System with Complex Coefficients to Boundary Value Problems

Consider the following exponential system:

$$\left\{A(t)\,e^{int};\,B(t)\,e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{10}$$

where $A(t)\equiv|A(t)|\,e^{i\alpha(t)}$; $B(t)\equiv|B(t)|\,e^{i\beta(t)}$ are complex-valued functions on $[-\pi,\pi]$. We will consider the completeness of the system (10) in the space $L_{p(\cdot);\rho}$. It is known [6] that the conjugate space of $L_{p(\cdot);\rho}$ is isometrically isomorphic to the space $L_{q(\cdot);\rho}$: $\frac{1}{p(t)}+\frac{1}{q(t)}\equiv 1$. Therefore, the completeness of the system (10) in $L_{p(\cdot);\rho}$ is equivalent to the equality to zero of any function $f(t)$ from the space $L_{q(\cdot);\rho}$ which satisfies the relations

$$\int_{-\pi}^{\pi}A(t)\,e^{int}\overline{f(t)}dt=0;\qquad\int_{-\pi}^{\pi}B(t)\,e^{-i(n+1)t}\overline{f(t)}dt=0,\,\forall n\in Z_+. \tag{11}$$

Assume that the following main condition is satisfied:

$$ess\sup_{[-\pi,\pi]}\left\{\,|A(t)|^{\pm1};\,|B(t)|^{\pm1}\right\}<+\infty. \tag{12}$$

From the first of equalities (11) we have

$$\int_{-\pi}^{\pi}A(t)\,e^{int}\overline{f}(t)\,dt=\frac{1}{i}\int_{\partial\omega}f^{+}(\tau)\,\tau^{n}d\tau=0,\forall n\in Z_+, \tag{13}$$

where $f^{+}(\tau)\equiv A(\arg\tau)\;\overline{f}(\arg\tau)\;\bar{\tau}\;,\;\;\tau\in\partial\omega$.

It is absolutely clear that $f^{+}(\tau)\in L_1(\partial\omega)$. Then it is well known (see [5], p.205) that the conditions (13) are equivalent to the existence of a function $F^{+}(z)$ from H_1^{+} whose nontangential boundary values on $\partial\omega$ coincide with $f^{+}(\tau): F^{+}(\tau)=f^{+}(\tau)$ a.e. on $\partial\omega$.

Similarly, from the second of equalities (11) we have

$$\int_{-\pi}^{\pi}\overline{B(t)}e^{i(n+1)t}f(t)\,dt=\frac{1}{i}\int_{\partial\omega}f^{-}(\tau)\,\tau^{n}d\tau=0,\forall n\in Z_+, \tag{14}$$

where $f^-(\tau) = \overline{B(\arg\tau)}\, f(\arg\tau)$, $\tau \in \partial\omega$. For the reason stated above, the equalities (14) are equivalent to the existence of a function $\Phi^+(z) \in H_1^+$ whose nontangential boundary values $\Phi^+(\tau)$ on $\partial\omega$ coincide with $f^-(\tau) : \Phi^+(\tau) = f^-(\tau)$ a.e. on$\partial\omega$.

It is absolutely clear that $F^+(\tau)\, ; \Phi^+(\tau) \in L_{q(\cdot);\rho}(\partial\omega)$. Consequently, if we additionally require that $p(t) \in WL$, then from theorem in [7] we obtain the inclusion $F^+(z)\, ; \Phi^+(z) \in H^+_{q(\cdot);\rho}$. Representing $f(t)$ in terms of $F^+(\tau)$ and $\Phi^+(\tau)$, we obtain the following conjugation problem:

$$F^+(\tau) - \frac{A(\arg\tau)}{B(\arg\tau)}\overline{\tau\,\Phi^+(\tau)} = 0, \tau \in \partial\omega.$$

Define the function $F^-(z)$ analytic outside the unit circle:

$$F^-(z) = \frac{1}{z}\,\overline{\Phi^+\left(\frac{1}{\bar{z}}\right)}, |z| > 1.$$

It is absolutely clear that $F^-(\infty) = 0$. Moreover, $F^-(\tau) = \bar{\tau}\,\overline{\Phi^+(\tau)}$, $\tau \in \partial\omega$. Then we arrive at the following Riemann problem:

$$\begin{cases} F^+(\tau) - G(\tau)\, F^-(\tau) = 0\ , & \tau \in \partial\omega, \\ F^-(\infty) = 0\,, \end{cases} \tag{15}$$

where

$$G(\tau) \equiv \frac{A(\arg\tau)}{B(\arg\tau)}, \tau \in \partial\omega.$$

By definition, we have $F^-(z) \in_{-1} H^-_{q(\cdot);\rho}$. Consequently, if the system (10) is incomplete in $L_{p(\cdot);\rho}$, then the Riemann problem (15) is non-trivially solvable in the classes $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$.

Now let's assume that the problem (15) is non-trivially solvable in the classes $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$, i.e. $F^+(z) \in H^+_{q(\cdot);\rho}$, $F^-(z) \in_{-1} H^-_{q(\cdot);\rho}$. Define

$\Phi_1^+(z) \equiv \overline{F^-\left(\frac{1}{\bar{z}}\right)}$ for $|z| < 1$.

We have $F^-(\tau) = \overline{\Phi_1^+(\tau)}$, $\tau \in \partial\omega$ and $\Phi^+(0) = 0$. Then it is clear that the function $\Phi^+(z) = z^{-1}\Phi_1^+(z)$ will be analytic when $|z| < 1$, and moreover, $\Phi^+(z) \in H^+_{q(\cdot);\rho}$. Thus,

$$F^+(\tau) - G(\tau)\ \overline{\tau\,\Phi^+(\tau)} = 0\ , \quad \tau \in \partial\omega,$$

or

$$\frac{F^+(\tau)}{A(\arg\tau)\,\bar{\tau}} = \frac{\overline{\Phi^+(\tau)}}{B(\arg\tau)}, \tau \in \omega.$$

Denote $f(t) = \frac{\overline{F^+(e^{it})}}{\overline{A(t)e^{it}}} = \frac{\Phi^+(e^{it})}{B(t)}$.

It is absolutely clear that $f(t) \in L_{q(\cdot);\rho}$. From $F^+(z)$, $\Phi^+(z) \in H_1^+$ we obtain the equalities

$$\int_{\partial\omega} F^+(\tau)\,\tau^n d\tau = 0; \int_{\partial\omega} \Phi^+(\tau)\,\tau^n d\tau = 0, \forall n \in Z_+.$$

Expressing $F^+(\tau)$ and $\Phi^+(\tau)$ in terms of $f(\arg\tau)$ as $\tau \in \partial\omega$, we have

$$\int_{\partial\omega} A(t)\,e^{-it}\overline{f(t)}e^{int}de^{it} = i\int_{-\pi}^{\pi} A(t)\,e^{int}\overline{f}(t)\,dt = 0, \forall n \in Z_+;$$

$$\int_{\partial\omega} \overline{B(t)}f(t)\,e^{int}de^{it} = i\int_{-\pi}^{\pi} \overline{B(t)}e^{i(n+1)t}f(t)\,dt = 0, \forall n \in Z_+.$$

Obviously, $f(t) \neq 0$ on $[-\pi,\pi]$. Then these relations imply that the system (10) is incomplete in $L_{p(\cdot);\rho}$. So we have the following:

Theorem 8. *Let $p:\ 1 < p^- \leq p^+ < +\infty$, $p(t) \in WL$, and complex-valued coefficients $A(t)$; $B(t)$ satisfy the condition* (12). *Then the exponential system* (10) *is complete in $L_{p(\cdot);\rho}$ only if the Riemann problem* (15) *is only trivially solvable in the classes* $\left(H^+_{q(\cdot);\rho};\ {}_{-1}H^-_{q(\cdot);\rho}\right)$.

5. Completeness of Exponential System with Complex Coefficients in $L_{p(\cdot);\rho}$

In this section, we apply the results of previous sections to obtain the sufficient conditions for the completeness of exponential system with complex coefficients in $L_{p(\cdot);\rho}$. So let's consider the system

$$\left\{A(t)\,e^{int};\ B(t)\,e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{16}$$

where $A(t) \equiv |A(t)|\,e^{i\alpha(t)}$; $B(t) \equiv |B(t)|\,e^{i\beta(t)}$ are complex-valued functions on $[-\pi,\pi]$. Assume that the following conditions are satisfied:

$$1)\sup_{[-\pi,\pi]} vrai \left\{\left(|A|^{\pm 1};\ |B|^{\pm 1}\right)\right\} < +\infty,$$

2) The function $\theta(t) \equiv \alpha(t) - \beta(t)$ is piecewise continuous on $[-\pi,\pi]$ with points of discontinuity $\{s_i\}_1^r:\ -\pi < s_1 < ... < s_r < \pi$. Let $\{h_k\}_1^r = \theta(s_k+0) - \theta(s_k-0)$, $k = \overline{1,r}$, be the jumps of the function $\theta(t)$ at these points and $h_0 = \theta(-\pi) - \theta(\pi)$.

3) $\frac{h_k}{2\pi} + \frac{1}{p(s_k)} \notin Z$ (Z is a set of all integers), where$h_k$is a jump of the function $\theta(t) \equiv \alpha(t) - \beta(t)$ at the discontinuity point s_k, $k = \overline{0,r}$; $s_0 = \pi$.

Define the integers n_i, $i = \overline{1,r}$, from the following inequalities:

$$\begin{cases} -\frac{1}{p(s_k)} < \frac{h_k}{2\pi} + n_k - n_{k-1} < \frac{1}{q(s_k)}\,,\ k = \overline{1,r}\,, \\ n_0 = 0\,. \end{cases} \tag{17}$$

Let

$$\Delta_r = \frac{1}{2\pi}\left[\alpha\left(-\pi\right) - \alpha\left(\pi\right) + \beta\left(\pi\right) - \beta\left(-\pi\right)\right] - n_r.$$

The following theorem is true:

Theorem 9. *Let the coefficients $A(t)$ and $B(t)$ of the system (16) satisfy the conditions 1)-3), where $G\left(e^{it}\right) \equiv \frac{A(t)}{B(t)}$, the integer n_r is defined by (17), $p(t) \in WL$, $1 < p^- \le p^+ < +\infty$. Then, if $\Delta_r \notin Z$ and $\Delta_r > -\frac{1}{p(\pi)}$, then the system (5.1) is complete in the space $L_{p(\cdot);\rho}$.*

Proof. Let the conditions 1)-3) be satisfied. We first assume that the inequalities (17) hold for $n_k = 0\,, k = \overline{1,r}$. Let $G(t) \equiv \frac{A(t)}{B(t)}$, $t \in [-\pi, \pi]$. As established above, the completeness of the system (16) in $L_{p(\cdot);\rho}$ is only equivalent to the trivial solvability of the Riemann problem

$$\begin{cases} F^+\left(e^{it}\right) - G\left(t\right) F^-\left(e^{it}\right) = 0\ , \ \ t \in \left[-\pi, \pi\right], \\ \\ F^+ \in H^+_{q(\cdot);\rho}\,;\ F^- \in_{-1} H^{-1}_{q(\cdot);\rho}\,. \end{cases} \tag{18}$$

It follows from (17) that $n_r = 0$, and hence $h_0 = 2\pi\Delta_r$. Suppose $-\frac{1}{p(\pi)} < \Delta_r < \frac{1}{q(\pi)}$. Then, by Corollary 3.1, the problem (18) has only the trivial solution. Consequently, by Theorem 4.1, the system (10) is complete in $L_{p(\cdot);\rho}$ in the considered case. Now let $\Delta_r \notin Z$ and $\Delta_r > \frac{1}{q(\pi)}$, for example, $\Delta_r \in \left[\frac{1}{q(\pi)}, \frac{1}{q(\pi)} + 1\right)$. Consider the system

$$\left\{A\left(t\right)e^{int};\, B\left(t\right)e^{-int}\right\}_{n\in N}. \tag{19}$$

Reduce it to the following form:

$$\left\{\tilde{A}\left(t\right)e^{int};\, B\left(t\right)e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{20}$$

where $\tilde{A}(t) \equiv A(t)\,e^{it} \equiv |A(t)|\,e^{i\tilde{\alpha}(t)}$ and $\tilde{\alpha}(t) \equiv t+\alpha(t)$. Calculate $\tilde{\Delta}_r$, which corresponds to the system (20). We have

$$\tilde{\Delta}_r = \frac{1}{2\pi}\left[\tilde{\alpha}\left(-\pi\right) - \tilde{\alpha}\left(\pi\right) + \beta\left(\pi\right) - \beta\left(-\pi\right)\right] = \Delta_r - 1.$$

Thus, $\tilde{\Delta}_r \in \left(-\frac{1}{p(\pi)}, \frac{1}{q(\pi)}\right)$. Then it follows from the previous discussion that the system (20), and hence the system (16), is complete in $L_{p(\cdot);\rho}$. Continuing this process, we obtain the completeness of the system (16) in $L_{p(\cdot);\rho}$ for $\Delta_r > -\frac{1}{p(\pi)}$.

Now let's consider the general case. Let all the conditions of the theorem be satisfied. Express the unit function $e(t)$ on $[-\pi, \pi]$ in the form $e(t) \equiv e^{i\arg e(t)}$:

$$\arg\, e\left(t\right) \equiv \begin{cases} 0, & -\pi < t \le s_1\,, \\ 2\pi n\,, & s_1 < t \le s_2\,, \\ \quad\vdots & \\ 2\pi n_r\,, & s_r < t \le \pi\,, \end{cases}$$

where n_k , $k=\overline{1,r}$ are the integers defined by (17). Replace the coefficient $A(t)$ with the function $A_0(t)$ which is equal to it: $A_0(t)=A(t)\cdot e(t)$. So, $\alpha_0(t)\equiv \arg A_0(t)=\alpha(t)+\arg e(t)$. Consider the system

$$\left\{A_0(t)\,e^{int};\ B(t)\,e^{-i(n+1)t}\right\}_{n\in Z_+}. \tag{21}$$

It is absolutely clear that the basis properties of the systems (16) and (21) in $L_{p(\cdot);\rho}$ are the same. We have

$$\theta_0(t)=\arg A_0(t)-\arg B(t)=\alpha(t)+\arg e(t)-\beta(t)=\theta(t)+\arg e(t).$$

It is clear that the points of discontinuity of the functions $\theta_0(t)$ and $\theta(t)$ are the same. We have

$$h_k^0=\theta_0(s_k+0)-\theta_0(s_k-0)=h_k+\arg e\,(s_k+0)-\arg e(s_k-0)=$$

$$=h_k+2\pi(n_k-n_{k-1}),\ k=\overline{1,r}\ ,$$

where $n_0=0$. Thus

$$\frac{h_0^0}{2\pi}=\frac{h_k}{2\pi}+n_k-n_{k-1}\,,\ k=\overline{1,r}.$$

On the other hand

$$h_0^0=\theta_0(-\pi)-\theta_0(\pi)=\theta(-\pi)-\theta(\pi)+$$

$$+\arg e(-\pi)-\arg e(\pi)=h_0-2\pi n_r\Rightarrow \tfrac{h_0^0}{2\pi}=\Delta_r.$$

Then it follows from the previous discussion that for $\Delta_r>-\frac{1}{p(\pi)}$ the system (21), and hence the system (16), is complete in $L_{p(\cdot);\rho}$.

Theorem is proved.

Let's apply the obtained theorem to the special case

$$\left\{e^{i[n+\alpha\, sign\, n]t}\right\}_{n\in Z}, \tag{22}$$

where $\alpha\in C$ is a complex parameter. Basis properties of the system (22) in the spaces $L_{p(\cdot);\rho}$ have been well studied. From Theorem 5.1 we have

Corollary 5.1. *Let* $p(t)\in WL$, $1<p^-\le p^+<+\infty$ *and* $Re\alpha\in Z$. *If* $Re\alpha<\frac{1}{2p(\pi)}$, *then the system* (22) *is complete in* $L_{p(\cdot);\rho}$.

References

[1] Bilalov B.T. Basicity properties of exponential systems in L_p. Sib.Mat.Journal, 2006, v. 47, №1 (*in Russian*).

[2] Veliev S.G. Boundary value problems and exponential bases, Baku, “ELM”, 2006, 282 p. (*in Russian*).

[3] Daniliuk I.I. Irregular boundary value problems in the plane, M., ”Nauka”, 1975, 256 p. (*in Russian*).

[4] Javadov M.G. On completeness of some part of eigenfunctions of non-self-adjoint differential operator. DAN SSSR, 1964, v.159, №4 (*in Russian*).

[5] Privalov I.I. Boundary properties of analytic functions. M.-L., Gostexizdat, 1950, 205 p. (*in Russian*).

[6] Sharapudinov I.I. On topology of the space $L^{p(x)}([0,1])$. Matem. zametki, 26:4 (1979), pp.613–632 (*in Russian*).

[7] Kokilashvili V., Paatashvili V., Samko S. Boundary value problems for analytic functions in the class of Caushy type integrals with density in $L^{p(\cdot)}()$. Bound. Value Probl., 2005, №1-2, pp. 43-71.

[8] Kovacik O., Rakosnik J. On spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. Czechoclovak Math. I., 1991, 41(116), pp. 592-618.

[9] Kokilashvili V., Paatashvili V. On Hardy classes of analytic functions with a variable exponent. Proc. Razmadze Math.Inc., 2006, 142, pp.134-137.

[10] Sharapudinov I.I. Some problems of approximation theory in spaces $L^{p(x)}$(E). Anal.Math., 33:2 (2007), 135-153.

[11] Kokilashvili V., Samko S. Singular integrals in weighted Lebesgue spaces with variable exponent. Georgian Math. J., 2003, v.10, №1, pp. 145-156.

S.I. Jafarova
Institute of Mathematics and Mechanics of NASA, Baku, Azerbaijan
E-mail:

Received 03 June 2021
Accepted 01 September 2021

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 9, No 2, 2021, December
ISSN 1560-4055

Conditions for the boundedness of the G-fractional integral and G-maximal function in modified G-Morrey spaces

G.A.Dadashova

Abstract. In this paper we find conditions for the strong and weak boundedness of the G-fractional integral and G-maximal operator.

Key Words and Phrases: G-fractional integral, G-maximal function, modified G-Morrey spaces, Hardy-Littlewood-Sobolev inequality.

2010 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

The study of boundedness of the fractional integral operator, singular integrals, maximal function were studied by lots of researchers in the last decodes. Morrey estimates of such kind of operators is a more recent problem and is still very popular. Just as an example we recall the study made in [1,3,7,8,10,11].

In this paper we introduce modified Gegenbauer Morrey space (G- Morrey space) and prove Adams type theorem on the boundedness of the G- fractional interal. The result obtained is an analog of the corresponding theorem obtained for Riesz potential in [4].

Let $1 \le p < \infty$ and $0 \le \lambda \le n$. The classical Morrey spaces is defined by

$$M_{p,\lambda}(R^n) = \left\{ f \in L^p_{loc}(R^n) : \|f\|_{L^{p,\lambda}} < \infty \right\}, \tag{1}$$

where

$$\|f\|_{L^{p,\lambda}} := \sup_{\theta} \left(\frac{1}{|\theta|^{\lambda/n}} \int_{\theta} |f(x)|^p \, dx \right)^{1/p},$$

the supremum, is taken over all cubes $Q \subset R^n$. It is well known that if $1 \le p < \infty$ then $M_{p,0}(R^n) = L^p(R^n)$ and $M_{p,n}(R^n) = L^\infty(R^n)$.

Morrey spaces were originally introduced by Morrey in [15] to study the local behavior of solutions to second-order

Morrey spaces were originally introduced by Morrey in [15] to study the local behavior of solutions to secon-order elliptic partial differential equations.

In [1], Adams for the Riesz potential

$$J_\alpha f(x) = \int_{R^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}},\ 0<\alpha<n$$

on the Morrey space proved the following theorem.

Theorem A. [1] Let $0<\alpha<n$ and let $0\le\lambda\le n$, $1\le p<(n-\lambda)/\alpha$.

1. If $1<p<(n-\lambda)/\alpha$ then the condition $1/p-1/q=\alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of J_α from $M_{p,\lambda}(R^n)$ to $M_{q,\lambda}(R^n)$.

2. If $p=1$, then the condition $1-1/q=\alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of J_α from $M_{1,\lambda}(R^n)$ to $M_{q,\lambda}(R^n)$.

In the work [12] is proved analog of this theorem for the Gegenbauer fractional integral on G- Morrey space.

The structure of the paper is as follows.

Section 1 is for informational purposes. In Section 2 are given some definition, notation and auxiliary results. In Section 4 is proved the theorem of strong and weak boundedness for maximal operator and also the Hardy-Littlewood-Sobolev type inequality for the G-fractional integral in modified G-Morrey spaces.

2. Definition, notation and auxiliary results

The generalized shift operator associated with the Gegenbauer differential operator G

$$G=G_\lambda=\left(x^2-1\right)^{\frac{1}{2}-\lambda}\frac{d}{dx}\left(x^2-1\right)^{\lambda+\frac{1}{2}}\frac{d}{dx},x\in(1,\infty),\ \lambda\in\left(0,\ \frac{1}{2}\right),$$

introduced in [7] has the form

$$A^\lambda_{ch\,y}f(ch\,x)=\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)}\int_0^\pi f(ch\,xch\,y-sh\,xsh\,y\cos\varphi)(\sin\varphi)^{2\lambda-1}\,d\varphi.$$

Let $L_p(R_+,G)\equiv L_{p,\lambda}(R_+)$, $1\le p\le\infty$, denote the space of $\mu_\lambda(x)=sh^{2\lambda}x$ measurable functions on $R_+=(0,\infty)$ with finite norm

$$\|f\|_{L_{p,\lambda}(R_+)}=\left(\int_0^\infty|f(ch\,x)|^p\,sh^{2\lambda}xdx\right)^{\frac{1}{p}},\ 1\le p<\infty,$$

$$\|f\|_{L_{\infty,\lambda}(R_+)}=\|f\|_{L_\infty}=\underset{x\in R_+}{es\sup}|f(ch\,x)|.$$

For all measurable sets $E\subset R_+$ put $\mu E=|E|_\lambda=\int_E sh^{2\lambda}xdx$.

