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# Uniform Convergence of Spectral Expansions for a Boundary Value Problem with a Boundary Condition Depending on the Spectral Parameter

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**Abstract.** In this paper, we consider the spectral problem for ordinary differential equations of fourth order with a spectral parameter contained in one of the boundary conditions. The uniform convergence of spectral expansions in terms of the system of eigenfunctions of this problem is studied.

**Key Words and Phrases**: spectral problem, eigenvalue, eigenfunction, Riesz basis, Fourier series

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## 1. Introduction

We consider the following eigenvalue problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < l, \tag{1.1}$$

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0, \tag{1.2a}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{1.2b}$$

$$(a\lambda + b)\, y'(l) + (c\lambda + d)\, y''(l) = 0, \tag{1.2c}$$

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{1.2d}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, $q$ is a positive absolutely continuous function on $[0, l]$, $\alpha$, $\beta$, $\delta$, $a$, $b$, $c$, $d$ are real constants such that $0 \le \alpha, \beta \le \pi/2$, $\pi/2 \le \delta < \pi$ (with the exception of the case $\beta = \delta = \pi/2$), $\sigma = bc - ad > 0$.

Note that problem (1.1), (1.2) for $\alpha = \beta = 0$ arises when describing small bending vibrations of an elastic cantilever homogeneous beam, in cross sections of which a longitudinal force acts, the left end of which is fixed, and a load is attached to the right end by means of a weightless rod, which is held in equilibrium by means of an elastic spring (see, e.g., [6, 17]).

 

The uniform convergence Fourier series expansions in the systems of root functions of Sturm-Liouville problems were studied in [7-9, 11-13, 15].

Problem (1.1), (1.2) in the case $\alpha = \beta = 0$ was studied in [1], where, in particular, it was proved that the eigenvalues of this problem are real and simple and form an infinitely increasing sequence. Moreover, the location of the eigenvalues on the real axis is studied, the oscillatory properties of the eigenfunctions are investigated, and the basis property in the space $L_p(0,l)$, $1 < p < \infty$, of the system of eigenfunctions of this problem with one arbitrary remote function is established.

The purpose of this paper is to study the uniform convergence of spectral expansions in terms of eigenfunctions of problem (1.1), (1.2).

## 2. Preliminary

Consider the boundary condition

$$y'(0)\cos\gamma + y''(0)\sin\gamma = 0, \tag{1.2$c'$}$$

where $\gamma \in \left[0, \frac{\pi}{2}\right]\big]$.

By following the argument in Theorem 5.2 of [4] we can prove that for each fixed $\alpha, \beta$ the eigenvalues of problem (1.1), (1.2a) (1.2b), (1.2c$'$), (1.2d) are real, simple and form infinitely increasing sequence $\{\lambda_k(\gamma,\delta)\}_{k=1}^{\infty}$ such that $\lambda_k(\gamma,\delta) > 0$ for $k \geq 2$, and for each $\gamma$ there exists $\delta_0(\gamma) \in \left[\frac{\pi}{2}, \pi\right)$ such that $\lambda_1(\gamma,\delta) > 0$ for $\delta \in [0, \delta_0(\gamma))$, $\lambda_1(\gamma,\delta) = 0$ for $\delta = \delta_0(\gamma)$, $\lambda_1(\gamma,\delta) < 0$ for $\delta \in (\delta_0(\gamma), \pi)$. Moreover, the eigenfunction $y_{k,\gamma,\delta}(x)$, corresponding to the eigenvalue $\lambda_k(\gamma,\delta)$, for $k \geq 2$ has exactly $k-1$ simple zeros, for $k = 1$ has no zeros if $\delta \in [0, \delta_0(\gamma)]$, has an arbitrary number of simple zeros in the interval $(0,1)$ if $\delta \in (\delta_0(\gamma), \pi)$.

For the study of spectral properties of problem (1.1), (1.2) we consider solutions of the initial-boundary problem (1.1), (1.2a) (1.2b), (1.2d).

**Theorem 2.1.** *For every fixed $\lambda \in \mathbb{C}$ there exists a unique non-trivial solution $y(x,\lambda)$ of problem (1.1), (1.2a) (1.2b), (1.2d) up to a constant multiplier.*

The proof of this lemma is similar to that of [1, Lemma 2.3] (see also [10, Theorem 2.1]).

**Remark 2.1.** Let $y(x,\lambda)$ be the solution of (1.1), (1.2a) (1.2b), (1.2d) normalized by the condition $|y(0)| + |Ty(0)| = 1$ for $\lambda > 0$, and $|y'(l)| + |y''(l)| = 1$ for $\lambda \leq 0$. Since Eq. (1.1) depends linearly of the parameter $\lambda$, it follows from the general theory of ordinary differential equations (see, e.g., [16, Ch. I]) that for every fixed $x \in [0,l]$ the function $y(x,\lambda)$ is an entire function of the parameter $\lambda$.

Let $\alpha, \beta \in [0, \pi/2]$ and $\delta \in [\pi/2, \pi)$ be arbitrary fixed, and let $\mathcal{B}_k = (\lambda_{k-1}(0,\delta), \lambda_k(0,\delta))$, $k = 1, 2, \ldots,$ where $\lambda_0(0,\delta) = -\infty$.

It is obvious that the eigenvalues $\lambda_k(0,\delta)$ and $\lambda_k(\pi/2,\delta)$, $k \in \mathbb{N}$, of problem (1.1), (1.2a) (1.2b), (1.2c$'$), (1.2d) for $\gamma = 0$ and $\gamma = \pi/2$ are zeros of entire functions $y'(l,\lambda)$ and $y''(l,\lambda)$ respectively. Note that the function $F(\lambda) = y''(l,\lambda)/y'(l,\lambda)$ is defined in

$\mathcal{B} \equiv \left(\bigcup_{k=1}^{\infty} \mathcal{B}_k\right) \cup (\mathbb{C}\backslash\mathbb{R})$ and is a meromorphic function of finite order, and the eigenvalues $\lambda_k(\pi/2, \delta)$ and $\lambda_k(0, \delta)$, $k \in \mathbb{N}$, are zeros and poles of this function respectively.

**Lemma 3.1.** *The following formula holds:*

$$\frac{dF(\lambda)}{d\lambda} = -\frac{1}{y'^2(l,\lambda)} \int_0^l y^2(x,\lambda)dx,\ \lambda \in \mathcal{B}. \tag{2.1}$$

The proof of this lemma literally repeats the proof of [11, formula (30)].

**Lemma 2.2** *The following limit relation holds:*

$$\lim_{\lambda \to -\infty} F(\lambda) = +\infty. \tag{2.2}$$

The proof of this lemma is similar to that of [1, Lemma 2.8].

In view of [5, Property 1], by (2.1) and (2.2) we get

$$\lambda_1(\pi/2, \delta) < \lambda_1(0, \delta) < \lambda_2(\pi/2, \delta) < \lambda_2(0, \delta) < \ \dots\ . \tag{2.3}$$

Let $m(\lambda) = ay'(l, \lambda) + cy''(l, \lambda)$.

**Remark 2.2.** If follows from boundary condition (1.2c) that if $\lambda$ is an eigenvalue of problem (1.1), (1.2), then $m(\lambda) \neq 0$.

We denote by $s(\lambda)$, $\lambda \in \mathbb{R}$, the number of zeros of the function $y(x, \lambda)$ contained in the interval $(0, l)$.

By following the arguments in Lemma 2.11 from [1] we verify that the following oscillation theorem is valid for the function $y(x, \lambda)$.

**Lemma 2.3.** *If $\lambda \in (\lambda_{k-1}(0, \delta), \lambda_k(\pi/2, \delta))$ for $k \geq 3$, then $k - 2 \leq s(\lambda) \leq k - 1$, and if $\lambda \in [\lambda_k(\pi/2, \delta), \lambda_k(0, \delta)]$ for $k \geq 3$, then $s(\lambda) = k - 1$. Moreover, if $\delta \in [\pi/2, \delta_0(0)]$, then $s(\lambda) = 0$ for $\lambda \in [0, \lambda_1(0, \delta)]$, $0 \leq s(\lambda) \leq 1$ for $\lambda \in (\lambda_1(0, \delta), \lambda_2(\pi/2, \delta))$ and $s(\lambda) = 1$ for $\lambda \in [\lambda_2(\pi/2, \delta), \lambda_2(0, \delta)]$, if $\delta \in [\delta_0(0), \delta_0(\pi/2))$, then $0 \leq s(\lambda) \leq 1$ for $\lambda \in [0, \lambda_2(\pi/2, \delta)$, $s(\lambda) = 1$ for $\lambda \in [\lambda_2(\pi/2, \delta), \lambda_2(0, \delta)]$, and if $\delta \in [\delta_0(\pi/2, \pi)$, then $s(\lambda) = 1$ for $\lambda \in [0, \lambda_2(0, \delta)]$.*

## 3. The properties of eigenvalues and eigenfunctions of problem (1.1), (1.2).

We introduce the following boundary condition

$$ay'(l) + cy''(l) = 0. \tag{1.2c''}$$

Note that, boundary condition (1.2c$''$) in the case $a = 0$ ($c = 0$) coincides with condition (1.2c$'$) for $\gamma = \pi/2$ ($\gamma = 0$). By [2, p. 768] the eigenvalues of problem (1.1), (1.2a) (1.2b), (1.2c″), (1.2d) for each fixed $\alpha$, $\beta$ and for $ac \neq 0$ are real, simple and form infinitely increasing sequence $\{\tau_k(\delta)\}_{k=1}^{\infty}$ such that for any fixed $\alpha$, $\beta$ and $\delta$ the relations hold

$$\lambda_1(\pi/2,\delta) < \tau_1(\delta) < \lambda_1(0,\delta) < \lambda_1(\pi/2,\delta) < \tau_2(\delta) < \lambda_1(0,\delta) < \ \ldots \qquad (3.1a)$$

in the case $a/c > 0$, and

$$\tau_1(\delta) < \lambda_1(\pi/2,\delta) < \lambda_1(0,\delta) < \tau_2(\delta) < \lambda_2(\pi/2,\delta) < \lambda_2(0,\delta) < \ \ldots \qquad (3.1b)$$

in the case $a/c < 0$.

For $a \neq 0$ ($c \neq 0$) we define the number $k_a$ ($k_c$) from the inequality

$$\lambda_{k_a-1} \leq -b/a < \lambda_{k_a} \, (\lambda_{k_c-1} < -d/c \leq \lambda_{k_c}).$$

**Remark 3.1.** If $ac \neq 0$, then $k_a \leq k_c + 1$ for $ac > 0$, and $k_a \geq k_c$ for $ac < 0$.

**Theorem 3.1.** *The eigenvalues of problem (1.1), (1.2) are real and simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ such $\lambda_k > 0$ for $k \geq 3 + \operatorname{sgn}|c|$. Moreover, for $k > k_1 = \max\{k_a, k_c\} + 2$, the eigenvalues have the following arrangement on the real axis:*

$$\lambda_{k-1}(0,\delta) < \lambda_k < \lambda_k(\pi/2,\delta) < \lambda_k(0,\delta), \ \textit{if} \ c = 0, \qquad (3.2a)$$

$$\lambda_{k-2}(0,\delta) < \tau_{k-1}(\delta) = \lambda_{k-1}(\pi/2,\delta) < \lambda_k < \lambda_{k-1}(0,\delta), \ \textit{if} \ a = 0, \qquad (3.2b)$$

$$\lambda_{k-2}(0,\delta) < \lambda_{k-1}(\pi/2,\delta) < \tau_{k-1}(\delta) < \lambda_k < \lambda_{k-1}(0,\delta), \ \textit{if} \ ac > 0, \qquad (3.2c)$$

$$\lambda_{k-2}(0,\delta) < \tau_{k-1}(\delta) < \lambda_k < \lambda_{k-1}(\pi/2,\delta) < \lambda_{k-1}(0,\delta), \ \textit{if} \ ac < 0. \qquad (3.2d)$$

**Remark 3.2.** Using Lemma 2.3 from Theorem 3.1 one can obtain the oscillatory properties of eigenfunctions corresponding to all positive eigenvalues. For example, if $c = 0$, then the function $y_k(x)$ ($k \geq 1$ for $\delta \leq \delta_0(\pi/2)$ and $k_a \geq 2$; $k \geq 2$ for $\delta \leq \delta_0(\pi/2)$ and $k_a = 1$, for $\delta_0(\pi/2) < \delta \leq \delta_0(0)$ and for $\delta > \delta_0(0)$ and $k_a \geq 3$; $k \geq 3$ for $\delta > \delta_0(0)$ and $k_a \leq 2$) has exactly $k-1$ simple zeros for $k < k_a$, has either $k-2$ or $k-1$ simple zeros in the interval $(0,1)$ for $k \geq k_a$.

## 4. Asymptotic formulas for eigenvalues and eigenfunctions of problems (1.1), (1.2a) (1.2b), (1.2c″), (1.2d) with $q \equiv 0$ and (1.1), (1.2)

**Lemma 4.1.** *Let $q \equiv 0$ in Eq. (1.1). Then the following asymptotic formulas for the eigenvalues and eigenfunctions of problem (1.1), (1.2a) (1.2b), (1.2c″), (1.2d) with $q \equiv 0$ are valid:*

$$\sqrt[4]{\tau_k}(\delta) = \left(k - \frac{1 + 3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l} + O\left(\frac{1}{k^2}\right), \ \textit{if}\ \alpha = 0,\ c = 0, \qquad (4.1a)$$

$$\sqrt[4]{\tau_k}(\delta) = \left(k - \frac{2 + 3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l} + \frac{(1+\mathrm{sgn}\beta)\cot\alpha}{2k\pi} + O\left(\frac{1}{k^2}\right), \ \textit{if}\ \alpha \in (0,\pi/2],\ c = 0, \qquad (4.1b)$$

$$\sqrt[4]{\tau_k}(\delta)=\left(k-\frac{2+3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l}+\frac{a/c}{k\pi}+O\left(\frac{1}{k^2}\right),\ \textit{if}\ \alpha=0,\ c\neq 0, \tag{4.1c}$$

$$\sqrt[4]{\tau_k}(\delta)=\left(k-\tfrac{3(1+\mathrm{sgn}\beta)}{4}\right)\tfrac{\pi}{l}+\tfrac{2a/c+(1+\mathrm{sgn}\beta)\cot\alpha}{2k\pi}+O\left(\tfrac{1}{k^2}\right), \quad \textit{if}\ \ \alpha\in(0,\pi/2]\,,\ c\neq 0, \tag{4.1d}$$

$$v_{k,\,\delta}(x)=\sqrt{\tfrac{1+\mathrm{sgn}\beta}{l}}\left\{(1-\mathrm{sgn}\,\beta)\sin\sqrt[4]{\tau_k}\,x-(-1)^{\mathrm{sgn}\,\beta}\cos\sqrt[4]{\tau_k}\,x+ (1-\mathrm{sgn}\,\beta)\,e^{-\sqrt[4]{\tau_k}\,x}+O\left(\tfrac{1}{k^2}\right)\right\},\ \textit{if}\ \alpha=0,\ c=0, \tag{4.2a}$$

$$v_{k,\,\delta}(x)=\sqrt{\tfrac{2-\,\mathrm{sgn}\beta}{l}}\left\{\sin\sqrt[4]{\tau_k}\,x-\mathrm{sgn}\,\beta\cdot\cos\sqrt[4]{\tau_k}\,x-\mathrm{sgn}\,\beta\ \cdot e^{-\sqrt[4]{\tau_k}\,x}+\right.$$
$$\mathrm{sgn}\beta\,\tfrac{\cot\alpha}{(2-\mathrm{sgn}\beta)\sqrt[4]{\tau_k}}\sin\sqrt[4]{\tau_k}\,x-(1+\mathrm{sgn}\beta)\tfrac{\cot\alpha}{2\sqrt[4]{\tau_k}}\cos\sqrt[4]{\tau_k}\,x+ \tag{4.2b}$$
$$\left.(1+\mathrm{sgn}\beta)\,\tfrac{\cot\alpha}{2\sqrt[4]{\tau_k}}\,e^{-\sqrt[4]{\tau_k}\,x}+O\left(\tfrac{1}{k^2}\right)\right\},\ \textit{if}\ \alpha\in(0,\pi/2]\,,\ c=0,$$

$$v_{k,\,\delta}(x)=\sqrt{\tfrac{1+\,\mathrm{sgn}\beta}{l}}\left\{(1-\mathrm{sgn}\beta)\sin\sqrt[4]{\tau_k}\,x-(-1)^{\mathrm{sgn}\,\beta}\ \cos\sqrt[4]{\tau_k}\,x+\right.$$
$$(1-\mathrm{sgn}\,\beta)\,e^{-\sqrt[4]{\tau_k}\,x}\ +(-1)^{k+1}\left(\tfrac{\sqrt{2}}{2}\right)^{\mathrm{sgn}\beta}e^{\sqrt[4]{\tau_k}\,(x-l)}+ \tag{4.2c}$$
$$\left.(-1)^{k+1}\left(\tfrac{\sqrt{2}}{2}\right)^{\mathrm{sgn}\beta}\tfrac{a/c}{\rho_k}\,e^{\sqrt[4]{\tau_k}\,(x-l)}+O\left(\tfrac{1}{k^2}\right)\right\},\ \textit{if}\ \alpha=0,\ c\neq 0,$$

$$v_{k,\,\delta}(x)=\sqrt{\tfrac{2-\,\mathrm{sgn}\beta}{l}}\left\{\sin\sqrt[4]{\tau_k}\,x-\mathrm{sgn}\,\beta\cdot\cos\sqrt[4]{\tau_k}\,x-\mathrm{sgn}\,\beta\cdot e^{-\sqrt[4]{\tau_k}\,x}+\right.$$
$$(-1)^{k+1-\,\mathrm{sgn}\,\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{1-\,\mathrm{sgn}\beta}e^{\sqrt[4]{\tau_k}\,(x-l)}\ -\ \mathrm{sgn}\beta\cdot\tfrac{\cot\alpha}{\rho_k}\sin\sqrt[4]{\tau_k}\,x-$$
$$\tfrac{\cot\alpha}{(2-\mathrm{sgn}\beta)\rho_k}\ \cos\sqrt[4]{\tau_k}\,x+\tfrac{\cot\alpha}{(2-\mathrm{sgn}\beta)\rho_k}\ e^{-\ \sqrt[4]{\tau_k}\,x}+ \tag{4.2d}$$
$$\left.+\ (-1)^{k+\mathrm{sgn}\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{1-\mathrm{sgn}\beta}\tfrac{a/c}{\rho_k}e^{\rho_k(x-l)}\ +\ O\left(\tfrac{1}{k^2}\right)\right\},\ \textit{if}\ \alpha\in\left(0,\,\tfrac{\pi}{2}\right]\,,\ c\neq 0,$$

*where relations (4.2a)-(4.2d) hold uniformly for* $x\in[0,1]$.

The proof of this lemma is similar to that of [3, Lemma 3.1].

By (4.1) from (4.2) by direct calculations we obtain

$$||v_{k,\,\delta}||_2^2=1+O\left(k^{-2}\right), \tag{4.3}$$

where $||\cdot||_2$ is the norm in $L_2(0,l)$.

We denote by $\Psi_k(x)$, $k\in\mathbb{N}$ the normalized eigenfunction, corresponding to the eigenvalue $\tau_k$ of problem (1.1), (1.2a) (1.2b), (1.2c′), (1.2d) with $q\equiv 0$, i.e. $\Psi_k(x)=\frac{v_{k,\,\delta}(x)}{||v_{k,\,\delta}||_2}$. Then by (4.3) for $\Psi_k(x)$ the asymptotic formulas (4.2a)-(4.2d) are valid.

The function $q_0(x)$, $x \in [0,l]$, and the number $q_0$ we define as follows:

$$q_0(x) = \int_0^x q(t)dt, \ q_0 = \int_0^l q(t)dt.$$

**Lemma 4.2.** *For the eigenvalues and eigenfunctions of problem (1.1), (1.2) we have the following asymptotic formulas:*

$$\sqrt[4]{\lambda_k} = \left(k - \frac{5+3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l} + \frac{q_0}{4k\pi} + O\left(\frac{1}{k^2}\right), \ \textit{if } \alpha = 0, \ c = 0, \tag{4.4a}$$

$$\sqrt[4]{\lambda_k} = \left(k - \tfrac{6+3\,\mathrm{sgn}\beta}{4}\right)\tfrac{\pi}{l} + \tfrac{q_0+2(1+\mathrm{sgn}\beta)\cot\alpha}{4k\pi} + O\left(\tfrac{1}{k^2}\right), \quad \textit{if } \alpha \in (0,\pi/2]\,, \ c = 0, \tag{4.4b}$$

$$\sqrt[4]{\lambda_k} = \left(k - \frac{6+3\,\mathrm{sgn}\beta}{4}\right)\frac{\pi}{l} + \frac{q_0+4a/c}{4k\pi} + O\left(\frac{1}{k^2}\right), \ \textit{if } \alpha = 0, \ c \neq 0, \tag{4.4c}$$

$$\sqrt[4]{\lambda_k} = \left(k - \tfrac{7+3\,\mathrm{sgn}\beta)}{4}\right)\tfrac{\pi}{l} + \tfrac{q_0+4a/c+2(1+\mathrm{sgn}\beta)\cot\alpha}{4k\pi} + O\left(\tfrac{1}{k^2}\right), \quad \textit{if } \alpha \in (0,\pi/2]\,, \ c \neq 0, \tag{4.4d}$$

$$y_k(x) = \sqrt{\tfrac{1+\mathrm{sgn}\,\beta}{l}}\left\{(1-\mathrm{sgn}\,\beta)\sin\sqrt[4]{\lambda_k}x - (-1)^{\mathrm{sgn}\beta}\cos\sqrt[4]{\lambda_k}x + (1-\mathrm{sgn}\,\beta)\,e^{-\sqrt[4]{\lambda_k}x} + (-1)^{\mathrm{sgn}\,\beta}\tfrac{(1-\mathrm{sgn}\,\beta)q_0 - q_0(x)}{4\varrho_k}\sin\sqrt[4]{\lambda_k}x - (-1)^{\mathrm{sgn}\,\beta}\tfrac{q_0+(1-\mathrm{sgn}\,\beta)q_0(x)}{4\varrho_k}\cos\sqrt[4]{\lambda_k}x + (1-\mathrm{sgn}\,\beta)\tfrac{q_0-q_0(x)}{4\varrho_k}\,e^{\sqrt[4]{\lambda_k}x} + O\left(\tfrac{1}{k^2}\right)\right\}, \quad \textit{if } \alpha = 0, \ c = 0, \tag{4.5a}$$

$$y_k(x) = \sqrt{\tfrac{2-\mathrm{sgn}\beta}{l}}\left\{\sin\sqrt[4]{\lambda_k}x - \mathrm{sgn}\,\beta\cdot\cos\sqrt[4]{\lambda_k}x - \mathrm{sgn}\,\beta\cdot e^{-\sqrt[4]{\lambda_k}x} - \mathrm{sgn}\,\beta\,\tfrac{q_0(x)+4\cot\alpha}{4\rho_k}\sin\sqrt[4]{\lambda_k}x - \tfrac{q_0(x)+2(1+\mathrm{sgn}\,\beta)\cot\alpha}{4\rho_k}\cos\sqrt[4]{\lambda_k}x + \tfrac{\mathrm{sgn}\,\beta\cdot q_0(x)+2(1+\mathrm{sgn}\,\beta)\cot\alpha}{4\rho_k}\,e^{-\sqrt[4]{\lambda_k}x} + O\left(\tfrac{1}{k^2}\right)\right\}, \ \textit{if } \alpha \in (0,\pi/2]\,, \ c = 0, \tag{4.5b}$$

$$y_k(x)=\sqrt{\tfrac{1+\text{sgn}\beta}{l}}\left\{(1-\operatorname{sgn}\beta)\sin\sqrt[4]{\lambda_k}x-(-1)^{\operatorname{sgn}\beta}\cos\sqrt[4]{\lambda_k}x+\right.$$

$$(1-\operatorname{sgn}\beta)\,e^{-\sqrt[4]{\lambda_k}x}+(-1)^{k+\text{sgn}\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{\text{sgn}\beta}e^{\sqrt[4]{\lambda_k}(x-l)}+$$

$$(-1)^{\operatorname{sgn}\beta}\frac{(1-\operatorname{sgn}\beta)(q_0+4a/c)-q_0(x)}{4\varrho_k}\sin\sqrt[4]{\lambda_k}x- \tag{4.5c}$$

$$(-1)^{\operatorname{sgn}\beta}\frac{q_0+4a/c+(1-\operatorname{sgn}\beta)\,q_0(x)}{4\varrho_k}\cos\sqrt[4]{\lambda_k}x+(1-\operatorname{sgn}\beta)\frac{q_0+4a/c-q_0(x)}{4\varrho_k}e^{-\sqrt[4]{\lambda_k}x}+$$

$$\left.(-1)^{k+\text{sgn}\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{\text{sgn}\beta}\frac{q_0(x)}{4\varrho_k}e^{\sqrt[4]{\lambda_k}(x-l)}+O\left(\tfrac{1}{k^2}\right)\right\},\ \textit{if } \alpha=0,\ c\neq 0,$$

$$y_k(x)=\sqrt{\tfrac{2-\text{sgn}\beta}{l}}\left\{\sin\sqrt[4]{\lambda_k}x-\operatorname{sgn}\beta\cdot\cos\sqrt[4]{\lambda_k}x-\operatorname{sgn}\beta\,\cdot\,e^{-\sqrt[4]{\lambda_k}x}+\right.$$

$$(-1)^{k+\text{sgn}\beta}\left(\tfrac{\sqrt{2}}{2}\right)^{\text{sgn}\beta}e^{\sqrt[4]{\lambda_k}(x-l)}-\operatorname{sgn}\beta\,\frac{4\cot\alpha+q_0(x)}{4\varrho_k}\sin\sqrt[4]{\lambda_k}x- \tag{4.5d}$$

$$\frac{q_0(x)+2(1+\operatorname{sgn}\beta)\cot\alpha}{4\varrho_k}\cos\sqrt[4]{\lambda_k}x+\frac{\text{sgn}\beta\cdot q_0(x)+4\cot\alpha}{4\varrho_k}e^{-\sqrt[4]{\lambda_k}x}$$

$$\left.+(-1)^{k+\text{sgn}\beta}\frac{q_0(x)-q_0+4a/c}{4\varrho_k}e^{\sqrt[4]{\lambda_k}(x-l)}+O\left(\tfrac{1}{k^2}\right)\right\},\ \textit{if } \alpha\in(0,\pi/2]\,,\ c\neq 0,$$

*where relations (4.5a)-(4.5d) hold uniformly for* $x\in[0,1]$.