Also but $WL_{p,\lambda}(R_+)$, $1 \le p < \infty$, denote the weak space $L_{p,\lambda}(R_+)$ of locally integrable functions $f(ch\,x)$, $x \in R_+$ with finite norm

$$\|f\|_{WL_{p,\lambda}(R_+)} = \sup_{r\in R_+} r\,|\{x \in R_+ : |f(chx)| > r\}|_\lambda^{\frac{1}{p}}$$

$$= \sup_{r\in R_+} r\left(\int_{\{x\in R_+:|f(chx)|>r\}} sh^{2\lambda}xdx\right)_\lambda^{\frac{1}{p}}.$$

In what follows, the expression $A \lesssim B$ will mean that there exist a constant C such that $0 < A \le CB$, where C may depend on some inessential parameters. If $A \lesssim B$ and $B \lesssim A$, the we write $A \approx B$ and say that A and B are equivalent.

Denote $H_r = (0,r) \subset R_+$. Further, we need the following relation (see [14] Lemma 2.3 by $x = 0$,$\gamma = 2\lambda$)

$$|H_r|_\lambda = \int_0^r sh^{2\lambda}tdt \approx \left(sh\frac{r}{2}\right)^\gamma, \tag{2}$$

where

$$\gamma = \gamma_\lambda(r) = \begin{cases} 2\lambda+1, & 0<r<2, \\ 4\lambda, & 2 \le r < \infty, \end{cases}$$

and $0 < \lambda < 1/2$.

In [10] the Gegenbauer maximal function (G-maximal function) is defined as follows:

$$M_G f(ch\,x) = \sup_{r>0}\frac{1}{|H_r|_\lambda}\int_{H_r} A^\lambda_{ch\,y}\,|f(ch\,x)|\,sh^{2\lambda}ydy.$$

In what follows we need the following Fefferman-Stein type inequality.

Theorem B ([9, Theorem 1.4]). For every $1 \le p < \infty$ and every $0 < t < \infty$ the inequality

$$\int_0^r A^\lambda_{ch\,y}\,(M_G f(ch\,x))^p\,sh^{2\lambda}ydy \le \int_0^r A^\lambda_{ch\,y}\,|f(ch\,x)|^p\,sh^{2\lambda}ydy$$

is true.

Theorem C ([10, Theorem 1.5]). The Chebyshev-type inequality

$$\Big|x \in (0,r) : A^\lambda_{ch\,y}M_G f(ch\,x) > \alpha\Big|_\lambda \le \frac{1}{\alpha}\int_0^r A^\lambda_{ch\,y}M_G f(ch\,x)\,sh^{2\lambda}ydy$$

is true for all $\alpha > 0$ and $t > 0$.

Theorem D [10] a) If $f \in L_{1,\lambda}(R)$, then for all $\alpha > 0$ the inequality

$$|\{x \in R_+ : M_G f(ch\,x) > \alpha\}|_\lambda \lesssim \frac{1}{\alpha}\,\|f\|_{L_{1,\lambda}(R_+)}.$$

b) If $f \in L_{p,\lambda}(R_+)$, $1 < p \le \infty$, then $M_G f(ch\,x) \in L_{p,\lambda}(R_+)$ and

$$\|M_G f\|_{L_{p,\lambda}(R_+)} \lesssim \|f\|_{L_{p,\lambda}(R_+)}.$$

Corollary E. If $f \in L_{p,\lambda}(R_+)$, $1 \le p \le \infty$, then

$$\lim_{r\to 0}\frac{1}{|H_r|_\lambda}\int_0^r A^\lambda_{ch\,y} f(ch\,x)\, sh^{2\lambda}y dy = f(ch\,x)$$

for a.e. $x \in R_+$.

3. Some embeddings into the G- Morrey and modified G-Morrey spaces.

We introduce the following nonation analogously in [8].

Definition 1. *Let $1 \le p < \infty$, $0 < \lambda < 1/2$, $0 \le \nu \le \gamma$, $\left[sh\frac{r}{2}\right]_1 = \min\left\{1,\, sh\frac{r}{2}\right\}$. We denote by $L_{p,\lambda,\nu}(R_+)$ the $G-$Morrey space, and by $\tilde{L}_{p,\lambda,\nu}(R_+)$ the modified $G-$Morrey space, as the set of locally integrable functions $f(ch\,x)$, $x \in R_+$ with the finite norms*

$$\|f\|_{L_{p,\lambda,\nu}(R_+)} = \sup_{x,r\in R_+}\left(\left(sh\frac{r}{2}\right)^{-\nu}\int_0^r A^\lambda_{ch\,y}|f(ch\,x)|^p\, sh^{2\lambda}ydy\right)^{\frac{1}{p}},$$

$$\|f\|_{\tilde{L}_{p,\lambda,\nu}(R_+)} = \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A^\lambda_{ch\,y}|f(ch\,x)|^p\, sh^{2\lambda}ydy\right)^{\frac{1}{p}},$$

respectively.

Note that

$$\tilde{L}_{p,\lambda,0}(R_+) = L_{p,\lambda,0}(R_+) = L_{p,\lambda}(R_+).$$

$$\tilde{L}_{p,\lambda,0}(R_+) \subset L_{p,\lambda,\nu}(R_+)\bigcap L_{p,\lambda}(R_+)$$

and

$$\max\left\{\|f\|_{L_{p,\lambda,\nu}},\, \|f\|_{L_{p,\lambda}}\right\} \le \|f\|_{\tilde{L}_{p,\lambda,\nu}}.$$

Definition 2. *Let $1 \le p < \infty$, $0 < \lambda < 1/2$, $0 \le \nu \le \gamma$. We denote by $WL_{p,\lambda,\nu}(R_+)$ the weak $G-$Morrey space and by $W\tilde{L}_{p,\lambda,\nu}(R_+)$ the modified weak $G-$Morrey space as the set of locally integrable functions $f(ch\,x)$, $x \in R_+$ with finite norms*

$$\|f\|_{WL_{p,\lambda,\nu}(R_+)} = \sup_{r\in R_+} r \sup_{x,t\in R_+}\left(\left(sh\frac{t}{2}\right)^{-\nu}\left|\left\{y\in(0,t) : A^\lambda_{ch\,y}|f(ch\,x)| > r\right\}\right|_\lambda\right)^{\frac{1}{p}}$$

$$= \sup_{r\in R_+} r \sup_{x,t\in R_+}\left(\left(sh\frac{t}{2}\right)^{-\nu}\int_{\left\{y\in(0,t):A^\lambda_{ch\,y}|f(ch\,x)|>r\right\}} sh^{2\lambda}ydy\right)^{\frac{1}{p}},$$

$$\|f\|_{W\tilde{L}_{p,\lambda,\nu}(R_+)} = \sup_{r\in R_+} r \sup_{x,t\in R_+}\left(\left[sh\frac{t}{2}\right]_1^{-\nu}\left|\left\{y\in(0,t) : A^\lambda_{ch\,y}|f(ch\,x)| > r\right\}\right|_\lambda\right)^{\frac{1}{p}}$$

$$= \sup_{r\in R_+} r \sup_{x,t\in R_+}\left(\left[sh\frac{t}{2}\right]_1^{-\nu}\int_{\left\{y\in(0,t):A^\lambda_{ch\,y}|f(ch\,x)|>r\right\}} sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

respectively.

Note that $WL_{p,\lambda,0}(R_+) = WL_{p,\lambda}(R_+) = W\tilde{L}_{p,\lambda,0}(R_+)$, $L_{p,\lambda,\nu}(R_+) \subset W\tilde{L}_{p,\lambda,\nu}(R_+)$ and $\|f\|_{WL_{p,\lambda,\nu}} \le \|f\|_{L_{p,\lambda,\nu}}$, $\tilde{L}_{p,\lambda,\nu}(R_+) \subset W\tilde{L}_{p,\lambda,\nu}(R_+)$ and $\|f\|_{W\tilde{L}_{p,\lambda,\nu}} \le \|f\|_{\tilde{L}_{p,\lambda,\nu}}$.

Lemma 1. *Let* $1 \le p < \infty$, $0 < \lambda < 1/2$, $0 \le \nu \le \gamma$. *Then*

$$L_{p,\lambda,\nu}(R_+) = L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$$

and

$$\|f\|_{\tilde{L}_{p,\lambda,\nu}} = \max\left\{\|f\|_{L_{p,\lambda,\nu}}, \|f\|_{L_{p,\lambda}}\right\}.$$

Proof. Let $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$. Then

$$\|f\|_{L_{p,\lambda}(R_+)} = \sup_{x,r\in R_+}\left(\int_0^r A^{\lambda}_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$\le \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A^{\lambda}_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$= \|f\|_{\tilde{L}_{p,\lambda,\nu}},$$

and

$$\|f\|_{L_{p,\lambda,\nu}(R_+)} = \sup_{x,r\in R_+}\left(\left(sh\frac{r}{2}\right)^{-\nu}\int_0^r A^{\lambda}_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$\le \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A^{\lambda}_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$= \|f\|_{\tilde{L}_{p,\lambda,\nu}}.$$

Therefore, $f \in L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$ and the embedding

$$\tilde{L}_{p,\lambda,\nu}(R_+) \subset L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$$

is valid.

Let $f \in L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$. Then

$$\|f\|_{\tilde{L}_{p,\lambda,\nu}(R_+)} = \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A^{\lambda}_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$= \max\left\{\sup_{x\in R_+, r\in(0,1]}\left(\left[sh\frac{r}{2}\right]^{-\nu}\int_0^r A^{\lambda}_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}},\right.$$

$$\left.\sup_{x\in R_+, r\in(1,\infty)}\left(\int_0^r A^{\lambda}_{ch\,y}|f(ch\,x)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}\right\}$$

$$\leq \max \left\{ \|f\|_{L_{p,\lambda,\nu}}, \|f\|_{L_{p,\lambda}} \right\}.$$

Therefore, $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$ and the embedding $L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+) \subset \tilde{L}_{p,\lambda,\nu}$ is valid. Thus $\tilde{L}_{p,\lambda,\nu}(R_+) = L_{p,\lambda,\nu}(R_+) \bigcap L_{p,\lambda}(R_+)$.

Let now $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$. Then

$$\|f\|_{L_{p,\lambda,\nu}(R_+)} = \sup_{x,r \in R_+} \left(\left(sh\frac{r}{2} \right)^{-\nu} \int_0^r A^{\lambda}_{ch\, y} |f(ch\, x)|^p \, sh^{2\lambda} y dy \right)^{\frac{1}{p}}$$

$$= \sup_{x,r \in R_+} \left(\frac{\left[sh\frac{r}{2} \right]_1}{sh\frac{r}{2}} \right)^{\frac{\nu}{p}} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^r A_{ch\, y} |f(ch\, x)|^p \, sh^{2\lambda} y dy \right)^{\frac{1}{p}}$$

$$= \|f\|_{\tilde{L}_{p,\lambda,\nu}},$$

since by $0 < r < 2arcsh\, 1$, $sh\frac{r}{2} < 1$ and $\left[sh\frac{r}{2} \right]_1 = sh\frac{r}{2}$. If $r \geq 2arcsh\, 1$, then $sh\frac{r}{2} \geq 1$ and we have

$$\frac{\left[sh\frac{r}{2} \right]_1}{sh\frac{r}{2}} = \frac{1}{sh\frac{r}{2}} \leq 1.$$

4. Hardy-Littlewood-Sobolev inequality in modified $G-$Morrey spaces

In this section we study the $\tilde{L}_{p,\lambda,\nu}$- boundedness of the $G-$maximal operator M_G.

Theorem 1. *1) If $f \in \tilde{L}_{1,\lambda,\nu}(R_+)$, $0 \leq \nu < \gamma$, then $M_G f \in W\tilde{L}_{1,\lambda,\nu}(R_+)$ and*

$$\|M_G f\|_{W\tilde{L}_{1,\lambda,\nu}} \lesssim \|f\|_{\tilde{L}_{1,\lambda,\nu}}.$$

2) If $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$, $1 < p < \infty$, then $M_G f \in W\tilde{L}_{p,\lambda,\nu}(R_+)$ and

$$\|M_G f\|_{\tilde{L}_{p,\lambda,\nu}} \lesssim \|f\|_{\tilde{L}_{p,\lambda,\nu}}.$$

Proof. 1) From the definition of weak modified $G-$Morrey spaces

$$\|M_G f\|_{W\tilde{L}_{1,\lambda,\nu}(R_+)} = \sup_{r \in R_+} r \sup_{x,t \in R_+} \left(\left[sh\frac{t}{2} \right]_1^{-\nu} \left| \left\{ y \in (0,t) : A^{\lambda}_{ch\, y} M_G f(ch\, x) > r \right\} \right|_{\lambda} \right)^{\frac{1}{p}}.$$

Applying the Theorem B and also Theorem A we get

$$\|M_G f\|_{W\tilde{L}_{1,\lambda,\nu}} \lesssim \sup_{x,t \in R_+} \left(\left[sh\frac{t}{2} \right]_1^{-\nu} \int_0^r A^{\lambda}_{ch\, y} |f(ch\, x)| \, sh^{2\lambda} y dy \right)$$

$$= \|f\|_{\tilde{L}_{1,\lambda,\nu}}.$$

Assertion 2) follows from Theorem A.

We consider $G-$fractional integral introduced in [14].

$$J_G^{\alpha} d\left(ch\, x\right) = \int_0^{\infty} \left|H_y\right|_{\lambda}^{\frac{\alpha}{\gamma}-1} A_{ch\, y}^{\lambda} f\left(ch\, x\right) sh^{2\lambda} y dy.$$

The following Hardy-Littlewood-Sobolev inequality in modified $G-$Morrey spaces is valid.

Theorem 2. *Let* $0 \leq \alpha < \gamma$, $0 \leq \nu < \gamma - \alpha p$ *and* $1 \leq p < \frac{\gamma-\nu}{\alpha}$.

1) If $1 < p < \frac{\gamma-\nu}{\alpha}$, *then the condition*

$$\frac{\alpha}{\gamma} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{\gamma-\nu}$$

is necessary and sufficient for the boundedness of the operator J_G^{α} *from* $\tilde{L}_{p,\lambda,\nu}\left(R_+\right)$ *to* $\tilde{L}_{q,\lambda,\nu}\left(R_+\right)$.

2) If $p = 1 < \frac{\gamma-\nu}{\alpha}$, *then the condition*

$$\frac{\alpha}{\gamma} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{\gamma-\nu}$$

is necessary and sufficient for the boundedness of the operator J_G^{α} *from* $\tilde{L}_{1,\lambda,\nu}\left(R_+\right)$ *to*$\tilde{L}_{q,\lambda,\nu}\left(R_+\right)$.

Proof.

1) Sufficiency. Let $0 \leq \alpha < \gamma$, $0 \leq \nu < \gamma - \alpha p$, $1 < p < \frac{\gamma-\nu}{\alpha}$ and $f \in \tilde{L}_{p,\lambda,\nu}\left(R_+\right)$. From (2), we have

$$\left|J_G^{\alpha} f\left(ch\, x\right)\right| \lesssim \left(\int_0^r + \int_r^{\infty}\right) \frac{A_{ch\, y}^{\lambda}\left|f\left(ch\, x\right)\right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} sh^{2\lambda} t dt$$

$$= A_1\left(x, r\right) + A_2\left(x, r\right). \tag{3}$$

We estimate $A_1\left(x, r\right)$. Let $0 < r < 2$, then by (2) we obtain

$$\left|A_1\left(x, r\right)\right| \lesssim \int_0^r \frac{A_{ch\, y}^{\lambda}\left|f\left(ch\, x\right)\right|}{\left(sh\,\frac{y}{2}\right)^{2\lambda+1-\alpha}} sh^{2\lambda} y dy \lesssim \sum_{j=0}^{\infty} \int_{r/2^{j+1}}^{r/2^j} \frac{A_{ch\, y}^{\lambda}\left|f\left(ch\, x\right)\right|}{\left(sh\frac{y}{2}\right)^{2\lambda+1-\alpha}} sh^{2\lambda} y dy$$

$$< \sum_{j=0}^{\infty} \left(sh\frac{r}{2^{j+1}}\right)^{\alpha} \left(sh\frac{r}{2^{j+2}}\right)^{-2\lambda-1} \int_0^{r/2^j} A_{ch\, y}^{\lambda}\left|f\left(ch\, x\right)\right| sh^{2\lambda} y dy.$$

Using the inequality (see [3], Lemma 2.2)

$$t \leq sh\, t \leq e^A t, \;\; A > 0 \tag{4}$$

and also $sh\, at \le a\, sh\, t$ at $0 \le a \le 1$, we have

$$|A_1(x,r)| \lesssim \left(sh\frac{r}{2}\right)^{\alpha} \sum_{j=0}^{\infty} \left(2^{-j\alpha}\right) \left(sh\frac{r}{2^{j+1}}\right)^{-2\lambda-1} \int_0^{r/2^j} A^{\lambda}_{ch\, y} \left|f(ch\, x)\right| sh^{2\lambda} y dy$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x) \sum_{j=0}^{\infty} 2^{-j\alpha}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x). \tag{5}$$

Let $2 \le r < \infty$. Then

$$A_1(x,r) \lesssim \int_0^r \frac{A^{\lambda}_{ch\, y} \left|f(ch\, x)\right|}{\left(sh\frac{y}{2}\right)^{4\lambda-\alpha}} sh^{2\lambda} y dy$$

$$\lesssim \sum_{j=0}^{\infty} \int_{r/2^{j+1}}^{r/2^j} \frac{A^{\lambda}_{ch\, y} \left|f(ch\, x)\right|}{\left(sh\frac{y}{2}\right)^{4\lambda-\alpha}} sh^{2\lambda} y dy$$

$$\lesssim \sum_{j=0}^{\infty} \left(sh\frac{r}{2^{j+1}}\right)^{\alpha} \left(sh\frac{r}{2^{j+1}}\right)^{-4\lambda} \int_0^{r/2^j} A^{\lambda}_{ch\, y} \left|f(ch\, x)\right| sh^{2\lambda} y dy$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x) \sum_{j=0}^{\infty} 2^{-j\alpha}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x). \tag{6}$$

Combining (5) and (6) we obtain

$$A_1(x,r) \lesssim \left(sh\frac{r}{2}\right)^{\alpha} M_G f(ch\, x),\ 0 < r < \infty. \tag{7}$$

Now consider $A_2(x,r)$. By Holders inequality we get

$$A_2(x,r) \lesssim \left(\int_r^{\infty} A^{\lambda}_{ch\, y} \left|f(ch\, x)\right|^p (sh\, y)^{-\beta} sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\times \left(\int_r^{\infty} (sh\, y)^{(\beta/p+\alpha-\gamma)p'} sh^{2\lambda} y dy\right)^{\frac{1}{p'}}$$

$$= A_{2\cdot 1} \cdot A_{2\cdot 2} \tag{8}$$

Let $\nu < \beta < \gamma - \alpha p$. Using the inequality [7]

$$\left\|A^{\lambda}_{ch\, y} f\right\|_{\tilde{L}_{p,\lambda,\nu}} \le \|f\|_{\tilde{L}_{p,\lambda,\nu}},$$

we obtain

$$A_{2\cdot 1} \lesssim \left(\sum_{j=0}^{\infty}\int_{2^j r}^{2^{j+1}r} A^{\lambda}_{ch\,y}\left|f\left(ch\,x\right)\right|^p \left(sh\,y\right)^{-\beta} sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\lesssim \left\|A^{\lambda}_{ch\,y} f\right\|_{\tilde{L}_{p,\lambda.\nu}} \left(\sum_{j=0}^{\infty}\frac{\left[2^{j+1} sh\frac{r}{2}\right]_1^{\nu}}{\left(sh\, 2^j r\right)^{\beta}}\right)^{\frac{1}{p}}$$

$$\lesssim \left[2sh\frac{r}{2}\right]_1^{\nu/p}\left(sh\,r\right)^{-\beta/p}\left(\sum_{j=0}^{\infty} 2^{j(\nu-\beta)}\right)^{\frac{1}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \left[sh\frac{r}{2}\right]_1^{\frac{\nu}{p}}\left(sh\frac{r}{2}\right)^{-\frac{\beta}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}} \tag{9}$$

since $sh\,ax \geq ashx$ at $a \geq 1$.

For $A_{2\cdot 2}$ we have

$$A_{2\cdot 2} = \left(\int_r^{\infty}\left(sh\,y\right)^{(\beta/p+\alpha-\gamma)p'} sh^{2\lambda} y dy\right)^{\frac{1}{p'}}$$

$$\lesssim \left(sh\,r\right)^{\beta/p+\alpha-\gamma+\gamma/p'} \lesssim \left(sh\,r\right)^{\beta/p+\alpha-\gamma+\gamma(1-1/p)}$$

$$\lesssim \left(sh\,r\right)^{\beta/p+\alpha-\gamma/p} \lesssim \left(sh\,\frac{r}{2}\right)^{\beta/p+\alpha-\gamma/p}. \tag{10}$$

Taking into account (9) and (10) in (8), we obtain

$$A_2\left(x,r\right) \lesssim \left[sh\frac{r}{2}\right]^{\nu/p}\left(sh\frac{r}{2}\right)^{\alpha-\gamma/p}\|f\|_{\tilde{L}_{p,\lambda,\nu}}. \tag{11}$$

Thus from (5) and (11), we get

$$\left|J_G^{\alpha} f\left(ch\,x\right)\right| \lesssim \left(\left[sh\frac{r}{2}\right]_1^{\nu/p}\left(sh\frac{r}{2}\right)^{\alpha-\gamma/p}\|f\|_{\tilde{L}_{p,\lambda,\nu}} + \left(sh\frac{r}{2}\right)^{\alpha} M_G f\left(chx\right)\right)$$

$$\lesssim \min\left\{\left(sh\frac{r}{2}\right)^{\alpha+(\nu-\gamma)/p}\|f\|_{\tilde{L}_{p,\lambda,\nu}} + \left(sh\frac{r}{2}\right)^{\alpha} M_G f\left(ch\,x\right),\right.$$

$$\left.\left(sh\frac{r}{2}\right)^{\alpha-\gamma/p}\|f\|_{\tilde{L}_{p,\lambda,\nu}} + \left(sh\frac{r}{2}\right)^{\alpha} M_G f\left(ch\,x\right)\right\}, \tag{12}$$

for all $r > 0$.

The right-hand side attains its minimum at

$$sh\frac{r}{2} = \left(\frac{\gamma-\alpha p}{\alpha p}\frac{\|f\|_{\tilde{L}_{p,\lambda,\nu}}}{M_G f\left(ch\,x\right)}\right)^{p/\gamma}, \tag{13}$$

and

$$sh\frac{r}{2}=\left(\frac{\gamma-\nu-\alpha p}{\alpha p}\frac{\|f\|_{\tilde{L}_{p,\lambda,\nu}}}{M_G f\left(ch\,x\right)}\right)^{\frac{p}{\gamma-\nu}}. \tag{14}$$

Applying (13) and (14) in (12), we get

$$\left|J_G^{\alpha} f\left(ch\,x\right)\right| \lesssim \min\left\{\left(\frac{M_G f\left(chx\right)}{\|f\|_{\tilde{L}_{p,\lambda,\nu}}}\right)^{1-\frac{\alpha p}{\gamma}},\left(\frac{M_G f\left(chx\right)}{\|f\|_{\tilde{L}_{p,\lambda,\nu}}}\right)^{1-\frac{\alpha p}{\gamma-\nu}}\right\}\|f\|_{\tilde{L}_{p,\lambda,\nu}}.$$

Then

$$\left|J_G^{\alpha} f\left(ch\,x\right)\right| \lesssim \left(M_G f\left(chx\right)\right)^{\frac{p}{q}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}^{1-\frac{p}{q}}.$$

Hence, by Theorem 1, we have

$$\int_0^r \left|J_G^{\alpha} f\left(ch\,x\right)\right|^q sh^{2\lambda}x dx \lesssim \|f\|_{\tilde{L}_{p,\lambda,\nu}}^{q-p}\int_0^r \left|M_G f\left(ch\,x\right)\right|^p sh^{2\lambda}x dx$$

$$\lesssim \left[sh\frac{r}{2}\right]_1^{\nu}\|f\|_{\tilde{L}_{p,\lambda,\nu}}^{q}.$$

From this it follows that

$$\left\|J_G^{\alpha} f\left(ch\,x\right)\right\|_{L_{q,\lambda,\nu}} \lesssim \|f\|_{\tilde{L}_{p,\lambda,\nu}}^{q},$$

i.e., J_G^{α} in bounded from $\tilde{L}_{p,\lambda,\nu}\left(R_+\right)$ to $\tilde{L}_{q,\lambda,\nu}\left(R_+\right)$.

Necessity. Let $1<p<\left(\gamma-\nu\right)/\alpha$, $f\in \tilde{L}_{p,\lambda,\nu}\left(R_+\right)$ and J_G^{α} is bounded from $\tilde{L}_{p,\lambda,\nu}\left(R_+\right)$ to $\tilde{L}_{q,\lambda,\nu}\left(R_+\right)$. Let the function $f\left(chx\right)$ be non-negative and monotonically on R_+. The delates function $f_t(chx)$ is defined as follows [6]:

$$f\left(ch\left(th\frac{t}{2}\right)x\right)\le f_t\left(chx\right)\le f\left(ch\left(cth\frac{t}{2}\right)x\right),\ 0<t<2$$

$$f\left(ch\left(th\frac{t}{2}\right)x\right)\le f_t\left(chx\right)\le f\left(ch\left(sh\frac{t}{2}\right)x\right),\ 2\le t<\infty \tag{15}$$

We suppose $\left[sh\frac{t}{2}\right]_{1,+}=\max\left\{1,sh\frac{t}{2}\right\}$. Let $0<t<2$. Using the symmetry of the operator A_{chy}^{λ} (see [7]) $A_{chx}^{\lambda}f\left(chy\right)=A_{chy}^{\lambda}f\left(chx\right)$ we will have

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}}=\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A_{chy}^{\lambda}\left|f_t\left(chy\right)\right|^p sh^{2\lambda}y dy\right)^{\frac{1}{p}}$$

$$\le \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^r A_{chx}^{\lambda}\left|f_t\left(ch\left(cth\frac{t}{2}\right)y\right)\right|^p sh^{2\lambda}y dy\right)^{\frac{1}{p}}$$

$$\left[\left(cth\frac{t}{2}\right)y=u,\ dy=\left(th\frac{t}{2}\right)du\right]$$

$$= \left(sh\frac{t}{2}\right)^{\frac{1}{p}} \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rcth\frac{t}{2}} A^{\lambda}_{chy}\left|f\left(chu\right)\right|^p sh^{2\lambda}\left(th\frac{t}{2}\right)udu\right)^{\frac{1}{p}}$$

$$= \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{p}} \sup_{r\in R_+}\left(\frac{\left[\left(sh\frac{r}{2}\right)cth\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{p}}$$

$$\times \sup_{x,r\in R_+}\left(\left[\left(sh\frac{r}{2}\right)cth\frac{t}{2}\right]_1^{-\nu}\int_0^{rcth\frac{t}{2}} A^{\lambda}_{chx}\left|f\left(chu\right)\right|^p sh^{2\lambda}udu\right)^{\frac{1}{p}}.$$

$$\leq \left(sh\frac{t}{2}\right)^{\frac{2\lambda+1}{p}}\left[cth\frac{t}{2}\right]_{1,+}^{\frac{\nu}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}} \leq \left(th\frac{t}{2}\right)^{\frac{2\lambda+1-\nu}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$= \left(\frac{sh\frac{t}{2}}{ch\frac{t}{2}}\right)^{\frac{2\lambda+1-\nu}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}} \lesssim \frac{1}{\left(ch\frac{t}{2}\right)^{\frac{2\lambda+1-\nu}{p}-\alpha}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-2\lambda-1}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}, 0<t<2. \tag{16}$$

On the other hand, by $0 < t < 2$, we get

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}} = \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{r} A^{\lambda}_{chy}\left|f_t\left(chx\right)\right|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$\geq \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{r} A^{\lambda}_{chx}\left|f_t\left(ch\left(th\frac{t}{2}\right)y\right)\right|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$\left[\left(th\frac{t}{2}\right)y=u,\, y=\left(cth\frac{t}{2}\right)u,\, dy=\left(cth\frac{t}{2}\right)du\right]$$

$$= \left(cth\frac{t}{2}\right)^{\frac{1}{p}} \sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rth\frac{t}{2}} A^{\lambda}_{chx}\left|f\left(chu\right)\right|^p sh^{2\lambda}\left(cth\frac{t}{2}\right)udu\right)^{\frac{1}{p}}$$

$$\geq \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{p}}\left(\sup_{r\in R_+}\frac{\left[\left(sh\frac{r}{2}\right)th\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{p}}\left[th\frac{t}{2}\right]_{1,+}^{\frac{\nu}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\geq \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{p}-\frac{\nu}{p}-\alpha}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1-\nu}{p}-\alpha} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\geq \left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}} \tag{17}$$

Combing (16) and (17), we obtain

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}, \; 0 < t < 2. \tag{18}$$

Now, let $2 \leq t < \infty$, then from (15) we have

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}} = \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r A_{ch\,y}^{\lambda} |f_t(ch\,x)|^p sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\geq \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r A_{ch\,x}^{\lambda} \left|f\left(ch\left(th\frac{t}{2}\right)y\right)\right|^p sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\left[\left(th\frac{t}{2}\right)y = u, \; y = \left(cth\frac{t}{2}\right)u, \; dy = \left(cth\frac{t}{2}\right)du\right]$$

$$= \left(cth\frac{t}{2}\right)^{\frac{1}{p}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{r\,th\frac{t}{2}} A_{ch\,y}^{\lambda} |f(ch\,u)|^p sh^{2\lambda}\left(cth\frac{t}{2}\right) udu\right)^{\frac{1}{p}}$$

$$\geq \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{p}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{r\,th\frac{t}{2}} A_{ch\,x}^{\lambda} |f(ch\,u)|^p sh^{2\lambda} udu\right)^{\frac{1}{p}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{p}} \sup_{x,r\in R_+} \left(\frac{\left[\left(sh\frac{r}{2}\right)th\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{p}} \left[th\frac{t}{2}\right]_1^{\frac{\nu}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}} \geq \left(cth\frac{t}{2}\right)^{\frac{4\lambda-\nu}{p}-\alpha} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\geq \left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-4\lambda}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}} = \left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}} \|f\|_{\tilde{L}_{p,\lambda,\nu}}, \; 2 \leq t < \infty. \tag{19}$$

On the other hand, at $2 \leq t < \infty$, we get

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}} = \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r A_{ch\,y}^{\lambda} |f_t(ch\,x)|^p sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\leq \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r A_{ch\,y}^{\lambda} \left|f\left(ch\left(sh\frac{t}{2}\right)y\right)\right|^p sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\left[\left(sh\frac{t}{2}\right)y=0,\ \ dy=\frac{du}{sh\frac{t}{2}}\right]$$

$$=\left(sh\frac{t}{2}\right)^{-\frac{1}{p}}\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rsh\frac{t}{2}}A^{\lambda}_{ch\,x}\left|f\left(ch\,u\right)\right|^p sh^{2\lambda}\frac{u}{sht}du\right)^{\frac{1}{p}}$$

$$\leq\left(sh\frac{t}{2}\right)^{-\frac{2\lambda+1}{p}}\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rsh\frac{t}{2}}A^{\lambda}_{ch\,x}\left|f\left(ch\,u\right)\right|^p sh^{2\lambda}udu\right)^{\frac{1}{p}}$$

$$\leq\left(sh\frac{t}{2}\right)^{-\frac{4\lambda}{p}}\left(\sup_{r\in R_+}\frac{\left[\left(sh\frac{r}{2}\right)sh\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$=\left(sh\frac{t}{2}\right)^{-\frac{4\lambda}{p}}\left[sh\frac{t}{2}\right]_1^{\frac{\nu}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\leq\left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-4\lambda}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$=\left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}},2\leq t<\infty. \qquad (20)$$

Combing (19) and (20), we obtain

$$\|f\|_{\tilde{L}_{p,\lambda,\nu}}\approx\left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}}, \qquad (21)$$

Thus from (18) and (21), we have

$$\|f_t\|_{\tilde{L}_{p,\lambda,\nu}}\approx\left(sh\frac{t}{2}\right)^{\alpha+\frac{\nu-\gamma}{p}}\|f\|_{\tilde{L}_{p,\lambda,\nu}},\ 0<t<\infty. \qquad (22)$$

From (2) $0<t<2$, we have

$$\|J^{\alpha}_G f_t\|_{\tilde{L}_{q,\lambda,\nu}}=\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{r}\left|J^{\alpha}_G f_t\left(ch\,y\right)\right|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\leq\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{r}\left|J^{\alpha}_G f\left(ch\left(cth\frac{t}{2}\right)y\right)\right|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\left[\left(cth\frac{t}{2}\right)y=z,\qquad dy=\left(th\frac{t}{2}\right)dz\right]$$

$$=\left(th\frac{t}{2}\right)^{\frac{1}{q}}\sup_{x,r\in R_+}\left(\left[sh\frac{r}{2}\right]_1^{-\nu}\int_0^{rcth\frac{t}{2}}\left|J^{\alpha}_G f\left(ch\,z\right)\right|^q sh^{2\lambda}\left(th\frac{t}{2}\right)zdz\right)^{\frac{1}{q}}$$

$$\leq \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{rcth\frac{t}{2}} |J_G^\alpha f\,(ch\,z)|^q\, sh^{2\lambda} zdz\right)^{\frac{1}{q}}$$

$$= \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \left(\sup_{r\in R_+} \frac{\left[\left(sh\frac{r}{2}\right) cth\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \left[cth\frac{t}{2}\right]_1^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\leq \left(cth\frac{t}{2}\right)^{-\frac{2\lambda+1}{q}} \left[cth\frac{t}{2}\right]_{1,+}^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\leq \left(cth\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\leq \left(sh\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}} \|J_G^\alpha\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha\|_{\tilde{L}_{q,\lambda,\nu}}\,, \quad 0 < t < 2. \tag{23}$$

On the other hand by $0 < t < 2$, we get

$$\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} = \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r |J_G^\alpha f_t\,(ch\,y)|^q\, sh^{2\lambda} ydy\right)^{\frac{1}{q}}$$

$$\geq \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^r \left|J_G^\alpha f\left(ch\left(th\frac{t}{2}\right)\,y\right)\right|^q sh^{2\lambda} ydy\right)^{\frac{1}{q}}$$

$$\left[\left(th\frac{t}{2}\right) y = z, \qquad dy = \left(cth\frac{t}{2}\right) dz\right]$$

$$= \left(cth\frac{t}{2}\right)^{\frac{1}{q}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2}\right]_1^{-\nu} \int_0^{rth\frac{t}{2}} |J_G^\alpha f\,(ch\,z)|^q\, sh^{2\lambda}\left(cth\frac{t}{2}\right) dz\right)^{\frac{1}{q}}$$

$$\geq \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \left(\sup_{r\in R_+} \frac{\left[\left(sh\frac{r}{2}\right) th\frac{t}{2}\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \left(th\frac{t}{2}\right)_1^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\geq \left(\frac{ch\frac{t}{2}}{sh\frac{t}{2}}\right)^{\frac{2\lambda+1-\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \geq \left(sh\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2} \right)^{\frac{\nu-\gamma}{q}} \|J_G^{\alpha} f\|_{\tilde{L}_{q,\lambda,\nu}}, \ 0 < t < 2. \tag{24}$$

Thus from (23) and (24), we obtain

$$\|J_G^{\alpha} f\|_{\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2} \right)^{\frac{\nu-\gamma}{q}} \|J_G^{\alpha} f\|_{\tilde{L}_{q,\lambda,\nu}}, \ \ 0 < t < 2. \tag{25}$$

Now we consider the case, then $2 \le t < \infty$. From (15), we have

$$\|J_G^{\alpha} f_t\|_{\tilde{L}_{q,\lambda,\nu}} = \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^r |J_G^{\alpha} f_t \, (ch\, y)|^q \, sh^{2\lambda} y dy \right)^{\frac{1}{q}}$$

$$\ge \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^r \left| J_G^{\alpha} f \left(ch \left(th\frac{t}{2} \right) \, y \right) \right|^q sh^{2\lambda} y dy \right)^{\frac{1}{q}}$$

$$\left[\left(th\frac{t}{2} \right) y = z, \qquad dy = \left(cth\frac{t}{2} \right) dz \right]$$

$$= \left(cth\frac{t}{2} \right)^{\frac{1}{q}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^{rth\frac{t}{2}} |J_G^{\alpha} f \, (ch\, z)|^q \, sh^{2\lambda} \left(cth\frac{t}{2} \right) zdz \right)^{\frac{1}{q}}$$

$$\ge \left(cth\frac{t}{2} \right)^{\frac{2\lambda+1}{q}} \left(\sup_{r\in R_+} \frac{\left[\left(sh\frac{r}{2} \right) th\frac{t}{2} \right]_1}{\left[sh\frac{r}{2} \right]_1} \right)^{\frac{\nu}{q}} \|J_G^{\alpha} f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2} \right)^{\frac{4\lambda}{q}} \left(th\frac{t}{2} \right)_1^{\frac{\nu}{q}} \|J_G^{\alpha} f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\ge \left(ch\frac{t}{2} \right)^{\frac{4\lambda-\nu}{q}} \|J_G^{\alpha} f\|_{\tilde{L}_{q,\lambda,\nu}} \ge \left(sh\frac{t}{2} \right)^{\frac{\nu-\gamma}{q}} \|J_G^{\alpha} f\|_{\tilde{L}_{q,\lambda,\nu}}, \, 2 \le t < \infty. \tag{26}$$

On the other and by $2 \le t < \infty$, we obtain

$$\|J_G^{\alpha} f_t\|_{\tilde{L}_{q,\lambda,\nu}} \le \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^r \left| J_G^{\alpha} f_t \left(ch \, \left(sh\frac{t}{2} \right) y \right) \right|^q sh^{2\lambda} y dy \right)^{\frac{1}{q}}$$

$$\left[\left(sh\frac{t}{2} \right) y = z, \qquad dz = \frac{dz}{sh\frac{t}{2}} \right]$$

$$= \left(sh\frac{t}{2} \right)^{-\frac{1}{q}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^r |J_G^{\alpha} f \, (cht)|^q \, sh^{2\lambda} \left(\frac{z}{sh\frac{t}{2}} \right) dz \right)^{\frac{1}{q}}$$

$$= \left(sh\frac{t}{2} \right)^{-\frac{2\lambda+1}{q}} \sup_{x,r\in R_+} \left(\left[sh\frac{r}{2} \right]_1^{-\nu} \int_0^{rsh\frac{t}{2}} |J_G^{\alpha} f_t \, (chz)|^q \, sh^{2\lambda} zdz \right)^{\frac{1}{q}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{-2\lambda+1}{q}} \left(\sup_{r\in R_+} \frac{\left[sh\frac{r}{2}\left(sh\frac{t}{2}\right)\right]_1}{\left[sh\frac{r}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\le \left(sh\frac{t}{2}\right)^{-\frac{4\lambda}{q}} \left[sh\frac{t}{2}\right]^{\frac{\nu}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$\le \left(sh\frac{t}{2}\right)^{\frac{\nu-4\lambda}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}, \ 2 \le t < \infty. \tag{27}$$

Combing (26) and (27), we have

$$\|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|f\|_{\tilde{L}_{q,\lambda,\nu}}, \ 2 < t < \infty. \tag{28}$$

Thus (23) and (28), we obtain

$$\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}}, \ 0 < t < \infty. \tag{29}$$

Since J_G^α is bounded from $\tilde{L}_{p,\lambda,\nu}(R_+)$ to $\tilde{L}_{q,\lambda,\nu}(R_+)$, i.e.

$$\|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \lesssim \|f\|_{\tilde{L}_{p,\lambda,\nu}},$$

then taking into account (18) and (29), we obtain

$$\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{\tilde{L}_{q,\lambda,\nu}} \lesssim \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|f_t\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\alpha+(\nu-\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{\tilde{L}_{p,\lambda,\nu}}$$

$$\lesssim \|f\|_{\tilde{L}_{p,\lambda,\nu}} \begin{cases} \left(sh\frac{t}{2}\right)^{\alpha-\gamma\left(\frac{1}{p}-\frac{1}{q}\right)}, \ 0 < t < 2 \\ \left(sh\frac{t}{2}\right)^{\alpha+(\nu-\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)}, \ 2 \le t < \infty \end{cases}.$$

If $\frac{1}{p} - \frac{1}{q} < \frac{\alpha}{\gamma}$, then at $t \to 0$ we have $\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} = 0$, for all $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$
As well as is $\frac{1}{p} - \frac{1}{q} > \frac{\alpha}{\nu-\gamma}$, then at $t \to \infty$ we get $\|J_G^\alpha f_t\|_{\tilde{L}_{q,\lambda,\nu}} = 0$, for all $f \in \tilde{L}_{p,\lambda,\nu}(R_+)$.
Therefore $\frac{\alpha}{\gamma} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{\gamma-\nu}$.