The proof of this lemma is similar to that of [3, Lemma 3.2].

## 5. Uniform convergence of expansions in Fourier series of subsystems of eigenfunctions of problem (1.1), (1.2)

Let

$$\delta_k=||y_k||_2^2+\sigma^{-1}m_k^2. \tag{5.1}$$

Since $\sigma>0$ and $m_k\neq 0$ it follows from (5.1) that

$$\delta_k>0,\ k\in\mathbb{N}. \tag{5.2}$$

**Theorem 5.1.** *Let r be the any fixed positive integer. Then the system* $\{y_k(x)\}_{k=1,\,k\neq r}^{\infty}$ *of eigenfunctions of problem (1.1), (1.2) forms a basis in* $L_p(0,l)$, $1<p<\infty$, *and for* $p=2$ *this basis is a Riesz basis. The system* $\{u_k(x)\}_{k=1,\,k\neq r}^{\infty}$, *conjugate to the system* $\{y_k(x)\}_{k=1,\,k\neq r}^{\infty}$, *is defined by the equality:*

$$u_k(x)=\delta_k^{-1}\left\{y_k(x)-m_k m_r^{-1}y_r(x)\right\},\ k\in\mathbb{N},\ k\neq r, \tag{5.3}$$

The proof of this theorem repeats the proof of Theorem 4.1 of [1].

If $r$ is an arbitrary fixed natural number, then by Theorem 5.1 the Fourier series expansion

$$f(x)=\sum_{k=1,\,k\neq r}^{\infty}(f,u_k)\,y_k(x), \tag{5.4}$$

in the system $\{y_k(x)\}_{k=1,\,k\neq r}^{\infty}$ of any continuous function $f(x)$ on $[0,1]$ converges in $L_p\,(0,l)$, $1<p<\infty$, and converges unconditionally for $p=2$.

The main result of this paper is the following theorem.

**Theorem 5.1.** *Let $r$ be the an arbitrary fixed positive integer, $f(x)$ is continuous function on the interval $[0,l]$ and has uniformly convergent on $[0,1]$ Fourier series in the system $\{\Psi_k(x)\}_{k=1}^{\infty}$. Then the series (5.4) converges uniformly on $[0,1]$.*

**Proof.** If $\alpha=\beta=0$ and $c=0$ in boundary conditions (1.2$a$)-(1.2$c$) and (1.2$c''$), then it follows from (4.1$a$) and (4.2$a$) that for eigenvalues and eigenfunctions of problem (1.1), (1.2a), (1.2b), (1.2c$''$), (1.2d) with $q\equiv 0$ the following asymptotic formulas hold:

$$\sqrt[4]{\tau_k(\delta)}=\left(k-\frac{1}{4}\right)\frac{\pi}{l}+O\left(\frac{1}{k^2}\right), \tag{5.5}$$

$$\Psi_k(x)=\sqrt{\tfrac{1}{l}}\left\{\sin\left(k-\tfrac14\right)\tfrac{\pi}{l}x-\cos\left(k-\tfrac14\right)\tfrac{\pi}{l}x+\,e^{-\left(k-\frac14\right)\frac{\pi}{l}\,x}+O\left(\tfrac{1}{k^2}\right)\right\}, \tag{5.6}$$

where (5.6) holds uniformly for $x\in[0,l]$.

It follows from (4.4a) and (4.5a) that for eigenvalues and eigenfunctions of problem (1.1), (1.2) with $\alpha=\beta=0$ and $c=0$ the asymptotic formulas are valid:

$$\sqrt[4]{\lambda_k}=\left(k-\frac{5}{4}\right)\frac{\pi}{l}+\frac{q_0}{4k\pi}+O\left(\frac{1}{k^2}\right), \tag{5.7}$$

$$\begin{gathered}y_k(x)=\sqrt{\tfrac{1}{l}}\left\{\sin\left(k-\tfrac54\right)\tfrac{\pi}{l}\,x-\cos\left(k-\tfrac54\right)\tfrac{\pi}{l}\,x+\,e^{-\left(k-\frac54\right)\frac{\pi}{l}\,x}+\right.\\ \tfrac{(q_0-\,q_0(x))l+q_0x}{4k\pi}\sin\left(k-\tfrac54\right)\tfrac{\pi}{l}\,x-\tfrac{(q_0+q_0(x))l-q_0x}{4k\pi}\cos\left(k-\tfrac54\right)\tfrac{\pi}{l}\,x+\\ \left.\tfrac{(q_0-q_0(x))l-q_0x}{4k\pi}\,e^{-\,\left(k-\frac54\right)\frac{\pi}{l}\,x}+O\left(\tfrac{1}{k^2}\right)\right\},\end{gathered} \tag{5.8}$$

where (5.8) holds uniformly for $x\in[0,l]$.

By asymptotic formulas (5.6) and (5.8) we have

$$\begin{gathered}y_k(x)=\Psi_{k-1}(x)+\sqrt{\tfrac{1}{l}}\left\{\tfrac{(q_0-\,q_0(x))l+q_0x}{4k\pi}\sin\left(k-\tfrac54\right)\tfrac{\pi}{l}\,x-\tfrac{(q_0+q_0(x))l-q_0x}{4k\pi}\times\right.\\ \left.\cos\left(k-\tfrac54\right)\tfrac{\pi}{l}\,x+\tfrac{(q_0-q_0(x))l-q_0x}{4k\pi}\,e^{-\,\left(k-\frac54\right)\frac{\pi}{l}\,x}+O\left(\tfrac{1}{k^2}\right)\right\}.\end{gathered} \tag{5.9}$$

In view of (5.8) we get

$$y_k'(l)=(-1)^k\sqrt{2}\sqrt{\frac{1}{l}}\left(\frac{q_0}{4}+O\left(\frac{1}{k}\right)\right),$$

$$y_k''(l) = (-1)^k \sqrt{2}\sqrt{\frac{1}{l}}\,\frac{k^2\pi^2}{l^2}\left(1 + \frac{q_0 l - 10\pi}{4k\pi} + O\left(\frac{1}{k^2}\right)\right),$$

which implies that

$$m_k = a y_k'(l) + c y_k''(l) = -\frac{b y_k'(l) + d y_k''(l)}{\lambda_k} = O\left(\frac{1}{k^2}\right). \tag{5.10}$$

Direct calculations show that

$$||y_k||_2^2 = 1 + O\left(\frac{1}{k^2}\right). \tag{5.11}$$

Then by (5.10) and (5.11) it follows from (5.1) that

$$\delta_k = ||y_k||_2^2 + \sigma^{-1} m_k^2 = 1 + O\left(\frac{1}{k^2}\right). \tag{5.12}$$

Let $r$ be the any fixed positive integer. By (5.10)-(5.12), from (5.3) we get

$$u_k(x) = \delta_k^{-1}\left\{y_k(x) - m_k\, m_r{}^{-1}\, y_r(x)\right\} = y_k(x) + O\left(\frac{1}{k^2}\right). \tag{5.13}$$

Note that for uniformly convergence of series (5.4) it is necessary and sufficient uniform convergence of the series

$$\sum_{k=r+1}^{\infty} (f, u_k) y_k(x). \tag{5.14}$$

By (5.13) we have

$$\sum_{k=r+1}^{\infty} (f, u_k) y_k(x) = \sum_{k=r+1}^{\infty} (f, y_k) y_k(x) + \sum_{k=r+1}^{\infty} O\left(\frac{1}{k^2}\right). \tag{5.15}$$

it follows from (5.9) that

$$y_k(x) = \Phi_{k-1}(x) + O\left(\frac{1}{k}\right). \tag{5.16}$$

According to (5.16) we have

$$\sum_{k=r+1}^{\infty} (f, y_k) y_k(x) = \sum_{k=r+1}^{\infty} (f, y_k)\Phi_{k-1}(x) + \sum_{k=r+1}^{\infty} (f, y_k) O\left(\tfrac{1}{k}\right). \tag{5.17}$$

Since $\{y_k(x)\}_{k=1,\,k\neq r}^{\infty}$ is a Riesz basis in $L_2(0,l)$ the following estimate holds

$$\sum_{k=l+1}^{\infty} \left|(f, y_k) O\left(\tfrac{1}{k}\right)\right| \le \text{const}\left\{\sum_{k=l+1}^{\infty} |(f, y_k)|^2 + \sum_{k=l+1}^{\infty} \tfrac{1}{k^2}\right\} < +\infty.$$

Hence to study the uniform convergence of the series (5.14), it suffices to study the uniform convergence of the series

$$\sum_{k=r+1}^{\infty}(f,y_k)\Psi_{k-1}(x) \tag{5.18}$$

Let

$$p_1(x)=\sqrt{\frac{1}{l}}\,\frac{(q_0-q_0(x))l+q_0x}{4\pi},\ p_2(x)=\sqrt{\frac{1}{l}}\,\frac{(q_0+q_0(x))l-q_0x}{4\pi},$$

$$p_3(x)=\sqrt{\frac{1}{l}}\,\frac{(q_0-q_0(x))l-q_0x}{4\pi}, e_{k,1}(x)=\sin\left(k-\frac{5}{4}\right)\frac{\pi}{l}\,x,\ \ e_{k,2}(x)=\cos\left(k-\frac{5}{4}\right)\frac{\pi}{l}\,x,$$

$$e_{k,3}(x)=e^{-\left(k-\frac{5}{4}\right)\frac{\pi}{l}\,x},\ x\in[0,l].$$

Then by (5.9) we get

$$y_k(x)=\Psi_{k-1}(x)+k^{-1}p_1(x)e_{k,1}(x)+k^{-1}p_2(x)\,e_{k,2}(x)+k^{-1}p_3(x)\,e_{k,3}(x)+O\left(\frac{1}{k^2}\right),$$

whence implies that

$$\sum_{k=r+1}^{\infty}(f,y_k)\Phi_{k-1}(x)=\sum_{k=r+1}^{\infty}(f,\Phi_{k-1})\,\Phi_{k-1}(x)+$$
$$\sum_{k=r+1}^{\infty}k^{-1}(fp_1,e_{k,1})\,\Phi_{k-1}(x)+\sum_{k=r+1}^{\infty}k^{-1}(fp_2,e_{k,2})\,\Phi_{k-1}(x)+$$
$$\sum_{k=r+1}^{\infty}k^{-1}(fp_3,e_{k,3})\,\Phi_{k-1}(x)+\sum_{k=r+1}^{\infty}O\left(k^{-2}\right)\Phi_{k-1}(x). \tag{5.19}$$

By virtue of [14, Lemma 5] each of the systems $\{e_{k,j}\}_{k=1}^{\infty}$, $j=1,\ 2.\,3$, is a Bessel system. Therefore, we have the following estimates

$$\sum_{k=l+1}^{\infty}\left|\frac{(fp_j,e_{k,\,j})}{k}\right|\le\text{const}\left(\sum_{k=l+1}^{\infty}\frac{1}{k^2}+\sum_{k=l+1}^{\infty}|(fp_j,e_{k,\,j})|^2\right)\le\text{const}\,(1+||f||_2^2),\ j=1,\ 2,\ 3.$$

By virtue of the condition of this theorem the series $\sum\limits_{k=r+1}^{\infty}(f,\Phi_{k-1})\,\Phi_{k-1}(x)$ converges uniformly on the interval $[0,1]$. Then, as seen from (5.19), the series (5.18) converges uniformly on $[0,1]$.

The rest cases are treated in a similar way. The proof of this theorem is complete.

## References

[1] Z.S. Aliyev, Basis properties of a fourth order differential operator with spectral parameter in the boundary condition, *Cent. Eur. J. Math.*, **8**(2) (2010), 378-388.

[2] Z.S. Aliev, Basis properties in $L_p$ of systems of root functions of a spectral problem with spectral parameter in a boundary condition, *Diff. Equ.*, **47**(6) (2011), 766-777.

[3] Z.S. Aliyev, K.F. Abdullayeva, Uniform convergence of spectral expansions in the terms of root functions of a spectral problem for the equation of a vibrating beam, *J. Math. Study* **54**(4) (2021), 435-450.

[4] J. Ben Amara, Sturm theory for the equation of vibrating beam, *J. Math. Anal. and Appl.* **349**(1) (2009), 1-9.

[5] D.O. Banks, G.J. Kurowski, A Prufer transformation for the equation of a vibrating beam subject to axial forces, *J. Differential Equations*, **24** (1977), 57-74.

[6] B.B. Bolotin, *Vibrations in technique: Handbook in 6 volumes, The vibrations of linear systems, I*, Engineering Industry, Moscow, 1978.

[7] N.Yu. Kapustin, On the uniform convergence of the Fourier series for a spectral problem with squared spectral parameter in a boundary condition. *Differ. Equ.* **46**(10) (2010), 1507-1510.

[8] N.Yu. Kapustin, On the uniform convergence in $C^1$ of Fourier series for a spectral problem with squared spectral parameter in a boundary condition. *Differ. Equ.* **47**(10), (2011), 1394-1399.

[9] N.Yu. Kapustin, E.I. Moiseev, A remark on the convergence problem for spectral expansions corresponding to a classical problem with spectral parameter in the boundary condition. *Differ. Equ.* **37**(2), (2001), 1677-1683.

[10] N.B. Kerimov, Z. S. Aliev, Basis properties of a spectral problem with spectral parameter in the boundary condition, *Sbornik: Mathematics*, **197**(10) (2006), 1467-1487.

[11] N.B. Kerimov, S. Goktas, E.A. Maris, Uniform convergence of the spectral expansions in terms of root functions for a spectral problem. *Electron. J. Differential Equations* **2016**(80) (2016), 1-14.

[12] N.B. Kerimov, E.A. Maris, On the uniform convergence of the Fourier Series for one spectral problem with a spectral parameter in a boundary condition. *Math. Methods Appl. Sci.* **39**(9), (2016), 2298-2309.

[13] N.B. Kerimov, E.A. Maris, On the uniform convergence of fourier series expansions for Sturm-Liouville problems with a spectral parameter in the boundary conditions, Results Math., **73**(3) (2018), 1-16.

[14] G.M. Keselman, On the unconditional convergence of eigenfunction expansions of certain differential operators. *Izv. Vyssh. Uchebn. Zaved. Mat.* **39**(2) (1964), 82-93 [in Russian].

[15] E.A. Maris, S.A. Goktas, A study on the uniform convergence of spectral expansions for continuous functions on a Sturm-Liouville problem, Miskolc Math. Notes, **20**(2) (2019), 1063-1081.

[16] M.A. Naimark, *Linear Differential Operators*, Ungar, New York, 1967.

[17] F. Roseau, *Theory of vibration in mechanical systems. Analytical methods and applications*, Berlin : Springer, 1987, 515 p.

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# An Inverse Boundary Value Problem for the Boussinesq-Love Equation with Periodic and Integral Condition

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**Abstract.** The work is devoted to the study of the solvability of an inverse boundary value problem with an unknown time-dependent coefficient for the Boussinesq-Love equation with Nonlocal Integral Condition . The goal of the paper consists of the determination of the unknown coefficient together with the solution. The problem is considered in a rectangular domain. The definition of the classical solution of the problem is given. First, the given problem is reduced to an equivalent problem in a certain sense. Then, using the Fourier method the equivalent problem is reduced to solving the system of integral equations. Thus, the solution of an auxiliary inverse boundary value problem reduces to a system of three nonlinear integro-differential equations for unknown functions. Concrete Banach space is constructed. Further, in the ball from the constructed Banach space by the contraction mapping principle, the solvability of the system of nonlinear integro-differential equations is proved. This solution is also a unique solution to the equivalent problem. Finally, by equivalence, the theorem of existence and uniqueness of a classical solution to the given problem is proved.
**Key Words and Phrases**: Inverse problems, hyperbolic equations, nonlocal integral condition; classical solution, existence, uniqueness.
**2010 Mathematics Subject Classifications**: Primary 35R30, 35L10, 35L70; Secondary 35A01, 35A02, 35A09.

## 1. Introduction*

There are many cases where the needs of the practice bring about the problems of determining coefficients or the right hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control in industry etc., which makes them an active field of contemporary mathematics. Inverse problems for various types of PDEs have been studied in many papers. Among them we should mention the papers of A.N. Tikhonov [1], M.M. Lavrentyev [2, 3], V.K. Ivanov [4] and their followers. For a comprehensive overview, the reader should see the monograph by A.M. Denisov [5].

In this paper we prove existence and uniqueness of the solution to an inverse boundary value problem for the Boussinesq-Love equation modeling the longitudinal waves in an elastic bar with the transverse inertia.

 

## 2. Problem statement and its reduction to an equivalent problem

Let $T>0$ be some fixed number and denote by $D_T:=\{(x,t):\ 0\le x\le 1,\ 0\le t\le T\}$. Consider the one-dimensional inverse problem of identifying an unknown pair of functions $\{u(x,t)\,,\,a(t)\}$ for the following Boussinesq-Love equation [6 ]

$$u_{tt}(x,t)-u_{ttxx}(x,t)-\alpha u_{txx}(x,t)-\beta u_{xx}(x,t)=a(t)u(t,x)+f(x,t) \qquad (1)$$

with the nonlocal initial conditions

$$u(x,0)=\varphi(x)+\int_0^T p(t)u(x,t)dt,\quad u_t(x,0)=\psi(x),\quad x\in[0,1], \qquad (2)$$

periodic boundary condition

$$u(0,t)=u(1,t),\quad t\in[0,T], \qquad (3)$$

nonlocal integral condition

$$\int_0^1 u(x,t)dx=0,\ \ t\in[0,T], \qquad (4)$$

and over determination condition

$$u(x_0,t)=h(t),\ \ t\in[0,T]. \qquad (5)$$

Where $x_0\in(0,1)$,$\alpha>0,\ \beta>0$ the given numbers,$f(x,t),\ \varphi(x),\ \psi(x),\ p(t)$ , and $h(t)$ are given sufficiently smooth functions of $x\in[0,1]$ and $t\in[0,T]$.
We introduce the following set of functions

$$\tilde{C}^{(2,2)}(D_T)=\left\{u(x,t):\ u(x,t)\in C^2(D_T),u_{xx}(x,t),u_{txx}(x,t),u_{ttxx}(x,t)\in C(D_T)\right\}.$$

**Definition 1.** *The pair* $\{u(x,t)\,,\,a(t)\}$ *is said to be a classical solution to the problem* (**??**)-(**??**)*, if the functions* $u(x,t)\in\tilde{C}^{(2,2)}(D_T)$ *and* $a(t)\in C[0,T]$ *satisfies an equation* (**??**)*in the region* $D_T$*, the condition* (**??**)*on* $[0,1]$*, and the statements* (**??**)-(**??**)*on the interval* $[0,T]$.

In order to investigate the problem (**??**)- (**??**), first we consider the following auxiliary problem

$$y''(t)=a(t)y(t),\ \ t\in[0,T], \qquad (6)$$

$$y(0)=\int_0^T p(t)y(t)dt,\ \ y'(0)=0, \qquad (7)$$

where $p(t),\ a(t)\in C[0,T]$ are given functions, and $y=y(t)$ is desired function. Moreover, by the solution of the problem (**??**), (**??**), we mean a function $y(t)$ belonging to $C^2[0,T]$ and satisfying the conditions (**??**), (**??**) in the usual sense.

**Lemma 1.** *[7] Assume that* $p(t) \in C[0,T]$, $a(t) \in C[0,T]$, $\|a(t)\|_{C[0,T]} \le R = const$, *and the condition*

$$\left(\|p(t)\|_{C[0,T]} + \frac{T}{2}R\right)T < 1$$

*hold. Then the problem* (**??**), (**??**) *has a unique trivial solution.*

Now along with the inverse boundary-value problem (**??**)-(**??**), we consider the following auxiliary inverse boundary-value problem: It is required to determine a pair $\{u(x,t), a(t)\}$ of functions $u(x,t) \in \tilde{C}^{(2,2)}(D_T)$ and $a(t) \in C[0,T]$, from relations (**??**)-(**??**), and

$$u_x(0,t) = u_x(1,t), \quad t \in [0,T]. \tag{8}$$

$$h''(t) - u_{ttxx}(x_0,t) - \alpha u_{txx}(x_0,t) - \beta u_{xx}(x_0,t) = a(t)h(t) + f(x_0,t), \ \ t \in [0,T]. \tag{9}$$

The following lemma is valid

**Theorem 1.** *Suppose that* $\varphi(x)$, $\psi(x) \in C^1[0,1]$, $\varphi(1) = \varphi'(0), \psi(1) = \psi'(0)$, $p(t) \in C[0,T]$ , $p(t) \le 0$, $t \in [0,T]$, $h(t) \in C^2[0,T]$, $h(t) \ne 0$, $t \in [0,T]$, $f(x,t) \in C(D_T)$, $\int_0^1 f(x,t)dx = 0$, $t \in [0,T]$, $\frac{\alpha^2}{4} - \beta > 0$, *and the compatibility conditions*

$$\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0, \tag{10}$$

$$h(0) = \int_0^T p(t)h(t)dt + \varphi(x_0), \quad h'(0) = \psi(0) \tag{11}$$

*holds. Then the following assertions are valid:*

1. *each classical solution* $\{u(x,t), a(t)\}$ *of the problem* (**??**)-(**??**) *is a solution of problem* (**??**)-(**??**), (**??**), (**??**), *as well;*
2. *each solution* $\{u(x,t), a(t)\}$ *of the problem* (**??**)-(**??**), (**??**), (**??**), *if*

$$\left(\|p(t)\|_{C[0,T]} + \frac{T}{2}\|a(t)\|_{C[0,T]}\right)T < 1, \tag{12}$$

*is a classical solution of problem* (**??**)-(**??**).