Sufficiency. Let $f \in \tilde{L}_{1,\lambda,\nu}(R_+)$, then

$$|\{x \in (0,r) : |J_G^\alpha f(ch\,x)| > 2\beta\}|_\lambda$$

$$\leq |\{x\in(0,r): A_1(x,r)>\beta\}|_\lambda + |\{x\in(0,r): A_2(x,r)>\beta\}|_\lambda .$$

Also

$$A_2(x,r)=\int_r^\infty A_{ch\,y}^\lambda \frac{f(ch\,x)\,sh^{2\lambda}ydy}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}}$$

$$\leq \sum_{j=0}^{\infty}\int_{2^j r}^{2^{j+1}r}\frac{A_{ch\,y}^\lambda |f(ch\,x)|\,sh^{2\lambda}ydy}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}}$$

$$\leq \left\|A_{ch\,y}^\lambda f\right\|_{\tilde{L}_{1,\lambda,\gamma}}\sum_{j=0}^{\infty}\frac{\left[2^{j+1}sh\frac{r}{2}\right]_1^\nu}{\left(2^j sh\frac{r}{2}\right)^{\gamma-\alpha}}\lesssim \left(sh\frac{r}{2}\right)^{\alpha-\gamma}\left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\alpha-\gamma}\left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}}. \tag{30}$$

$$\lesssim \|f\|_{\tilde{L}_{1,\lambda,\gamma}}\begin{cases}\left(sh\frac{r}{2}\right)^{\alpha+\nu-\gamma}, & if\ 0<r<2arsh\,1,\\ \left(sh\frac{r}{2}\right)^{\alpha-\gamma}, & if\ 2arcsh\,1\leq r<\infty.\end{cases} \tag{31}$$

According the inequality (7) and Theorem C, we obtain

$$|\{x\in(0,r): A_1(x,r)>\beta\}|_\lambda$$

$$\lesssim \left|\left\{x\in(0,r): M_G f(chx)>\frac{\beta}{Csh^\alpha\frac{r}{2}}\right\}\right|_\lambda \lesssim$$

$$\lesssim \frac{1}{\beta}\left(sh^\alpha\frac{r}{2}\right)\left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}},\ \ 0<r<\infty. \tag{32}$$

If $\left(sh\frac{r}{2}\right)^{\alpha-\gamma}\left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}}=\beta$, then from (30) we obtain $|A_2(x,r)|\lesssim\beta$ and consequently, $|\{x\in(0,r): A_2(x,r)>\beta\}|_\lambda=0$. Then by $0<r<2arcsh\,1$ $\left(sh\frac{r}{2}\right)^{\alpha+\nu-\gamma}\|f\|_{\tilde{L}_{1,\lambda,\gamma}}=\beta$ and from (31), we have

$$|\{x\in(0,r): |J_G^\alpha f(ch\,x)|>2\beta\}|_\lambda \lesssim \frac{1}{\beta}\left(sh^\alpha\frac{r}{2}\right)\left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}}$$

$$=\left(sh\frac{r}{2}\right)^{\alpha-\gamma}\left[sh\frac{r}{2}\right]_1^\nu=\left(\beta^{-1}\|f\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{\frac{\gamma-\nu}{\gamma-\nu-\alpha}}\left[sh\frac{r}{2}\right]_1^\nu. \tag{33}$$

And for $2arcsh\,1<r<\infty$, $\beta=\left(sh\frac{r}{2}\right)^{\alpha-\gamma}\|f\|_{\tilde{L}_{1,\lambda,\gamma}}$ and from (32), we have

$$|\{x\in(0,r): |J_G^\alpha f(ch\,x)|>2\beta\}|_\lambda \lesssim \frac{1}{\beta}\left(sh^\alpha\frac{r}{2}\right)\left[sh\frac{r}{2}\right]_1^\nu \|f\|_{\tilde{L}_{1,\lambda,\gamma}}$$

$$=\left(sh\frac{r}{2}\right)^{\alpha}\left[sh\frac{r}{2}\right]_1^\nu=\left(\beta^{-1}\|f\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{\frac{\gamma}{\gamma-\alpha}}\left[sh\frac{r}{2}\right]_1^\nu. \tag{34}$$

Finally from (33) and (34), we obtain

$$|\{x\in(0,r): |J_G^\alpha f(ch\,x)|>2\beta\}|_\lambda$$

$$\lesssim \left[sh\frac{r}{2}\right]_1^{\nu} \min\left\{\left(\beta^{-1}\|f\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{\frac{\gamma}{\gamma-\alpha}}, \left(\beta^{-1}\|f\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{\frac{\gamma-\nu}{\gamma-\nu-\alpha}}\right\}$$

$$\lesssim \left[sh\frac{r}{2}\right]_1^{\nu} \left(\beta^{-1}\|f\|_{\tilde{L}_{1,\lambda,\gamma}}\right)^{q},$$

where by condition of the theorem

$$\frac{\gamma}{\gamma-\alpha} \le q \le \frac{\gamma-\nu}{\gamma-\nu-\alpha} \Leftrightarrow \frac{\alpha}{\gamma} \le 1-\frac{1}{q} \le \frac{\alpha}{\gamma-\nu}.$$

Necessity. Preliminarily we established the estimates for $\|J_G^{\alpha} f_t\|_{W\tilde{L}_{q,\lambda,\nu}}$. From (15) for $0<t<2$, we have

$$\|J_G^{\alpha} f_t\|_{W\tilde{L}_{q,\lambda,\nu}} = \sup_{r\in R_+}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{y\in(0,u):A^{\lambda}_{ch\,y}\left|J_G^{\alpha} f_t(ch\,x)\right|>r\right\}} sh^{2\lambda} y dy\right)^{\frac{1}{q}}$$

$$\ge \sup_{r\in R_+} r \sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{y\in(0,u):A^{\lambda}_{ch\,x}\left|J_G^{\alpha} f_t\left(ch\left(th\frac{t}{2}\right)y\right)\right|>r\right\}} sh^{2\lambda} y dy\right)^{\frac{1}{q}}$$

$$\left[\left(th\frac{t}{2}\right)y=z,\ dy=\left(cth\frac{t}{2}\right)dz\right]$$

$$=\left(cth\frac{t}{2}\right)^{1+\frac{1}{q}}\times$$

$$\times\sup_{r\in R_+}\left(rth\frac{t}{2}\right)\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):A^{\lambda}_{ch\,x}\left|J_G^{\alpha} f(ch\,)\right|>rth\frac{t}{2}\right\}} sh^{2\lambda}\left(\left(cth\frac{t}{2}\right)z\right)dz\right)^{\frac{1}{q}}$$

$$\ge\left(cth\frac{t}{2}\right)^{\frac{1}{q}}\sup_{r\in R_+}\left(rth\frac{t}{2}\right)\sup_{u\in R_+}\left(\frac{\left[\left(sh\frac{u}{2}\right)th\frac{t}{2}\right]_1}{\left[sh\frac{u}{2}\right]_1}\right)^{\frac{\nu}{q}}$$

$$\times\sup_{x,u\in R_+}\left(\left[\left(sh\frac{u}{2}\right)th\frac{t}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):A^{\lambda}_{ch\,x}\left|J_G^{\alpha} f(czh\,)\right|>rth\frac{t}{2}\right\}} sh^{2\lambda}\left(cth\frac{t}{2}\right)zdz\right)^{\frac{1}{q}}$$

$$\ge\left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}}\left[th\frac{t}{2}\right]_1^{\frac{\nu}{q}}\times$$

$$\times\sup_{r\in R_+}\left(\left(sh\frac{r}{2}\right)th\frac{t}{2}\right)\sup_{x,u\in R_+}\left(\left[\left(sh\frac{u}{2}\right)th\frac{t}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):A^{\lambda}_{ch\,x}\left|J_G^{\alpha} f(ch\,)\right|>rth\frac{t}{2}\right\}} sh^{2\lambda} zdz\right)^{\frac{1}{q}}$$

$$\ge\left(th\frac{t}{2}\right)^{-\frac{2\lambda-1}{q}}\left[th\frac{t}{2}\right]_1^{\frac{\nu}{q}}\|J_G^{\alpha} f_t\|_{W\tilde{L}_{q,\lambda,\nu}}.$$

$$\geq \left(th\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}. \tag{35}$$

On the other hand at $0 < r < 2$, we have

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} = \sup_{r\in R_+}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{y\in(0,u):A^\lambda_{ch\,x}\left|J_G^\alpha f_t\left(ch\left(cth\,\frac{x}{t2}\right)y\right)\right|>r\right\}} sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\left[\left(cth\frac{t}{2}\right)y = z,\, dy = \left(th\frac{t}{2}\right)dz\right]$$

$$= \left(th\frac{t}{2}\right)^{\frac{1}{q}} \sup_{r\in R_+}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,ucth\frac{t}{2}\right):A^\lambda_{ch\,x}\left|J_G^\alpha f(ch\,z)\right|>r\right\}} sh^{2\lambda}\left(th\frac{t}{2}\right)zdz\right)^{\frac{1}{q}}$$

$$\leq \left(th\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \sup_{u\in R_+}\left(\frac{\left[\left(sh\frac{u}{2}\right)cth\frac{t}{2}\right]_1}{\left[sh\frac{u}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$\lesssim \left(th\frac{t}{2}\right)^{\frac{2\lambda+1-q}{\nu}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}} \lesssim \left(sh\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}. \tag{36}$$

From (35) and (36) it follows that

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\gamma}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}, 0 < r < 2. \tag{37}$$

Now we consider the case then $2 \leq t < \infty$. From (15), we get

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \geq \sup_{r\in R_+} r \sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{y\in(0,u):\left|A^\lambda_{ch\,x}J_G^\alpha f_t\left(ch\left(th\frac{t}{2}\right)y\right)\right|>r\right\}} sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\left[\left(th\frac{t}{2}\right)y = z,\, dy = \left(cth\frac{t}{2}\right)dz\right]$$

$$= \left(cth\frac{t}{2}\right)^{\frac{1}{q}} \sup_{r\in R_+}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):\left|A^\lambda_{ch\,x}J_G^\alpha f(ch\,z)\right|>r\right\}} sh^{2\lambda}\left(cth\frac{t}{2}\right)zdz\right)^{\frac{1}{q}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}+1} \sup_{r\in R_+} rth\frac{t}{2}\sup_{x,u\in R_+}\left(\left[sh\frac{u}{2}\right]_1^{-\nu}\int_{\left\{z\in\left(0,\,uth\frac{t}{2}\right):\left|A^\lambda_{ch\,x}J_G^\alpha f(ch\,z)\right|>rth\frac{t}{2}\right\}} sh^{2\lambda}zdz\right)^{\frac{1}{q}}$$

$$\geq \left(cth\frac{t}{2}\right)^{\frac{2\lambda+1}{q}} \sup_{r\in R_+} \left(\frac{\left[\left(sh\frac{u}{2}\right) th\frac{t}{2}\right]_1}{\left[sh\frac{u}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(cth\frac{t}{2}\right)^{\frac{4\lambda}{q}} \left[th\frac{t}{2}\right]_1^{\frac{\nu}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$\geq \left(th\frac{t}{2}\right)^{\frac{\nu-4\lambda}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}} \geq \left(sh\frac{t}{2}\right)^{\frac{\nu-4\lambda}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}, \ \ 2 \leq t < \infty. \tag{38}$$

On the other than, we have

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \leq \sup_{r\in R_+} r \sup_{x,u\in R_+} \left(\left[sh\frac{u}{2}\right]_1^{-\nu} \int_{\left\{y\in(0,u):\left|A^\lambda_{ch\,x} J_G^\alpha f_t\left(ch\left(sh\frac{t}{2}\right)y\right)\right|>r\right\}} sh^{2\lambda} y dy\right)^{\frac{1}{q}}$$

$$\left[\left(sh\frac{t}{2}\right) y = z,\, dy = \frac{dz}{sh\frac{t}{2}}\right]$$

$$= \left(sh\frac{t}{2}\right)^{-\frac{1}{q}} \sup_{r\in R_+} r \sup_{x,u\in R_+} \left(\left[sh\frac{u}{2}\right]_1^{-\nu} \int_{\left\{y\in\left(0,ush\frac{t}{2}\right):\left|A^\lambda_{ch\,x} J_G^\alpha f(chz)\right|>rsh\frac{t}{2}\right\}} sh^{2\lambda} y dy\right)^{\frac{1}{q}}$$

$$\leq \frac{\left(sh\frac{t}{2}\right)^{-\frac{2\lambda+1}{q}}}{sh\frac{t}{2}} \sup_{r\in R_+} \left(rsh\frac{t}{2}\right) \sup_{u\in R_+} \left(\frac{\left[\left(sh\frac{u}{2}\right) sh\frac{t}{2}\right]_1}{\left[sh\frac{u}{2}\right]_1}\right)^{\frac{\nu}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\frac{\nu-2\lambda-1}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \lesssim \left(sh\frac{t}{2}\right)^{\frac{\nu-4\lambda}{q}} \|J_G^\alpha f\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{\nu-\lambda}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}, \ \ 2 \leq t < \infty. \tag{39}$$

According to (38) and (39), we obtain

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\lambda}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}, \ \ 2 \leq t < \infty. \tag{40}$$

Thus from (37) and (40), we have

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \approx \left(sh\frac{t}{2}\right)^{\frac{\nu-\gamma}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}, \ \ 0 < t < \infty. \tag{41}$$

From the boundedness J_G^α from $\tilde{L}_{1,\lambda,\nu}(R_+)$ to $W\tilde{L}_{q,\lambda,\nu}(R_+)$ and from (22) and (41), we have

$$\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}} \le \left(sh\frac{t}{2}\right)^{\frac{\gamma-\nu}{q}} \|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\frac{\gamma-\nu}{q}} \|f_t\|$$

$$\lesssim \left(sh\frac{t}{2}\right)^{\frac{\gamma-\nu}{q}} \left(sh\frac{t}{2}\right)^{\alpha+\nu-\gamma} \|f\|_{\tilde{L}_{1,\lambda,\nu}}$$

$$= \left(sh\frac{t}{2}\right)^{\alpha-(\gamma-\nu)\left(1-\frac{1}{q}\right)} \|f\|_{\tilde{L}_{1,\lambda,\nu}}$$

$$\lesssim \|f\|_{\tilde{L}_{1,\lambda,\gamma}} \begin{cases} \left(sh\frac{t}{2}\right)^{\alpha-\gamma\left(1-\frac{1}{q}\right)}, & if\ 0<t<2arcsh\,1, \\ \left(sh\frac{t}{2}\right)^{\alpha-(\gamma-\nu)\left(1-\frac{1}{q}\right)}, & if\ 2arcsh\,1<t<\infty. \end{cases}$$

If $1-\frac{1}{q}<\frac{\alpha}{\gamma}$, then at $t\to 0$, we have $\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}=0$ for all $f\in\tilde{L}_{1,\lambda,\nu}(R_+)$. Similarly, if $1-\frac{1}{q}>\frac{\alpha}{\gamma-\nu}$, then at $t\to\infty$ we obtain $\|J_G^\alpha f_t\|_{W\tilde{L}_{q,\lambda,\nu}}=0$ for all $f\in\tilde{L}_{1,\lambda,\nu}(R_+)$. Therefore, $\frac{\alpha}{\gamma}\le 1-\frac{1}{q}\le\frac{\alpha}{\gamma-\nu}$.

References

[1] D. R. Adams, “A note on Riesz potentials,” Duke Mathematical Journal, vol. 42, no. 4, pp. 765–778, 1975.

[2] L. Durand, P. M. Fishbane, L. M. Simmons, Expansion formulas and addition theorems for Gegenbauer functions., J. Math. Phys. 17, 1976, 1933–1948.

[3] G.A.Dadashova, On some properties of the Lorentz-Gegenbauer spaces

[4] V.S. Guliyev, J. Hasanov, Necessary and sufficient conditions for the boundedness of B-Riesz potential in the B-Morrey spaces. J. of Math.Anal. and App., 347 (2008), 113-122.

[5] V.S. Guliyev, E.J. Ibrahimov, S.Ar. Jafarova, Gegenbauer harmonic analysis and approximation of functions on the half line. Advances in Analysis, Vol. 2, No. 3, 2017, 167

[6] E. Ibrahimov, V.S. Guliyev, Conditions for the Lp, λ-Boundedness of the Riesz Potential Generated by the Gegenbauer Differential Operator, Mathematical Notes 105 (5-6) (2019), 674-683

[7] Vagif Guliyev , Elman Ibrahimov, Generalized Gegenbauer shift and some problems of the theory of approximation of functions on the metric of , Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. 35(4) (2015), pp. 19-51.

[8] V.S. Guliyev, J. Hasanov, Yusuf Zeren, Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces. Journal of Mathematical Inequalities, 5(4) 2011, 491-506

[9] V.I. Burenkov, V.S. Guliyev, Necessary and sufficient condi-tions for boundedness of the Riesz potential in the local Morrey-type spaces, Potential Analysis, 30 (2009), no. 3, 211-249.

[10] V.S. Guliyev, E. Ibrahimov, Necessary and sufficient conditions for the boundedness of the Gegenbauer-Riesz potential on Morrey spaces. Georgian Mathematical Journal, 25 (2018), no. 2, 235-248

[11] E.J. Ibrahimov, A Akbulut, The Hardy–Littlewood–Sobolev theorem for Riesz potential generated by Gegenbauer operator, Transactions of A. Razmadze Mathematical Institute 170 (2), 166-199

[12] E.J. Ibrahimov, SA Jafarova, On boundedness of the Riesz potential generated by Gegenbauer differential operator on Morrey spaces, Trans Natl Acad Sci Azerb Ser Phys-Tech Math Sci 37(4), 49-70

[13] E.J. Ibrahimov, GA Dadashova, SE Ekincioglu, On the boundedness of the G-maximal operator and G-Riesz potential in the generalized G-Morrey spaces, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci 40(1), 111-125

[14] E. J. Ibrahimov, V. S. Guliyev, Saadat A. Jafarova, Weighted Boundedness of the Fractional Maximal Operatorand Riesz Potential Generated by Gegenbauer differential Operator

[15] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations. Trans. Amer. Math. Soc. 43, 1938, 126-166 .

G. A. Dadashova
Institute of Mathematics and Mechanics of NASA, Baku, Azerbaijan
E-mail: gdova@mail.ru

Received 10 July 2021
Accepted 17 October 2021

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 9, No 2, 2021, December
ISSN 1560-4055

Mixed problem for systems of semilinear hyperbolic equations with anisotropic elliptic part nonlinear dissipations

Gunay Yusifova

Abstract. In this paper we investigate the mixed problem for some class of quasi linear hyperbolic equations with nonlinear dissipation and with anisotropic elliptic part. The theorems of local solution and global solution are proved.

Key Words and Phrases: global solution, hyperbolic system, existence, anisotropic elliptic parts.

2010 Mathematics Subject Classifications: 35L40, 35L56, 35L76

1. Introduction

The solution of a series of technical problems is brought to non-stationary equations with derivatives of a different order by space variables [1, 2]. For these equations, the problem with initial conditions in time reduces to abstract hyperbolic equations in some function spaces. Those terms of these equations in which only derivatives with respect to space variables participate are called the anisotropic elliptic part.

In this paper, we study a mixed problem for systems of hyperbolic equations with an anisotropic elliptic part in a certain cylinder whose base is a certain three-dimensional cube. The existence and uniqueness of local and global solutions of this mixed problem with Dirichlet boundary conditions are proved.

2. Statement of the problem and main results

Let us introduce the following notation: $x = (x_1, x_2, x_3) \in \Pi_3$,

$$x_1(a) = (a, x_2, x_3), x_2(a) = (x_1, a, x_3), \quad x_3(a) = (x_1, x_2, a).$$

Let us also introduce the notation:

$$\langle u, v\rangle = \int_{\Pi_3} u\,(x)\,v\,(x)\,dx, u, v\in L_2\,(\Pi_3)\,, \|u\| = \sqrt{\langle u, u\rangle}.$$

Let us consider the mixed problem for systems of semilinear equations

$$\left.\begin{array}{l} u_{1tt}+\sum_{k=1}^{3}(-1)^{L_{1k}}D_{x_k}^{2\Lambda_{1k}}u_1+|u_{1t}|^{r_1-1}u_{1t}=g_1(u_1,u_2) \\ u_{2tt}+\sum_{k=1}^{3}(-1)^{L_{2k}}D_{x_k}^{2\Lambda_{2k}}u_2+|u_{2t}|^{r_2-1}u_{2t}=g_2(u_1,u_2) \end{array}\right\} \tag{1}$$

with boundary conditions

$$D_{x_k}^{\beta_k}u_1(t,x_k(0))=D_{x_k}^{\beta_k}u_1(t,x_k(1))=0,\quad \beta_k=0,1,...,\Lambda_{ik}-1, i=1,2,\quad k=1,2,3, \tag{2}$$

and initial conditions

$$u_i(0,x)=\varphi_i(x), u_{it}(0,x)=\psi_i(x),\quad x\in\Pi_3, i=1,2. \tag{3}$$

where $\Lambda_{ik}\in N,\ i=1,2,\ \ k=1,2,3,\ \ g_1$ and g_2 are the following non-linear functions

$$g_1(u_1,u_2)=a_1|u_1+u_2|^{p_1+p_2}(u_1+u_2)+b_1|u_1|^{p_1-1}|u_2|^{p_2+1}u_1,$$

$$g_2(u_1,u_2)=a_2|u_1+u_2|^{p_1+p_2}(u_1+u_2)+b_2|u_1|^{p_1+1}|u_2|^{p_2-1}u_2,$$

a_1,a_2,b_1,b_2,p_1,p_2 are real constants and

$$p_1\geq 0,\ \ p_2\geq 0. \tag{4}$$

We introduce the notation: $\left|\overrightarrow{\Lambda_i}^{-1}\right|=\sum_{k=1}^{3}\frac{1}{\Lambda_{ik}}$, where $\overrightarrow{\Lambda_i}=(\Lambda_{i1},\Lambda_{i2},\Lambda_{i3})$. Let us denote the anisotropic Sobolev space by $W_2^{\overrightarrow{\Lambda_i}}$, i.e.

$$W_2^{\overrightarrow{\Lambda_i}}=W_2^{\overrightarrow{\Lambda_i}}(\Pi_3)=\left\{v:\ v,D_{x_k}^{\Lambda_{ik}}v\in L_2(\Pi_3)\right\},$$

$$\|v\|_{W_2^{\overrightarrow{\Lambda_i}}}=\left[\|v\|_{L_2(\Pi_3)}^2+\sum_{k=1}^{n}\left\|D_{x_k}^{\Lambda_{ik}}v\right\|_{L_2(\Pi_3)}^2\right]^{1/2}.$$

Denote by $\hat{W}_2^{\overrightarrow{\Lambda_i}}$ the next subspace of $W_2^{\overrightarrow{\Lambda_i}}$:

$$\hat{W}_2^{\overrightarrow{\Lambda_i}}=\left\{u:\ u\in W_2^{\overrightarrow{\Lambda_i}},D_{x_k}^{\beta_k}u(t,x_k(0))=D_{x_k}^{\beta_k}u(t,x_k(1))=0,\beta_k=0,1,...,\Lambda_{ik}-1\right\}.$$

Let X be some Banach space and denote by $C([0,T];X)$ the set of continuous functions acting from $[0,T]$ to X: $\|u(t)\|_{C([0,T];X)}=\max\limits_{0\leq t\leq T}\|u(t)\|_X$.

Denote by $C^k([0,T];X)$ the set of continuously differentiable functions of order k acting from $[0,T]$ to X: $\|u(t)\|_{C^k([0,T];X)}=\sum_{i=0}^{k}\left\|u^{(i)}(t)\right\|_{C([0,T];X)}$.

Denote by $C_w([0,T];X)$ the set of weakly continuous functions acting from $[0,T]$ to X.