*Proof.* Let $\{u(x,t), a(t)\}$ be any classical solution to problem (**??**)-(**??**). By integrating both sides of equation (**??**) with respect to $x$ from 0 to 1, we find

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx - (u_{ttx}(1,t) - u_{ttx}(0,t)) - \alpha(u_{tx}(1,t) - u_{tx}(0,t)) - \beta(u_x(1,t) - u_x(0,t))$$

$$= a(t)\int\limits_0^1 u(x,t)dx + \int\limits_0^1 f(x,t)dx, \quad t \in [0,T]. \tag{13}$$

Using the fact that $\int\limits_0^1 f(x,t)dx = 0\,,\ \ t \in [0,T]$, and the conditions (**??**), (**??**), we find that :

$$u_{ttx}(1,t) - u_{ttx}(0,t) + \alpha\left(u_{tx}(1,t) - u_{tx}(0,t)\right) + \beta\left(u_x(1,t) - u_x(0,t)\right) = 0, t \in [0,T]. \tag{14}$$

It's obvious that the general solution of equation (2.14) has the form:

$$u_x(1,t) - u_x(0,t) = c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t}, \tag{15}$$

where $c_1$, $c_2$ are the unknown number and

$$\mu_1 = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \beta} \quad , \ \mu_2 = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \beta}\,.$$

By (**??**) and $\varphi'(\mathbf{??}) = \varphi'(0)$, $\ \psi'(\mathbf{??}) = \psi'(0)$ we obtain:

$$u_x(1,0) - u_x(0,0) - \int\limits_0^T p(t)\left(u_x(1,t) - u_x(0,t)\right)dt$$

$$= u_x(1,0) - \int\limits_0^T p(t)u_x(1,t)dt - \left(u_x(0,0) - \int\limits_0^T p(t)u_x(0,t)dt\right)$$

$$= \varphi'(1) - \varphi'(0) = 0,\ u_{tx}(1,0) - u_{tx}(0,0) = \psi'(1) - \psi'(0) = 0. \tag{16}$$

Using (**??**) and (**??**) we obtain

$$c_1 + c_2 - \int\limits_0^T p(t)\left(c_1 e^{\mu_1 t} + c_2\, e^{\mu_2 t}\right)dt = 0,\ c_1\mu_1 + c_2\mu_2\ = 0\,.$$

Hence we find:

$$c_2 = -\frac{\mu_1}{\mu_2}c_1,\ c_1\left(\mu_2 - \mu_1\ - \int\limits_0^T p(t)\left((\mu_2 - \mu_1)e^{\mu_1 t} - \mu_1(e^{\mu_2 t} - e^{\mu_1 t})\right)dt\right) = 0.$$

By $p(t) \le 0$ , $\mu_2 - \mu_1\ = 2\sqrt{\frac{\alpha^2}{4} - \beta} > 0$, from the latter relations we have $c_1 = c_2 = 0$.

Putting the value of $c_1 = c_2 = 0$ in (2.15), we get that the problem (2.14), (2.16) has only the trivial solution, i.e. we conclude that the statement (**??**) is true.

Substituting $x = x_0$ in equation (**??**)we find:

$$u_{tt}(x_0,t) - u_{ttxx}(x_0,t) - \alpha u_{txx}(x_0,t) - \beta u_{xx}(x_0,t) = a(t)u(x_0,t) + f(x_0,t)\,,\quad t \in [0,T], \tag{17}$$

Assume now that $h(t) \in C^2[0,T]$. Differentiating (**??**)twice, we get

$$u_{tt}(x_0,t) = h''(t)\,, \;\; t \in [0,T]. \tag{18}$$

From (2.17), taking into account (**??**) and (2.18), we conclude that the relation (**??**) is fulfilled.

Now, assume that $\{u(x,t)\,,\, a(t)\}$ is the solution to problem (**??**)-(**??**), (2.8), (2.9). Then from (**??**), taking into account the condition $\int\limits_0^1 f(x,t)dx = 0, \;\; t \in [0,T]$ and relations (**??**), (**??**) we have

$$\frac{d^2}{dt^2}\int\limits_0^1 u(x,t)dx = a(t)\int\limits_0^1 u(x,t)dx, \;\; t \in [0,T]. \tag{19}$$

Furthermore, from (**??**)and (**??**)it is easy to see that

$$\int\limits_0^1 u(x,0)dx - \int\limits_0^T p(t)\left(\int\limits_0^1 u(x,t)dx\right)dt$$

$$= \int\limits_0^1\left(u(x,0) - \int\limits_0^T p(t)u(x,t)dt\right)dx = \int\limits_0^1 \varphi(x)dx = 0,$$

$$\int\limits_0^1 u_t(x,0)dx = \int\limits_0^1 \psi(x)dx = 0. \tag{20}$$

Since, by Lemma **??**, problem (2.19), (2.20) has only a trivial solution. It means that $\int\limits_0^1 u(x,t)dx = 0, \;\; t \in [0,T]$, i.e. the condition (**??**) is satisfied.

Next, from (**??**) and (2.17), we obtain

$$\frac{d^2}{dt^2}(u(x_0,t) - h(t)) = a(t)(u(x_0,t) - h(t)), \;\; 0 \le t \le T. \tag{21}$$

By virtue of (**??**)and the compatibility conditions (**??**), we have

$$u(x_0,0) - h(0) - \int\limits_0^T p(t)(u(x_0,t) - h(t))dt$$

$$= u(x_0,0) - \int\limits_0^T p(t)u(x_0,t)dt - \left(h(0) - \int\limits_0^T p(t)h(t)dt\right)$$

$$= \varphi(x_0) - \left( h(0) - \int\limits_0^T p(t)h(t)dt \right) = 0, u_t(x_0, 0) - h'(0) = 0. \qquad (22)$$

Using Lemma **??**, and relations (2.21), (2.22), we conclude that condition (**??**) is satisfied. The theorem is proved.

## 3. Existence and uniqueness of the classical solution to the inverse boundary value problem

It is known [8] that the system

$$1, \cos \lambda_1 x, \sin \lambda_1 x, ..., \cos \lambda_k x, \sin \lambda_k x, ... \qquad (23)$$

is a basis in $L_2(0,1)$, where $\lambda_k = 2k\pi \ (k = 1, 2, ...)$.

Then the first component of classical solution $\{u(x,t), a(t)\}$ of the problem (**??**)-(**??**), (**??**), (**??**) has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t) \sin \lambda_k x, \quad \lambda_k = 2\pi k, \qquad (24)$$

where

$$u_{10}(t) = \int\limits_0^1 u(x,t)dx,$$

$$u_{1k}(t) = 2 \int\limits_0^1 u(x,t) \cos \lambda_k x dx, \quad k = 1, 2, ...,$$

$$u_{2k}(t) = 2 \int\limits_0^1 u(x,t) \sin \lambda_k x dx, \quad k = 1, 2, .....$$

Then, applying the formal scheme of the Fourier method, from (**??**)-(**??**)we have

$$u''_{10}(t) = F_{10}(t; u, a), \quad 0 \le t \le T, \qquad (25)$$

$$(1 + \lambda_k^2)\, u''_{ik}(t) + \alpha \lambda_k^2 u'_{ik}(t) + \beta \lambda_k^2 u_{ik}(t) = F_{ik}(t; u, a), \qquad (26)$$
$$i = 1, 2; \ k = 0, 1, 2 \ldots; \ 0 \le t \le T,$$

$$u_{10}(0) = \varphi_{10} + \int\limits_0^T p(t) u_{10}(t) dt, \ , \quad u'_{10}(0) = \psi_{10} \ , \qquad (27)$$

$$u_{ik}(0) = \varphi_{ik} + \int_0^T p(t)u_{ik}(t)dt, \quad u'_{ik}(0) = \psi_{ik}, \;\; i = 1, 2; \;\; k = 1, 2 \ldots. \tag{28}$$

where

$$F_{1k}(t) = a(t)u_{1k}(t) + f_{1k}(t), \quad k = 0, 1, \ldots,$$

$$f_{10}(t) = \int_0^1 f(x, t)dx, \quad f_{1k}(t) = 2\int_0^1 f(x, t)\cos\lambda_k x dx, \quad k = 1, 2, \ldots,$$

$$\varphi_{10} = \int_0^1 \varphi(x)dx, \quad \psi_{10} = 2\int_0^1 \psi(x)dx,$$

$$\varphi_{1k} = 2\int_0^1 \varphi(x)\cos\lambda_k x dx, \quad \psi_{1k} = 2\int_0^1 \psi(x)\cos\lambda_k x dx, \quad k = 0, 1, \ldots,$$

$$F_{2k}(t) = a(t)u_{2k}(t) + f_{2k}(t), \quad f_{2k}(t) = 2\int_0^1 f(x, t)\sin\lambda_k x dx, \quad k = 1, 2, \ldots,$$

$$\varphi_{2k} = 2\int_0^1 \varphi(x)\sin\lambda_k x dx, \quad k = 1, 2, \ldots, \quad \psi_{2k} = 2\int_0^1 \psi(x)\sin\lambda_k x dx, \quad k = 1, 2, \ldots.$$

It is obvious that $\lambda_k^2 < 1 + \lambda_k^2 < 2\lambda_k^2 \, (k = 1, 2, \ldots)$. Therefore

$$\frac{\alpha^2}{8} - \beta < \frac{\alpha^2\lambda_k^2}{4(1 + \lambda_k^2)} - \beta < \frac{\alpha^2}{4} - \beta \;\; (k = 1, 2, \ldots).$$

Now suppose that $\frac{\alpha^2}{8} - \beta > 0$. Solving the problem (3.3)–(3.6), we fin

$$u_{10}(t) = \varphi_{10} + \int_0^T p(t)u_{10}(t)dt + t\psi_{10} + \int_0^t (t - \tau)F_{10}(\tau; \, u, \, a)d\tau, \tag{29}$$

$$u_{ik}(t) = \frac{1}{\gamma_k}\left[\left(\mu_{2k}e^{\mu_{1k}t} - \mu_{1k}e^{\mu_{2k}t}\right)\left(\varphi_{ik} + \int_0^T p(t)u_{ik}(t)dt\right) + \left(e^{\mu_{2k}t} - e^{\mu_{1k}t}\right)\psi_{ik}\right.$$

$$\left. + \frac{1}{1 + \lambda_k^2}\int_0^t F_{ik}(\tau; u, a)\left(e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)}\right)d\tau\right] \tag{30}$$

$$(0 \le t \le T; \;\; i = 1, 2; \; k = 1, 2, \ldots),$$

where

$$\mu_{1k} = -\frac{\alpha\lambda_k^2}{2(1+\lambda_k^2)} - \lambda_k\sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}} \quad ,$$

$$\mu_{2k} = -\frac{\alpha\lambda_k^2}{2(1+\lambda_k^2)} + \lambda_k\sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}} \quad ,$$

$$\gamma_k = \mu_{2k} - \mu_{1k} = 2\lambda_k\sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}} \, .$$

Differentiating (3.8) twice, we get:

$$u'_{ik}(t) = \frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\left(e^{\mu_{1k}t} - e^{\mu_{2k}t}\right)\left(\varphi_{ik} + \int_0^T P_2(t)u_{ik}(t)dt\right) + \left(\mu_{2k}e^{\mu_{2k}t} - \mu_{1k}e^{\mu_{1k}t}\right)\psi_{ik}\right.$$

$$\left. + \frac{1}{1+\lambda_k^2}\int_0^t F_{ik}(\tau;u,a)\left(\mu_{2k}e^{\mu_{2k}(t-\tau)} - \ \mu_{1k}e^{\mu_{1k}(t-\tau)}\right)d\tau\right] \tag{31}$$

$$(0 \le t \le T;\ i = 1,2;\ k = 1,2,...),$$

$$u''_{ik}(t) = \frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\left(\mu_{1k}e^{\mu_{1k}t} - \mu_{2k}e^{\mu_{2k}t}\right)\left(\varphi_{ik} + \int_0^T P_2(t)u_{ik}(t)dt\right)\right.$$

$$+ \left(\mu_{2k}^2 e^{\mu_{2k}t} - \mu_{1k}^2 e^{\mu_{1k}t}\right)\psi_{ik} + \frac{1}{1+\lambda_k^2}\int_0^t F_{ik}(\tau;u,a,b)\left(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)}\right.$$

$$\left.\left. - \ \mu_{1k}^2 e^{\mu_{1k}(t-\tau)}\right)d\tau\right] + \frac{1}{1+\lambda_k^2}F_{ik}(\,t;u,a,b) \quad (\,k = 1,2,...). \tag{32}$$

To determine the first component of the classical solution to the problem (**??**)-(**??**), (**??**), (**??**) we substitute the expressions $u_{ik}(t)$ $(i = 1,2; k = 0,\ 1, ...)$ into (3.2) and obtain

$$u(x,t) = \varphi_{10} + t\left(\psi_{10} + \int_0^T p(t)u_{10}(t)dt\right) + \int_0^t (t-\tau)F_{10}(\tau\,;u,a)d\tau$$

$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_k}\left[\left(\mu_{2k}e^{\mu_{1k}t} - \mu_{1k}e^{\mu_{2k}t}\right)\left(\varphi_{1k} + \int_0^T P_2(t)u_{1k}(t)dt\right) + \left(e^{\mu_{2k}t} - e^{\mu_{1k}t}\right)\psi_{1k}\right.\right.$$

$$\left.\left. + \frac{1}{1+\lambda_k^2}\int_0^t F_{1k}(\tau;u,a)\left(e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)}\right)d\tau\right]\right\}\cos\lambda_k x$$

$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_k}\left[\left(\mu_{2k}e^{\mu_{1k}t}-\mu_{1k}e^{\mu_{2k}t}\right)\left(\varphi_{2k}+\int_0^T P_2(t)u_{2k}(t)dt\right)+\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\psi_{2k}\right.\right.$$

$$\left.\left.+\frac{1}{1+\lambda_k^2}\int_0^t F_{2k}(\tau;u,a)\left(e^{\mu_{2k}(t-\tau)}-e^{\mu_{1k}(t-\tau)}\right)d\tau\right]\right\}\sin\lambda_k x. \tag{33}$$

It follows from (**??**) and (3.2) that

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(x_0,\,t)+\sum_{k=1}^{\infty}(\lambda_k^2u''_{1k}(t)+\alpha\,\lambda_k^2u'_{1k}(t)+\beta\,\lambda_k^2u_{1k}(t))\cos\lambda_k x_0\right.$$

$$\left.+\sum_{k=1}^{\infty}(\lambda_k^2u''_{2k}(t)+\alpha\,\lambda_k^2u'_{2k}(t)+\beta\,\lambda_k^2u_{2k}(t))\sin\lambda_k x_0\right\}. \tag{34}$$

By (3.4) and (3.10) we have:

$$\lambda_k^2\,u''_{ik}(t)+\alpha\lambda_k^2u'_{ik}(t)+\beta\lambda_k^2u_{ik}(t)=F_{ik}(\,\tau;u,a)-\,u''_{ik}(t)$$

$$=-\frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\,\left(\mu_{1k}e^{\mu_{1k}t}-\mu_{2k}e^{\mu_{2k}t}\right)\left(\varphi_{ik}+\int_0^T p(t)u_{ik}(t)dt\right)+\left(\mu_{2k}^2e^{\mu_{2k}t}-\mu_{1k}^2e^{\mu_{1k}t}\right)\psi_{ik}\right.$$

$$\left.+\frac{1}{1+\lambda_k^2}\int_0^t F_{ik}(\tau;u,a)\left(\mu_{2k}^2e^{\mu_{2k}(t-\tau)}-\;\mu_{1k}^2e^{\mu_{1k}(t-\tau)}\right)d\tau\right]$$

$$+\frac{\lambda_k^2}{1+\lambda_k^2}F_{ik}(\,t;u,a)\qquad(0\le t\le T;\;i=1,2;\;k=1,2,...). \tag{35}$$

By substituting expression (3.13) into (3.12), we obtain the equation for the second component of the solution to problem (**??**)- (**??**), (**??**), (**??**):

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(x_0,\,t)\right.$$

$$-\sum_{k=1}^{\infty}\left[\frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\;\left(\mu_{1k}e^{\mu_{1k}t}-\mu_{2k}e^{\mu_{2k}t}\right)\left(\varphi_{1k}+\int_0^T p(t)u_{1k}(t)dt\right)\right.\right.$$

$$\left.+\left(\mu_{2k}^2e^{\mu_{2k}t}-\mu_{1k}^2e^{\mu_{1k}t}\right)\psi_{1k}+\frac{1}{1+\lambda_k^2}\int_0^t F_{1k}(\tau;u,a)\left(\mu_{2k}^2e^{\mu_{2k}(t-\tau)}-\;\mu_{1k}^2e^{\mu_{1k}(t-\tau)}\right)d\tau\right]$$

$$\left.-\frac{\lambda_k^2}{1+\lambda_k^2}F_{1k}(\,t;u,a)\right]\cos\lambda_k x_o-\sum_{k=1}^{\infty}\left[\frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\;\left(\mu_{1k}e^{\mu_{1k}t}-\mu_{2k}e^{\mu_{2k}t}\right)\right.\right.$$

$$\times\left(\varphi_{2k}+\int\limits_0^T p(t)u_{2k}(t)dt\right)+\left(\mu_{2k}^2 e^{\mu_{2k}t}-\mu_{1k}^2 e^{\mu_{1k}t}\right)\psi_{2k}$$

$$\left.\left.+\frac{1}{1+\lambda_k^2}\int\limits_0^t F_{2k}(\tau;u,a)\left(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)}-\ \mu_{1k}^2 e^{\mu_{1k}(t-\tau)}\right)d\tau\right]\ -\ \frac{\lambda_k^2}{1+\lambda_k^2}F_{2k}(\,t;u,a)\right]\sin\lambda_k x_o\ . \tag{36}$$

Thus, the solution of problem (**??**)- (**??**), (**??**), (**??**)was reduced to the solution of system (3.11), (3.14) with respect to unknown functions $u(x,t)$ and $a(t)$.

**Lemma 2.** *If* $\{u(x,t),a(t)\}$ *is any solution to problem* (**??**)- (**??**), (**??**), (**??**), *then the functions*

$$u_{10}(t)=\int\limits_0^1 u(x,t)dx,$$

$$u_{1k}(t)=2\int\limits_0^1 u(x,t)\cos\lambda_k x dx,\quad k=1,2,...,$$

$$u_{2k}(t)=2\int\limits_0^1 u(x,t)\sin\lambda_k x dx,\quad k=1,2,.....,$$

*satisfies the system (3.7), (3.8) in* $C[0,T]$.

It follows from Lemma **??** that

**Corollary 1.** *Let system (3.11), (3.14) have a unique solution. Then problem* (**??**)- (**??**), (**??**), (**??**)*cannot have more than one solution, i.e. if the problem* (**??**)- (**??**), (**??**), (**??**)*has a solution, then it is unique.*

With the purpose to study the problem (**??**)- (**??**), (**??**), (**??**), we consider the following functional spaces.

Denote by $B_{2,T}^3$ [9] a set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_k x,\quad \lambda_k=2\pi k,$$

considered in the region $D_T$, where each of the function $u_{1k}(t)$ $(k=0,\,1,\,2,...)$, $u_{2k}(t)$ $(k=1,\,2,...)$ is continuous over an interval $[0\,,\,T]$ and satisfies the following condition:

$$J(u)\equiv\|\,u_{10}(t)\,\|_{C[0\,,\,T]}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\ \|\,u_{1k}(t)\,\|_{C[0\,,\,T]}\right)^2\right\}^{\frac{1}{2}}$$

$$+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3 \parallel u_{2k}(t)\parallel_{C[0\,,T]}\right)^2\right\}^{\frac{1}{2}} < +\infty\,.$$

The norm in this set is defined by

$$\parallel u(x,t)\parallel_{B^3_{2,T}} = J(u)\,.$$

It is known that $B^3_{2,T}$ is Banach .

Obviously, $E^3_T = B^3_{2,T}\times C[0,T]$ with the norm $\|z(x,t)\|_{E^3_{,T}} = \|u(x,t)\|_{B^3_{2,T}} + \|a(t)\|_{C[0,T]}$ is also Banach space.

Now consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a)\,,\,\Phi_2(u,a)\,\}\,,$$

in the space $E^3_T$, where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty}\tilde{u}_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty}\tilde{u}_{2k}(t)\sin\lambda_k x\,, \Phi_2(u,a) = \tilde{a}(t)$$

and the functions $\tilde{u}_{10}(t)$ ,$\tilde{u}_{ik}(t)$, $i = 1,2;\ k = 1,\,2,\dots,$ and $\tilde{a}(t)$ are equal to the right-hand sides of (3.7), (3.8), and (3.14), respectively.
It is easy to see that

$$\mu_{ik} < 0, \quad e^{\mu_{ik}t} < 1,\ e^{\mu_{ik}(t-\tau)} < 1\ \ , (i = 1,2;\ 0 \le t \le T;\ 0 \le \tau \le t)$$

$$|\mu_{ik}| \le \lambda_k\left(\frac{\alpha\lambda_k}{2(1+\lambda_k^2)} + \sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}}\right) \le \frac{\alpha\lambda_k}{1+\lambda_k^2} \le \alpha\ \ (i\ =\ 1,2),$$

$$|\mu_{1k}\mu_{2k}| \le \frac{\beta\lambda_k}{1+\lambda_k^2} \le \beta, \frac{1}{\gamma_k} = \frac{1}{2\sqrt{\frac{\lambda_k^2}{1+\lambda_k^2}\left(\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \beta\right)}} \le \frac{1}{2\sqrt{\frac{1}{2}\left(\frac{\alpha^2}{8} - \beta\right)}} \equiv \gamma_0\,.$$

Taking into account these relations, by means of simple transformations we find:

$$\|\tilde{u}_{10}(t)\|_{C[0,T]} \le |\varphi_{10}| + T\,\|p(t)\|_{C[0,T]}\,\|u_{10}(t)\|_{C[0,T]} + T\,|\psi_{10}|$$

$$+T\sqrt{T}\left(\int_0^T |f_{10}(\tau)|^2\,d\tau\right)^{\frac{1}{2}} + T^2\,\|a(t)\|_{C[0,T]}\,\|u_{10}(t)\|_{C[0,T]}\,, \tag{37}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{5}\alpha\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{5}\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_{ik}\right|)^2\right)^{\frac{1}{2}}+\gamma_0\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\left|f_{ik}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}$$

$$+\sqrt{5}T\gamma_0\left(\|p(t)\|_{C[0,T]}+\|a(t)\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{38}$$

$$\|\tilde{a}(t)\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(x_0,t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\sum_{i=1}^{2}\left[2\alpha\beta\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_{ik}\right|)^2\right)^{\frac{1}{2}}\right.$$

$$+2\alpha^2\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_{ik}\right|)^2\right)^{\frac{1}{2}}+2\alpha^2\gamma_0T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\alpha^2\gamma_0\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\left|f_{ik}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+2\alpha^2\gamma_0T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$\left.\left.+\left(\sum_{k=1}^{\infty}(\lambda_k\left\|f_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{39}$$

Suppose that the data for problem (**??**)-(**??**), (**??**), (**??**) satisfy the assumptions:

1. $\varphi(x)\in C^2[0,1]\,,\ \varphi'''(x)\in L_2(0,1),\ \varphi(0)=\varphi(\mathbf{??})\ ,\varphi'(0)=\varphi'(\mathbf{??})\,,\varphi''(0)=\varphi''(\mathbf{??})\,;$
2. $\psi(x)\in C^2[0,1]\,,\ \psi'''(x)\in L_2(0,1),\ \ \psi(0)=\psi(\mathbf{??})\ ,\psi'(0)=\psi'(\mathbf{??})\,,\psi''(0)=\psi''(\mathbf{??})\,;$
3. $f(x,t),f_x(x,t),f_{xx}(x,t)\in C(D_T)\,,f_{xxx}(x,t)\in L_2(D_T),\ f(0,t)=f(1,t)\,,$

$$f_x(0,t)=f_x(1,t)\,f_{xx}(0,t)=f_{xx}(1,t)\,,\ 0\leq t\leq T;$$

4. $p(t)\in C[0,T]\,,h(t)\in C^2[0,T]\,,\quad h(t)\neq 0,\quad 0\leq t\leq T.$
5. $\alpha>0,\ \beta>0,\ \frac{\alpha^2}{8}-\beta>0.$