Let us define the following spaces of functions

$$H_T^1=C\left([0,T];\hat{W}_2^{\overline{\Lambda}_1}\times\hat{W}_2^{\overline{\Lambda}_2}\right)\cap C^1([0,T];L_2(\Pi_3)\times L_2(\Pi_3)),$$

$$H^1_{T,\infty}=\left\{u:\quad u\in L_\infty\left(0,T;\hat{W}_2^{\overline{\Lambda}_1}\times\hat{W}_2^{\overline{\Lambda}_2}\right),u_t\in L_\infty\left(0,T;L_2\left(\Pi_3\right)\times L_2\left(\Pi_3\right)\right)\right\},$$

$$H^2_{T,w}=\{u:\quad u\in C_w\left(\left[0,T\right];\hat{W}_2^{2\overline{\Lambda}_1}\times\hat{W}_2^{2\overline{\Lambda}_2}\right),\quad u_t\in C_w\left(\left[0,T\right];\hat{W}_2^{\overline{\Lambda}_1}\times\hat{W}_2^{\overline{\Lambda}_2}\right)\quad,$$
$$u_{tt}\in C_w\left(\left[0,T\right];L_2\left(\Pi_3\right)\times L_2\left(\Pi_3\right)\right)\},$$

$$H^2_{T,\infty}=\left\{u:\quad u\in L_\infty\left(0,T;\hat{W}_2^{2\overline{\Lambda}_1}\times\hat{W}_2^{2\overline{\Lambda}_2}\right),u_t\in L_\infty\left(0,T;\hat{W}_2^{\overline{\Lambda}_1}\times\hat{W}_2^{\overline{\Lambda}_2}\right)\right.$$
$$\left.u_{tT}\in L_\infty\left(0,T;L_2\left(\Pi_3\right)\times L_2\left(\Pi_3\right)\right)\right\}.$$

It is clear from the expression of the functions $g_i(u_1,u_2)$, that

$$\left|g_i\left(u_1,u_2\right)\right|\le c\left[\,\left|u_1\right|^{p_1+p_2+1}+\left|u_2\right|^{p_1+p_2+1}\right],\; i=1,2,\; c>0. \tag{5}$$

A strong solution of problem (1) - (3) is a pair of function $s\left(u_1\left(\cdot\right),u_2\left(\cdot\right)\right)\in H^2_{T,\infty}$, such that for all $(\eta_1(\cdot),\eta_2(\cdot))\in\hat{W}_2^{\overline{\Lambda}_1}\times\hat{W}_2^{\overline{\Lambda}_2}$ the following equalities hold

$$a)\ \frac{d}{dt}\left\langle u_{1t}\left(t,\cdot\right),\eta_1\left(\cdot\right)\right\rangle+\sum_{k=1}^{3}\left\langle D_{x_k}^{\Lambda_{1k}}u_1\left(t,\cdot\right),D_{x_k}^{\Lambda_{1k}}\eta_1\left(\cdot\right)\right\rangle+$$
$$+\left\langle\left|u_{1t}\left(t,\cdot\right)\right|^{r_1-1}u_{1t}\left(t,\cdot\right),\eta_1\left(\cdot\right)\right\rangle=\left\langle g_1(u_1\left(t,\cdot\right),u_2\left(t,\cdot\right),\eta_1\left(\cdot\right)\right\rangle, \tag{6}$$

$$\frac{d}{dt}\left\langle u_{2t}\left(t,\cdot\right),\eta_2\left(\cdot\right)\right\rangle+\sum_{k=1}^{3}\left\langle D_{x_k}^{\Lambda_{2k}}u_2\left(t,\cdot\right),D_{x_k}^{\Lambda_{1k}}\eta_2\left(\cdot\right)\right\rangle+$$
$$+\left\langle\left|u_{2t}\left(t,\cdot\right)\right|^{r_1-1}u_{2t}\left(t,\cdot\right),\eta_2\left(\cdot\right)\right\rangle=\left\langle g_2(u_1\left(t,\cdot\right),u_2\left(t,\cdot\right),\eta_2\left(\cdot\right)\right\rangle,$$
$$\text{almost all } t\in(0,T), \tag{7}$$

$$b)\ \lim_{t\to+0}\left\|\sum_{k=1}^{3}D_{x_k}^{\Lambda_{ik}}\left[u_i\left(t,\cdot\right)-\varphi_i\left(\cdot\right)\right]\right\|_{L_2(\Pi_3)}=0, i=1,2, \tag{8}$$

$$c)\ \lim_{t\to+0}\int_{\Pi_3}\sum_{k=1}^{3}D_{x_k}^{\Lambda_{ik}}\left[u_{it}\left(t,x\right)-\psi_i\left(x\right)\right]D_{x_k}^{\Lambda_{ik}}\eta_i\left(x\right)dx=0,\quad i=1,2. \tag{9}$$

By a weak solution to problem (1) - (3) we mean the functions $\left(u_1\left(\cdot\right),u_2\left(\cdot\right)\right)\in H^1_{T,\infty}$ such that for all $\left(\eta_1\left(\cdot\right),\eta_2\left(\cdot\right)\right)\in H^1_{T,\infty}$, $\eta_i\left(x,T\right)=0, i=1,2$ the following equalities hold

$$a)\int_0^T\left[\left\langle u_{it}\left(t,\cdot\right),\eta_{it}\left(t,\cdot\right)\right\rangle+\sum_{k=1}^{3}\left\langle D_{x_k}^{\Lambda_{1k}}u_i\left(t,\cdot\right),D_{x_k}^{\Lambda_{1k}}\eta_i\left(\cdot\right)\right\rangle\right]dt+$$
$$+\int_0^T\left\langle\left|u_{it}\left(t,\cdot\right)\right|^{r_i-1}u_{it}\left(t,\cdot\right),\eta_i\left(t,\cdot\right)\right\rangle dt=$$

$$= \int_0^T \langle g_i (u_1 (t, \cdot), u_2 (t, \cdot)), \eta_i (t, \cdot)\rangle dt + \langle \psi_i (\cdot), \eta_i (0, \cdot)\rangle, i = 1, 2, \quad (10)$$

$$b) \lim_{t \to 0} \langle u_i (\cdot, t) - \phi_i (\cdot), \eta_1 (t, \cdot)\rangle_{\hat{W}_2^{\overline{\Lambda}_i}} = 0, \quad i = 1, 2. \quad (11)$$

It is known that under the condition

$$min \left\{ \left|\overline{\Lambda}_1^{-1}\right|, \left|\overline{\Lambda}_2^{-1}\right| \right\} > 2, \quad (12)$$

the embedding

$$\hat{W}_2^{\overline{\Lambda}_i} \subset C(\overline{\Pi}_3), \quad i = 1, 2, \quad \text{(see [3])} \quad (13)$$

is valid. The following theorems on the existence of a local solution of the problem (1) - (3) are true.

Theorem 1. *Suppose that the conditions* (4), (5) *and* (12) *are satisfied. Then for any initial data* $(\phi_1, \phi_2) \in \hat{W}_2^{2\overline{\Lambda}_1} \times \hat{W}_2^{2\overline{\Lambda}_2}$, $(\psi_1, \psi_2) \in \hat{W}_2^{\overline{\Lambda}_1} \times \hat{W}_2^{\overline{\Lambda}_2}$ *there exists* $T' > 0$ *such that the problem* (1) - (4) *has a unique solution* $(u_1, u_2) \in H^2_{T',w}$.

In addition, if $T_{\max} = \max T'$ is the length of the maximum interval of the existence of this solution, then one of the following statements is true:

$$\lim_{t \to T_{max}} \sum_{i=1}^{2} \left[\|u_{it}(t, \cdot)\|^2 + \|u_1(t, \cdot)\|^2_{\hat{W}_2^{\overline{\Lambda}_i}} \right] = +\infty; \quad (14)$$

or

$$T_{\max} = +\infty. \quad (15)$$

Theorem 2. *Suppose that the conditions* (4), (5) *and* (12) *are satisfied. Then for any initial data* $(\phi_1, \phi_2) \in \hat{W}_2^{\overline{\Lambda}_1} \times \hat{W}_2^{\overline{\Lambda}_2}$, $(\psi_1, \psi_2) \in L_2(\Pi_3) \times L_2(\Pi_3)$ *there exists* $T' > 0$ *such that the problem* (1) - (3) *has a unique solution* $(u_1, u_2) \in H^1_{T',w}$.

In addition, if $T_{\max} = \max T'$ is the length of the maximum interval of the existence of this solution, then one of the relations (14) and (15) is true.

In some cases, for any $T > 0$, the local solutions defined by Theorem 1 can be distributed over the entire $[0, T] \times \Pi_3$ region. According to Theorem 1, this is possible if the following a priori estimate is true for local solutions

$$\sum_{i=1}^{2} \left[\|u_{it}(t, x)\|^2 + \left\| \sum_{k=1}^{3} D_{x_k}^{\Lambda_{ik}} u_i(t, x) \right\|^2 \right] \le c, 0 \le t \le T \quad (16)$$

We get this estimate if

$$\lambda = \frac{a_1 (p_1 + 1)}{b_1} = \frac{a_2 (p_2 + 1)}{b_2}, \quad (17)$$

$$a_i \le 0, b_i \le 0, \quad i = 1, 2. \quad (18)$$

When these conditions are met, the following theorem on the global solvability of the problem (1) - (3) is proved.

Theorem 3. *Suppose that the conditions* (4) , (5), (17) *and* (18) *are satisfied, then for any* $T>0$, $(\phi_1,\phi_2)\in \hat{W}_2^{\overline{\Lambda}_1}\times \hat{W}_2^{\overline{\Lambda}_2}$ *and* $(\psi_1,\psi_2)\in L_2(\Pi_3)\times L_2(\Pi_3)$ *the problem* (1) - (3) *has a unique solution* $(u_1(\cdot),u_2(\cdot))\in C\left([0,T];\hat{W}_2^{\overline{\Lambda}_1}\times \hat{W}_2^{\overline{\Lambda}_2}\right)\cap C^1([0,T];L_2(\Pi_3)\times L_2(\Pi_3))$.

3. Proof of Theorem 1

We will prove the theorem using Galyorkin's method. Let $e_j(x)$, $j=1,2,...$-denote the solutions of the following problem:

$$\sum_{k=1}^{3}(-1)^{\Lambda_{ik}}D_{x_k}^{2\Lambda_{ik}}e_{ij}(x)=\lambda_{ij}e_{ji}(x),x\in\Pi_3,$$

$$D_{x_k}^{\beta_k}e_j(x_k(0))=D_{x_k}^{\beta_k}e_j(x_k(1))=0,\beta_k=0,1,...,\Lambda_{ik}-1,k=1,...,n,\ i=1,2.$$

In other words, $e_{ij}(x)$ $,x\in\Pi_3,$ $j=1,2,$ $i=1,2,...$ are eigenfunctions of the operator $\overrightarrow{\mathcal{L}}=\sum_{k=1}^{3}(-1)^{\Lambda_{ik}}D_{x_k}^{2\Lambda_{ik}}$ with the Dirichlet boundary condition (see [4, 5]).

We approximate the functions $\varphi_i(x)$ and $\psi_i(x)$ and the functions $\varphi_{im}(x)$ and $\psi_{im}(x)$, $i=1,2,$ $m=1,2,...$ respectively. So that,

$$\phi_{im}=\sum_{r=1}^{m}a_{irm}e_{ir}(x)\to\phi_i,\text{ in }\hat{W}_2^{2\overline{\Lambda}_i}\text{ as }m\to\infty,i=1,2,\tag{19}$$

$$\psi_{im}=\sum_{r=1}^{m}b_{irm}e_{ir}(x)\to\psi_i,\text{ in }\hat{W}_2^{\overline{\Lambda}_i}\text{ as }m\to\infty,i=1,2.\tag{20}$$

We are looking for approximate solutions of problem (1) - (3) as follows

$$u_{im}(t,x)=\sum_{r=1}^{m}C_{irm}(t)\,e_{ir}(x),\quad i=1,2,$$

so that the functions $C_{irm}(t)$, $i=1,2,$ $r=1,...,m$ are the solutions of the following Cauchy problem for the system of ordinary differential equations

$$\langle u_{im_{tt}}(t,x),e_{ir}(x)\rangle+\sum_{k=1}^{3}\left\langle D_{x_k}^{\Lambda_{ik}}u_{im}(t,x),D_{x_k}^{\Lambda_{ik}}e_{ir}(x)\right\rangle+$$

$$+\int_{\Pi_3}|u_{im_t}(t,x)|^{r_1-1}u_{im_t}(t,x)\,e_{ir}(x)\,dx=\langle g_1(u_{1m}(t,x),u_{2m}(t,x)),e_{ir}(x)\rangle,$$

$$r=1,\ldots,m,\quad i=1,2,\tag{21}$$

$$u_{im}(0,x)=\phi_{im}(x),u_{im_t}(0,x)=\psi_{im}(x),\quad x\in\Pi_n,\quad i=1,2.\tag{22}$$

According to Cauchy-Picard theorem [6], on the existence of a solution of the Cauchy problem for a system of ordinary differential equations, problem (21) - (22) has a solution in some half-interval $[0,t_m)$.

Multiplying both side of each equation (21) by the function $C'_{ir}(t)$, and summing up the resulting equalities, we obtain

$$\langle u_{im_{tt}}(t,x), u_{imt}(t,x)\rangle + \sum_{k=1}^{3}\left\langle D_{x_k}^{\Lambda_{ik}} u_{im}(t,x), D_{x_k}^{\Lambda_{ik}} u_{imt}(t,x)\right\rangle$$

$$+\int_{\Pi_3} |u_{im_t}(t,x)|^{r_1+1} dx =$$

$$= \langle g_i(u_{1m}(t,x), u_{2m}(t,x)), u_{im_t}(t,x)\rangle, i=1,2. \tag{23}$$

It is obvious that

$$\langle u_{im_{tt}}(t,x), u_{im_t}(t,x)\rangle = \frac{1}{2}\frac{d}{dt}\|u_{im_t}(t,\cdot)\|^2, \quad i=1,2, \tag{24}$$

$$\sum_{k=1}^{3}\left\langle D_{x_k}^{\Lambda_{ik}} u_{1m}(t,x), D_{x_k}^{\Lambda_{ik}} u_{1mt}(t,x)\right\rangle = \frac{1}{2}\frac{d}{dt}\sum_{k=0}^{3}\left\|D_{x_k}^{\Lambda_{ik}} u_{im}(t,\cdot)\right\|^2, \ i=1,2. \tag{25}$$

Summing equalities (23) and taking into account (24) and (25) , we obtain:

$$\frac{d}{dt}\sum_{i=1}^{2}\left[\|u_{im_t}(t,\cdot)\|^2 + \sum_{k=1}^{3}\left\|D_{x_k}^{\Lambda_{ik}} u_{im}(t,\cdot)\right\|^2\right] + \sum_{i=1}^{2}\int_{\Pi_n} |u_{im_t}(t,x)|^{r_i+1} dx =$$

$$= \sum_{i=1}^{2}\int_{\Pi_3} g_i(u_{1m}(t,x), u_{2m}(t,x)), u_{im_t}(t,x) dx. \tag{26}$$

Using the Hölder's and Young's inequalities, we obtain the following:

$$\left|\int_{\Pi_3} g_i(u_{1m}(t,x), u_{2m}(t,x)), u_{im_t}(t,x) dx\right| \le$$

$$\le \left(\frac{1}{(r_i+1)\varepsilon}\right)^{\frac{1}{r_i}} \int_{\Pi_3} |g_i(u_{1m}(t,x), u_{2m}(t,x))|^{\frac{r_i+1}{r_i}} dx + \varepsilon\int_{\Pi_3} |u_{im_t}(t,x)|^{r_i+1} dx.$$

Using (5) we have

$$\int_{\Pi_3} |g_i(u_{1m}(t,x), u_{2m}(t,x))|^{\frac{r_i+1}{r_i}} dx \le$$

$$\le C\left[\int_{\Pi_3} |u_1|^{(p_1+p_2+1)\frac{r_i+1}{r_i}} dx + \int_{\Pi_3} |u_2|^{(p_1+p_2+1)\frac{r_i+1}{r_i}} dx\right] \le$$

$$\le C\left[\|u_1\|_{C(\overline{\Pi}_3)}^{(p_1+p_2+1)\frac{r_i+1}{r_i}} + \|u_2\|_{C(\overline{\Pi}_3)}^{(p_1+p_2+1)\frac{r_i+1}{r_i}}\right] \le C\sum_{i=1}^{2}\|u_i\|_{\hat{W}_2^{\bar{\Lambda}_i}}^{(p_1+p_2+1)\frac{r_i+1}{r_i}}. \tag{27}$$

It follows from (26) and (27) that

$$\frac{d}{dt}\sum_{i=1}^{2}\left[\|u_{im_t}(t,\cdot)\|^2+\sum_{k=1}^{3}\left\|D_{x_k}^{\Lambda_{ik}}u_{im}(t,\cdot)\right\|^2\right]+(1-\varepsilon)\sum_{i=1}^{2}\int_{\Pi_3}|u_{im_t}(t,x)|^{r_i+1}dx\leq$$

$$\leq C\sum_{i=1}^{2}\|u_i\|_{\hat{W}_2^{\overline{\Lambda}_i}}^{(p_1+p_2+1)\frac{r_i+1}{r_i}}. \tag{28}$$

Hence, for

$$y=y(t)=\sum_{i=1}^{2}\left[\|u_{im_t}(t,\cdot)\|^2+\sum_{k=1}^{3}\left\|D_{x_k}^{\Lambda_{ik}}u_{im}(t,\cdot)\right\|^2\right]$$

we obtain the following inequality

$$y^{'}\leq C\sum_{i=1}^{2}y^{(p_1+p_2+1)\frac{r_i+1}{r_i}}.$$

From here we obtain the following inequality

$$z'\leq C_1z^p, z(0)=z_0=y_0+1, \tag{29}$$

where $z=z(t)=y(t)+1$, $p=(p_1+p_2+1)\,,\max\left\{\frac{r_1+1}{r_1},\frac{r_2+1}{r_2}\right\}$.
From inequality (29) we obtain that

$$y\leq\frac{y_0+1}{\left[1-c_1(p-1)(y_0+1)^{p-1}t\right]^{\frac{1}{p-1}}}-1.$$

It follows that

$$y(t)\leq 2(y_0+1)\,,0\leq t\leq T', \tag{30}$$

where $T'=\frac{1}{2c_1(p-1)(y_0+1)^{p-1}}$.
From (30) we obtain the following a priori estimate:

$$\sum_{i=1}^{2}\left[\|u_{im_t}(t,\cdot)\|^2+\sum_{k=1}^{3}\left\|D_{x_k}^{\Lambda_{ik}}u_{im}(t,\cdot)\right\|^2\right]\leq$$

$$\leq c_1\sum_{i=1}^{2}\left[\|\psi_{im}\|^2+\sum_{k=0}^{n}\left\|D_{x_k}^{\Lambda_{ik}}\phi_{im}\right\|^2\right],0\leq t\leq T^{'}. \tag{31}$$

According to (19), (20), we get

$$\sum_{i=1}^{2}\left[\|\psi_{im}\|^2+\sum_{k=0}^{3}\left\|D_{x_k}^{\Lambda_{ik}}\phi_{im}\right\|^2\right]\leq c_2. \tag{32}$$

From (31) and (32) it follows that

$$\sum_{i=1}^{2}\left[\left\|u_{im_t}(t,\cdot)\right\|^2+\sum_{k=1}^{3}\left\|D_{x_k}^{\Lambda_{ik}}u_{im}(t,\cdot)\right\|^2\right]\le c_3, \tag{33}$$

where $c_3>0$ is a constant independent of m.
It follows from (28) and (33) that

$$\sum_{i=1}^{2}\int_0^t\int_{\Pi_3}\left|u_{im_s}(s,x)\right|^{r_i+1}dxds\le c_4, 0\le t\le T, \tag{34}$$

where $c_i>0,\ i=1,2,3$ are constants that do not depend on m.
Multiplying both sides of (21) by the function $C_{ik}^{''}(t)$, summing over $k=1$ to m, we get that:

$$\begin{gathered}\left\|u_{im_{tt}}(t,\cdot)\right\|^2\le\left\|u_{im}(t,\cdot)\right\|_{\hat{W}_2^{2\vec{\Lambda}}}\cdot\left\|u_{imtt}(t,x)\right\|+\\+\left(\int_{\Pi_3}\left|u_{im_t}(t,x)\right|^{2r_1}dx\right)^{\frac{1}{2}}\left\|u_{imtt}(t,x)\right\|+\\+\left(\int_{\Pi_3}\left|g_i\left(u_{1m}(t,x),u_{2m}(t,x)\right)\right|^2dx\right)^{\frac{1}{2}}\left\|u_{imtt}(t,x)\right\|\le\\\le\left\|u_{im}(t,\cdot)\right\|_{\hat{W}_2^{2\vec{\Lambda}}}\cdot\left\|u_{imtt}(t,x)\right\|+\\+\left(\max_{x\in\overline{\Pi}_3}\lceil u_{im_t}(t,x)\rceil\right)^{r_1}\left\|u_{imtt}(t,x)\right\|+\\+\max_{x\in\overline{\Pi}_3}\lceil g_i\left(u_{1m}(t,x),u_{2m}(t,x)\right)\rceil\ \left\|u_{imtt}(t,x)\right\|\le\\\le\delta\left\|u_{imtt}(t,x)\right\|+c_\delta\left\|u_{im}(t,\cdot)\right\|_{\hat{W}_2^{2\vec{\Lambda}}}.\end{gathered}$$

From this relation it follows that

$$\left\|u_{im_{tt}}(0,\cdot)\right\|\le C\|\phi_{im}\|_{\hat{W}_2^{2\vec{\Lambda}}}, i=1,2. \tag{35}$$

We differentiate both parts (21) - by t. Then we multiply each of the obtained equations by $c_{ikmtt}(t)$ and add them. Then we will get the following equality

$$\begin{gathered}\left\langle u_{imttt}(t,\cdot),u_{1m_{tt}}(t,\cdot)\right\rangle+\sum_{k=1}^{3}\left\langle D_{x_k}^{\Lambda_{ik}}u_{1mt}(t,x),D_{x_k}^{\Lambda_{ik}}u_{1mtt}(t,x)\right\rangle+\\+\int_{\Pi_3}\frac{\partial}{\partial t}\left(\left|u_{im}(t,x)\right|^{r_i-1}u_{im}(t,x)\right)u_{im_{tt}}(t,x)\,dx=\\=\sum_{j=1}^{2}\left\langle g_{iu_j}\left(u_{1m}(t,x),u_{2m}(t,x)\right)u_{jm_t}(t,x),u_{jm_{tt}}(t,x)\right\rangle.\end{gathered} \tag{36}$$

Since $\beta(s) = |s|^{\gamma-1} s$ is a monotonically increasing function, therefore

$$\int_{\Pi_3} \frac{\partial}{\partial t} \left(|u_{im}(t,x)|^{r_i-1} u_{im}(t,x) \right) u_{im_{tt}}(t,x)\, dx \geq 0. \tag{37}$$

If we evaluate the right side of the equality (36) from above, we get that:

$$|J_j| = \left| \left\langle g_{iu_j}(u_{1m}(t,x), u_{2m}(t,x))\, u_{jm_t}(t,x), u_{jm_{tt}}(t,x) \right\rangle \right| \leq$$

$$\leq \left(\int_{\Pi_3} \left| g_{iu_j}(u_{1m}(t,x), u_{2m}(t,x)) \right|^2 |u_{jm_t}(t,x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Pi_3} |u_{jm_{tt}}(t,x)|^2 dx \right)^{\frac{1}{2}}. \tag{38}$$