Then from (3.15)-(3.17) we correspondingly find

$$\|\tilde{u}_0(t)\|_{C[0,T]}\leq\|\varphi(x)\|_{L_2(0,1)}+T\,\|\psi(x)\|_{L_2(0,1)}+T\sqrt{T}\,\|f(x,t)\|_{L_2(0,1)}$$

$$+T\,\|p(t)\|_{C[0,T]}\,\|u_{10}(t)\|_{C[0,T]}+T^2\,\|a(t)\|_{C[0,T]}\,\|u_{10}(t)\|_{C[0,T]}\,, \tag{40}$$

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\parallel\tilde{u}_{ik}(t)\parallel_{C[0\,,\,T]}\right)^2\right\}^{\frac{1}{2}}\leq 2\sqrt{5}\alpha\gamma_0\left\|\varphi'''(x)\right\|_{L_2(0,1)}$$

$$+2\sqrt{5}\gamma_0 \left\|\psi'''(x)\right\|_{L_2(0,1)} + 2\gamma_0\sqrt{5T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}$$

$$+\sqrt{5}T\gamma_0 \left\|p(t)\right\|_{C[0,T]} \sum_{i=1}^{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3 \left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{5}T\gamma_0 \left\|a(t)\right\|_{C[0,T]} \sum_{i=1}^{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3 \left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{41}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left\{\left\|h''(t) - f(0,t)\right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\right.$$

$$\times\left[4\alpha\beta\gamma_0\left\|\varphi'''(x)\right\|_{L_2(0,1)} + 4\alpha^2\gamma_0\left\|\psi'''(x)\right\|_{L_2(0,1)} + 4\alpha^2\gamma_0\sqrt{T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}\right.$$

$$+4\alpha^2\gamma_0 T\left(\left\|p(t)\right\|_{C[0,T]} + \left\|a(t)\right\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$\left.\left.+2\left\|\left\|f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)} + 2\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{42}$$

It follows from (3.18) and (3.19) that

$$\left\|\tilde{u}(x,t)\right\|_{B_{2,T}^3} \le A_1(T) + B_1(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3} + C_1(T)\left\|u(x,t)\right\|_{B_{2,T}^3}, \tag{43}$$

where

$$A_1(T) = \left\|\varphi(x)\right\|_{L_2(0,1)} + T\left\|\psi(x)\right\|_{L_2(0,1)} + T\sqrt{T}\left\|f(x,t)\right\|_{L_2(D_T)}$$

$$+2\sqrt{5}\alpha\gamma_0\left\|\varphi'''(x)\right\|_{L_2(0,1)} + 2\sqrt{5}\gamma_0\left\|\psi'''(x)\right\|_{L_2(0,1)} + 2\gamma_0\sqrt{5T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T) = T^2 + \gamma_0\sqrt{5}T,$$

$$C_1(T) = T(T + \sqrt{5}\gamma_0)\left\|p(t)\right\|_{C[0,T]}.$$

Further from (3.20), we may also write

$$\left\|\tilde{a}(t)\right\|_{C[0,T]} \le A_2(T) + B_2(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3} + C_2(T)\left\|u(x,t)\right\|_{B_{2,T}^3}, \tag{44}$$

where

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t) - f(0,t))\right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[4\alpha\beta\gamma_0\left\|\varphi'''(x)\right\|_{L_2(0,1)}\right.\right.$$

$$\left.\left.+4\alpha^2\gamma_0\left\|\psi'''(x)\right\|_{L_2(0,1)} + 4\alpha^2\gamma_0\sqrt{T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)} + 2\left\|\left\|f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\right]\right\},$$

$$B_2(T) = \| [h(t)]^{-1}\|_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^{-2})\right)^{\frac{1}{2}} \left(2\alpha^2\gamma_0 T + 1\right) \ ,$$

$$C_2(T) = \| [h(t)]^{-1}\|_{C[0,T]} \left(\sum_{k=1}^{\infty}(\lambda_k^{-2})\right)^{\frac{1}{2}} T \, \|p(t)\|_{C[0,T]} \, .$$

From the inequalities (3.21) and (3.22), we conclude that

$$\begin{aligned} &\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \\ &\leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C(T)\,\|u(x,t)\|_{B^3_{2,T}} \, , \end{aligned} \tag{45}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T), \;\; C(T) = C_1(T) + C_2(T).$$

Thus, we can prove the following theorem:

**Theorem 2.** *Assume that statements A)-E) and the condition*

$$(B(T)(A(T)+2) + C(T))(A(T)+2) < 1 \ , \tag{46}$$

*holds, then problem* (**??**)-(**??**), (**??**), (**??**)*has a unique solution in the ball* $K = K_R(\|z\|_{E^3_T} \leq R \leq A(T)+2)$ *of the space* $E^3_T$.

*Proof.* In the space $E^3_T$, consider the operator equation

$$z = \Phi z \, , \tag{47}$$

where $z = \{u, a\}$, and the components $\Phi_i(u,a) \ (i = 1,2)$, of operator $\Phi(u,a)$ defined by the right sides of (3.11) and (3.14), respectively and the following inequalities hold:

$$\|\Phi z\|_{E^3_T} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C(T)\,\|u(x,t)\|_{B^3_{2,T}}$$

$$\leq A(T) + B(T)R^2 + C(T)R = A(T) + (B(T)(A(T)+2) + C(T))(A(T)+2) \, , \tag{48}$$

$$\begin{aligned} \|\Phi z_1 - \Phi z_2\|_{E^3_T} \leq B(T)R(\|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}} + \|a_1(t) - a_2(t)\|_{C[0,T]}) \\ + C(T)\,\|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}} \ , \end{aligned} \tag{49}$$

Then it follows from (3.24), (3.26), and (3.27) that the operator $\Phi$ acts in the ball $K = K_R$, and satisfy the conditions of the contraction mapping principle. Therefore the operator $\Phi$ has a unique fixed point $\{z\} = \{u \, , \, a\}$ in the ball $K = K_R$, which is a solution of equation (3.25); i.e. the pair $\{u, a\}$ is the unique solution of the systems (3.11) and (3.14) in $K = K_R$.

Then the function $u(x,t)$ as an element of space $B^3_{2,T}$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in $D_T$.

Now from (3.9) it is obvious that $u'_{ik}(t)\,(i=1,2;\ k=1,2,...)$ is continuous in $[0,T]$ and from the same relation we get:

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u'_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\leq 2\sqrt{5}\beta\gamma_0\left\|\varphi'''(x)\right\|_{L_2(0,1)}$$

$$+2\sqrt{5}\alpha\left\|\psi'''(x)\right\|_{L_2(0,1)}+2\alpha\sqrt{5T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}$$

$$+2\alpha\sqrt{5}T\left(\|p(t)\|_{C[0,T]}+\|a(t)\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}.$$

Hence, it follows that $u_t(x,t),u_{tx}(x,t),u_{txx}(x,t)$ are continuous in $D_T$ .

Next, from (3.4) it follows that $u''_{ik}(t)\,(i=1,2;\ k=1,2,...)$ is continuous in $[0,T]$ and consequently we have:

$$\left(\sum_{k=1}^{\infty}(\lambda_k\left\|u''_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\leq 2\alpha\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u'_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\beta\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+2\left\|\left\|f_x(x,t)+a(t)u_x(x,t)\right\|_{C[0,T]}\right\|_{L(0,1)}.$$

From the last relation it is obvious that $u_{tt}(x,t),u_{ttx}(x,t),u_{ttxx}(x,t)$are continuous in $D_T$.

It is easy to verify that equation (**??**)and conditions (**??**), (**??**), (**??**), (**??**)satisfy in the usual sense. So, $\{u(x,t),a(t)\}$ is a solution of (**??**)-(**??**), (**??**), (**??**), and by Lemma **??** it is unique in the ball $K=K_R$. The proof is complete.

In summary, from Theorem **??** and Theorem **??**, straightforward implies the unique solvability of the original problem (**??**)- (**??**).

**Theorem 3.** *Suppose that all assumptions of Theorem* **??**, *and the conditions*

$$\int_0^1\varphi(x)dx=0,\quad\int_0^1\psi(x)dx=0,\int_0^1 f(x,t)dx=0,\quad t\in[0,T],p(t)\leq 0,\ t\in[0,T],$$

$$h(0)=\int_0^T p(t)h(t)dt+\varphi(x_0),\quad h'(0)=\psi(0)$$

$$\left(\|p(t)\|_{C[0,T]}+\frac{1}{2}(A(T)+2)\right)T<1,$$

*holds. Then problem* (**??**)- (**??**)*has a unique classical solution in the ball* $K=K_R(\|z\|_{E_T^3}\leq A(T)+2)$ *of the space* $E_T^3$.

## References

[1] A.N.Tikhonov, On stability of inverse problems, Dokl. AN SSSR, 39(1943), no. 5, 195–198 (in Russian).

[2] M.M.Lavrent'ev, On an inverse problem for a wave equation, Dokl. AN SSSR, 157(1964), no. 3, 520–521 (in Russian).

[3] M.M.Lavrent'ev, V.G.Romanov, S.T.Shishatsky Ill-posed problems of mathematical physics and analysis, M., Nauka, 1980 (in Russian).

[4] V.K.Ivanov, V.V.Vasin, V.P.Tanana, Theory of linear ill-posed problems and its applications, M., Nauka, 1978 (in Russian).

[5] A.M.Denisov, Introduction to theory of inverse problems, M., MGU, 1994 (in Russian).

[6] Dzh.Uizem, Linear and Nonlinear Waves, M., Mir, 1977 (in Russian).

[7] Ya. T. Megraliev, F. Kh. Alizade, "Inverse boundary value problem for a Boussinesq type equation of fourth order with nonlocal time integral conditions of the second kind", *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, 26:4 (2016), 503–514(in Russian).

[8] B.M.Budak, A.A.Samarskii, A.N.Tikhonov, A Collection of Problems in Mathematical Physics, M., Nauka, 1972 (in Russian).

[9] Ya.T.Mehraliyev, On solvability of an inverse boundary value problem for a second order elliptic equation, Vest. Tverskogo Gos. Univ., Ser. prikladnaya matematika, (2011), no. 23, 25–38 (in Russian).

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# On the Basicity of Eigenfunctions of a Non-self-adjoint Spectral Problem with a Spectral Parameter in the Boundary Condition in Lebesgue Spaces

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**Abstract.** In this work we consider the following spectral problem

$$-y'' + q(x)\, y = \lambda y, \quad x \in (0,1),$$

$$\left.\begin{array}{c} y(0) = 0 \\ y'(0) = (a\lambda + b) y(1) \end{array}\right\},$$

where $q(x)$ is a complex-valued summable function, $\lambda$ is a spectral parameter, $a$ and $b$ are arbitrary complex numbers ( $a \neq 0$). We prove theorems on the basicity of eigenfunctions and associated functions of the spectral problem in the Lebesgue spaces $L_p(0,1) \oplus C$ and $L_p(0,1), 1 < p < \infty$, as well as in their weighted analogs with a general weight function satisfying the Mackenhaupt condition.

**Key Words and Phrases**: eigenvalues, eigenunctions, complete and minimal system, basicity.

**2010 Mathematics Subject Classifications**: 34B05, 34B24, 34L10, 34L20

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## 1. Introduction

Consider the following spectral problem:

$$-y'' + q(x)\, y = \lambda y, \quad x \in (0,1), \tag{1}$$

$$\left.\begin{array}{c} y(0) = 0, \\ y'(0) = (a\lambda + b) y(1), \end{array}\right\} \tag{2}$$

where $q(x)$ is a complex-valued summable function, $\lambda$ is a spectral parameter, $a$ and $b$ are arbitrary complex numbers ( $a \neq 0$). In this work it is proved the theorems on the basicity of eigenfunctions and associated functions of the spectral problem in the Lebesgue spaces $L_p(0,1) \oplus C$ and $L_p(0,1), 1 < p < \infty$, as well as in their weighted analogs with a general weight function satisfying the Mackenhaupt condition. Numerous works are devoted to

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spectral problems for ordinary differential operators with a spectral parameter in boundary conditions (see, e.g., [1-16]). Of the latter, let us note the works [17-26]. The works [8,9,14,25,26,27,28] are directly related to our work. The case $q(x) \equiv 0,\ b = 0$ is considered in [8,9]. The case $b = 0$, was considered in [14], and other generalizations of boundary conditions (2) were considered in [25,26]. Note that the theorems on the basicity in $L_p(0,1)$ under the additional assumption $q(x) = q(1-x)$, and the theorems on the uniform convergence of spectral expansions for the potential $q(x)$ from class $L_2(0,1)$ were proved in [14,25,26]. In [27], asymptotic formulas for the eigenvalues (1),(2) and eigenfunctions were found, and in [28], theorems on the completeness and minimality of the root functions of problem (1),(2) in the Lebesgue spaces $L_p(0,1) \oplus C$ and $L_p(0,1)$, $1 < p < \infty$ were proved.

## 2. Needed information and preliminary results

In obtaining the main results, we need some concepts and facts from the theory of bases in a Banach space.

**Definition 1.** *A basis $\{u_n\}_{n\in N}$ of a space $X$ is called a p-basis, if for any $x \in X$ the condition*

$$\left(\sum_{n=1}^{\infty} |\langle x, \vartheta_n\rangle|^p\right)^{\frac{1}{p}} \leq M\,\|x\|\,,$$

*is fulfilled, where $\{\vartheta_n\}_{n\in N}$ is a biorthogonal system to $\{u_n\}_{n\in N}$.*

**Definition 2.** *Sequences $\{u_n\}_{n\in N}$ and $\{\phi_n\}_{n\in N}$ of a Banach space $X$ are said to be p close if the condition*

$$\sum_{n=1}^{\infty} \|u_n - \phi_n\|^p < \infty,$$

*is fulfilled.*

Let us recall that two systems in a Banach space are said to be isomorphic (or equivalent) if there exists a bounded linear operator in this space with a bounded inverse that maps one of these systems to the other. We will also use the following result from [29], which is a Banach analogue of the well-known theorem of N.K. Bari [30].

**Theorem 1.** *([29]) Let $\{x_n\}_{n\in N}$ be a q-basis of a Banach space $X$, and let the system $\{y_n\}_{n\in N}$ be a p-close to $\{x_n\}_{n\in N}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then the following properties are equivalent:*

*a) $\{y_n\}_{n\in N}$ is complete in $X$;*
*b) $\{y_n\}_{n\in N}$ is minimal in $X$;*
*c) $\{y_n\}_{n\in N}$ is $\omega-$ linearly independent in $X$;*
*d) $\{y_n\}_{n\in N}$ forms a basis for $X$;*
*e) $\{y_n\}_{n\in N}$ forms a basis in $X$, isomorphic to the system $\{x_n\}_{n\in N}$;*
*f) $\{y_n\}_{n\in N}$ forms a q-basis for $X$.*

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system, and $\left\{\widehat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ is its biorthogonal system:

$$\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm}); \qquad \widehat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm}).$$

Let $J = \{n_1, ..., n_m\}$ be some set of $m$ natural numbers. Assume

$$\delta = det \left\|\beta_{n_i j}\right\|_{i,j=\overline{1,m}}.$$

The following theorem was proved in [31] (see also [32]).

**Theorem 2.** *Let the system $\{\hat{u}_n\}_{n\in N}$ form a basis for $X_1$. For the system $\{u_n\}_{n\in N_J}$, where $N_J = N\backslash J$, to be a basis in $X$, it is necessary and sufficient that the condition $\delta \neq 0$ be satisfied. In this case, the system biorthogonal to $\{u_n\}_{n\in N_J}$ is defined by the equality*

$$\vartheta_n^* = \frac{1}{\delta}\begin{vmatrix} \vartheta_n & \vartheta_{n_1} & \dots & \vartheta_{n_m} \\ \beta_{n1} & \beta_{n_1 1} & \dots & \beta_{n_m 1} \\ \dots & \dots & \dots & \dots \\ \beta_{nm} & \beta_{n_1 m} & \dots & \beta_{n_m m} \end{vmatrix}.$$

*In particular, if $X$ is a Hilbert space and $\{\hat{u}_n\}_{n\in N}$ is a Riesz basis in $X_1$, then under the condition $\delta \neq 0$, the system $\{u_n\}_{n\in N_J}$ also forms a Riesz basis for $X$.*

*For $\delta = 0$ the system $\{u_n\}_{n\in N_J}$ is neither complete nor minimal in $X$.*

We will need some results from [27,28]. In [27] it was proved that the eigenvalues of problem (1),(2) are asymptotically simple and have the form $\lambda_n = \rho_n^2, n = 0, 1, 2, ...$, where the following asymptotic formula holds for the numbers $\rho_n$:

$$\rho_n = \pi n + O\left(\frac{1}{n}\right), \tag{3}$$

and for the eigenfunctions and associated functions $y_n(x)$ of problem (1),(2) corresponding to the eigenvalues $\lambda_n, n = 0, 1, 2, ...$, the asymptotic formula

$$y_n(x) = \sin \pi n x + O\left(\frac{1}{n}\right), \tag{4}$$

is valid, moreover, the problem can have only a finite number of associated functions, and the eigenvalues are numbered taking into account their multiplicities.

The conjugate spectral problem has the form

$$-z'' + \overline{q(x)}z = \lambda z, \quad x \in (0,1), \tag{5}$$

$$\left.\begin{array}{c} z(1) = 0, \\ z'(1) + \left(\overline{a}\lambda + \overline{b}\right) z(0) = 0. \end{array}\right\} \tag{6}$$

The spectral problem (1),(2) is reduced to a spectral problem $L\hat{y} = \lambda\hat{y}$ for the operator $L$, acting in the space $L_p(0,1) \oplus C$. The operator $L$ is defined as follows:

$$D(L) = \left\{\hat{y} = (y(x), ay(1)), y(x) \in W_p^2(0,1), l(y) \in L_p(0,1), y(0) = 0\right\},$$

$$\forall \hat{y} \in D(L) : L\hat{y} = \left(l(y), y'(0) - by(1)\right).$$

It was proved in [28] that the operator $L$ is densely defined in $L_p(0,1) \oplus C$ as a closed operator with a compact resolvent. The eigenvalues of the operator $L$ and problem (1),(2) coincide, and each eigen(or associated) function $y(x)$ of problem (1),(2) corresponds to an eigen(or associated) vector $\hat{y} = (y(x), ay(1))$ of the operator $L$. The adjoint operator $L^*$ is defined as the operator generated in the space $L_q(0,1) \oplus C$, $\frac{1}{p} + \frac{1}{q} = 1$, by problem (5),(6). The eigenfunctions and associated functions of problem (5),(6) satisfy the asymptotic formulas

$$z_n(x) = 2\sin \pi nx + O\left(\frac{1}{n}\right), \quad n = 0, 1, 2, ..., \tag{7}$$

where the eigenfunctions and associated functions $z_n(x)$ are normalized so that the biorthogonality conditions are satisfied

$$\langle \hat{y}_n, \hat{z}_k \rangle = \int_0^1 y_n(x)\overline{z_k(x)}dx + a^2 y_n(1)\overline{z_k(0)} = \delta_{nk},$$

where $\hat{z}_k = (z_k(x), \bar{a}z_k(0))$ are the eigenvectors and associated vectors of the adjoint operator $L^*$, and $\delta_{nk}$ is the Kronecker symbol.

The following theorems are also true.

**Theorem 3.** *([28]) The root vectors of the operator $L$ form a complete and minimal system in the space $L_p(0,1) \oplus C, 1 < p < \infty$.*

**Theorem 4.** *([28]) The system $\{y_n(x)\}_{n=0, n\neq n_0}^{\infty}$ of eigenfunctions and associated functions of problem (1),(2) with one rejected eigenfunction $y_{n_0}(x)$, corresponding to a simple eigenvalue $\lambda_{n_0}$, forms a complete and minimal system in the space $L_p(0,1), 1 < p < \infty$. The corresponding biorthogonal system is $\{\vartheta_n(x)\}_{n=0, n\neq n_0}^{\infty}$, where*

$$\vartheta_n(x) = z_n(x) - \frac{z_n(0)}{z_{n_0}(0)} z_{n_0}(x). \tag{8}$$

## 3. Main results

### 3.1. Basicity in spaces $L_p(0,1) \oplus C$ and $L_p(0,1)$.

Let $e_n(x) = sin\pi nx, n \in \mathbb{N}$ and introduce the following system in space $L_p(0,1) \oplus C$:

$$\hat{e}_0 = (0, 1), \quad \hat{e}_n = (e_n(x), 0), n \in \mathbb{N}.$$

The following theorem is true.

**Theorem 5.** *The system* $\{\hat{y}_n\}_{n=0}^{\infty}$ *, eigenvectors and associated vectors of the operator* $L$ *forms a basis for* $L_p(0,1) \oplus C$, $1 < p < \infty$, *isomorphic to the system* $\{\hat{e}_n\}_{n=0}^{\infty}$.

*Proof.* From the formula (4) it follows

$$y_n = e_n + O\left(\frac{1}{n}\right), \quad y_n(1) = O\left(\frac{1}{n}\right).$$

On the other hand

$$\hat{y}_n = \left(y_n\left(x\right),\ ay_n\left(1\right)\right) = \hat{e}_n + \ O\left(\frac{1}{n}\right).$$

Therefore, for any $r > 1$ :

$$\sum_{n=0}^{\infty} \left\|\hat{y}_n - \hat{e}_n\right\|^r < +\infty, \tag{9}$$

i.e. the system $\{\hat{y}_n\}_{n=0}^{\infty}$ is $r-$ close to the system $\{\hat{e}_n\}_{n=0}^{\infty}$, and by Theorem 3 the system $\{\hat{y}_n\}_{n=0}^{\infty}$ is complete and minimal in $L_p(0,1) \oplus C$.

Let $1 < p \leq 2$, and $q$ – be its conjugate number: $\frac{1}{p} + \frac{1}{q} = 1$. By the Hausdorff-Young inequality [33] for any function $f \in L_p(0,1)$

$$\left(\sum_{n=1}^{\infty} |\langle f, e_n\rangle|^q\right)^{\frac{1}{q}} \leq C\|f\|_{L_p}.$$

Then for any element $\hat{f} = (f,\ \beta) \in L_p(0,1) \oplus C$ we have

$$\left(\sum_{n=0}^{\infty} \left|\left\langle \hat{f}, \hat{e}_n\right\rangle\right|^q\right)^{\frac{1}{q}} \leq |\beta| + \left(\sum_{n=1}^{\infty} |\langle f, e_n\rangle|^q\right)^{\frac{1}{q}} \leq |\beta| + \|f\|_{L_p} \leq C_1 \left\|\hat{f}\right\|_{L_p \oplus C}.$$

Consequently, the system $\{\hat{e}_n\}_{n=0}^{\infty}$ is a $q$ - basis in $L_p(0,1) \oplus C$. Now, choosing $r = p$ in (9) we get that all the conditions of Theorem 2 are satisfied, therefore, the system $\{\hat{y}_n\}_{n=0}^{\infty}$ forms a basis for $L_p(0,1) \oplus C$, equivalent to the system $\{\hat{e}_n\}_{n=0}^{\infty}$.

Let, now $p > 2$. Then $1 < q < 2$ and the embedding

$$L_p(0,1) \subset L_q(0,1)$$

or

$$L_p(0,1) \oplus C \subset L_q(0,1) \oplus C$$

holds, and for $\hat{f} \in L_p(0,1) \oplus C$ we have

$$\left(\sum_{n=0}^{\infty} \left|\left\langle \hat{f}, \hat{e}_n\right\rangle\right|^p\right)^{\frac{1}{p}} \leq \ \mathrm{c}\left\|\hat{f}\right\|_{L_q \oplus C} \leq \mathrm{c}_1 \left\|\hat{f}\right\|_{L_p \oplus C}$$

i.e. the system $\{\hat{e}_n\}_{n=0}^{\infty}$ is a p-basis in $L_p\left(0,1\right) \oplus C$. Choosing $r = q$ we get that all the conditions of Theorem 2 are satisfied, which means that in this case the system $\{\hat{y}_n\}_{n=0}^{\infty}$ forms a basis for $L_p\left(0,1\right) \oplus C,$ equivalent to the system$\{\hat{e}_n\}_{n=0}^{\infty}$. Theorem is proved.

**Corollary 1.** *In the case p=2 the system* $\{\hat{y}_n\}_{n=0}^{\infty}$ *forms a Riesz basis for* $L_2(0,1) \oplus C$.

**Theorem 6.** *In order for the system* $\{y_n(x)\}_{n=0,\ \ n\neq n_0}^{\infty}$ *of root functions of problem* (1) *and* (2) *with one remote function* $y_{n_0}(x)$ *to form a basis for* $L_p(0,1)$, $1<p<\infty$, *it is necessary and sufficient that the condition* $z_{n_0}(0) \neq 0$ *be satisfied. If* $z_{n_0}(0)=0$, *then the system* $\{y_n(x)\}_{n=0,\ \ n\neq n_0}^{\infty}$ *is not complete and minimal, and even more so is not a basis in* $L_p(0,1)$.

The *proof* follows from Theorem 5 followed by the application of Theorems 2 and 4.

**Theorem 7.** *The eigenfunctions and associated functions* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *of problem* (1) *and* (2) *with one remote eigenfunction* $y_{n_0}(x)$, *corresponding to a simple eigenvalue* $\lambda_{n_0}$ *forms a basis for* $L_p(0,1)$, $1<p<\infty$, *isomorphic to the trigonometric system* $\{sin\pi nx\}_{n=1}^{\infty}$.

*Proof.* If $\lambda_{n_0}$ is a simple eigenvalue, then it corresponds to one eigenfunction $y_{n_0}(x)$ and $z_{n_0}(x)$ is the corresponding eigenfunction of the adjoint problem (5), (6). It should be noted that for all eigenfunctions $z_n(x)$ of the adjoint problem, the condition $z_n(0) \neq 0$ is satisfied. Indeed, let $z_n(0)=0$, then from the second boundary condition (6) we obtain $z_n^{'}(1)=0$, and this together with the first boundary condition $z_n(1)=0$ means that $z_n(x)$ is the solution of Cauchy problem

$$-z^{''}+q(x)\,z=\lambda z,$$

$$z(1)=z^{'}(1)=0,$$

which has only the trivial solution $z(x)\equiv 0$. And this contradicts the fact that $z_n(x)$ is an eigenfunction. Thus $z_{n_0}(1)\neq 0$. Then, by Theorem 6, the system $\{y_n(x)\}_{n=0,\ \ n\neq n_0}^{\infty}$ forms a basis for $L_p(0,1)$. It follows from the asymptotic formulas (4) that $\forall r\in(1;+\infty)$

$$\sum_{n=n_0+1}^{\infty}\|y_n-e_n\|^r<+\infty$$

i.e. the system $\{y_n(x)\}_{n=0,\ ,\ n\neq n_0}^{\infty}$ is $r-$ close to the system $\{e_n\}_{n=1}^{\infty}$ $(e_n(x)=\sin\pi nx)$. Choosing $r=min\,\{p,q\}$, and taking into account that the system $\{e_n\}_{n=1}^{\infty}$ is an $r^{'}-$basis in $L_p(0,1)$ for the system $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ ($r'=max\,\{p,q\}$, $\frac{1}{r}+\frac{1}{r'}=1$), we find that all conditions of Theorem 1 are satisfied and, therefore, it is isomorphic to the system $\{\sin\pi nx\}_{n=1}^{\infty}$. Theorem is proved.

**Corollary 2.** *Under the conditions of Theorem 7, the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *forms a* $r-$ *basis for* $L_p(0,1)$, $1<p<\infty$, *where* $r=max\,\{p,q\}$.

**Corollary 3.** *In the case* $p=2$ *the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *forms a Riesz basis for* $L_2(0,1)$.

### 3.2. Basicity in spaces

**$L_{p,\omega}\ (0,1) \oplus C$ and $L_{p,\omega}(0,1)$.**

Denote by $L_{p,\omega}\ (0,1)$ the weighted Lebesgue space with the norm

$$\|f\|_{L_{p,\omega}} = \left( \int_0^1 |f\,(x)|^p \omega\,(x)\,dx \right)^{\frac{1}{p}},$$

where the weight function $\omega\,(x)$ belongs to the Mackenhaupt class $A_p$, i.e. satisfies the condition

$$\sup_{I\subset(0,1)} \left( \frac{1}{|I|} \int_I \omega\,(x)\,dx \right) \left( \frac{1}{|I|} \int_I (\omega\,(x))^{-\frac{1}{p-1}} dx \right)^{p-1} < +\infty.$$

It was proved in [34] that if $\omega\,(x) \in A_p$, then there exists a number $r \in (1,p)$ such that $\omega\,(x) \in A_r$. Using this fact, we prove the following

**Lemma 1.** *Let the weight function $\omega\,(x)$ belong to the class $A_p$, $1 < p < \infty$. Then there exists a number $p_0$: $1 < p_0 < p$, such that a continuous embedding $L_{p,\omega}\ (0,1) \subset L_{p_0}\ (0,1)$ holds.*

*Proof.* Let $f \in L_{p,\omega}\ (0,1)$. Assume $p_0 = \frac{p}{r}$. Then $|f\,(x)|^{p_0} = |f\,(x)|^{p_0} \omega^{\frac{p_0}{p}}(x) \omega^{-\frac{p_0}{p}}(x)$ and from belonging of the function $|f\,(x)|^{p_0} \omega^{\frac{p_0}{p}}(x)$ to the class $L_{\frac{p}{p_0}}\ (0,1)$, and also from belonging of the function $\omega^{-\frac{p_0}{p}}(x)$ to the class $\left(L_{\frac{p}{p_0}}\ (0,1)\right)^* = L_{\frac{p}{p-p_0}}\ (0,1)$, and using the Hölder inequality, we obtain

$$\|f\|_{L_{p_0}\ (0,1)} = \left( \int_0^1 |f\,(x)|^{p_0} dx \right)^{\frac{1}{p_0}} = \left( \int_0^1 |f\,(x)|^{p_0} \omega^{\frac{p_0}{p}}(x) \omega^{-\frac{p_0}{p}}(x) dx \right)^{\frac{1}{p_0}} \leq$$

$$\leq \left( \int_0^1 |f\,(x)|^p \omega\,(x)\,dx \right)^{\frac{1}{p}} \left( \int_0^1 \omega^{-\frac{p_0}{p-p_0}}(x)\,dx \right)^{\frac{p-p_0}{pp_0}} = \|f\|_{L_{p,\omega}\ (0,1)} \left( \int_0^1 \omega^{-\frac{1}{r-1}}(x)\,dx \right)^{\frac{r-1}{p}} =$$

$$= K_{p,r}(\omega) \|f\|_{L_{p,\omega}\ (0,1)}.$$

Since $\omega^{-1} \in L_{\frac{1}{r-1}}(0,1)$, then the quantity $K_{p,r}(\omega) = \left( \int_0^1 \omega^{-\frac{1}{r-1}}(x)\,dx \right)^{\frac{r-1}{p}}$ has a finite value. Consequently, $f \in L_{p_0}\ (0,1)$.

**Corollary 4.** *If $f \in L_{p,\omega}\ (0,1)$, then $\forall s \in (0,p_0]$, i.e. $\forall s \in \left(0, \frac{p}{r}\right] : f \in L_s\ (0,1)$.*

**Lemma 2.** *Let $\omega \in A_p\,(0,1)$. Then each of the systems $\{sin\pi nx\}_{n=1}^{\infty}$ and $\{cos\pi nx\}_{n=0}^{\infty}$ forms a basis for $L_{p,\omega}\ (0,1)$.*

*Proof.* Denote by $\widetilde{\omega}\,(x)$ the even extension of the function $\omega\,(x)$ to $[-1,1]$, i.e. for $x \in [-1,0]$ $\widetilde{\omega}\,(x) = \omega\,(-x)$, or $x \in [0,1]$ $\widetilde{\omega}\,(x) = \omega\,(x)$. Then it is evident that $\widetilde{\omega}\,(x) \in A_p(-1,1)$. Let $f \in L_{p,\omega}\ (0,1)$. Let's extend it to $[-1,1]$ in an odd way, i.e.

$$\tilde{f}\,(x) = \begin{cases} f\,(x)\,, & x \in [0,1]\,, \\ -f\,(-x)\,, & x \in [-1,0]\,. \end{cases}$$

Then $\tilde{f}(x) \in L_{p,\widetilde{\omega}}\ (-1,1)$. We expand this function in the basis $\left\{e^{\ i\pi nx}\right\}_{n=-\infty}^{+\infty}$:

$$\tilde{f}\,(x) = \sum_{n=-\infty}^{+\infty} a_n e^{\ i\pi nx}, a_n = \frac{1}{2}\int_{-1}^{1} \tilde{f}\,(x) e^{-i\pi nx} dx.$$

It is obvious that

$$a_n = \frac{1}{2}\int_{0}^{1} f\,(x) e^{-i\pi nx} dx - \frac{1}{2}\int_{-1}^{0} f\,(-x) e^{-i\pi nx} dx =$$

$$= \frac{1}{2}\int_{0}^{1} f\,(x)\,(e^{-i\pi nx} - e^{i\pi nx}) dx = \frac{1}{i}\int_{0}^{1} f\,(x) \sin\pi nx\ dx.$$

In addition $a_{-n}$=-$a_n$, $a_0 = 0$. Taking into account these relations, we get

$$\sum_{n=-m}^{m} a_n e^{i\pi nx} = \sum_{n=1}^{m} a_n (e^{i\pi nx} - e^{-i\pi nx}) =$$

$$= 2i \sum_{n=1}^{m} a_n \sin\pi nx \ = \sum_{n=1}^{m} \langle f,\ 2\sin\pi nt\ \rangle \sin\pi nx\ .$$

Hence

$$\left\| \tilde{f}\,(x) - \sum_{n=-m}^{m} a_n e^{i\pi nx} \right\|_{L_{p,\omega}\ (-1,1)} = \left\| \tilde{f}\,(x) - \sum_{n=1}^{m} \langle f,\ 2\sin\pi nt\ \rangle \sin\pi nx\ \right\|_{L_{p,\omega}\ (-1,1)} =$$

$$= 2^{\frac{1}{p}} \left\| f\,(x) - \sum_{n=1}^{m} \langle f,\ 2\sin\pi nt\ \rangle \sin\pi nx\ \right\|_{L_{p,\omega}\ (0,1)}.$$

The left side of the last equality tends to zero as $m \to \infty$, which means that the right side tends to zero as $m \to \infty$, and it means that the system $\{\sin\pi nx\ \}_{n=1}^{\infty}$ forms a basis for $L_{p,\omega}\ (0,1)$.

The basicity of the system $\{cos\pi nx\ \}_{n=0}^{\infty}$ in $L_{p,\omega}\ (0,1)$ is proved similarly. To do this, it suffices to take an even extension of the function $f\,(x)$ to $[-1,1]$.

**Theorem 8.** *The system $\{\hat{y}_n\}_{n=0}^{\infty}$ of root vectors of the operator $L$ forms a basis for $L_{p,\omega}\ (0,1) \oplus C$ isomorphic to the system $\{\hat{e}_n\}_{n=0}^{\infty}$.*

*Proof.* From the continuity of the embedding

$$L_{p,\omega}\ (0,1) \oplus C \subset L_{p_0}\ (0,1) \oplus C,$$

and also from the minimality of the system $\{\hat{y}_n\}_{n=0}^{\infty}$ (according to Theorem 3) in the space $L_{p_0}\ (0,1) \oplus C$ it follows that this system is also minimal in $L_{p,\omega}\ (0,1) \oplus C$. It follows from asymptotic formulas (4) that

$$y_n\,(x) = \ e_n\,(x) + \varepsilon_n\,(x) \tag{10}$$

where for $\varepsilon_n(x)$ uniformly with respect to $x \in [0,1]$ the estimate

$$|\varepsilon_n(x)| \leq \frac{const}{n}, \tag{11}$$

is valid. Taking into account estimate (11), from (10) we obtain

$$\|\hat{y}_n - \hat{e}_n\|_{L_{p,\omega}\ (0,1) \oplus C} = \left( \int_0^1 |\varepsilon_n(\mathrm{x})|^{\mathrm{p}} \omega(\mathrm{x})\, \mathrm{dx} \right)^{\frac{1}{\mathrm{p}}} \leq \frac{const}{n}.$$

Consequently, $\forall \tau \in (1; +\infty)$

$$\sum_{n=0}^{\infty} \|\hat{y}_n - \hat{e}_n\|^{\tau} < +\infty, \tag{12}$$

i.e. the system $\{\hat{y}_n\}_{n=0}^{\infty}$ is $\tau-$close to the system $\{\hat{e}_n\}_{n=0}^{\infty}$ for any $\tau \in (1; +\infty)$. On the other hand, according to Corollary 3, a continuous embedding

$$L_{p,\omega}\ (0,1)\ \oplus C \subset L_s\ (0,1)\ \oplus C,$$

holds, $\forall s \in (1,\ p_0]$. Then, choosing $1 < s < min\,\{2, p_0\}$ and applying the Hausdorff-Young inequality for the system $\{e_n(x)\}_{n=1}^{\infty}$ $(e_n(x) = \sin\pi nx\ )$, we obtain $\forall \hat{f} = (f(x),\ \beta) \in L_{p,\omega}(0,1) \oplus C$

$$\left( \sum_{n=0}^{\infty} \left| \left\langle \hat{f}, \hat{e}_n \right\rangle \right|^{\mathrm{s}'} \right)^{\frac{1}{\mathrm{s}'}} \leq |\beta| + \left( \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^{\mathrm{s}'} \right)^{\frac{1}{\mathrm{s}'}} \leq$$

$$\leq |\beta| + c_2 \|f\|_{L_s} \leq c_3 \left\| \hat{f} \right\|_{L_s\ (0,1)\ \oplus C} \leq c_4 \|f\|_{L_{p,\omega}\ \ \oplus C}.$$

The latter means that the system $\{\hat{e}_n\}_{n=0}^{\infty}$ forms an $\mathrm{s}'$–basis for $L_{p,\omega}\ (0,1)$, where $\mathrm{s}' = s/(s-1)$. Choosing $\tau = s$, in (12) we obtain that all the conditions of Theorem 1 are satisfied, therefore, the system $\{\hat{y}_n\}_{n=0}^{\infty}$ forms a basis for $L_{p,\omega}\ (0,1) \oplus C$ isomorphic to the system $\{\hat{e}_n\}_{n=0}^{\infty}$ .

Similarly to the previous section, we prove that the following theorems and corollaries are true.

**Theorem 9.** *For the basicity of the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *of eigenfunctions and associated functions of problem* (1), (2) *with one remote function* $y_{n_0}(x)$ *in* $L_{p,\omega}\ (0,1)$ *it is necessary and sufficient that the condition* $z_{n_0}(0) \neq 0$ *be satisfied. For* $z_{n_0}(0) = 0$ *the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *does not form a basis in the space* $L_{p,\omega}\ (0,1)$. *Moreover, in this case the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *is neither complete nor minimal in* $L_{p,\omega}\ (0,1)$.

**Theorem 10.** *The system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *corresponding to eigenfunctions and associated functions of problem* (1), (2) *with one removed function* $y_{n_0}(x)$, *corresponding to a simple eigenfunction value* $\lambda_{n_0}$, *forms a basis for* $L_{p,\omega}\ (0,1)$, $1 < p < \infty$, *isomorphic to the trigonometric system* $\{sin\pi nx\ \}_{n=1}^{\infty}$.

**Corollary 5.** *Under the conditions of Theorem 10, the system* $\{y_n(x)\}_{n=0,\ n\neq n_0}^{\infty}$ *forms an s–basis in* $L_{p,\omega}\ (0,1)$ *for some* $s > 2$.

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## References

[1] Atkinson F.B. Discrete and continuous boundary value problems. Moscow, Mir, 1968, 749 p.

[2] Tikhonov A.N., Samarskii A.A. Equations of Mathematical Physics. Moscow, Nauka, 1977, 766 p.

[3] Kapustin N.Yu., Moiseev E.I. Spectral problems with the spectral parameter in the boundary condition. Diff. Uravn., 1997, v. 33, No. 1, pp. 115-119.

[4] Gomilko A.M., Pivovarchik V.N. On bases of eigenfunctions of boundary problem associated with small vibrations of damped non smooth inhomogeneous string. Asympt. Anal. 1999., v. 20 No. 3-4, pp. 301-315.

[5] Kerimov N.B., Mirzoev V.S. On the Basis Properties of One Spectral Problem with a Spectral Parameter in a Boundary Condition. Sib. Math. J. 2003, v.44, No 5, pp.1041-1045.

[6] Kerimov N.B., Poladov R.G. On basicity in $L_p$ of the system of eigenfunctions of one boundary value problem II. Proc. IMM NAS Azerb., 2005, v. 23, pp. 65-76.

[7] Kerimov N.B., Aliyev Z.S. On the basis properties of eigenfunctions of a spectral problem with a spectral parameter in the boundary condition. Dokl. RAN, 2007, v. 412, No 1, pp. 18-21.

[8] Marchenkov D.B. On the convergence of spectral expansions of functions for problems with a spectral parameter in a boundary condition. Differential Equations, 2005, v. 41, No. 10, pp. 1496-1500.

[9] Marchenkov D.B. Basis property of the system of eigenfunctions corresponding to a problem with a spectral parameter in the boundary condition. Differential Equations, 2006, v. 42, No. 6, pp. 905-908.

[10] Gasymov T.B., Mammadova Sh.J. On convergence of spectral expansions for one discontinuous problem with spectral parameter in the boundary condition. Trans. Of NAS of Azerb. 2006,vol. XXVI, No. 4, pp. 103-116.

[11] Gasymov T.B., Huseynli A.A. The basis properties of eigenfunctions of a discontinuous differential operator with a spectral parameter in boundary condition // Proc. of IMM of NAS of Azerb., v. XXXV(XLIII), 2011, pp. 21-32.

[12] Shahriari M., Akbarfam J.A., Teschl G. Uniqueness for inverse Sturm-Liouville problems with a finite number of transmission conditions. J. Math. Anal. Appl., v. 395, 2012, pp. 19-29.

[13] Shahriari M. Inverse Sturm-Liouville Problem with Eigenparameter Dependent Boundary and Transmission Conditions. Azerb. J. Math., v.4, No. 2, 2014, 16-30.

[14] Kerimov N.B., Maris E.A. On the basis properties and convergence of expansions in terms of eigenfunctions for a spectral problem with a spectral parameter in the boundary condition. Proceedings of IMM of NAS of Azerbaijan, 2014, v.40, Special Issue, pp. 245-258.

[15] Bilalov B.T., Gasymov T.B. On bases for direct decomposition. Doklady Mathematics. v. 93, No. 2, 2016, pp 183-185.

[16] Bilalov B.T., Gasymov T.B. On basicity a system of eigenfunctions of second order discontinuous differential operator. Ufa Mathematical Journal, 2017, v. 9, No 1, pp. 109-122.

[17] Maharramova G.V. On the completeness of eigenfunctions of a second-order differential operator, Journal of Contemporary Applied Mathematics, v. 8, No. 2, 2018, pp. 45-55.

[18] Gasymov T.B., Maharramova G.V. The stability of the basis properties of multiple systems in a Banach space with respect to certain transformations. Caspian Journal of Applied Mathematics, Ecology and Economics. V. 6, No 2, 2018, December, pp. 66-77.

[19] Gasymov T.B, Maharramova G.V., Mammadova N.G. Spectral properties of a problem of vibrations of a loaded string in Lebesgue spaces. Transactions of NAS of Azerbaijan, Issue Mathematics, 38(1), 62-68(2018), Series of Physical-Technical and Mathematical Sciences.

[20] Gasymov T.B, Maharramova G.V., Jabrailova A.N. Spectral properties of the problem of vibration of a loaded string in Morrey type spaces. Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, v. 44, No 1, 2018, pp. 116-122.

[21] Bilalov B.T., Gasymov T.B., Maharramova G.V. Basis Property of Eigenfunctions in Lebesgue Spaces for a Spectral Problem with a Point of Discontinuity. Diff. Uravn., 2019, v.55, No 12, pp.1544-1553.

[22] Maharramova G.V. Properties of Eigenvalues and Eigenfunctions of a Spectral Problem with Discontinuity Point. Caspian Journal of Applied Mathematics, Ecology and Economics V. 7, No 1, 2019, July, pp. 114-125.

[23] Gasymov T.B., Maharramova G.V., T.F. Kasimov, Completeness and minimality of eigenfunctions of a spectral problem in spaces $L_p \oplus C$ and $L_p$. Journal of Contemporary Applied Mathematics, v.10, No 2, 2020, pp. 84-100.

[24] Gasymov T.B., Akhtyamov A.M., Ahmedzade N.R. On the basicity of eigenfunctions of a second-order differential operator with a discontinuity point in weighted Lebesgue spaces. Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, v. 46, No 1, 2020, pp. 32–44.

[25] Maris E.A., Goktaş S. On the spectral properties of a Sturm-Liouville problem with eigenparameter in the boundary condition. Hacet. J. Math. Stat., v. 49, No. 4, 2020, pp. 1373 – 1382.

[26] Goktaş S., Maris E.A. The uniform convergence of Fourier series expansions of a Sturm-Liouville problem with boundary condition which contains the eigenparameter. Commun. Fac. Sci. Univ. Ank. Ser. A1 M ath. Stat., v. 70, No. 1, 2021, pp. 205- 215.

[27] Kasimov T.F. On the Completeness and Minimality of Eigenfunctions of a Non-self-adjoint Spectral Problem With Spectral Parameter in the Boundary Condition, Caspian Journal of Applied Mathematics, Ecology and Economics, V. 8, No 2, 2020, December, pp. 61-75.

[28] Kasimov T.F. Asymptotics of the Eigenvalues and Eigenfunctions of a Non-Self-Adjoint Problem with a Spectral Parameter in the Boundary Condition. Journal of Contemporary Applied Mathematics V. 11, No 2, 2021, December, pp. 11-22.

[29] Bilalov, B.T., Bases of exponentials, cosines, and sines formed by eigenfunctions of differential operators, Differ. Uravn., v. 39, No. 5, 2003, pp.652-657.

[30] Gohberg I.C., Krein M.G. Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space, AMS, 1969, 394p.

[31] Gasymov T.B. On necessary and sufficient conditions of basicity of some defective systems in Banach spaces. Trans. NAS Azerb., ser. phys.-tech. math. sci., math.mech., v.26, No. 1, 2006, pp. 65-70.

[32] Gasymov T.B., Garayev T.Z. On necessary and sufficient conditions for obtaining the bases of Banach spaces. Proc. of IMM of NAS of Azerb., v. XXVI(XXXIV), 2007, pp. 93-98.

[33] Sigmund A. Trigonometric series. v 2, M.: Mir, 1965, 537 p.

[34] Hunt R., Muckenhoupt B., Wheeden R. Weighted norm inequalities for the conjugate function and Holder transform. Transactions of the American Mathematical Society, v. 176, February 1973, pp. 227-251.

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# Inverse Boundary Value Problem for a Third-Order Partial Differential Equation with an Additional Integral Condition

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**Abstract.** In the article the author analyses one inverse boundary problem for a partial differential equation of third order with an Additional Integral Condition. First, an original problem is reduced to the equivalent problem, the theorem of existence and uniqueness of solution is proved for the latter. Then, using these facts the author proves existence and uniqueness of classical solution of the original problem.

**Key Words and Phrases**: inverse problem, differential equations, existence, uniqueness, classical solution.

**2010 Mathematics Subject Classifications**: 65N06, 31A25, 35A09

## 1. Introduction

Inverse problems are an actively developing branch of modern mathematics. Recently, inverse problems have arisen in various fields of human activity, such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc., which puts them among the actual problems of modern mathematics. Various inverse problems for certain types of partial differential equations have been studied in many works.

Let us note here, first of all, the works of A.N. Tikhonov [1], M.M. Lavrentev [2, 3], V.K. Ivanov [4] and their students. More details about this can be found in the monograph by A.M. Denisov [5].