In view of the embedding theorems, using (12) , we obtain that $g_{iu_j}(u_{1m}(t,x), u_{2m}(t,x)) \in C\left(\overline{\Pi}_3\right)$. That is why

$$|J_j| = \sup_{x \in \overline{\Pi}_3} \left| g_{iu_j}(u_{1m}(t,x), u_{2m}(t,x)) \right| \left(\int_{\Pi_3} |u_{jm_t}(t,x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\overline{\Pi}_3} |u_{im_{tt}}(t,x)|^2 dx \right)^{\frac{1}{2}} \leq$$

$$\leq c \left(\sup_{x \in \widehat{\overline{\Pi}}_3} |u_{1m}(t,x)|^{p_1+p_2} + \sup_{x \in \widehat{\overline{\Pi}}_3} |u_{2m}(t,x)|^{p_1+p_2} \right) \|u_{im_t}(t,\cdot)\| \cdot \ \|u_{im_{tt}}(t,\cdot)\| \leq$$

$$\leq C \sum_{i=1}^{2} \|u_{im}(t,\cdot)\|_{\hat{W}_2^{\overline{\Lambda}_i}} \ \|u_{im_t}(t,\cdot)\| \, \|u_{im_{tt}}(t,\cdot)\|. \tag{39}$$

Taking into account (37) and (39) in (36), we obtain the inequality

$$\frac{\partial}{\partial t} \sum_{i=1}^{2} \left[\|u_{im_{tt}}(t,\cdot)\|^2 + \|u_{imt}(t,\cdot)\|_{\hat{W}_2^{\overline{\Lambda}_i}} \right] \leq c \sum_{i=1}^{2} \|u_{im_{tt}}(t,\cdot)\|^2. \tag{40}$$

From here we get

$$\sum_{i=1}^{2} \{ \|u_{im_{tt}}(t,\cdot)\|^2 + \|u_{imt}(t,\cdot)\|_{\hat{W}_2^{\overline{\Lambda}_i}} \} \leq c. \tag{41}$$

If we multiply both side (21) by $\lambda_j c_{jkm}(t,\cdot)$ and sum over $j = 1, ..., m$ and $k = 1,2,3$, we get the following equality

$$\left\langle u_{im_{tt}}(t,x), \sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{1m}(t,x) \right\rangle + \left\langle \sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{im}(t,x), \sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{im}(t,x) \right\rangle +$$

$$+ \int_{\Pi_3} |u_{im_t}(t,x)|^{r_i-1} u_{im_t}(t,x), \sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{im}(t,x) dx =$$

$$= \left\langle g_i(u_{1m}(t,x), u_{2m}(t,x)), \sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{im}(t,x) \right\rangle. \tag{42}$$

From here, using the Hölder inequality, we obtain that

$$\left\|\sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{im}(t,x)\right\|^2 \leq \|u_{im_{tt}}(t,\cdot)\| \left\|\sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{im}(t,x)\right\| +$$

$$+\left\|\sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{im}(t,x)\right\| \left(\int_{\Pi_3} |u_{1m_t}(t,x)|^{2r_1} dx \right)^{1/2} +$$

$$+\left\|\sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{1m}(t,x)\right\| \left(\int_{\Pi_3} |g_1(u_{1m}(t,x), u_{2m}(t,x))|^2 dx \right)^{1/2}.$$

Taking into account a priori estimates (41) from here we get

$$\left\|\sum_{k=1}^{3} D_{x_k}^{2\Lambda_{ik}} u_{1m}(t,\cdot)\right\| \leq c. \tag{43}$$

By virtue of (41) - (43) there is a subsequence of $\{u_{1m_k}, u_{2m_k}\}$ which we will denote by $\{u_{1m}, u_{2m}\}$, where

$$u_{im} \to u_i \text{ *-weak in } L_\infty\left(0,T;\hat{W}_2^{2\overline{\Lambda}_i}\right), i=1,2, \tag{44}$$

$$u_{im_t} \to u_{it} \text{ *-weak in } L_\infty\left(0,T;\hat{W}_2^{\overline{\Lambda}_i}\right), i=1,2. \tag{45}$$

$$u_{im_t} \to u_{it} \text{ *-weak in } L^{r_i+1}\left((0,T)\times\Pi_3\right), i=1,2, \tag{46}$$

$$u_{im_{tt}} \to u_{itt} \text{ *-weak in } L_\infty\left(0,T;L_2(\Pi_3)\right), i=1,2. \tag{47}$$

It follows from (44) and (45) that

$$u_i \in C\left([0,T];\hat{W}_2^{\overline{\Lambda}_i}\right), i=1,2. \tag{48}$$

On the other hand, it is known that if $u_1, u_2 \in C\left([0,T];\hat{W}_2^{\overline{\Lambda}_i}\right) \cap L_\infty\left(0,T;\hat{W}_2^{2\overline{\Lambda}_i}\right)$, where $(u_1(\cdot), u_2(\cdot))$ is a solution of the problem (1)-(3) then $u_1, u_2 \in C_w\left([0,T];\hat{W}_2^{2\overline{\Lambda}_i}\right)$ (see [4, 7]). Similarly, we can show that

$$u_{1t} \in C_w\left([0,T];\hat{W}_2^{\overline{\Lambda}_i}\right) \text{ and } u_{itt} \in C_w\left([0,T];L_2(\Pi_3)\right). \tag{49}$$

If in (19) we pass to the limit as $m \to \infty$, then we obtain that the functions (u_1, u_2) satisfy the systems (1).

According to (48), (49), these functions also satisfy the initial conditions (2), (3).

4. Proof of Theorem 2

We choose such functions $\phi_{ik} \in \hat{W}_2^{2\overline{\Lambda}_i}$, $\psi_{ik} \in \hat{W}_2^{\overline{\Lambda}_i}$, $i = 1, 2,\ k = 1, 2, ...$ that

$$\left.\begin{array}{lll} \phi_{ik} \to \phi_i & in & \hat{W}_2^{\overline{\Lambda}_i} \\ \psi_{ik} \to \psi_i & in & L_2\left(\Pi_3\right) \end{array}\right\} \tag{50}$$

as $k \to \infty$.
Then, according to Theorem 1, there exist functions $(u_{1r}, u_{2r}) \in H^2_{T_r}$, $r = 1, 2, ...$ such that

$$\left.\begin{array}{l} u_{1rtt} + \sum_{k=1}^{n} (-1)^{\Lambda_{1k}} D_{x_k}^{2\Lambda_{1k}} u_{1r} + |u_{1rt}|^{r_1 - 1} u_{1rt} = g_1\left(u_{1r}, u_{2r}\right) \\ u_{2rtt} + \sum_{k=1}^{n} (-1)^{\Lambda_{2k}} D_{x_k}^{2L_{2k}} u_{2r} + |u_{2rt}|^{r_2 - 1} u_{2rt} = g_2\left(u_{1r}, u_{2r}\right) \end{array}\right\}, \tag{51}$$

$$D_{x_k}^{\beta_k} u_{1r}\left(t, x_k\left(0\right)\right) = D_{x_k}^{\beta_k} u_{1r}\left(t, x_k\left(1\right)\right) = 0, \quad \beta_k = 0, 1, \ldots, \Lambda_{ik} - 1, i = 1, 2,$$

$$k = 1, ..., n, \quad r = 1, 2, ..., \tag{52}$$

$$u_{ir}\left(0, x\right) = \phi_{ir}\left(x\right), u_{irt}\left(0, x\right) = \psi_{ir}\left(x\right), \quad x \in \Pi_3, i = 1, 2, \quad r = 1, 2, ... \tag{53}$$

are satisfied. In addition, the following a priori estimate is true

$$\sum_{i=1}^{2} \left[\left\| u_{irt}\left(t, \cdot\right) \right\|^2 + \left\| \sum_{k=1}^{3} D_{x_k}^{L_{ik}} u_{ir}(t, x) \right\|^2 \right] \leq c_r, 0 \leq t \leq T_r, \tag{54}$$

$$\int_0^t \int_{\Pi_3} |u_{ir_s}|^{r_i + 1} dx ds \leq c_r, \tag{55}$$

where $c_r = c\left(\sum_{i=1}^{2}\left[\left\|\sum_{k=1}^{3} D_{x_k}^{\Lambda_{ik}} \phi_{ir}\right\|^2 + \|\psi_{ir}\|^2\right]\right)$,

$$T_r = \frac{1}{2\left(p-1\right)\left(\sum_{i=1}^{2}\left[\left\|\sum_{k=1}^{3} D_{x_k}^{\Lambda_{ik}} \phi_{ir}\right\|^2 + \|\psi_{ir}\|^2\right] + 1\right)}. \tag{56}$$

By virtue of (50)

$$c_r \leq c, \quad r = 1, 2, ..., \tag{57}$$

where c_r depends only on the expression $\sum_{i=1}^{2}\left[\left\|\sum_{k=1}^{3} D_{x_k}^{ik} \phi_{ir}\right\|^2 + \|\psi_{ir}\|^2\right]$.
By virtue of (56) and (57) there exists $N_0 \in \{1, 2, ...\}$ such that for $r \geq N_0$ the following inequalities hold

$$T_r \geq T = \frac{1}{4\left(p-1\right)\left(\sum_{i=1}^{2}\left[\sum_{i=1}^{2}\left[\left\|\sum_{k=1}^{3} D_{x_k}^{\Lambda_{ik}} \phi_i\right\|^2 + \|\psi_i\|^2\right] + 1\right] + 1\right)}.$$

Hence the sequence $\{u_{ir}(t,\cdot), u_{ir_t}(t,\cdot)\}$ is bounded in the space

$$L_\infty\left(0,T;\hat{W}_2^{\overline{\Lambda}_i}\right)\times L_\infty\left(0,T;L_2\left(\Pi_3\right)\right).$$

Then from this sequence, we can choose subsequence which we will again denote by $\{u_{1r}(t,\cdot), u_{2r}(t,\cdot)\}$, such that as $k\to\infty$

$$u_{ir}\to u_i \quad \text{*-weakly in } L_\infty\left(0,T;\hat{W}_2^{\overline{\Lambda}_i}\right),\quad i=1,2; \tag{58}$$

$$u_{ir_t}\to u_{it} \quad \text{*-weakly in } L_\infty\left(0,T;L_2\left(\Pi_3\right)\right), i=1,2; \tag{59}$$

$$u_{ik_t}\to u_{it} \quad \text{*-weakly in } L_{r_i+1}\left(\left[0,T\right]\times\Pi_3\right), i=1,2. \tag{60}$$

From (58) and (59) it follows that

$$u_{ik}\to u_i \text{ in } C\left(\left[0,T\right];L_2\left(\Pi_3\right)\right), i=1,2. \tag{61}$$

Let us investigate whether the function $g_i(u_{1r},u_{2r})$ is converted to the function $g_i(u_1,u_2)$, $i=1,2$.

Using Lagrange's Mean Value Theorem, we obtain that

$$J_k=\left\|g_1\left(u_{1r},u_{2r}\right)-g_1\left(u_1,u_2\right)\right\|^2=$$

$$=\int_{\Pi_3}\left|\int_0^1\left(g_{1u_1}\left(u_1+\tau\left(u_{1r}-u_1\right),u_2+\tau\left(u_{2r}-u_2\right)\right)\left(u_{1r}-u_1\right)+\right.\right.$$
$$\left.+g_{2u_2}\left(u_1+\tau\left(u_{1r}-u_1\right),u_2+\tau\left(u_{2r}-u_2\right)\right)\left(u_{2r}-u_2\right)\right)d\tau|^2dx.$$

According to the embedding theorem, the following relations are true.

$$0\le J_k\le \sup_{x\in\overline{\Pi}_3}\left|g_{1u_1}\left(u_1+\tau\left(u_{1r}-u_1\right),u_2+\tau\left(u_{2r}-u_2\right)\right)\right|^2\left\|u_{1r}-u_1\right\|^2+$$

$$+\sup_{x\in\overline{\Pi}_3}\left|g_{1u_2}\left(u_1+\tau\left(u_{1r}-u_1\right),u_2+\tau\left(u_{2r}-u_2\right)\right)\right|^2\left\|u_{2r}-u_2\right\|^2\le$$

$$\le c\left(\left\|u_1\right\|_{C\left(\overline{\Pi}_n\right)},\left\|u_2\right\|_{C\left(\overline{\Pi}_n\right)}\right)\left[\left\|u_{1r}-u_1\right\|^2+\left\|u_{2r}-u_2\right\|^2\right]\le$$

$$\le c\left(\left\|u_1\right\|_{\hat{W}_2^{\overline{\Lambda}_1}},\left\|u_2\right\|_{\hat{W}_2^{\overline{\Lambda}_2}}\right)\left[\left\|u_{1r}-u_1\right\|^2+\left\|u_{2r}-u_2\right\|^2\right].$$

Then it follows from (61) that

$$\lim_{k\to\infty}J_k=0. \tag{62}$$

Thus, according to the relations (58) - (62), if we pass to the limit in the equation (51), we will get that , $(u_1,\ u_2)$ satisfies the problem (1) - (3), so that

$$u_i\left(\cdot\right)\in L_\infty\left(0,T;\hat{W}_2^{\overline{L}_i}\right), u_{it}\left(\cdot\right)\in L_\infty\left(0,T;L_2\left(\Pi_3\right)\right)\cap L_{r_i+1}\left(\left(0,T\right)\times\Pi_3\right),\quad i=1,2,$$

It follows that $h_i(t,x)= \ g_1(u_1,u_2)-|u_{1t}|^{r_1-1}u_{1t}\in L_2((0,T)\times\Pi_3), \quad i=1,2.$
It is obvious that the functions $u_1,\ u_2$ are a solution of the mixed problem

$$u_{itt}+\sum_{k=1}^{3}(-1)^{\Lambda_{ik}}D_{x_k}^{2\Lambda_{ik}}u_i=h_i(t,x),$$

$$D_{x_k}^{\beta_k}u_1(t,x_k(0))=D_{x_k}^{\beta_k}u_1(t,x_k(1))=0, \quad \beta_k=0,1,...,\Lambda_{ik}-1, i=1,2, \quad k=1,...,n,$$

$$u_i(0,x)=\phi_i(x), u_{it}(0,x)=\psi_i(x), \quad x\in\Pi_3, i=1,2.$$

It is known that if the solutions of the problem (1) - (3) satisfy the condition

$$u_i(\cdot)\in L_\infty\left(0,T;\hat{W}_2^{\overline{\Lambda}_i}\right), \quad u_{it}(\cdot)\in L_\infty(0,T;L_2(\Pi_3)), i=1,2,$$

then

$$u_i(\cdot)\in C\left([0,T];\hat{W}_2^{\overline{\Lambda}_i}\right), \quad u_{it}(\cdot)\in C^1([0,T];L_2(\Pi_3)), i=1,2.$$

(see [4, 7]).

5. The existence of a global solution

In some cases, for any $T>0$, the local solutions defined by Theorem 1 can be distributed over the entire $[0,T]\times\Pi_3$ region. According to Theorem 1, this is possible if the following a priori estimate is true for local solutions.

$$\sum_{i=1}^{2}\left[\|u_{it}(t,x)\|^2+\left\|\sum_{k=1}^{3}D_{x_k}^{\Lambda_{ik}}u_i(t,x)\right\|^2\right]\leq c, 0\leq t\leq T. \tag{63}$$

We get this estimate if

$$\lambda=\frac{a_1(p_1+1)}{b_1}=\frac{a_2(p_2+1)}{b_2}, \tag{64}$$

$$a_i\leq 0, b_i\leq 0, \quad i=1,2. \tag{65}$$

Theorem 4. *Suppose that conditions* (4), (12), (64) *and* (65) *are satisfied, then for any* $T>0$, $(\phi_1,\phi_2)\in\hat{W}_2^{\overline{\Lambda}_1}\times\hat{W}_2^{\overline{\Lambda}_2}$ *and* $(\psi_1,\psi_2)\in L_2(\Pi_3)\times L_2(\Pi_3)$ *the problem* (1) - (3) *has a unique solution* $(u_1(\cdot),u_2(\cdot))\in C\left([0,T];\hat{W}_2^{\overline{\Lambda}_1}\times\hat{W}_2^{\overline{\Lambda}_2}\right)\cap C^1([0,T];L_2(\Pi_3)\times L_2(\Pi_3))$.

Proof of the Theorem 4. Assume that $(u_1(\cdot),u_2(\cdot))$ is a local solution of the problem (1)-(3) in the domain $[0,T_{max}]\times\Pi_3$ defined by Theorem 2. Denote $b_i'=-b_i$, $i=1,2$, and multiply both sides of equation (1) by the function $\frac{p_i+1}{b'_i}u_{i_t}(t,x)$.

Integrating the resulting equality over the area $[0,T]\times\Pi_3$, we obtain

$$\frac{p_i+1}{b'_i}\int_0^t\int_{\Pi_3}u_{i_{ss}}(s,x)\,u_{i_s}(s,x)\,dxds+$$

$$+\frac{p_i+1}{b'_i}\int_0^t\int_{\Pi_3}\sum_{k=1}^{3}(-1)^{\Lambda_{ik}}D_{x_k}^{2\Lambda_{ik}}u_i(s,x)\,u_{i_s}(s,x)\,dxds+$$

$$+\frac{a_i(p_i+1)}{b'_i}\int_0^t\int_{\Pi_3}|u_{i_s}(s,x)|^{r_i+1}dxds=$$

$$=\frac{p_i+1}{b'_i}\int_0^t\int_{\Pi_3}g_i(u_1(s,x),u_2(s,x))\,u_{i_s}(s,x)\,dxds,$$

if we use integration by parts and sum the resulting equalities, we get the following:

$$\sum_{i=1}^{2}\frac{p_i+1}{2b'_i}\left[\int_{\Pi_3}|u_{i_t}(t,x)|^2dx+\int_{\Pi_3}\left|D_{x_k}^{\Lambda_{ik}}u_i(s,x)\right|^2dx+\right.$$

$$\left.+\,2\int_0^t\int_{\Pi_3}|u_{i_s}(s,x)|^{r_i+1}dxds\right]+$$

$$+\sum_{i=1}^{2}\frac{p_i+1}{b_i}\int_0^t\int_{\Pi_3}g_i(u_1(s,x),u_2(s,x))\,u_{i_s}(s,x)\,dxds=$$

$$=\sum_{i=1}^{2}\frac{p_i+1}{2b'_i}\left[\int_{\Pi_3}|\psi_i(x)|^2dx+\sum_{k=1}^{n}\int_{\Pi}\left|D_{x_k}^{\Lambda_{ik}}\phi_i(x)\right|^2dt\right]. \tag{66}$$

On the other hand, if we use the expression of the functions $g_1(u_1,u_2)$,$g_2(u_1,u_2)$ and the condition (64) , we get that

$$\sum_{i=1}^{2}\frac{p_i+1}{b_i}\int_0^t\int_{\Pi_3}g_i(u_1(s,x),u_2(s,x))\,u_{i_s}(s,x)\,dxds=$$

$$=\frac{\lambda}{p_1+p_2+2}\int_{\Pi_3}|u_1+u_2|^{p_1+p_2+2}dx+\int_{\Pi_3}|u_1|^{p_1+1}|u_2|^{p_2+1}dx-$$

$$-\frac{\lambda}{p_1+p_2+2}\int_{\Pi_3}|\phi_1+\phi_2|^{p_1+p_2+2}dx-\int_{\Pi_3}|\phi_1|^{p_1+1}|\phi_2|^{p_2+1}dx. \tag{67}$$

Considering (65) and (67) in (66), we obtain the following:

$$\sum_{i=1}^{2}\frac{p_i+1}{2b'_i}\left[\int_{\Pi_3}|u_{i_t}(t,x)|^2dx+\int_{\Pi_3}\left|D_{x_k}^{\Lambda_{ik}}u_i(s,x)\right|^2dx+\right.$$

$$\left.+\,2\int_0^t\int_{\Pi_3}|u_{i_s}(s,x)|^{r_i+1}dxds+\frac{\lambda}{p_1+p_2+2}\int_{\Pi_3}|u_1+u_2|^{p_1+p_2+2}dx+\int_{\Pi_3}|u_1|^{p_1+1}|u_2|^{p_2+1}dx\right]=$$

$$=\sum_{i=1}^{2}\frac{p_i+1}{2b'_i}\left[\int_{\Pi_3}|\psi_i(x)|^2dx+\sum_{k=1}^{n}\int_{\Pi_3}\left|D_{x_k}^{\Lambda_{ik}}\phi_i(x)\right|^2dt\right]+$$

$$+\frac{\lambda}{p_1+p_2+2}\int_{\Pi_n}|\phi_1(x)+\phi_2(x)|^{p_1+p_2+2}dx+\int_{\Pi_3}|\phi_1(x)|^{p_1+1}|\phi_2(x)|^{p_2+1}dx.$$

From this we obtain the a prior estimate (1).

Acknowledgement

The author would like to express her deepest gratitude to Associate Professor A.B. Aliyev for his guidance and valuable comments.

References

[1] S.S. Kokmanyuk, E.T. Yanyugin, L.G. Romanenko, *Oscillations of deformable systems under impulsive and moving loads*, Naukovo dumka, Kiev, 1980, pp 231.

[2] V. Komkov, *Optimal Control Theory for the Damping of Vibrations of Simple Elastic Systems*, Mir, Moscow, 1975 (in Russian).

[3] O.V. Besov, V.P. Ilin, S.M. Nikolski, *Integral Representation of Functions and Embedding Theorem*, V.H. Wilson and Sons, Washington, DC, 1978.

[4] J.L. Lions, *Quelques methods de resolution des problemes aux limites non lineare*, Paris, 1969.

[5] S.Y. Yakubov, Y.Y. Yakubov, *Differential-Operator Equations, Ordinary and Partial Differential Equations*, Chapman and Hall/CRC Press, New York, 2000.

[6] I.Q. Petrovski, *Ordinary Differential Equations*, Hardcover, 1966.