The purpose of this work is to prove the existence and uniqueness of solutions of one inverse boundary value problem for a third order differential equation with partial derivatives with an integral condition of the first kind.

In this work, using Fourier method and contraction mapping principle, we prove the existence and uniqueness of the solution of the nonlocal inverse boundary value problem for a third order two-dimensional pseudo parabolic equation.

 

## 2. Formulation of the inverse boundary value problem

Consider for the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial t}\left(a(t)\frac{\partial^2 u(x,t)}{\partial x^2}\right) = p(t)u(x,t) + f(x,t) \tag{1}$$

in the domain $D_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$ inverse boundary value problem with initial conditions

$$u(x,0) = \varphi(x)\,,\ u_t(x,0)\ = \psi(x)\ \ (0 \le x \le 1)\ , \tag{2}$$

with Neumann boundary conditions

$$u_x(0,t) = u(1,t) = 0\ \ (0 \le t \le T)\,, \tag{3}$$

and with an additional integral condition

$$\int_0^1 g(x)u(x,t)dx = 0\ (0 \le t \le T) \tag{4}$$

where $a(t) > 0,\ f(x,t),\ \varphi(x),\ \psi(x),\ \omega(x)\ ,\ h(t)$ are given functions, and $u(x,t)$ and $p(t)$ are unknown functions.

Let us introduce the notation

$$\tilde{C}^{2,2}(D_T) = \{u(x,t) : u(x,t) \in C^2(D_T),\ u_{txx}(x,t) \in C(D_T)\}\,.$$

**Definition 1.** *Under the classical solution of the inverse boundary value problem* (1)-(4) *we mean a pair* $\{u(x,t), p(t)\}$ *of functions* $u(x,t),\ p(t)$ *, if* $u(x,t) \in \tilde{C}^{2,2}(D_T),\ p(t) \in C[0,T]$ *and the relations* (1)-(4) *are satisfied in the usual sense .*

The following theorem is true.

**Theorem 1.** *Let* $f(x,t) \in C(D_T),\ \ \psi(x) \in C[0,\ 1],\ \varphi(x) \in C[0,\ 1],\ h(t) \in C^2[0,T]\ ,$ $0 < a(t) \in C^1[0,\ T],\ h(t) \neq 0\ \ (0 \le t \le T)$ *and the matching conditions are met*

$$\int_0^1 g(x)\varphi(x)dx = 0, \quad \int_0^1 g(x)\psi(x)dx = 0.$$

*Then the problem of finding a classical solution to problem* (1)-(4) *is equivalent to the problem of determining functions* $u(x,t) \in \tilde{C}^{2,2}(D_T),\ p(t) \in C[0,T]$*, satisfying equation* (1)*, conditions* (2)*,* (3) *and conditions*

$$h''(t) - \frac{d}{dt}\left(a(t)\int_0^1 g(x)\frac{\partial^2 u(x,t)}{\partial x^2}dx\right) = p(t)h(t) + \int_0^1 g(x)f(x,t)dx\ (0 \le t \le T). \tag{5}$$

*Proof.* Let $\{u(x,t), p(t)\}$ be a classical solution to problem (1)-(4). Since $h(t) \in C^2[0,T]$, we differentiate (4) twice with respect to $t$, we get:

$$\int_0^1 g(x)u_t(x,t)dx = h'(t)\ ,\ \ \int_0^1 g(x)u_{tt}(x,t)dx = h''(t) \quad (0 \le t \le T). \tag{6}$$

We multiply equation (1) by the function $g(x)$ and integrate the resulting equality from 0 to 1 with respect to $x$, we have:

$$\frac{d^2}{d\,t^2}\int_0^1 g(x)u(x,t)dx - \frac{d}{dt}\left(a(t)\int_0^1 g(x)\frac{\partial^2 u(x,t)}{\partial x^2}dx\right) =$$

$$= p(t)\int_0^1 g(x)u(x,t)dx + \int_0^1 g(x)f(x,t)dx \ (\ 0 \le t \le T). \tag{7}$$

Hence, taking into account (4) and (6), we easily arrive at the fulfillment of (5).

Now, suppose that $\{u(x,t), p(t)\}$ is a solution to problem (1)-(3), (5).

Then from (5) and (7) we get:

$$\frac{d^2}{dt^2}\int_0^1 g(x)u(x,t)dx = p(t)\int_0^1 g(x)u(x,t)dx \ (0 \le t \le T). \tag{8}$$

Due to (2) and $\int_0^1 g(x)\varphi(x)dx = 0,\quad \int_0^1 g(x)\psi(x)dx = 0$ it is clear that

$$\int_0^1 g(x)u(x,0)dx = \int_0^1 g(x)\varphi(x)dx = 0, \int_0^1 g(x)u_t(x,0)dx = \int_0^1 g(x)\psi(x)dx = 0. \tag{9}$$

From (8) and (9) we conclude that condition (4) is satisfied. Theorem is proved.

## 3. On the solvability of the inverse boundary value problem

The first component $u(x,t)$ of the solution $\{u(x,t), p(t)\}$ of problem (1)-(3), (5) will be sought in the form:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right), \tag{10}$$

where

$$u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx \ \ (k = 1, 2, ...).$$

Then, applying the formal scheme of the Fourier method, from (1), (2), we obtain:

$$u_k''(t) + \lambda_k^2(a(t)u_k(t))' = F_k(t; u, p) \ \ (k = 1, 2, ...;\ 0 \le t \le T), \tag{11}$$

$$u_k(0) = \varphi_k, u_k'(0) = \psi_k \ (k = 1, 2, ...), \tag{12}$$

where

$$F_k(t; u, p) = f_k(t) + p(t)u_k(t), f_k(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx, ,$$

$$\varphi_k = 2\int_0^1 \varphi(x)\cos\lambda_k x dx, \psi_k = 2\int_0^1 \psi(x)\cos\ \lambda_k x dx \ \ (k = 1, 2, ...).$$

Solving problem (11), (12) we find:

$$u_k(t) = \varphi_k \left( e^{-\lambda_k^2 \int_0^t a(s)ds} + \lambda_k^2 a(0) \int_0^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau \right) + \psi_k \int_0^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau +$$

$$+ \int_0^t F_k(\eta; u, p) \left( \int_\eta^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau \right) d\eta \quad (k = 1, 2, ...). \tag{13}$$

Differentiating twice (21) we obtain:

$$u_k'(t) = -\lambda_k^2 \varphi_k \left( a(t) e^{-\lambda_k^2 \int_0^t a(s)ds} - a(0) \left( 1 - \lambda_k^2 a(t) \int_0^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau \right) \right) +$$

$$+\psi_k \left( 1 - \lambda_k^2 a(t) \int_0^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau \right) +$$

$$+ \int_0^t F_k(\eta; u, p, q) \left( 1 - \lambda_k^2 a(t) \int_\eta^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau \right) d\eta \quad (k = 1, 2, ...), \tag{14}$$

$$u_k''(t) = -\lambda_k^2 \varphi_k \left( (a'(t) - \lambda_k^2 a^2(t)) e^{-\lambda_k^2 \int_0^t a(s)ds} + \lambda_k^2 a(0) (a'(t) - \right.$$

$$\left. -\lambda_k^2 a^2(t)) \int_0^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau \right) - \lambda_k^2 \psi_k (a'(t) - \lambda_k^2 a^2(t)) \int_0^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau -$$

$$-\lambda_k^2 \int_0^t F_k(\eta; u, p) \left( (a'(t) - \lambda_k^2 a^2(t)) \int_0^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau + a(t) \right) d\eta +$$

$$+F_k(t; u, p) \, (k = 1, 2, ...). \tag{15}$$

After substituting the expression $u_k(t)$ $(k = 1, 2, ...)$ from (13) into (10), to determine the the component $u\,(x, t)$ of the solution of problem (5) we obtain:

$$u(x, t) = \sum_{k=1}^{\infty} \left\{ \varphi_k \left( e^{-\lambda_k^2 \int_0^t a(s)ds} + \lambda_k^2 a(0) \int_0^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau \right) + \right.$$

$$\left. +\psi_k \int_0^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau + \int_0^t F_k(\eta; u, p) \left( \int_\eta^t e^{-\lambda_k^2 \int_\tau^t a(s)ds} d\tau \right) d\eta \right\} \cos \lambda_k x \,. \tag{16}$$

Now from (5), taking into account (10), we get:

$$p(t) = [h(t)]^{-1} \left\{ h''(t) - \int_0^1 g(x) f(x, t) dx + \sum_{k=1}^{\infty} n_k \lambda_k^2 (a(t) u_k(t))' \right\}, \tag{17}$$

where

$$n_k = \int_0^1 g(x) \cos \lambda_k x dx. \tag{18}$$

Further, from (18), by virtue of (25) we find:

$$\lambda_k^2(a(t)u_k(t))' = -u_k''(t) + F_k(t;u,p,q) =$$

$$= \lambda_k^2\varphi_k\left((a'(t)-\lambda_k^2a^2(t))\left(e^{-\lambda_k^2\int_0^t a(s)ds} + \lambda_k^2 a(0)\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)\right)+$$

$$+\lambda_k^2\psi_k(a'(t)-\lambda_k^2a^2(t))\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau+$$

$$+\lambda_k^2\int_0^t F_k(\eta;u,p)\left((a'(t)-\lambda_k^2a^2(t))\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau + a(t)\right)d\eta\,(k=1,2,\dots) \qquad (19)$$

In order to obtain an equation for the second component $p(t)$ of the solution $\{u(x,t),p(t)\}$ of problem (1)-(3), (5) we substitute the expression $\lambda_k^2(a(t)u_k(t))'\,(k=1,2,...)$ from (19) to (17) . We have:

$$p(t) = [h(t)]^{-1}\left\{h''(t) - \int_0^1 g(x)f(x,t)dx + \right.$$

$$+\sum_{k=1}^{\infty} n_k\left[\lambda_k^2\varphi_k\left((a'(t)-\lambda_k^2a^2(t))\left(e^{-\lambda_k^2\int_0^t a(s)ds} + \lambda_k^2 a(0)\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\right)\right)+\right.$$

$$+\lambda_k^2\psi_k(a'(t)-\lambda_k^2a^2(t))\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau+$$

$$\left.\left.+\lambda_k^2\int_0^t F_k(\eta;u,p)\,\left((a'(t)-\lambda_k^2a^2(t))\int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau + a(t)\right)d\eta\right]\right\}, \qquad (20)$$

Thus, the solution of problem (1)-(3), (5) is reduced to the solution of system (16), (20) with respect to unknown functions $u(x,t)$ and $p(t)$.

To study the question of the uniqueness of the solution of problem (1) - (3), (5), the following lemma plays an important role.

**Lemma 1.** *If* $\{u(x,t),p(t)\}$ *- be any solution to the problem* (1)-(3), (5), *then the functions*

$$u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx \;\; (k=1,2,...)$$

*satisfy the system consisting of equations* (13).

It is obvious that if $u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx(k=1,2,...)$ is a solution to system (20) and (21), then the pair $\{u(x,t),p(t)\}$ of a function $u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x$ and $p(t)$ are solutions to system (16), (20) .

It follows from Lemma 1 that the following corollary holds.

**Corollary 1.** *Let system* (16), (20) *have a unique solution. Then problem* (1)-(3), (5) *cannot have more than one solution, i.e. if problem* (1)-(3), (5) *has a solution, then it is unique.*

1. Denote by $B_{2,T}^3$, the set of all functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\cos\lambda_k x\ ,$$

considered in $D_T$, where each of the functions $u_k(t)\,(k=1,2,...)$ is continuous on $[0,T]$ and

$$I(u)\equiv\left\{\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}}<+\infty.$$

We define the norm on this set as follows:

$$\|u(x,t)\|_{B_{2,T}^3}=I(u).$$

2. Denote by $E_T^3$ the space consisting of the topological product

$$B_{2,T}^3\times C[0,T]\,.$$

The norm of an element $z=\{u,p\}$ is defined by the formula

$$\|z\|_{E_T^3}=\|u(x,t)\|_{B_{2,T}^3}+\|p(t)\|_{C[0,T]}\,.$$

It is known that $B_{2,T}^3$ and $E_T^3$ are Banach spaces.
Now consider in space $E_T^3$ the operator

$$\Phi(u,a)=\{\Phi_1(u,p),\Phi_2(u,p)\}\ ,$$

where

$$\Phi_1(u,p)=\tilde{u}(x,t)=1\sum_{k=0}^{\infty}\tilde{u}_k(t)\cos\lambda_k x,\Phi_2(u,p)=\tilde{p}(t).$$

and $\tilde{u}_k(t)$ and $\tilde{p}(t)$ are equal to the right-hand sides of (13) and (20), respectively.
It is easy to see that

$$\int_0^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\leq\frac{1}{m\lambda_k^2},\ \ \int_\eta^t e^{-\lambda_k^2\int_\tau^t a(s)ds}d\tau\leq\frac{1}{m\lambda_k^2},$$

where $m=\min\limits_{0\leq t\leq T}a(t)$.
Considering these relations, we find:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\ \|\tilde{u}_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq 2\left(1+\frac{a(0)}{m}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\ \ |\varphi_k|\right)^2\right)^{\frac{1}{2}}+$$

$$+\frac{2}{m}\left(\sum_{k=1}^{\infty}(\lambda_k\ \ |\psi_k|)^2\right)^{\frac{1}{2}}+\frac{2\sqrt{T}}{m}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\ |f_k(\tau)|)^2\ d\tau\right)^{\frac{1}{2}}+$$

$$+\frac{2T}{m}\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\ \left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}, \tag{21}$$

$$\left\|\tilde{p}(t)\right\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-\int_0^1 g(x)f(x,t)dx\right\|_{C[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left(\left\|a'(t)\right\|_{C[0,T]}+\left\|a^2(t)\right\|_{C[0,T]}\right)\times$$

$$\times\left\|g(x)\right\|_{L_2(0,1)}\left[\left(1+\frac{a(0)}{m}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\ \left|\varphi_k\right|\right)^2\right)^{\frac{1}{2}}+\frac{1}{m}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\ \left|\psi_k\right|\right)^2\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+\frac{\sqrt{T}}{m}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\ \left|f_k(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}+\frac{T}{m}\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\ \left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\right\}, \tag{22}$$

Let us assume that the data of problem (1)-(3), (5) satisfy the following conditions:

1) $\varphi(x)\in C^4[0,1],\varphi^{(5)}(x)\in L_2(0,1)$,

$$\varphi'(0)=\varphi(1)=\varphi'''(0)=\varphi''(1)=\varphi^{(4)}(1)=0\ ,$$

2) $\psi(x)\in C^2[0,1],\psi'''(x)\in L_2(0,1),\ \psi'(0)=\psi(1)=\psi''(1)=0$ .

3) $f(x,t),f_x(x,t),\ f_{xx}(x,t)\in C(D_T),f_{xxx}(x,t)\in L_2(D_T)$,

$$f_x(0,t)=f(1,t)=f_{xx}(1,t)=0\ \ (0\le t\le T),$$

4) $b(x)\in L_2(0,1),0<a(t)\in C^1[0,T],\ h(t)\in C^2[0,T]\ ,\ h(t)\ne 0\,(0\le t\le T)$.

Then from (21) , (22), we get:

$$\left\|\tilde{u}(x,t)\right\|_{B_{2,T}^3}\ \le A_1(T)+B_1(T)\left\|p(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3}, \tag{23}$$

$$\left\|\tilde{p}(t)\right\|_{C[0,T]}\ \le A_2(T)+B_2(T)\left\|p(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3}. \tag{24}$$

where

$$A_1(T)=\left\|\varphi(x)\right\|_{L_2(0,1)}+T\left\|\psi(x)\right\|_{L_2(0,1)}+T\sqrt{T}\left\|f(x,t)\right\|_{L_2(D_T)}+$$

$$A_1(T)=2\left(1+\frac{a(0)}{m}\right)\left\|\varphi'''(x)\right\|_{L_2(0,1)}+\frac{2}{m}\left\|\psi'(x)\right\|_{L_2(0,1)}+\frac{2\sqrt{T}}{m}\left\|f_x(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T)=\frac{2T}{m},$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-\int_0^1 g(x)f(x,t)dx\right\|_{C[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left(\left\|a'(t)\right\|_{C[0,T]}+\left\|a^2(t)\right\|_{C[0,T]}\right)\|g(x)\|_{L_2(0,1)}\times$$

$$\times\left[\left(1+\frac{a(0)}{m}\right)\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\frac{1}{m}\left\|\psi'''(x)\right\|_{L_2(0,1)}+\frac{\sqrt{T}}{m}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}\right]\Bigg\}$$

$$B_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left(\left\|a'(t)\right\|_{C[0,T]}+\left\|a^2(t)\right\|_{C[0,T]}\right)\frac{T}{m}.$$

From inequalities (23), (24) we conclude:

$$\|\tilde{u}(x,t)\|_{B_{2,T}^3}+\|\tilde{p}(t)\|_{C[0,T]}\le A(T)+B(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3}\,. \tag{25}$$

where

$$A(T)=A_1(T)+A_2(T),\quad B(T)=B_1(T)+B_2(T)\,.$$

So, we can prove the following theorem:

**Theorem 2.** *Let conditions 1)-4) be satisfied and*

$$B(T)(A(T)+2)^2<1\ , \tag{26}$$

*Then problem* (1)-(3), (5) *has a unique solution in the ball* $K=K_R(\|z\|_{E_T^3}\le R=$ $=A(T)+2)$ *of space* $E_T^3$.

*Proof.* In space $E_T^3$ consider the following equation

$$z=\Phi z, \tag{27}$$

where $z=\{u,p\}$, components $\Phi_i(u,p\,)(i=1,2)$ of the operator $(u,p)$ are defined by the right-hand sides of equations (16), (20), respectively. Consider the operator $\Phi(u,p)$ in the ball $K=K_R(\|z\|_{E_T^3}\le R=A(T)+2)$ from $E_T^3$.

Similarly to (23), we obtain that for any $z,z_1,z_2\in K_R$ the estimates

$$\|\Phi z\|_{E_T^3}\le A(T)+B(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3}\le A(T)+B(T)(A(T)+2)^2\ , \tag{28}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^3}\le\le B(T)R\,\left(\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3}+\|p_1(t)-p_2(t)\|_{C[0,T]}\right). \tag{29}$$

are valid. Then from the estimates (28) and (29), taking into account (26), it follows that the operator $\Phi$ acts in the ball $K=K_R$and is contractive. Therefore, in the ball $K=K_R$ the operator $\Phi$ has a unique fixed point $\{u,p\}$, which is the only solution of the equation (27), i.e. is the unique solution in the ball $K=K_R$ of the system. The function $u(x,t)$, as an element of space $B_{2,T}^3$, is continuous and has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$, in $D_T$.

It can be shown that $u_{tt}(x,t)$,$u_{txx}(x,t)$, are continuous in $D_T$.

It is easy to check that equation (1) and conditions (2), (3), and (5) are satisfied in the usual sense. Consequently, $\{u(x,t), p(t)\}$ is a solution to problem (1)-(3), (5), and, by virtue of the corollary of Lemma 1, it is unique in the ball $K = K_R$. Theorem is proved.

Using Theorem 1, we prove the following.

**Theorem 3.** *Let all conditions of Theorem 2 be satisfied and*

$$\int_0^1 g(x)\varphi(x)dx = h(0)\,,\ \int_0^1 g(x)\psi(x)dx = h'(0)\ .$$

*Then problem* (1)-(4) *has a unique classical solution in the ball* $K = K_R(\|z\|_{E_T^3} \le R = A(T) + 2)$ *of space* $E_T^3$.

## References

[1] Tikhonov A.N. On the stability of inverse problems // DAN USSR, 1943, vol. 39, no. 5, pp. 195-198.

[2] Lavrentiev M.M. On some ill-posed problems of mathematical physics. Novosibirsk, Nauka, 1962, 92 p.

[3] Ivanov V.K. On linear ill-posed problems // DAN USSR, 1962, vol. 145, no. 2, pp. 270-272.

[4] Denisov A.M. Introduction to the theory of inverse problems. - M: Moscow State University. 1994. 206 p.

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# On Absolute and Uniform Convergence of Biorthogonal Expansion in Root Functions of a Second Order Discontinuous Differential Operator

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**Abstract.** A second order differential operator is considered. Theorems on absolute and uniform convergence of expansions of discontinuous functions by the root functions of the given are proved. Uniform convergence rate of these expansions is also studied.

**Key Words and Phrases**: discontinuous operator, biorthogonal expansions, root functions, uniform convergence.

## 1. Introduction

In V.A.Il'ins paper [1], a second order discontinuous operator was investigated and constructive necessary and sufficient conditions of Riesz basicity of the system of root functions of the considered operator arc established. These investigations played an important roll in studying absolute and uniform convergence of biorthogonal expansions of functions from the class $W_2^1$ by the root functions of the Schredinger operator with multipoint boundary conditions [2,3]. In the present paper, we consider a second order discontinuous operator, issues of absolutes and uniform convergence and also on uniform convergence rate of biorthogonal expansions of discontinuous functions by root functions are studied.

## 2. The Basic Concepts and Formulation of Results

Let $G = \bigcup\limits_{l=1}^{m} G_l$, $G_l = (\xi_{l-1}, \xi_l)$, $a = \xi_0 < ... < \xi_m = b$, where $-\infty < a < b < +\infty$.

By $\widetilde{W}_r^n(G)$, $1 \le r \le \infty$, denote a class of functions possessing the property: if $f(x) \in \widetilde{W}_r^n(G)$, then for each $G_l$, $l = \overline{1, m}$ there exists the function $f_l(x)$ from the Sobolev space $W_r^n(G_l)$ such that $f(x) = f_l(x)$ for $\xi_{l-1} < x < \xi_l$.

On the set $G$ consider the formal operator

$$Lu = u'' + q(x)u \tag{1}$$

 

with complex-valued coefficients $q(x) \in L_1(G)$.

Under the system of the root functions of the operator we'll understand an arbitrary system $\{u_k\}_{k=1}^{\infty}$ of complex-valued non identical zero functions from $\widetilde{W}_r^2(G)$, satisfying almost everywhere in $G$ the equation

$$Lu_k + \lambda_k u_k = \theta_k u_{k-1},$$

where $\theta_k$ either equals zero (in this case the function uk is an eigenfunction) or unit (in this case we require $\lambda_k = \lambda_{k-1}$ and call $u_k$ an associated function of order $j$, where $\theta_k = \theta_{k-1} = ... = \theta_{k-j+1} = 1$, $\theta_{k-j} = 0$).

Denote $\mu_k = \sqrt{\lambda_k}$, $\arg \mu_k \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Later on, we'll additionally suppose that the elements of the considered system of the root functions $\{u_k(x)\}_{k=1}^{\infty}$ are determined at the points $\xi_l$, $l = \overline{1,m}$ and are continuous from the left at these points, but at the point $\xi_0 = a$ they are continuous from the right. The expanded function $f(x) \in \widetilde{W}_p^1(G)$, $1 \leq p \leq \infty$ will also satisfy these requirements.

We'll require that the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfies V.A. Ilin's following conditions: 1) the system $\{u_k(x)\}_{k=1}^{\infty}$ complete and minimal in $L_2(G)$; 2) the inequalities are

$$|Im\mu_k| \leq C_0, \tag{2}$$

$$\sum_{\tau \leq Re\mu_k \leq \tau+1} \leq C_1, \quad \forall \tau \geq 0; \tag{3}$$

fulfilled. 3) there exists a constant $C_2 > 0$ such half

$$\|u_k\|_2 \|u_k\|_2 \leq C_2, \quad k = 1, 2, ..., \tag{4}$$

where $\{v_k\}_{k=1}^{\infty}$ is a biorthogonall adjoint system to the system $\{u_k(x)\}_{k=1}^{\infty}$ and consists of the root functions of the formally adjoint operator $L^* = \frac{d^2}{dx^2} + \overline{q(x)}$, $x \in G$ (i.e. $L^* v_k + +\overline{\lambda}_k v_k = \theta_{k+1} v_{k+1}$).

For an arbitrary function $f(x) \in \widetilde{W}_p^1(G)$, $p \geq 1$ compose a partial sum of its biorthogonal expansion by the system $\{u_k\}_{k=1}^{\infty}$

$$\sigma_\nu(x, f) = \sum_{\rho_k \leq \nu} (f, v_k) u_k(x), \nu \geq 1,$$

where

$$(f, v_k) = \int_G f(x) \overline{v_k(x)} dx.$$

Assume

$$R_\nu(x, f) = f(x) - \sigma_\nu(x, f)$$

and denote

$$Q_m(f, v_k) = f(b) \overline{v_k'(b-0)} - f(a) \overline{v_k'(a+0)} +$$

$$+\sum_{l=1}^{m-1}\left[f\left(\xi_l-0\right)\overline{v_k'\left(\xi_l-0\right)}-f\left(\xi_l+0\right)\overline{v_k'\left(\xi_l+0\right)}\right].$$

The main results of this section consist of the following theorems:

**Theorem 1.** Let the system $\{u_k(x)\}_{k=1}^{\infty}$ of the root functions of the operator (1) satisfy conditions 1)-3). Then, a biorthogonal expansion of any function $f(x)\in\widetilde{W}_p^1(G)$, $p\geq 1$ satisfying the condition

$$|Q_m(f,v_k)|\leq C_1(f)\,\|v_k\|_2\,,\quad k=1,2,....,\tag{5}$$

converges absolutely and uniformly on $\overline{G}=[a,b]$ and the following the relations are valid

$$f(x)=\sum_{k=1}^{\infty}(f,v_k)\,u_k(x)\,,\qquad x\in\overline{G},\tag{6}$$

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|\leq const\left\{\nu^{-\delta}\,\|f\|_{\widetilde{W}_p^1(G)}+\nu^{-1}\left[C_1(f)+\|q\|_1\,\|f\|_\infty\right]\right\},\tag{7}$$

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|=o\left(\nu^{-\delta}\right),\nu\to+\infty,\tag{8}$$

where $\delta=\min\left\{\frac{1}{2},\frac{1}{q}\right\}$, $p^{-1}+q^{-1}=1$; symbol "o" depends on the expanded function $f(x)$; const is independent of the expanded function $f(x)$;

$$\|f\|_{\widetilde{W}_p^1(G)}=\|f\|_{L_p(G)}+\left\|f'\right\|_{L_p(G)}.$$

**Theorem 2.** Let the function $f(x)$ belong to the class $\widetilde{W}_1^1(G)$ conditions l)-3) and condition(5) be fulfilled for the system of the root functions $\{u_k(x)\}_{k=1}^{\infty}$ of operator (1) and the number series

$$\sum_{n=n_0}^{\infty}n^{-1}\omega_1\left(f',n^{-1}\right)<\infty,\tag{9}$$

converge. Here $n_0>2(b-a)$, $\omega_1(\cdot,\delta)$ is a modulus of continuity in $L_1(G)$.

Then a biorthogonal expansion of the function $f(x)$ by the system $\{u_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on $\overline{G}$, equality (6) and the estimation

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|\leq constK(\nu,f)\,,\tag{10}$$

are valid. Here the const is independent on the expanded function $f(x)$; the function $K(\nu,f)$ is determined by the formula

$$K(\nu,f)=\sum_{n=[\nu]}^{\infty}n^{-1}\omega_1\left(f',n^{-1}\right)+\nu^{-1}\left[C_1(f)+\|q\|_1\,\|f\|_\infty+\left\|f'\right\|_1\right],$$

$$\nu > 4\pi / \left( \min_{1 \le l \le m} |G_l| \right).$$

By $\widetilde{H}_1^{\alpha}(G)$ denote a space of functions $f(x)$ such that for each $l$ $\left(l = \overline{1,m}\right)$ there exists the function $f_l(x)$ of Nikolskiy class $\widetilde{H}_1^{\alpha}(G)$, such that $f(x) = f_l(x)$ for $\xi_{l-1} < x < \xi_l$. We define the norm in $\widetilde{H}_1^{\alpha}(G)$ by the equality

$$\|f\|_{\widetilde{H}_1^{\alpha}(G)} = \|f\|_{1,G}^{\alpha} = \|f\|_{L_1(G)} + \max_{1 \le l \le m} \sup_{\delta > 0} \frac{\omega_1(f,\delta)_{G_l}}{\delta^{\alpha}},$$

where

$$\omega_1(f,\delta)_{G_l} = \sup_{0 < h \le \delta} \int\limits_{\xi_{l-1}}^{\xi_l - h} |f(x+h) - f(x)|\, dx.$$

**Corollary from theorem 2.** If in theorem 2 we additionally require $f(x) \in \widetilde{H}_1^{\alpha}(G)$, $0 < \alpha < 1$, the estimation

$$\sup_{x \in \overline{G}} |R_{\nu}(x,f)|_{C\left(\overline{G}\right)} = O\left(\nu^{-\alpha}\right)$$

is fulfilled.

**Remark 1.** The condition (5) required from the bilinear functional $Q_m(f, v_k)$, is natural. Usually, the class of functions $f(x)$, for which $Q_m(f, v_k) = 0$, $k = 1, 2, ...$, should be determined from the considered space. In many cases, condition (5) is fulfilled for all the functions $f(x)$ from the space under consideration. For example, in the case when the boundary conditions for $v_k(x)$ are two -point and the principal minor of the matrix that corresponds to boundary conditions are non zero, i. e. if

$$\alpha_i v'(\alpha) + \beta_i v'(b) + c_i v(a) + d_i v(b) = 0,$$

$$i = 1, 2;\;\; \alpha_1 \beta_2 - \alpha_2 \beta_1 \ne 0.$$

**Remark 2.** Estimation (7) is unimprovable in the considered class of functions.

## 3. Representation for the Fourier coefficients from the class $\widetilde{W}_p^1(G)$, $1 \le p \le \infty$.

Before proving theorems 1 and 2, we get some representations for the Fourier coefficients of the functions from the class $\widetilde{W}_p^1(G)$, $1 \le p \le \infty$.

**Lemma 1.** The following representations are valid for the Foruier coefficients of the function $f(x) \in \widetilde{W}_p^1(G)$.

$$(v_k, f) = -\frac{\overline{Q_m(f, v_k)}}{\overline{\mu}_k^2} + \frac{(v_k', f')}{\overline{\mu}_k^2} - \frac{1}{\overline{\mu}_k^2}(v_k, qf) + \frac{\theta_{k+1}}{\overline{\mu}_k^2}(v_{k+1}, f) \qquad (11)$$

$$(v_k, f) = -\sum_{j=0}^{n_k} \frac{\overline{Q_m(f, v_{k+j})}}{\overline{\mu}_k^{2(j+1)}} - \sum_{j=0}^{n_k} \frac{(v_{k+j}, qf)}{\overline{\mu}_k^{2(j+1)}} +$$

$$+\sum_{j=0}^{n_k}\sum_{l=1}^{m} \frac{v'_{k+j}(\xi_{l-1})}{\overline{\mu}_k^{2(j+1)}} \left(\cos\overline{\mu}_k(t-\xi_{l-1}), f'(t)\right)_{G_l} -$$

$$-\sum_{j=0}^{n_k}\sum_{l=1}^{m} \frac{v_{k+j}(\xi_{l-1})}{\overline{\mu}_k^{2j+1}} \left(\sin\overline{\mu}_k(t-\xi_{l-1}), f'(t)\right)_{G_l} -$$

$$-\sum_{j=0}^{n_k}\sum_{l=1}^{m} \frac{1}{\overline{\mu}_k^{2(j+1)}} \int_{\xi_{l-1}}^{\xi_l} \overline{q(\tau)} v_{k+j}(\tau) \left(\int_{\tau}^{\xi_l} \overline{f'(t)} \cos\overline{\mu}_k(t-\tau)\, dt\right) d\tau +$$

$$+\sum_{j=0}^{n_k}\sum_{l=1}^{m} \frac{\theta_{k+j+1}}{\overline{\mu}_k^{2(j+1)}} \int_{\xi_{l-1}}^{\xi_l} v_{k+j+1}(\tau) \left(\int_{\tau}^{\xi_l} \overline{f'(t)} \cos\overline{\mu}_k(t-\tau)\, dt\right) d\tau, \qquad (12)$$

therewith $n_k$ is the order of the associated function $v_k$; $\theta_{k+n_k+1} = 0$;

$$v_{k+j}^{(i)}(\xi_{l-1}) \overset{def}{=} v_{k+j}^{(i)}(\xi_{l-1}+0),$$

$$l = \overline{1,m}, \quad i = \overline{0,1}; \quad (f,g)_{G_l} = \int_{G_l} f(t)\overline{g(t)}dt.$$

**Proof.** It is seen from the definition of the function $v_k(x)$ that

$$v''_k(x) + \overline{q(x)} v_k(x) + \overline{\lambda}_k v_k(x) = \theta_{k+1} v_{k+1}(x), \quad x \in G_l, l = 1, m.$$

Therefore, multiplying by $\overline{f}(x)$ and integrating with respect to $x$ from $\xi_{l-1}$ to $\xi_l$, from the last equality we get

$$\int_{\xi_{l-1}}^{\xi_l} \overline{f(x)} v_k(x)\, dx = -\frac{1}{\overline{\lambda}_k} \int_{\xi_{l-1}}^{\xi_l} \overline{f(x)} v''_k(x)\, dx -$$

$$-\frac{1}{\overline{\lambda}_k} \int_{\xi_{l-1}}^{\xi_l} \overline{q(x)} v_k(x) \overline{f(x)} dx + \frac{\theta_{k+1}}{\overline{\lambda}_k} \int_{\xi_{l-1}}^{\xi_l} \overline{f(x)} v_{k+1}(x)\, dx.$$

Conducting integration by parts in the integral containing $v''_k(\xi)$, we find

$$\int_{\xi_{l-1}}^{\xi_l} \overline{f(x)} v_k(x)\, dx = -\frac{1}{\overline{\lambda}_k} \left[\overline{f(\xi_l - 0)} v'_k(\xi_l - 0) - \overline{f(\xi_{l-1}+0)} v'_k(\xi_l + 0)\right] +$$

$$+\frac{1}{\overline{\lambda}_k}\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(x)}v'_k(x)\,dx-\frac{1}{\overline{\lambda}_k}(v_k,qf)+\frac{\theta_{k+1}}{\overline{\lambda}_k}\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f(x)}v_{k+1}(x)\,dx.$$

Summing these equalities over $l$ from 1 to $m$ and taking into account $\lambda_k=\mu_k^2$, we arrive at representation (11).

Now, derive formula (12). It follows from (11) that if we consider this formula as a recurrent relation for the Fourier coefficients, then

$$(v_k,f)=-\frac{\overline{Q_m(f,v_k)}}{\overline{\mu}_k^2}+\frac{1}{\overline{\mu}_k^2}\left(v'_k,f'\right)-\frac{1}{\overline{\mu}_k^2}(v_k,qf)+$$

$$+\frac{\theta_{k+1}}{\overline{\mu}_k^2}\left[-\frac{\overline{Q_m(f,v_{k+1})}}{\overline{\mu}_k^2}+\frac{\left(v'_{k+1},f'\right)}{\overline{\mu}_k^2}-\frac{1}{\overline{\mu}_k^2}(v_{k+1},qf)+\frac{1}{\overline{\mu}_k^2}(v_{k+2},f)\right]$$

Continuing this substitution process to the eigenfunction $v_{k+n_k}$, for $(v_k,f)$ we get the following representation

$$(v_k,f)=-\sum_{j=0}^{n_k}\frac{\overline{Q_m(f,v_{k+j})}}{\overline{\mu}_k^{2(j+1)}}+\sum_{j=0}^{n_k}\frac{\left(v'_{k+j},f'\right)}{\overline{\mu}_k^{2(j+1)}}-\sum_{j=0}^{n_k}\frac{(v_{k+j},qf)}{\overline{\mu}_k^{2(j+1)}}. \tag{13}$$

Transform the expression $\left(v'_{k+j},f'\right)$. For that we use the mean value formula for the function $v'_{k+j}(t)$ for $t\in(\xi_{-1},\xi_l)$:

$$v'_{k+j}(t)=-\overline{\mu}_k v_{k+j}(\xi_{l-1}+0)\sin\overline{\mu}_k(t-\xi_{l-1})+v'_{k+j}(\xi_{l-1}+0)\times$$

$$\times\cos\overline{\mu}_k(t-\xi_{l-1})-\int\limits_{\xi_{l-1}}^{t}\overline{q}(\tau)\,v_{k+j}(\tau)\cos\overline{\mu}_k(t-\tau)\,d\tau+$$

$$+\theta_{k+j+1}\int\limits_{\xi_{l-1}}^{t}v_{k+j+1}(\tau)\cos\overline{\mu}_k(t-\tau)\,d\tau,$$

$$j=\overline{0,n_k},\qquad \theta_{k+n_k+1}=0.$$

As a result of substitution of the expression $v'_{k+j}(t)$ in $\left(v'_{k+j},f'\right)$ we get

$$\left(v'_{k+j},f'\right)=\sum_{l=1}^{m}\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}v'_{k+j}(t)\,dt=\sum_{l=1}^{m}\left\{-\overline{\mu}_k\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}\sin\overline{\mu}_k(t-\xi_{l-1})\,dt\times\right.$$

$$\times v_{k+j}(\xi_{l-1}+0)+\int\limits_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}\cos\overline{\mu}_k(t-\xi_{l-1})\,dt\,v'_{k+j}(\xi_{l-1}+0)-$$

$$-\int_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}\left(\int_{\xi_{l-1}}^{t}\overline{q(\tau)}v_{k+j}(\tau)\cos\overline{\mu}_k(t-\tau)\,d\tau\right)dt+$$

$$+\theta_{k+j+1}\int_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}\left(\int_{\xi_{l-1}}^{t}\overline{q(\tau)}v_{k+j+1}(\tau)\cos\overline{\mu}_k(t-\tau)\,d\tau\right)dt\Bigg\}.$$

Change the integration order in double integrals, take into account the obtained expression for $\left(v'_{k+j},f'\right)$ in formula (13), and get the formula

$$(v_k,f)=-\sum_{j=0}^{n_k}\frac{\overline{Q_m(f,v_{k+j})}}{\overline{\mu}_k^{2(j+1)}}-\sum_{j=0}^{n_k}\frac{(v_{k+j},qf)}{\overline{\mu}_k^{2(j+1)}}+$$

$$+\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{v'_{k+j}(\xi_{l-1})}{\overline{\mu}_k^{2(j+1)}}\left(\cos\overline{\mu}_k(t-\xi_{l-1}),f'(t)\right)_{G_l}-$$

$$-\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{v_{k+j}(\xi_{l-1})}{\overline{\mu}_k^{2j+1}}\left(\sin\overline{\mu}_k(t-\xi_{l-1}),f'(t)\right)_{G_l}-$$

$$-\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{1}{\overline{\mu}_k^{2(j+1)}}\int_{\xi_{l-1}}^{\xi_l}\overline{q(\tau)}v_{k+j}(\tau)\left(\int_{\tau}^{\xi_l}\overline{f'(t)}\cos\mu_k(t-\tau)\,dt\right)d\tau+$$

$$+\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{\theta_{k+j+1}}{\overline{\mu}_k^{2(j+1)}}\int_{\tau}^{\xi_l}\overline{q(\tau)}v_{k+j+1}(\tau)\left(\int_{\tau}^{\xi_l}\overline{f'(t)}\cos\mu_k(t-\tau)\,dt\right)d\tau,$$

where under $v_{k+j}^{(i)}(\xi_{l-1})$, $i=0,1$, we mean $v_{k+j}^{(i)}(\xi_{l-1}+0)$. Besides $\theta_{k+n_k+1}=0$, $n_k$ - is the order of the associated function $v_k(x)$. Lemma 1 is proved.

**Proof of Theorem 1.**

It suffices to consider the case $1<p\le 2$. Estimate the series

$$\sum_{k=1}^{\infty}|(f,v_k)|\,|u_k(x)|,\quad x\in\overline{G}.$$

Represent this series in the form

$$\sum_{k=1}^{\infty}|(f,v_k)|\,|u_k(x)|=\sum_{0\le Re\mu_k<1}|(f,v_k)|\,|u_k(x)|+\sum_{Re\mu_k\ge 1}|(f,v_k)|\,|u_k(x)|.$$

Estimate each of the sums in the right hand side of the last equality

$$\sum_{0\le Re\mu_k<1}|(f,v_k)|\,|u_k(x)|\le\sum_{0\le Re\mu_k<1}\|f\|_1\,\|v_k\|_\infty\,|u_k(x)|.$$

Apply here the estimates (2)-(4) and (see [4])

$$\sup_{x\in\overline{G}_l}|u_k(x)|\leq C(l)\|u_k\|_{L_2(\xi_{l-1},\xi_l)} \tag{14}$$

$$\sup_{x\in\overline{G}_l}|v_k(x)|\leq C(l)\|v_k\|_{L_2(\xi_{l-1},\xi_l)}\,. \tag{15}$$

As a result we have:

$$\sum_{0\leq Re\mu_k<1}|(f,v_k)|\,|u_k(x)|\leq C\sum_{0\leq Re\mu_k<1}\|u_k\|_2\,\|v_k\|_2\,\|f\|_1\leq$$

$$\leq C\,\|f\|_1\sum_{0\leq Re\mu_k<1}1\leq C\,\|f\|_1\,,$$

where $C$ is a positive number. For estimating the series

$$\sum_{Re\mu_k\geq 1}|(f,v_k)|\,|u_k(x)|$$

we apply formula (11) for the Fourier coefficients $(f,v_k)$:

$$\sum_{Re\mu_k\geq 1}|(f,v_k)|\,|u_k(x)|\leq\sum_{|\mu_k|\geq 1}|(f,v_k)|\,|u_k(x)|\leq\sum_{|\mu_k|\geq 1}\frac{\overline{|Q_m(f,v_k)|}}{|\mu_k|^2}|u_k(x)|+$$

$$+\sum_{|\mu_k|\geq 1}\frac{1}{|\mu_k|^2}\left|\left(v_k',f'\right)\right|\,|u_k(x)|+\sum_{|\mu_k|\geq 1}\frac{|u_k(x)|}{|\mu_k|^2}(v_k,qf)+$$

$$+\sum_{|\mu_k|\geq 1}\frac{\theta_{k+1}}{|\mu_k|^2}|(v_{k+1},f)|\,|u_k(x)|\,. \tag{16}$$

For estimating the first sum in the right hand side of relation (16) we apply inequalities (5), (14), (15) and then (2)-(4). As a result, we find

$$\sum_{|\mu_k|\geq 1}\frac{\overline{|Q_m(f,v_k)|}}{|\mu_k|^2}|u_k(x)|\leq C_1(f)\sum_{|\mu_k|\geq 1}\frac{\|v_k\|_2}{|\mu_k|^2}\|u_k\|_2\leq$$

$$\leq C_2(f)\sum_{|\mu_k|\geq 1}\frac{1}{|\mu_k|^2}\leq C_2(f)\sum_{n=1}^{\infty}\left(\sum_{n\leq|\mu_k|\leq n+1}\frac{1}{|\mu_k|^2}\right)\leq$$

$$\leq C_2(f)\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{n\leq|\mu_k|\leq n+1}1\right)\leq C(f)\,,$$

where $C(f)$ in a number dependent on $f(x)$.

Estimate the second sum in the right hand side of (16)

$$\sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|^2} \left|\left(f', v_k'\right)\right| |u_k(x)| = \sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|} \left|\left(f', v_k' \|v_k\|_q^{-1} |\mu_k|^{-1}\right)\right| \|v_k\|_q |u_k(x)|,$$

$$p^{-1} + q^{-1} = 1.$$

Applying estimates (14), (15) and (4), we get that the left hand side of the last relation is majorized from above by the quantity

$$C \sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|} \left|\left(f', v_k' \|v_k\|_q^{-1} |\mu_k|^{-1}\right)\right|.$$

Since the system $\left\{v_k'(x) \|v_k\|_q^{-1} |\mu_k|^{-1}\right\}$ is a Riesz system (see [8]) and taking into account $f'(x) \in L_p(a,b)$, we can apply the Riesz inequality.

Consequently, the second sum in (16) is bounded from above by the quantity:

$$C \left(\sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|^p}\right)^{1/p} \left(\sum_{|\mu_k|\geq 1} \left|\left(f', v_k' \|v_k\|_q^{-1} |\mu_k|^{-1}\right)\right|^q\right)^{1/q}.$$

By the condition (4) the sum

$$\sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|^p}, \quad p > 1,$$

converges by the Riesz inequality

$$\left(\sum_{|\mu_k|\geq 1} \left|\left(f', v_k' \|v_k\|_q^{-1} |\mu_k|^{-1}\right)\right|^q\right)^{1/q} \leq C \|f'\|_p.$$

Thus, the second series in the right hand side of relation (16) also converges and its sum doesn't exceed the quantity $C \|f'\|_p$, where $C$ is independent of the function $f(x)$.

Estimate the third sum in the right hand side of (16):

$$\sum_{|\mu_k|\geq 1} \frac{|u_k(x)|}{|\mu_k|^2} |(v_k, qf)| \leq \sum_{|\mu_k|\geq 1} \|q\|_1 \|v_k\|_\infty |u_k(x)| |\mu_k|^{-2} \|f\|_\infty.$$

Apply estimates (14), (15), and then (2)-(4). As a result we find

$$\sum_{|\mu_k|\geq 1} \frac{|u_k(x)|}{|\mu_k|^2} |(v_k, qf)| \leq C_1 \|q\|_1 \|f\|_\infty \sum_{|\mu_k|\geq 1} \frac{1}{|\mu_k|^2} \leq$$

$$\leq C_1 \|q\|_1 \|f\|_\infty \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sum_{n \leq |\mu_k| \leq n+1} 1 \right) \leq C \|q\|_1 \|f\|_\infty ,$$

where $C$ is a constant independent of $f(x)$.

Now, prove the convergence of the last series in the right hand side of (16)

$$\sum_{|\mu_k| \geq 1} \frac{\theta_{k+1}}{|\mu_k|^2} |(v_{k+1}, f)| |u_k(x)| =$$

$$= \sum_{|\mu_k| \geq 1} \frac{1}{|\mu_k|} \left| \left( f, \theta_{k+1} v_{k+1} \|v_{k+1}\|_q^{-1} \right) \right| \frac{1}{|\mu_k|} \|v_{k+1}\|_q |u_k(x)| .$$

Here, applying estimation (14)

$$\|v_{k+1}\|_q \leq C |\mu_k| \|v_k\|_2 , \tag{17}$$

(see [5]) and the Holder inequality, we have:

$$\sum_{|\mu_k| \geq 1} \frac{\theta_{k+1}}{|\mu_k|^2} |(v_{k+1}, f)| |u_k(x)| \leq$$

$$\leq C \sum_{|\mu_k| \geq 1} \frac{1}{|\mu_k|} \left| \left( f, \theta_{k+1} v_{k+1} \|v_{k+1}\|_q^{-1} \right) \right| \leq$$

$$\leq C \left( \sum_{|\mu_k| \geq 1} \frac{1}{|\mu_k|^p} \right)^{1/p} \left( \sum_{|\mu_k| \geq 1} \left| \left( f, \theta_{k+1} v_{k+1} \|v_{k+1}\|_q^{-1} \right) \right|^q \right)^{1/q} , \quad \frac{1}{p} + \frac{1}{q} = 1.$$

By the convergence of the series $\sum\limits_{|\mu_k| \geq 1} \frac{1}{|\mu_k|^p}$, $p > 1$ of the system $\left\{ \theta_{k+1} v_{k+1} \|v_{k+1}\|_q^{-1} \right\}$, $|\mu_k| \geq 1$ (see [9]), we get

$$\sum_{|\mu_k| \geq 1} \frac{\theta_{k+1}}{|\mu_k|^2} |(v_{k+1}, f)| |u_k(x)| \leq const \|f\|_p .$$

Consequently, the series $\sum\limits_{k=1}^{\infty} |(f, v_k)| |u_k(x)|$ converges uniformly with respect to $x \in \overline{G}$. Prove the validity of representation (6). Let the series in the right hand side of (6) converge on $\overline{G}$ to some function $g(x)$. By the uniformity of the convergence the function $g(x)$ will be continuous on each of intervals $[\xi_0, \xi_1]$, $(\xi_{l-1}, \xi_l]$, $l = \overline{2, m}$. From the representation

$$g(x) = \sum_{k=1}^{\infty} (f, v_k) u_k(x) , \quad x \in \overline{G}$$

we get that $(f,v_k) = (g,v_k)$, $k = 1,2,\ldots$. Hence, by the completeness of the system $\{v_k(x)\}$ in $L_2(G)$, it follows that $f(x) = g(x)$ almost everywhere at each of intervals $[\xi_0,\xi_1]$, $(\xi_{l-1},\xi_l]$, $l=\overline{2,m}$. Since the both functions $f(x)$ and $g(x)$ are continuous in these intervals, they coincide everywhere on $\overline{G}$. Consequently, representation (6) is true.