[7] E. Vitillaro, *Global existence of the wave equation with nonlinear boundary damping and source terms*, Journal of Differential Equations, **186**, 2002, 259-298.

Gunay I. Yusifova
Ganja State University, Ganja, Azerbaijan
E-mail: yusifova81@inbox.ru

Received 01 August 2021
Accepted 21 October 2021

Caspian Journal of Applied Mathematics, Ecology and Economics
V. 9, No 2, 2021, December
ISSN 1560-4055

On the Completeness and Minimality of Eigenfunctions of a Non-self-adjoint Spectral Problem With Spectral Parameter in the Boundary Condition

Tehran Gasimov

Abstract. The article considers the following spectral problem:

$$-y'' + q(x)\, y = \lambda y, \quad x \in (0,1),$$

$$\left.\begin{array}{c} y(0) = 0, \\ y'(0) = (a\lambda + b) y(1), \end{array}\right\}$$

where $q(x)$ is a complex-valued summable function, λ is a spectral parameter, a and b are arbitrary complex numbers ($a \neq 0$.) The theorems on completeness and minimality of eigenunctions of a spectral problem in $L_p(0,1) \oplus C$ and $L_p(0,1)$ are proved.

Key Words and Phrases: eigenvalues, eigenunctions, complete and minimal system.

2010 Mathematics Subject Classifications: 34B05, 34B24, 34L10, 34L20

---

1. Introduction

Consider the following spectral problem:

$$-y'' + q(x)\, y = \lambda y, \quad x \in (0,1), \tag{1}$$

$$\left.\begin{array}{c} y(0) = 0, \\ y'(0) = (a\lambda + b) y(1), \end{array}\right\} \tag{2}$$

where $q(x)$ is a complex-valued function, λ is a spectral parameter, a and b are arbitrary complex numbers ($a \neq 0$.) The purpose of this article is to prove the corresponding theorems on the completeness and minimality of a system of eigenfunctions of the spectral problem (1), (2) in the spaces $L_p(0,1) \oplus C$ and $L_p(0,1)$. There are numerous articles and monographs on the study of the spectral properties of problems posed for ordinary differential operators and including spectral parameters in the boundary conditions (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). One can cite articles from recent works [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. Special mention should be made of the works [8, 9, 14, 25, 26] directly related to our paper. So, the case $q(x) \equiv 0$, $b = 0$ is considered in [8, 9], and in [14] under the additional condition $q(x) = q(1-x)$, it is considered the case $b = 0$. Other generalizations of the boundary conditions (2) are also found in [25, 26], where questions of the uniform convergence of spectral expansions and also, under an additional condition $q(x) = q(1-x)$, the basis properties of eigenfunctions in the spaces $L_p(0,1)$ are studied.

2. Auxiliary facts and initial results

In order to obtain the main results, we need some abstract results on complete and minimal systems in the direct sum of a Banach space with a finite-dimensional one. A system $\{u_n\}_{n\in N}$ of a Banach space X is called complete in X if the closure of the linear span of this system coincides with the entire space X, and is called minimal if no element of this system is included in the closed linear span of other elements of this system. Recall also that a system is complete in X if and only if there is no nonzero linear continuous functional that annihilates all elements of this system. A system is minimal in X if and only if it has a biorthogonal system.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system, and $\left\{\widehat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ is its biorthogonal system :

$$\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm}); \qquad \widehat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm}).$$

Let $J = \{n_1, ..., n_m\}$ be some set of m distinct natural numbers and $N_J = N \backslash J$. Assume

$$\delta = det \left\| \beta_{n_i j} \right\|_{i,j=\overline{1,m}}.$$

The following theorem is true.

Theorem 1. *[27, 28] Let $\{\hat{u}_n\}_{n\in N}$ be minimal in X_1 with conjugated system $\left\{\widehat{\vartheta}_n\right\}_{n\in N} \subset X_1^*$. If $\delta \neq 0$, then the system $\{u_n\}_{n\in N_J}$ is minimal in X. In this case, the orthogonally conjugate system has the form*

$$\vartheta_n^* = \frac{1}{\delta} \begin{vmatrix} \vartheta_n & \vartheta_{n1} & \ldots & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & \ldots & \beta_{n_m 1} \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{nm} & \beta_{n_1 m} & \cdots & \beta_{n_m m} \end{vmatrix}.$$

If $\{\hat{u}_n\}_{n\in N}$ is complete and minimal in X_1 and $\delta \neq 0$, then $\{u_n\}_{n\in N_0}$ is complete and minimal in X. If the system $\{\hat{u}_n\}_{n\in N}$ is complete and minimal in X_1 and $\delta = 0$, then the system $\{u_n\}_{n\in N_0}$ is not complete in X.

Accept $\lambda = -\rho^2$. Let us denote the forms included in the boundary conditions (2) as follows:

$$\left.\begin{aligned} U_1(y) &= y(0), \\ U_2(y) &= y'(0) - \left(a\rho^2 + b\right) y(1). \end{aligned}\right\} \tag{3}$$

After these notation, the problem (1), (2) can be written as follows:

$$-y'' + q(x) y + \rho^2 y = 0, x \in (0,1), \tag{4}$$

$$\left.\begin{aligned} U_1(y) &= 0, \\ U_2(y) &= 0. \end{aligned}\right\} \tag{5}$$

It is known that there is a system of fundamental solutions $y_1(x)$ and $y_2(x)$ of the equation (4) in the interval $(0,1)$ and these solutions are regular functions of ρ and at large values of $|\rho|$ with respect to the variable $x \in [0,1]$ uniformly satisfy the following asymptotic relationships:

$$\left.\begin{aligned} y_1^{(j)}(x) &= (\rho\omega_1)^j e^{\rho\omega_1 x}\left[1 + O\left(\tfrac{1}{\rho}\right)\right], \\ y_2^{(j)}(x) &= (\rho\omega_2)^j e^{\rho\omega_2 x}\left[1 + O\left(\tfrac{1}{\rho}\right)\right], \end{aligned}\right\} \tag{6}$$

here $j = 0,1;$ ρ belongs to one of the four S -sectors [29, p. 62], and $\omega_1, \ \omega_2$ are different square roots of -1, numbered so that for $\rho \in S$ the inequality $Re(\rho\omega_1) \leq Re(\rho\omega_2)$ holds. For example, for the sector $S_0 = \left\{\rho : 0 \leq arg\ \rho \leq \frac{\pi}{2}\right\}$ we have $\omega_1 = i, \ \omega_2 = -i$.

The solution of the equation (1) (or (4)) should be in the form of

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Let us choose the constants c_1 and c_2 so that the function $y(x)$ satisfies the boundary conditions (5). Then to find the constants $c_1, \ c_2$ we get the following system of algebraic equations:

$$\left.\begin{aligned} c_1 U_1(y_1) + c_2 U_1(y_2) &= 0, \\ c_1 U_2(y_1) + c_2 U_2(y_2) &= 0. \end{aligned}\right\}$$

It is known that there is a non-trivial solution of this system of algebraic equations when its main determinant (characteristic determinant) $\Delta(\rho)$ equals zero. Thus, the number $\lambda = \rho^2$ is a eigen value of the spectral problem (1) - (2) if and only if it is a solution of the following equation:

$$\Delta(\rho) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix} = U_1(y_1) U_2(y_2) - U_2(y_1) U_1(y_2) = 0. \tag{7}$$

Considering the asymptotic formulas (6) in the expressions of U_1 and U_2 in (3), we obtain the following asymptotic relations:

$$U_1(y_1) = 1 + O(\frac{1}{\rho}), \qquad U_1(y_2) = 1 + O(\frac{1}{\rho}),$$

$$U_2(y_1) = i\rho\left[1 + O(\frac{1}{\rho})\right] - (a\rho^2 + b)e^{i\rho}\left[1 + O(\frac{1}{\rho})\right],$$

$$U_2(y_2) = -i\rho\left[1 + O(\frac{1}{\rho})\right] - (a\rho^2 + b)e^{-i\rho}\left[1 + O(\frac{1}{\rho})\right].$$

By substituting these asymptotic relations in the expression of $\Delta(\rho)$ in (7) and using Birkhof's sign $[A] = A + O\left(\frac{1}{\rho}\right)$ we obtain:

$$\Delta(\rho) = \begin{vmatrix} [1] & [1] \\ i\rho[1] - (a\rho^2 + b)e^{i\rho}[1] & -i\rho[1] - (a\rho^2 + b)e^{-i\rho}[1] \end{vmatrix}$$

Calculating the determinant $\Delta(\rho)$ and consider that when the complex number ρ enters the sector $Re\rho \geq 0$, $Im\rho \geq 0$, the inequality $Re(i\rho) \leq 0 \leq Re(-i\rho)$ satisfies, then we get the following asymptotic relation:

$$\Delta(\rho) = \left(a\rho^2 + b\right)e^{-i\rho}\left[e^{2i\rho} - 1 + O\left(\frac{1}{\rho}\right)\right] - 2i\rho[1] =$$

$$= \left(a\rho^2 + b\right)e^{-i\rho}\left[e^{2i\rho} - 1 - \frac{2i\rho}{a\rho^2 + b}e^{i\rho} + O(\frac{1}{\rho})\right]. \tag{8}$$

So, the eigen values of the problem (1),(2) are the root of the equation

$$\Delta_0(\rho) = e^{2i\rho} - 1 - \frac{2i\rho}{a\rho^2 + b}e^{i\rho} + O(\frac{1}{\rho}) = 0. \tag{9}$$

Note that the number $\lambda = -\frac{b}{a}$ (i.e. $\rho = \pm\sqrt{\frac{a}{b}}i$) cannot be an eigen value, because in this case the function $y(x)$ satisfies the initial conditions $y(0) = 0$, $y'(0) = 0$, from which $y(x) \equiv 0$ is obtained . The roots of the equation $f(\rho) = e^{2i\rho} - 1 = 0$ are the numbers $\widetilde{\rho}_k = \pi k, k = 0, \pm 1,....$. Since $\Delta(\rho)$ is an even function, we will consider only the roots of this function in the right hemisphere. Draw a circle γ_k with the same radius δ $(0 < \delta < \frac{\pi}{2})$ around each point $\widetilde{\rho}_k$. If we denote the region outside these circles by Q_δ, then the function $f(\rho) = e^{2i\rho} - 1$ in this region is bounded by a definite positive constant from below. Indeed, since the function $f(\rho)$ is a periodic function with a period π, it suffices to investigate this function in a vertical stripe bounded by the straight lines $Rez = \pm\frac{\pi}{2}$. While in this stripe the following relations

$$\lim_{Im\rho\to-\infty}|f(\rho)| = +\infty,$$

$$\lim_{Im\rho\to+\infty}|f(\rho)| = 1,$$

are true. Since the function $f(\rho)$ does not vanish outside the circle γ_0 in this band, it is bounded from below by an absolute value positive number α outside the circle γ_0. At large values of $|\rho|$ the inequality $\left|O(\frac{1}{\rho})\right| < \alpha$ is also satisfied. Therefore, according to Rouché's theorem, at sufficiently large values of k, equation (9) has only one root inside the circle γ_k, and if we denote it by ρ_k, then from equation (9) we get the asymptotic formula

$$\rho_k = \pi k + O\left(\frac{1}{k}\right). \tag{10}$$

In addition, since the function $f(\rho) = e^{2i\rho} - 1$ is bounded below by a certain positive number in the domain Q_δ it follows that for sufficiently large $|\rho|$ the function $\Delta_0(\rho) = e^{2i\rho} - 1 - \frac{2i\rho}{a\rho^2+b}e^{i\rho} + O(\frac{1}{\rho})$ is also bounded below by a certain positive number in domain $S_0 \cap Q_\delta$.

Taken into account the asymptotic formula (6) for sufficiently large $|\rho|$ we get the inequality

$$|\Delta(\rho)| \geq M_\delta |\rho|^2 e^\tau, \tag{11}$$

where the constant M_δ independent of ρ, only depends on the number $\delta > 0$.

Thus, the following theorem is proved.

Theorem 2. *The characteristic determinant $\Delta(\rho)$ of the spectral problem* (1),(2) *has the following properties:*

1. *there exists a positive number M_δ such that, in domain $S_0 \cap Q_\delta$ for the sufficiently large $|\rho|$ the inequality $|\Delta(\rho)| \geq M_\delta |\rho|^2 e^\tau$ holds;*
2. *The zeros of the function $\Delta(\rho)$are asymptotically simple and have asymptotics as follows:*

$$\rho_k = \pi k + O\left(\frac{1}{k}\right), k = 0, 1, 2, ...$$

3. Construction of the Green function of the spectral problem (1), (2)

To construct the Green function of the problem (1), (2), it is necessary to obtain an integral representation for the solution of the corresponding non-homogeneous equation. Let us write the non-homogeneous equation as follows

$$-y'' + q(x)y = \lambda y + f(x), x \in (0, 1). \tag{12}$$

When the number λ is not an eigenvalue, if we apply the method of variation of the constant to find the solution of equation (12) that satisfies the boundary conditions (2), we obtain the following formula for the solution $y(x)$ of this equation:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_0^1 g(x, \xi) f(\xi) d\xi,\ x \in (0, 1), \tag{13}$$

where

$$g(x,\xi)=\begin{cases} \frac{1}{2}\frac{1}{W(\xi)}\left(y_1\left(\xi\right)y_2\left(x\right)-y_2\left(\xi\right)y_1\left(x\right)\right), & x>\xi, \\ -\frac{1}{2}\frac{1}{W(\xi)}\left(y_1\left(\xi\right)y_2\left(x\right)-y_2\left(\xi\right)y_1\left(x\right)\right), x<\xi, \end{cases}$$

$W(x)$ is the Wronskan of the functions $y_1(x), y_2(x)$, i.e.

$$W(\xi)=\begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_1'(\xi) & y_2'(\xi) \end{vmatrix}.$$

Let us claim that the general solution (13) of equation (12) satisfies the boundary conditions (2), i.e. is the solution of the boundary value problem (12), (2). This means that the constants c_1, c_2 must be solutions of the following non-homogeneous system of algebraic equations:

$$\begin{cases} c_1U_1(y_1)+c_2U_1(y_2)+\int_0^1 U_1(g)f(\xi)d\xi=0. \\ c_1U_2(y_1)+c_2U(y_2)+\int_0^1 U_2(g)f(\xi)d\xi=0. \end{cases}$$

Since λ is not an eigenvalue, the main determinant of this system is differ from zero, and therefore there exists only one solution. Solving this system and substituting the found values of the constants c_1 and c_2 in equation (13), we obtain the following formula:

$$y\left(x\right)=\int_0^1 G\left(x,\xi,\rho\right)f\left(\xi\right)d\xi. \tag{14}$$

In formula (14) $G(x,\xi,\rho)$ is a Green's function and defined as follows:

$$G\left(x,\xi,\rho\right)=\frac{1}{\Delta\left(\rho\right)}\begin{vmatrix} y_1\left(x\right) & y_2\left(x\right) & g\left(x,\xi\right) \\ U_1\left(y_1\right) & U_1\left(y_2\right) & U_1\left(g\right) \\ U_2\left(y_1\right) & U\left(y_2\right) & U_2\left(g\right) \end{vmatrix}, x,\xi\in[0,1], \tag{15}$$

where

$$g(x,\xi)=\begin{cases} \frac{1}{2}\left(z_1\left(\xi\right)y_2\left(x\right)+z_2\left(\xi\right)y_1\left(x\right)\right), & x\geq\xi, \\ -\frac{1}{2}(z_1(\xi)y_2(x)+z_2(\xi)y_1(x), & x<\xi, \end{cases}$$

$$z_1\left(\xi\right)=\frac{y_2(\xi)}{W(\xi)},\ z_2\left(\xi\right)=-\frac{y_1(\xi)}{W(\xi)},$$

$$U_1\left(g\right)=-\frac{1}{2}\left(U_1\left(y_2\right)z_1\left(\xi\right)+U_1\left(y_1\right)z_2\left(\xi\right)\right),$$

$$U_2\left(g\right)=-\frac{1}{2}\left(z_1\left(\xi\right)y_2'\left(0\right)+z_2\left(\xi\right)y_1'\left(0\right)\right)-\left(a\rho^2+b\right)\frac{1}{2}\left(z_1\left(\xi\right)y_2\left(1\right)+z_2\left(\xi\right)y_1\left(1\right)\right).$$

So, the following lemma is proved.

Lemma 1. *The Green function of the spectral problem* (1),(2) *is defined by the formula* (15).

4. Evaluation of the linearized operator's resolvent. Theorems on the completeness

Let us now reduce the study of the spectral problem (1), (2) to the study of the spectral problem $L\hat{y} = \lambda\hat{y}$ for an operator L acting in the space $L_p(0,1) \oplus C$. The operator L is defined as follows:

$$D(L) = \left\{\hat{y} \in L_p \oplus C : \hat{y} = (y(x), ay(1)), y \in W_p^2(0,1), l(y) \in L_p(0,1), y(0) = 0\right\},$$

for $\hat{y} \in D(L)$ it is true $L\hat{y} = (l(y); y^{'}(0) - by(1))$.

Lemma 2. *Operator L is a closed operator with a compact resolvent and is dense everywhere in the domain $L_p(0,1) \oplus C$. The eigenvalues of the operator L coincide with the eigenvalues of problem* (1), (2)*. Each eigen or associated function $y(x)$ of the problem* (1), (2) *corresponds to an eigen or associated vector $\hat{y} = (y(x), ay(1))$ of the operator L.*

Proof. Let's define the function $F\hat{y} = y(0)$ for the vector $\hat{y} = (y\left(x\right), ay\left(1\right)),\ y\left(x\right) \in W_p^2\left(0,1\right)$. It can be easily checked that the functional F is bounded in space $W_p^2(0,1) \oplus C$ and unbounded in space $L_p(0,1) \oplus C$. Then the considered operator L is a finite-dimensional contraction of the maximum operator $\tilde{L}$, defined as follows:

$$\tilde{L} : L_p \oplus C \to L_p \oplus C,$$

$$D\left(\tilde{L}\right) = \left\{\hat{y} \in L_p \oplus C : \hat{y} = \left(y\left(x\right), ay\left(1\right)\right), y \in W_p^2\left(0,1\right), l\left(y\right) \in L_p\left(0,1\right)\right\},$$

$$\tilde{L}\hat{y} = \left(l\left(y\right), y^{'}(0) - by\left(1\right)\right), \quad \forall\ \hat{y} \in D(\tilde{L}).$$

Then (see [30, 31]) we obtain that the operator L is a closed operator with a compact resolvent and its domain is dense everywhere. The second part of the lemma is examined directly.

Note that since the operator L is closed and dense defined everywhere, it has an adjoint, and the adjoint operator L^* will be the linear operator generated by the spectral problem

$$-z^{''} + \overline{q(x)}z = \lambda z, \tag{16}$$

$$\left.\begin{array}{r} z(1) = 0, \\ z^{'}(1) = -(\overline{a}\lambda + \overline{b})z(0), \end{array}\right\} \tag{17}$$

in the space $L_q(0,1) \oplus C$, where $q = \frac{p}{p-1}$.

To construct the resolvent operator $R\left(\lambda\right) = (L - \lambda I)^{-1}$ take an arbitrary element $\tilde{f} = (f(x), \beta) \in L_p(0,1) \oplus C$ and consider the operator equation $(L - \lambda I)\hat{y} = \hat{f}$. To solve this equation, it is necessary to find a solution to equation (12) that satisfies condition

$$\left.\begin{array}{r} y(0) = 0, \\ y'(0) - (a\lambda + b)y(1) = \beta. \end{array}\right\}. \tag{18}$$

It is obvious that, for each regular number λ the element $\hat{y} = (y(x,\lambda), ay(1,\lambda)) \in D(L)$ will be the solution of the equation $L\hat{y} - \lambda\hat{y} = \tilde{f}$ if and only if the function $y(x)$ will be a solution of the non-homogeneous equation (12), (18). We can present the solution $y(x,\lambda)$ of equations (12), (18) in the form of the sum of two functions:

$$y(x,\lambda) = \phi(x,\lambda) + h(x,\lambda)$$

thus, $\phi(x,\lambda)$ is the solution of the problem (12), (18), and $h(x,\lambda)$ is the solution of the problem (1), (18). The representation (14) for the function $\phi(x,\lambda)$ has already been obtained. Now let's take a representation for the function $h(x,\lambda)$ Let's denote it briefly by $h(x)$. Then let us seek it in the form

$$h(x) = a_1 y_1(x) + a_2 y_2(x), x \in (0,1), \tag{19}$$

where the constants a_1, a_2 must be the solution of the following system of algebraic equations:

$$\begin{cases} U_1(h) = 0, \\ U_2(h) = \beta, \end{cases}$$

or

$$\begin{cases} a_1 U_1(y_1) + a_2 U_1(y_2) = 0, \\ a_1 U_2(y_1) + a_2 U_2(y_2) = \beta. \end{cases}$$

By solving this system of equations, we have

$$a_1 = -\frac{\beta}{\Delta(\rho)} U_1(y_2), \quad a_2 = \frac{\beta}{\Delta(\rho)} U_1(y_1).$$

If we substitute them in (18), we obtain

$$h(x) = \frac{\beta}{\Delta(\rho)} (-U_1(y_2) y_1(x) + U_1(y_1) y_2(x)). \tag{20}$$

Thus, if the number λ is a regular point of the operator L, then we obtain the following representation for the solution $y(x,\lambda)$ of the problem (12), (18):

$$y(x,\lambda) = \int_0^1 G(x,\xi,f) f(\xi) d\xi + \frac{\beta}{\Delta(\lambda)} (-U_1(y_2) y_1(x) + U_1(y_1) y_2(x)) \tag{21}$$

where $G(x,\xi,\lambda)$ is a Green function and is determined by equation (15) .

Now we can proceed to a direct estimate of the resolvent $R(\lambda) = (L - \lambda I)^{-1}$. Let Ω_δ be the image of the domain Q_δ in the complex λ -plane under the mapping $\lambda = \rho^2$.