Now, establish estimation (7) for $1 < p \le 2$. It follows from representations (6) and (11) that for each $x \in \overline{G}$

$$|R_\nu(x,f)| = |f(x) - \sigma_\nu(x,f)| \le \sum_{Re\mu_k>\nu} |(f,v_k)|\,|u_k(x)| \le$$

$$\le \sum_{|\mu_k|\ge\nu} |(f,v_k)|\,|u_k(x)| \le \sum_{|\mu_k|\ge\nu} \frac{\overline{|Q_m(f,v_k)|}}{|\mu_k|^2}|u_k(x)| +$$

$$+\sum_{|\mu_k|\ge\nu}\frac{1}{|\mu_k|^2}\left|(v_k',f')\right||u_k(x)| + \sum_{|\mu_k|\ge\nu}\frac{1}{|\mu_k|^2}(v_k,qf)\,|u_k(x)| +$$

$$+\sum_{|\mu_k|\ge\nu}\frac{\theta_{k+1}}{|\mu_k|^2}|(v_{k+1},f)|\,|u_k(x)| \le constC_1(f)\,\nu^{-1}+$$

$$+const\nu^{-1/q}\left\|f'\right\|_p + const\,\|q\|_1\,\|f\|_\infty\,\nu^{-1} + const\nu^{-1/q}\,\|f\|_p \le$$

$$\le const\left\{\nu^{-1/q}\,\|f\|_{\widetilde{W}_p^1(G)} + \nu^{-1}\left[C_1(f) + \|q\|_1\,\|f\|_\infty\right]\right\}.$$

Prove estimation (8) for $1 < p \le 2$. For that, we again use representation (6), (11) and behave as in the estimation of the right hand side of relation (16).

$$|R_\nu(x,f)| = |f(x) - \sigma_\nu(x,f)| = \left|\sum_{Re\mu_k>\nu}(f,v_k)\,u_k(x)\right| \le$$

$$\le \sum_{|\mu_k|\ge\nu}|(f,v_k)|\,|u_k(x)| \le (CC_1(f) + \|q\|_1\,\|f\|_\infty)\,\nu^{-1}+$$

$$+C\left(\sum_{|\mu_k|\ge\nu}\frac{1}{|\mu_k|^p}\right)^{1/p}\left(\sum_{|\mu_k|\ge\nu}\left|\left(f',v_k'\,\|v_k\|_q^{-1}\,|\mu_k|^{-1}\right)\right|^q\right)^{1/q}+$$

$$+\left(\sum_{|\mu_k|\ge\nu}\frac{1}{|\mu_k|^p}\right)^{1/p}\left(\sum_{|\mu_k|\ge\nu}\left|\left(f',\frac{\theta_{k+1}v_{k+1}\,\|v_{k+1}\|_q^{-1}}{|\mu_k|}\right)\right|^q\right)^{1/q} =$$

$$= O\left(\nu^{-1}\right) + O\left(\nu^{-1/q}\right)\left(\sum_{|\mu_k|\ge\nu}\left|\left(f',v_k'\,\|v_k\|_q^{-1}\,|\mu_k|^{-1}\right)\right|^q\right)^{1/q}+$$

$$+O\left(\nu^{-1/q}\right)\left(\sum_{|\mu_k|\geq\nu}\left|\left(f',\theta_{k+1}v_{k+1}\left\|v_{k+1}\right\|_q^{-1}\right)\right|^q\right)^{1/q}.$$

Since the residual of the converging series tends to zero, then

$$\left(\sum_{|\mu_k|\geq\nu}\left|\left(f',v_k'\left\|v_k\right\|_q^{-1}\left|\mu_k\right|^{-1}\right)\right|^q\right)^{1/q}=o(1)$$

$$\left(\sum_{|\mu_k|\geq\nu}\left|\left(f',\theta_{k+1}v_{k+1}\left\|v_{k+1}\right\|_q^{-1}\right)\right|^q\right)^{1/q}=o(1),$$

as $\nu\to\infty$.

Consequently, for any $x\in\overline{G}$ and $1<p\leq 2$

$$|R_\nu(x,f)|=O\left(\nu^{-1}\right)+O\left(\nu^{-1/q}\right)o(1)=o\left(\nu^{-1/q}\right),\quad \nu\to+\infty$$

in the case $1<p\leq 2$ . Theorem 1 is proved.

The case $p>2$ is reduced to the case $p=2$, or $\widetilde{W}_p^1(G)$ is embedded into $\widetilde{W}_2^1(G)$. Theorem 1 is completely proved.

**Proof of theorem 2.**

Let $f(x)\in\widetilde{W}_1^1(G)$. Prove the uniform convergence of the series $\sum\limits_{k=1}^{\infty}|(f,v_k)|\,|u_k(x)|$ on $\overline{G}$. For that, we again represent it in the from (see the proof of theorem 1)

$$\sum_{k=1}^{\infty}|(f,v_k)|\,|u_k(x)|=\sum_{0\leq Re\mu_k<\alpha}+\sum_{Re\mu_k\geq\alpha}\leq C_1\left\|f\right\|_1+$$

$$+\sum_{Re\mu_k\geq\alpha}|(f,v_k)|\,|u_k(x)|,$$

where $\alpha=\max\limits_{1\leq l\leq m}\frac{4\pi}{|G_l|}$.

For investigating a uniform convergence in $\overline{G}$ of the series

$$\sum_{Re\mu_k\geq\alpha}|(f,v_k)|\,|u_k(x)|$$

we use representation (12) for the Fourier coefficient $(f,v_k)$. This leads us to studying the uniform convergence of the series

$$\sum_{Re\mu_k\geq\alpha}\sum_{j=0}^{n_k}\frac{\overline{|Q_m(f,v_{k+j})|}}{|\mu_k|^{2(j+1)}}|u_k(x)|,\tag{18}$$

$$\sum_{Re\mu_k\geq\alpha}\sum_{j=0}^{n_k}\frac{|(v_{k+j},qf)|}{|\mu_k|^{2(j+1)}}|u_k(x)|, \tag{19}$$

$$\sum_{Re\mu_k\geq\alpha}\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{\left|v'_{k+j}(\xi_{l-1})\right|}{|\mu_k|^{2(j+1)}}|u_k(x)|\left|\left(\cos\overline{\mu}_k(t-\xi_{l-1}),f'(t)\right)_{G_l}\right|, \tag{20}$$

$$\sum_{Re\mu_k\geq\alpha}\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{|v_{k+j}(\xi_{l-1})|}{|\mu_k|^{2j+1}}|u_k(x)|\left|\left(\sin\overline{\mu}_k(t-\xi_{l-1}),f'(t)\right)_{G_l}\right|, \tag{21}$$

$$\sum_{Re\mu_k\geq\alpha}\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{|u_k(x)|}{|\mu_k|^{2(j+1)}}\times$$

$$\times\left|\int_{\xi_{l-1}}^{\xi_l}\overline{q(\tau)}v_{k+j}(\tau)\left(\int_{\xi_{l-1}}^{\xi_l}\overline{f'(t)}\cos\overline{\mu}_k(t-\tau)\,dt\right)d\tau\right|, \tag{22}$$

$$\sum_{Re\mu_k\geq\alpha}\sum_{j=0}^{n_k}\sum_{l=1}^{m}\frac{|\theta_{k+j+1}u_k(x)|}{|\mu_k|^{2(j+1)}}\times$$

$$\times\left|\int_{\xi_{l-1}}^{\xi_l}v_{k+j+1}(\tau)\left(\int_{\tau}^{\xi_l}\overline{f'(t)}\cos\overline{\mu}_k(t-\tau)\,dt\right)d\tau\right|. \tag{23}$$

By inequalities (2), (3), (6), (14), (17) and (4) the series (18) is majorized by the number series

$$C_1(f)\sum_{Re\mu_k\geq\alpha}\frac{1}{|\mu_k|^2},$$

that converges by condition (3).

By inequalities (2), (3), (15), (17) and (4) series (19) is majorized by the converging number series

$$C\,\|q\|_1\,\|f\|_\infty\sum_{Re\mu_k\geq\alpha}\frac{1}{|\mu_k|^2}.$$

Since (see [4])

$$\left|v'_{k+j}(\xi_{l-1})\right|\leq C\,|\mu_k|\,\|v_{k+j}\|_2$$

then by (2), (3), (14), (17) and (4) series (20) is majorized by the number series

$$C\sum_{Re\mu_k\geq\alpha}\frac{1}{|\mu_k|}\sum_{l=1}^{m}\left|\left(\cos\overline{\mu}_k(t-\xi_{l-1}),f'(t)\right)_{G_l}\right|. \tag{24}$$

Obviously, the system $\{\cos \overline{\mu}_k (t-\xi_{l-1})\}_{k=1}^{\infty}$ is a system of eigenfunctions of the operator $-y = \overline{\mu}^2 y$. Therefore, by lemma 7 of the paper [9]

$$\left| \left(\cos \overline{\mu}_k (t-\xi_{l-1}), f'(t)\right)_{G_l} \right| \leq C \left[ \omega_1 \left( f', \frac{1}{Re\mu_k} \right) + \frac{\|f'\|_1}{Re\mu_k} \right]. \tag{25}$$

Taking this into account in (25), we get that series (20) is majorized by the number series

$$C \sum_{Re\mu_k \geq \alpha} \frac{1}{Re\mu_k} \left[ \omega_1 \left( f', \frac{1}{Re\mu_k} \right) + \frac{\|f'\|_1}{Re\mu_k} \right],$$

that converges by conditions (3) and (10). Allowing for estimation (17), a uniform convergence on $\overline{G}$ of series (21) is proved in the same way. It follows from (25) that

$$\left| \int_{\tau}^{\xi_l} \overline{f'(t)} \cos \overline{\mu}_k (t-\tau)\, dt \right| \leq C \left\{ \omega_1 \left( g_\tau, \frac{1}{Re\mu_k} \right)_{G_l} + \frac{\|f'\|_1}{Re\mu_k} \right\}, \quad l = \overline{1,m},$$

where

$$g_\tau(z) = \begin{cases} f'(z), & z \geq \tau \\ 0, & z < \tau \end{cases} \quad \xi_{l-1} \leq \tau \leq \xi_l.$$

On the other hand, (see [6]) it is known that for $Re\mu_k \geq \frac{4\pi}{|G_l|}$

$$\omega_1 \left( g_\tau, \frac{1}{Re\mu_k} \right)_{G_l} \leq const \left\{ \omega_1 \left( f', \frac{1}{Re\mu_k} \right) + \frac{\|f'\|_1}{Re\mu_k} \right\}.$$

Consequently,

$$\left| \int_{\tau}^{\xi_l} \overline{f'(t)} \cos \mu_k (t-\tau)\, dt \right| \leq \left\{ \omega_1 \left( f', \frac{1}{Re\mu_k} \right) + \frac{\|f'\|_1}{Re\mu_k} \right\}. \tag{26}$$

Taking into account (26), (2), (3), (14), (15), (17) and inequality (4), we majorize series (22) from above by the number series

$$C \sum_{Re\mu_k \geq \alpha} \frac{1}{Re\mu_k} \left[ \omega_1 \left( f', \frac{1}{Re\mu_k} \right) + \frac{\|f'\|_1}{Re\mu_k} \right],$$

that converges by (3) and (9).

For establishing uniforn convergence of series (23) it suffices to take into account

$$\left| \int_{\xi_{l-1}}^{\xi_l} \overline{q(t)} v_{k+j}(\tau) \left( \int_{\tau}^{\xi_l} \overline{f'(t)} \cos \overline{\mu}_k (t-\tau)\, dt \right) d\tau \right| \leq$$

$$\leq C \|q\|_1 \left\|f'\right\|_1 \sup_{G_l} |v_{k+j}(x)|$$

and apply the estimations (14), (15), (17) and (4). As a result, series (23) will be majorized from above by a converging series

$$C\left\|f'\right\|_1 \|q\|_1 \sum_{Re\mu_k \geq \alpha} \frac{1}{|\mu_k|^2}.$$

Thus, the series

$$\sum_{k=1}^{\infty} |(f, v_k)| \, |u_k(x)|$$

converges uniformly on $\overline{G}$.

By unconditional basicity in $L_2(G)$ of the system $\{u_k(x)\}$ the following representation will be walid:

$$f(x) = \sum_{k=1}^{\infty} (f, v_k) \, u_k(x).$$

Now, estimate the residual $R_\nu(x, f)$, $x \in \overline{G}$, $\nu \geq \frac{4\pi}{\min\limits_{1\leq l\leq m}\{|G_l|\}}$:

$$|R_\nu(x, f)| \leq \sum_{Re\mu_k \geq \nu} |(f, v_k)| \, |u_k(x)|.$$

Substituting here the expression $(f, v_k)$, from (12) we conclude that for estimating the residual $R_\nu(x, f)$ it is enough to estimate the residuals of the series (18)-(23). Therefore, in the estimations obtained above for the series (18)-(23) we must put $a = \nu$. Consequently, the inequality

$$|R_\nu(x, f)| \leq CC_1(f) \sum_{Re\mu_k \geq \nu} \frac{1}{|\mu_k|^2} + C\|q\|_1 \|f\|_\infty +$$

$$+ \sum_{Re\mu_k \geq \nu} \frac{1}{|\mu_k|^2} + C \sum_{Re\mu_k \geq \nu} \frac{1}{Re\mu_k}\left[\omega_1\left(f', \frac{1}{Re\mu_k}\right) + \frac{\|f'\|_1}{Re\mu_k}\right] +$$

$$+ C\left\|f'\right\|_\infty \|q\|_1 \sum_{Re\mu_k \geq \nu} \frac{1}{|\mu_k|^2}$$

will be valid. Here, taking into account

$$\sum_{Re\mu_k \geq \nu} \frac{1}{|\mu_k|^2} \leq \sum_{Re\mu_k \geq \nu} \frac{1}{(Re\mu_k)^2} = O\left(\nu^{-1}\right),$$

$$\sum_{Re\mu_k \geq \nu} \frac{1}{Re\mu_k} \omega_1\left(f', \frac{1}{Re\mu_k}\right) \leq$$

$$\leq \sum_{n=[\nu]}^{\infty} \sum_{n \leq Re\mu_k \leq n+1} \frac{1}{Re\mu_k} \omega_1\left(f', \frac{1}{Re\mu_k}\right) \leq$$

$$\leq \sum_{n=[\nu]}^{\infty} \frac{1}{n}\omega_1\left(f', \frac{1}{n}\right) \sum_{n\leq Re\mu_k \leq n+1} 1 \leq C\sum_{n=[\nu]}^{\infty} \frac{1}{n}\omega_1\left(f', \frac{1}{n}\right),$$

we get

$$\sup_{x\in\overline{G}} |R_\nu(x,f)| \leq const\left\{\nu^{-1}C_1(f) + \|q\|_1 \|f\|_\infty + \left\|f'\right\|_1 [1+\|q\|_1]\right\} +$$

$$+\sum_{n=[\nu]}^{\infty} n^{-1}\omega_1\left(f', n^{-1}\right) \leq constK(\nu, f)$$

Theorem 2 proved.

**Remark 3.** *It is seen from the proof of theorems 1 and 2 that if the system $\{u_k(x)\}$ is not biorthogonally adjoint to $\{u_k(x)\}$ and the remaining conditions of these theorems are fulfilled, then the*

$$\sum_{k=1}^{\infty} (f, v_k)\, u_k(x)$$

*uniformly and absolutely converges on $\overline{G}$ and estimations (7), (8) and (10) are valid for the residual* $\sum\limits_{Re\mu_k\geq\nu} (f, v_k)\, u_k(x)$.

## 4. Some applications of the theorems 1 and 2.

1. Consider the operator $Lu = u''$ on $G = \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ with the conditions $u(0) = 0$, $u'(1) = 0$, $u\left(\frac{1}{2}+0\right) - u\left(\frac{1}{2}\right) = u(1)$, $u'\left(\frac{1}{2}+0\right) - u'\left(\frac{1}{2}-0\right) = 0$.

For this problem, it is well known [1] that $\mu_k = 4\pi k$ for $k = 0, 1, 2, ...$ and $\widetilde{\mu}_k = 4\pi k/3$ for $k = 1, 2, 4, 5, 6, 7, 8...$ ($k$ is not multiply of 3). The eigenvalue $\lambda_0 = \mu_0^2 = 0$ is associated with the single eigenfunction

$$\overset{\circ}{u}_0(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{8}{3} & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Each eigenvalue $\lambda_k = \mu_k^2 = (4\pi k)^2$, $k = 1, 2, ...$ corresponds to a single eigen-function and a single associated function:

$$\overset{\circ}{u}_k(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 32\cos(4\pi k(1-x)) & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

$$\overset{1}{u}_k(x) = \begin{cases} -\frac{2}{3\pi k}\sin(4kx) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{4}{3\pi k}(1-x)\sin(4\pi k(1-x)) & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Each eigenvalue $\widetilde{\lambda}_k = (\widetilde{\mu}_k)^2 = \left(\frac{4\pi k}{3}\right)^2$, $k = 1, 2, 4, 5, 6, 7, 8...$ corresponds to a single eigenfunction

$$\overset{0}{\widetilde{u}}_k(x) = \begin{cases} \frac{8}{3}\sin\left(\frac{4\pi k}{3}x\right) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{8}{3}ctg\frac{2l\pi}{3}\cos\left(\frac{4\pi k}{3}(1-x)\right) & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

The biorthogonally adjoint system consists of root functions of the operator $Lv = v''$ on (0,1) with the conditions $v(0) = 0$ and $v'(1) - v'\left(\frac{1}{2}\right) = 0$.

The corresponding eigen and associated functions have the form

$$\overset{0}{v}_0(x) = x; \quad \overset{0}{v}_k(x) = 4\pi k \sin(4\pi k x);$$

$$\overset{1}{v}_k(x) = -\frac{x}{2}\cos(4\pi k x), \quad k = 1, 2, 3, ...;$$

$$\overset{0}{\widetilde{v}}_k(x) = \sin(4\pi k x/3), \quad k = 1, 2, 4, 5, 6, 7, 8....$$

For the problem in question,

$$Q_2(f, v) = -f(0)\overline{v'(0)} + \left[f\left(\frac{1}{2}\right) - f\left(\frac{1}{2} + 0\right) + f(1)\right]\overline{v'\left(\frac{1}{2}\right)}.$$

Condition (5) implies that

$$f(0) = 0, f\left(\frac{1}{2} + 0\right) - f\left(\frac{1}{2}\right) = f(1). \tag{27}$$

Since conditions 1) - 3) are fulfilled for this example,Theorem 1 holds for any function $f(x)$ from $\widetilde{W}_p^1(G)$, $p > 1$ that satisfies conditions (27). Moreover. Theorem 2 holds for any function $f(x)$ from $\widetilde{W}_1^1(G)$, that satisfies conditions (9), (27).

**2. Consider the eigenvalue problem**

$$u''(x) + \lambda u(x) = 0, \quad x \in (-1, 0) \cup (0, 1) \tag{28}$$

$$\begin{gathered} u(-1) = u(1) = 0 \\ u(-0) = u(+0) \\ u'(-0) - u'(+0) = \lambda m u(0), m \neq 0. \end{gathered} \tag{29}$$

It is well known [7] that eigenvalue of this problem are simple and form two series

$$\lambda_{1,k} = \mu_{1,k}^2 = (\pi k)^2, \quad k = 1, 2, ...;$$

$$\lambda_{2,k} = \mu_{2,k}^2, \; \mu_{2,k} = \pi k + \frac{2}{\pi m k} + O(k^{-2}), k = 0, 1, ...$$

The eigenfunctions have the form

$$u_{2k-1}(x) = \sin \pi k x, \quad k = 1, 2, ...,$$

$$u_{2k}(x) = \begin{cases} \sin \mu_{2,k}(1 + x) & \text{if} \quad x \in [-1, 0], \\ \sin \mu_{2,k}(1 - x) & \text{if} \quad x \in (0, 1], \quad k = 0, 1, .... \end{cases}$$

The system $\{u_k(x)\}_{k=0, k\neq k_0}^{\infty}$, where $k_0$ is an arbitrary fixed even number, forms a basis in $L_p(-1, 1)$ $1 < p < \infty$ (a Riesz basis for $p = 2$).

The biorthogonally adjoint system $\{u_k(x)\}_{k=0,k\neq k_0}^{\infty}$ has the form

$$v_k(x) = \varphi_k(x) - \frac{\varphi_k(0)}{\varphi_{k_0}(0)}\varphi_{k_0}(x),$$

where

$$\varphi_{2k-1}(x) = \sin \pi k x, \quad k = 1, 2, ...,$$

$$\varphi_{2k}(x) = \begin{cases} c_{2k} \sin \mu_{2,k}(1+x) & \text{if} \quad x \in [-1, 0] \\ c_{2k} \cos \mu_{2,k}(1-x) & \text{if} \quad x \in [0, 1] \end{cases}$$

$$c_{2k} = 1 + O\left(k^{-2}\right).$$

Obviously, the system $\{u_k(x)\}_{k=0,k\neq k_0}^{\infty}$ satisfies conditions Remark 3. For problem (28)-(29) the application of Theorems 1 and 2 (Remark 3) leads to the conditions

$$f(-1) = f(1) = f(0) = f(+0) = 0 = (f, \varphi_{k_0}). \tag{30}$$

Therefore statements of Theorem 1 holds for any function $f(x) \in \widetilde{W}_p^1(G)$, $p > 1$ satisfying conditions (30),while statements of Theorem 2 holds for any function $f(x) \in \widetilde{W}_1^1(G)$ satisfying condition (9) and (30).

## References

[1] V.A. Il'in Necessary and sufficient conditions for being a Riesz basis of root vectors of second-order discontinuous operators . Differentsialnye Uravneniya 22 (1986), No 12, 2059-2071. (Russian)

[2] T.A. Samarskaya Absolute and uniform convergence of expansions by root 'functions of I kaid non-local boundary value problem. Differentsialnye Uravneniya 25(1989), No 7, 1152-1160. (Russian)

[3] I.S. Lomov Uniform convergence of biorthogonal series for the Schrodinger operator with multidot boundary conditions. Differentsialnye Uravneniya 38 (2002). No 7, 890-896. (Russian)

[4] V.V. Tikhomirov Exact estimate of regular solutions of the one-dimensional Schrodinger equation with spectral parameter. Sovet math, dokl,273(1983), No 4, 807-810. (Russian)

[5] V.M. Kurbanov Equiconvergence of biorthogonal expansions by root functions of differential operators. I. Differentsialnye Uravneniya 35(1999), 1597-1609 (Russian).

[6] V.M. Kurbanov, R.A. Safarov Uniform convergence of biorthogonal expansion responding to the Schrodinger operator. Proc.of IMM of NAS . Azebaijan, XX(2004), 63-70.

[7] T.B. Gasymov, Sh.J. Mammadova On convergence of spectral expansions for one discontinuous problem with spectral parameter in the boundary condition. Trans, of NAS of Azerbaijan. 2006. No 4. 103-116.

[8] E.J. Ibadov On absolute and uniform convergence of biorthogonal expansion responding to Schrodinger discontinuous operator . Trans, of NAS of Azerbaijan. XXVIII (2008), No 4. 51-58.

[9] E.J. Ibadov On distribution of eigenvalues of a second order discontinuous differential operator. Trans, of NAS of Azerbaijan. XXVIII (2008), No 1. 77-84.

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