Theorem 3. *For the resolvent of the operator L, which linearizes the spectral problem (1),(2), in the domain Ω_δ for large values of $|\lambda|$ the following estimate is valid*

$$\|R(\lambda)\| \le \frac{M_\delta}{|\lambda|^{\frac{1}{2}}}. \tag{22}$$

Proof. Let $\hat{f} = (f(x), \beta \in L_p \oplus C$ be an arbitrary fixed element. To estimate the resolvent it is necessary to estimate the vector $(y(x), ay(1)) \in L_p \oplus C$. Let us show that if $\rho \in Q_\delta$, $Im\rho \geq 0$, then for sufficiently large $|\rho|$, for the solution $y(x,\rho)$ of the problem (12), (18) uniformly with respect to the variable $x \in [0,1]$ the following inequality

$$|(y(x,\rho)| \leq \frac{C}{|\rho|}$$

is true; where the constant C is independent of ρ, but depend only on element $\hat{f} \in L_p(0,1) \oplus C$ and δ. Let us accept $\lambda = -\rho^2, \rho = s + i\tau, \tau \geq 0$. Then according to (21) the following representation

$$y(x,\rho) = \int_0^1 G(x,\xi,\rho) f(\xi)\, d\xi + \frac{\beta}{\Delta(\rho)} (-U_1(y_2)y_1(x) + U_1(y_1)y_2(x)) =$$

$$= \int_0^1 G(x,\xi,\rho) f(\xi) d\xi + h(x,\rho))$$

is true, where

$$h(x,\rho) = \beta \frac{-U_1(y_2)y_1(x) + U_2(y_1)y_2(x)}{\Delta(\rho)}. \tag{23}$$

Using asymptotic formulas (3), we can write the following:

$$U_1(y_2)y_1(x) = e^{i\rho x}[1]\cdot[1] = e^{i\rho x}[1] = O(1), (Rei\ \rho \leq 0),$$

$$U_2(y_1)y_2(x) = e^{-i\rho x}[1](i\rho\,[1] - \left(a\rho^2 + b\right) e^{i\rho[1]} = a\rho^2 e^{-i\rho(1-x)}\,[1] = O\left(e^{\tau}\right).$$

Note that these asymptotic formulas uniformly satisfy with respect to the variable $x \in [0,1]$. Considering these asymptotic formulas in (23), we obtain that as $|\rho| \to \infty$, the increase in the numerator of the fraction in (23) is like $O(e^{\tau})$. On the other hand, taking into account the inequality (11) and the above estimate of the increase in the numerator of the fraction in (23), for sufficiently large $|\rho|$ in the domain Q_δ the following estimate

$$|h(x,\rho)| \leq M_\delta' e^{-\tau} \leq \frac{M_\delta'}{|\rho|^2} \tag{24}$$

is obtained, here the constant M_δ' is independent of ρ.

Now, let us estimate the function $\phi(x,\rho)$. Taking into account the asymptotic formulas (3) in the following expressions

$$z_1(\xi) = \frac{y_1(\xi)}{W(\xi)}, \quad z_2(\xi) = -\frac{y_2(\xi)}{W(\xi)},$$

we have:

$$z_1(\xi) = \frac{y_2(\xi)}{\begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_2'(\xi) & y_2'(\xi) \end{vmatrix}} = \frac{e^{-i\rho\xi}[1]}{\begin{vmatrix} e^{i\rho\xi}[1] & e^{-i\rho\xi}[1] \\ i\rho e^{i\rho\xi}[1] & -i\rho e^{-i\rho\xi}[1] \end{vmatrix}} = \frac{e^{-i\rho\xi}\,[1]}{i\rho \begin{vmatrix} [1] & [1] \\ [1] & [1] \end{vmatrix}} = \frac{e^{-i\rho\xi}}{-2i\rho}[1] = \frac{i}{2\rho} e^{-i\rho\xi}[1] \tag{25}$$

$$z_2(\xi)=-\frac{y_1(\xi)}{\begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_1^{'}(\xi) & y_2^{'}(\xi)\end{vmatrix}}=-\frac{e^{i\rho\xi}[1]}{\begin{vmatrix} e^{i\rho\xi}[1] & e^{-i\rho\xi}[1] \\ i\rho e^{i\rho\xi}[1] & -i\rho e^{-i\rho\xi}[1]\end{vmatrix}}=-\frac{e^{-i\rho\xi}\,[1]}{i\rho\begin{vmatrix} [1] & [1] \\ [1] & [1]\end{vmatrix}}=-\frac{i}{2\rho}e^{i\rho\xi}[1] \tag{26}$$

Consider the function $G(x,\xi,\rho)$ in the case of $x\geq\xi$ (the case of $x<\xi$ is considered similarly).

The determinant (12), which determines the function $G\ (x,\xi,\ \rho)$, can be transformed as follows: multiply the first column of the determinant by $\frac{1}{2}z_2(\xi)$, and the second column by $-\frac{1}{2}z_1(\xi)$ and add to last column. Using asymptotic formulas (3), (25), (26), we obtain the following formulas for the elements of the last column of the determinant in (15)

$$P_1=g\,(x,\xi)+\frac{1}{2}y_1\,(x)\,z_2\,(\xi)-\frac{1}{2}y_2\,(x)\,z_1\,(\xi)=$$

$$=\frac{1}{2}z_1(\xi)y_2(x)+\frac{1}{2}z_2(\xi)y_1(x)+\frac{1}{2}y_1(x)z_2(\xi)-\frac{1}{2}y_2\,(x)\,z_1\,(\xi)=$$

$$=y_1\,(x)\,z_2\,(\xi)=e^{i\rho x}\,[1]\left(\frac{-i}{2\rho}e^{-i\rho\xi}\right)[1]=-\frac{i}{2\rho}e^{i\rho(x-\xi)}\,[1]\,, \tag{27}$$

$$P_2=U_1\,(g)+\frac{1}{2}z_2\,(\xi)\,U_1\,(y_1)-\frac{1}{2}z_1\,(\xi)\,U_1\,(y_2)=$$

$$=-\frac{1}{2}z_1(\xi)U_1(y_2)-\frac{1}{2}z_2(\xi)U_1(y_1)+\frac{1}{2}z_2(\xi)U_1(y_1)-$$

$$-\frac{1}{2}z_1\,(\xi)\,U_1\,(y_2)=-z_1\,(\xi)\,U_1\,(y_2)=-\frac{i}{2\rho}e^{-i\rho\xi}\,[1]\cdot[1]=-\frac{i}{2\rho}e^{i\rho\xi}\,[1]\,, \tag{28}$$

$$P_3=-\frac{1}{2}z_1\,(\xi)\,y_2^{'}(0)-\frac{1}{2}z_2\,(\xi)\,y_1^{'}(0)-\frac{1}{2}\left(a\rho^2+b\right)z_1\,(\xi)\,y_2\,(1)$$

$$-\frac{1}{2}(a\rho^2+b)z_2(\xi)y_1(1)+\frac{1}{2}z_2\,(\xi)\,y_1^{'}\,(0)$$

$$-\frac{1}{2}z_2(\xi)(a\rho^2+b)y_1(1)-\frac{1}{2}z_1(\xi)y_2^{'}(0)+\frac{1}{2}(a\rho^2+b)z_1(\xi)y_2(1)=$$

$$=-z_1\,(\xi)\,y_2^{'}(0)-\left(a\rho^2+b\right)z_2\,(\xi)\,y_1\,(1)=i\rho\,[1]\,\frac{i}{2\rho}e^{i\rho\xi}\,[1]-$$

$$(a\rho^2+b)e^{i\rho}[1](-\frac{i}{2\rho}e^{-i\rho\xi}[1])=-\frac{1}{2}e^{i\rho\xi}[1]+\frac{(a\rho^2+b)i}{2\rho}e^{i\rho(1-\xi)} \tag{29}$$

Substituting formulas (27), (28), (29) into the formula (15) of the Green function, we obtain:

$$G(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} y_1(x) & y_2(x) & P_1 \\ U_1(y_1) & U_1(y_2) & P_2 \\ U_2(y_1) & U_2(y_2) & P_3\end{vmatrix}=\frac{e^{i\rho}}{(a\rho^2+b)\Delta_0(\rho)}\times$$

$$\times\begin{vmatrix} e^{i\rho x}[1] & e^{-i\rho x}[1] & -\frac{i}{2\rho}e^{i\rho(x-\xi)}[1] \\ i\rho[1]-(a\rho^2+b)e^{i\rho}[1] & -i\rho[1]-(a\rho^2+b)e^{-i\rho}[1] & \frac{a\rho^2+b}{2\rho}ie^{i\rho(1-\xi)}[1]-\frac{1}{2}e^{i\rho\xi}[1] \end{vmatrix}.$$

Since the last formula contains $0 \le x \le 1$, $0 \le \xi \le 1$, $x \ge \xi$ and $Re(\ i\rho) \le 0$ the powers of the exponents included in the determinant are complex numbers, the real part of which is not positive. We have shown that the function $\Delta_0(\rho)$ is bounded below by some positive number. Thus, the function $G(x,\xi,\rho)$ for large values of $\rho \in S_0 \cap Q_\delta, 0 \le \xi \le x \le 1$, and $|\rho|$ satisfies the following inequality

$$|G(x,\xi,\rho)| \le \frac{C}{|\rho|}; \tag{30}$$

this inequality is satisfied uniformly with respect to the variables x and ξ. Now, taking into account the inequalities (20) and (30), we obtain the following estimate for the solution $y(x,\rho)$ of equations (12), (18) for the fixed element $\hat{f} \in L_p(0,1) \oplus C$:

$$\begin{aligned} |y(x,\rho| &= \left|\int_0^1 G(x,\xi,\rho)\, f(\xi)\, d\xi + h(x,\rho)\right| \le \\ &\le \int_0^1 |G(x,\xi,\rho)|\, |f(\xi)|\, d\xi + |h(x,\rho| \le \\ &\le \frac{C}{|\rho|}(\int_0^1 |f(\xi)|\, d\xi + |\beta|) \le \\ &\le \frac{C}{|\rho|}\left\|\hat{f}\right\|_{L_p \oplus C}. \end{aligned} \tag{31}$$

Hence, we have the inequality

$$\|y\|_{L_p} \le \frac{C}{|\rho|}\left\|\hat{f}\right\|_{L_p \oplus C}.$$

Since the estimate (31) is satisfied uniformly with respect to the variable $x \in [0,1]$, the estimate for $|y(1)|$ is obtained by writing $x = 1$ in (31). Thus, the inequality (22) is true for each $\lambda \in \Omega_\delta$.

Theorem is proved.

Using the Theorem 3, let's prove the following theorem, which is the main result of this section.

Theorem 4. *The system of eigen and associated elements of the operator L is a complete and minimal system in the space $L_p(0,1) \oplus C$, $1 < p < \infty$.*

Proof. The minimality of the system of eigenvectors and associated vectors of the operator L in the space $L_p(0,1) \oplus C$, $1 < p < \infty$, is a consequence of the fact that the resolvent of the operator L is a compact operator in this space [32]. Therefore, we prove the completeness of this system. According to Theorem 2, the resolvent of the operator

L satisfies estimate (22) . This estimate means that the resolvent $R(\lambda) = (L - \lambda I)^{-1}$ satisfies the inequality

$$\left\|R\left(\rho^2\right)\right\| \leq \frac{C_\delta}{|\rho|},\ \rho \in Q_\delta,\ |\rho| \geq r_0. \tag{32}$$

Let us assume that the system of root vectors of the operator L is not complete in space $L_p(0,1) \oplus C$. Then there exists a vector $\hat{g} \in L_q(0,1) \oplus C$ orthogonal to all root subspaces of the operator L, i.e.

$$\left\langle Q_n \hat{f}, \hat{g} \right\rangle = 0, \forall \hat{f} \in L_p(0,1) \oplus C,\ n = 0,1,2,...,$$

and hence $Q_n^* \hat{g} = 0,\ n = 0,1,2,...$; here Q_n denotes the Riesz projectors of the operator L :

$$Q_n = \frac{1}{2\pi i} \oint_{|\lambda - \lambda_n| = r} R(\lambda) d\lambda.$$

In this case it is obvious that $Q_n^*,\ n \in N_0,\ (N_0 = N \cup \{0\})$, will be the Riesz projectors of the adjoint operator L^*. It follows that $R(\lambda, L^*)\hat{g}$ will be an entire function in the entire λ - plane. On the other hand, based on estimate (32), the inequality

$$\|R(\lambda, L^*)\| \leq \frac{C_\delta}{|\lambda|^{\frac{1}{2}}},\ \lambda \in \Omega_\delta,\ |\lambda| \geq r_0^2, \tag{33}$$

is true. Then, by the maximum principle, inequality (33) is satisfied in the entire λ-plane and $R(\lambda, L^*)\hat{g} \to 0$ as $|\lambda| \to \infty$, and by Liouville's theorem this means that an entire function $R(\lambda, L^*)\hat{g}$ is a constant function. Then differentiating this function and taking into account that $\frac{d}{d\lambda} R(\lambda, L^*) = R^2(\lambda, L^*)$ we obtain that $R^2(\lambda, L^*)\hat{g} = 0$. Since for all $\lambda \in \rho(L^*)$ the operator $R(\lambda, L^*)$ is single-valued, we obtain that $\hat{g} = 0$, which means that the root vectors of the operator L form a complete system in the space $L_p(0,1) \oplus C$. Theorem is proved.

From Theorem 4 it also follows that the system of eigenfunctions and associated functions of the spectral problem (1),(2) is overflowing in space $L_p(0,1)$, and in this system one function is superfluous. Therefore, we clarify the question of which function can be excluded from this system while maintaining the completeness and minimality properties. Let the system $\{\hat{z}_n\}_{n=0}^\infty$ be biorthogonal system to $\{\hat{y}_n\}_{n=0}^\infty$. It is a system of root vectors of the adjoint operator L^* moreover $\hat{z}_n = (z_n(x), \bar{a} z_n(0))$, where $z_n(x)$ is an eigenfunction or an associated function of the adjoint spectral problem (16),(17).

The following theorem is true.

Theorem 5. *The system* $\{y_n(x)\}_{n=0,n\neq n_0}^\infty$*, obtained from the system of eigen and associated functions* $\{y_n(x)\}_{n=0,n\neq n_0}^\infty$*, of the spectral problem (1),(2) after removing any eigenfunction* $y_{n_0}(x)$*, corresponding to a simple eigenvalue, is complete and minimal in the space* $L_p(0,1)$, $1 < p < \infty$. *In this case, the biorthogonal system has the form* $\{\vartheta_n(x)\}_{n=0,n\neq n_0}^\infty$*, where*

$$\vartheta_n(x) = z_n(x) - \frac{z_n(0)}{z_{n_0}(0)} z_{n_0}(x).$$

Proof. As follows from Theorem 1, a sufficient condition for the completeness and minimality of the system $\{y_n(x)\}_{n=0,n\neq n_0}^{\infty}$ is the condition $z_{n_0}(0) \neq 0$. For any simple eigenvalue λ_{n_0} this condition is satisfied, because, otherwise, we get that the function $z_{n_0}(x)$ is a solution to equation (16), satisfying the initial conditions $z_{n_0}(1) = 0$, $z'_{n_0}(1) = 0$, so this solution is trivial, i.e. $z_{n_0}(x) \equiv 0$, which contradicts the fact that it is an eigenfunction. Thus, the assertion of the theorem follows from Theorem 1.

Acknowledgement

The author expresses his gratitude to his supervisor, MD. T.B. Kasumov for setting of the problem and for support in its implementation.

References

[1] Atkinson F.B. Discrete and continuous boundary value problems. Moscow, Mir, 1968, 749 p.

[2] Tikhonov A.N., Samarskii A.A. Equations of Mathematical Physics. Moscow, Nauka, 1977, 766 p.

[3] N.Yu. Kapustin, E.I. Moiseev Spectral problems with the spectral parameter in the boundary condition // Diff. Uravn. , 1997, v.33, No. 1, p. 115-119.

[4] Gomilko A.M., Pivovarchik V.N. On bases of eigenfunctions of boundary problem associated with small vibrations of damped non smooth inhomogeneous string // Asympt. Anal. 1999., v.20 No. 3-4, p.301-315.

[5] N. B. Kerimov, V. S. Mirzoev On the Basis Properties of One Spectral Problem with a Spectral Parameter in a Boundary Condition // Sib. Math. J. 2003, v.44, No 5, pp.1041-1045.

[6] Kerimov N.B., Poladov R.G. On basicity in of the system of eigenfunctions of one boundary value problem II // Proc. IMM NAS Azerb., 2005, v.23, p. 65-76.

[7] Kerimov N.B., Aliyev Z. S.On the basis properties of eigenfunctions of a spectral problem with a spectral parameter in the boundary condition // Dokl. RAN, 2007, v.412, No 1, pp. 18-21.

[8] Marchenkov D.B. On the convergence of spectral expansions of functions for problems with a spectral parameter in a boundary condition// Differential Equations, 2005, vol. 41, No. 10, pp. 1496-1500.

[9] Marchenkov D.B. Basis property of the system of eigenfunctions corresponding to a problem with a spectral parameter in the boundary condition// Differential Equations, 2006, vol. 42, No. 6, pp. 905-908.

[10] Gasymov T.B., Mammadova Sh.J. On convergence of spectral expansions for one discontinuous problem with spectral parameter in the boundary condition // Trans. Of NAS of Azerb. 2006,vol. XXVI, No4, p. 103-116.

[11] Gasymov T.B., Huseynli A.A. The basis properties of eigenfunctions of a discontinuous differential operator with a spectral parameter in boundary condition // Proc. of IMM of NAS of Azerb. vol. XXXV(XLIII), 2011, pp. 21-32.

[12] Shahriari M., Akbarfam J.A., Teschl G. Uniqueness for inverse Sturm-Liouville problems with a finite number of transmission conditions. J.Math. Anal. Appl. 395. 19-29(2012).

[13] Shahriari M. Inverse Sturm-Liouville Problem with Eigenparameter Dependent Boundary and Transmission Conditions. Azerb. J. Math., 4(2) (2014), 16-30.

[14] Kerimov N.B., Maris E.A. On the basis properties and convergence of expansions in terms of eigenfunctions for a spectral problem with a spectral parameter in the boundary condition// Proceedings of IMM of NAS of Azerbaijan, 2014, v.40, Special Issue, pp. 245-258.

[15] Bilalov B.T., Gasymov T.B. On bases for direct decomposition. Doklady Mathematics. 93(2) (2016).pp 183-185.

[16] Bilalov B.T., Gasymov T.B. On basicity a system of eigenfunctions of second order discontinuous differential operator // Ufa Mathematical Journal, 2017, v. 9, No 1, p.109-122.

[17] G.V. Maharramova. On the completeness of eigenfunctions of a second-order differential operator, Journal of Contemporary Applied Mathematics, v.8, No. 2, 2018, c.45-55.

[18] Gasymov T.B., G.V. Maharramova. The stability of the basis properties of multiple systems in a Banach space with respect to certain transformations // Caspian Journal of Applied Mathematics, Ecology and Economics. V. 6, No 2, 2018, December, p. 66-77.

[19] Gasymov T.B, Maharramova G.V., Mammadova N.G. Spectral properties of a problem of vibrations of a loaded string in Lebesgue spaces. Transactions of NAS of Azerbaijan, Issue Mathematics, 38(1), 62-68(2018), Series of Physical-Technical and Mathematical Sciences.

[20] Gasymov T.B, Maharramova G.V., Jabrailova A.N. Spectral properties of the problem of vibration of a loaded string in Morrey type spaces. Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, Volume 44, Number 1, 2018, Pages 116-122.

[21] Bilalov B.T., Gasymov T.B., Maharramova G.V. Basis Property of Eigenfunctions in Lebesgue Spaces for a Spectral Problem with a Point of Discontinuity, Diff. Uravn., 2019, v.55, No 12, pp. 1-10.

[22] Maharramova G.V. Properties of Eigenvalues and Eigenfunctions of a Spectral Problem with Discontinuity Point // Caspian Journal of Applied Mathematics, Ecology and Economics V. 7, No 1, 2019, July, pp.114-125.

[23] Gasymov T.B., Maharramova G.V., T.F. Kasimov, Completeness and minimality of eigenfunctions of a spectral problem in spaces $L_p \oplus C$ and L_p Journal of Contemporary Applied Mathematics, v.10, No 2, 2020, pp.84-100.

[24] T.B.Gasymov, A.M.Akhtyamov, N.R.Ahmedzade. On the basicity of eigenfunctions of a second-order differential operator with a discontinuity point in weighted Lebesgue spaces // Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, Volume 46, Number 1, 2020, Pages 32–44.

[25] Maris E.A., Goktaş S. On the spectral properties of a Sturm-Liouville problem with eigenparameter in the boundary condition. Hacet. J. Math. Stat., Volume 49 (4) (2020), 1373 – 1382

[26] Goktaş S., Maris E.A. The uniform convergence of Fourier series expansions of a Sturm-Liouville problem with boundary condition which contains the eigenparameter. Commun.Fac.Sci.Univ.Ank.Ser. A1 M ath. Stat., Volume 70, Number 1, Pages 205-215 (2021)

[27] Gasymov T.B. On necessary and sufficient conditions of basicity of some defective systems in Banach spaces // Trans. NAS Azerb., ser. phys.-tech. math. sci., math.mech., 2006, v.26, No 1, p.65-70.

[28] Gasymov T.B., Garayev T.Z. On necessary and sufficient conditions for obtaining the bases of Banach spaces // Proc. of IMM of NAS of Azerb.2007. vol XXVI(XXXIV). P. 93-98.

[29] M. A. Naimark, Linear Differential Operators. Moscow: Nauka, 1969, 526 p.

[30] Kato T. Perturbation Theory for Linear Operators. Moscow: Mir, 1972, 740 p.

[31] V.E. Lyantse, O.G. Storozh, Methods of the theory of unbounded operators, Naukova Dumka, Kyiv, 1983. 212 p.

[32] Neilson Dunford and Jacob T. Schwartz Linear Operators, Part 3: Spectral Operators, Moscow: Mir, 1971, 661 p.

Tehran Gasimov
Azerbaijan Technical University, Ganja, Azerbaijan
E-mail: tehransend@mail.ru

Received 26 June 2021
Accepted 25 October 2021