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On the Completeness and Minimality of Eigenfunctions of the Indefinite Sturm-Liouville Problem with Conjugation Condition

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Abstract. In this work we consider the following spectral problem:

$$-y'' = \lambda\omega(x)\,y,\quad x \in (-1,0)\cup(0,1),$$

$$\left.\begin{array}{c} y(-1) = y(1) = 0, \\ y(-0) = ay(+0) \\ y'(-0) = by'(+0) \end{array}\right\}$$

where a weight function $\omega(x)$ is in the following form:

$$\omega(x) = \begin{cases} -\alpha^2, & x \in (-1,0), \\ 1, & x \in (0,1), \end{cases}$$

$\alpha > 0$ is a given number, λ is a spectral parameter, a and b are arbitrary complex numbers. The theorem on the completeness and minimality of the eigenfunctions and associated functions of the spectral problem in the spaces $L_p(-1,1)$ is proved.

Key Words and Phrases: completeness, minimality, eigenfunctions, indefinite Sturm-Liouville problem.

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1. Introduction

Consider the spectral problem for the differential equation

$$-y'' = \lambda\omega(x)\,y,\quad x \in (-1,0)\cup(0,1), \tag{1}$$

with boundary conditions

$$y(-1) = y(1) = 0, \tag{2}$$

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and with conjugation conditions

$$\left\{ \begin{array}{l} y\left(-0\right) = ay\left(+0\right), \\ y'\left(-0\right) = by'\left(+0\right), \end{array} \right. \tag{3}$$

where $\omega(x)-$ is a sign-alternating weight function,

$$\omega\left(x\right) = \left\{ \begin{array}{cc} -\alpha^2, & x \in \left(-1, 0\right), \\ 1, & x \in \left(0, 1\right), \end{array} \right.$$

$\alpha > 0$ is a given number, λ is a spectral parameter, a and b are non-zero arbitrary complex numbers. Our goal in this work is to find asymptotic formulas for eigenvalues, to prove theorems on the completeness and minimality of eigenfunctions and associated functions of problem (1)-(3) in the spaces $L_p\left(-1,1\right)$. Previously, such problems were studied in the case $a = b = 1$, i.e. at the discontinuity point of the weight function, as a conjugation condition the continuity of the solution and its derivative are required. In works [1, 2, 3, 4, 5, 6], numerous applications of such problems are given, results are obtained in the case $p = 2, \quad \alpha = 1$. The results of these works are based on the theory of self-adjoint operators. Considering the case $p \neq 2$, in work [7] the methods of [8] are used, and also the methods of the theory of functions of a complex variable, in particular, the results of Paley-Wiener [9] and Levinson [10] on nonharmonic Fourier series are used. We also note the works [11, 12, 13], where ordinary differential operators of arbitrary order with a indefinite weight function are studied, asymptotic formulas for eigenvalues are found, and questions of convergence of expansions in eigenfunctions are investigated.

Recently, interest in spectral problems with a indefinite weight function has increased in connection with attempts to solve the Dirichlet problems for the Lavrent'ev–Bitsadze equation by the method of separation of variables. It is known [14, p. 303] that the problem of transition through the sound barrier of steady two-dimensional irrotational flows of an ideal gas in nozzles, when supersonic waves adjoin the nozzle walls near the minimum cross section, is reduced to the Dirichlet problem for equations of mixed type. In [15, 16], the Dirichlet problem for a mixed-type equation with one internal line of power degeneracy and degeneracy at the boundary in a rectangular domain was studied, a uniqueness criterion was established using spectral analysis methods, and the solution was constructed as the sum of a series over a system of eigenfunctions. In [17], for the first time the Dirichlet problem was studied for the Lavrent'ev–Bitsadze equation with two type-change internal lines in a rectangular domain. A uniqueness criterion is established and the solution of the problem is constructed as the sum of a series in a biorthogonal system of two mutually conjugate spectral conjugation problems for a second-order ordinary differential operator with a discontinuous coefficient at the highest derivative. The uniqueness of the solution of the stated problem is proved based on completeness of the biorthogonal system in the space $L_2\left(-1,1\right)$.

In [18, 19, 20] the problem for a discontinuous second-order differential operator with a constant coefficient at the highest derivative and with a spectral parameter under conjugation conditions was studied, a system of eigenfunctions was found and investigated for completeness and basicity in the spaces $L_p \oplus C$ and L_p.

2. Asymptotics of eigenvalues

Let $\lambda = \rho^2$. We also denote the linear forms included in the boundary conditions (2), (3) as follows:

$$\left.\begin{array}{ll} U_{11}(y) = y(-1), & U_{12}(y) \equiv 0 \\ U_{21}(y) \equiv 0, & U_{22}(y) = y(1) \\ U_{31}(y) = y(-0), & U_{32}(y) = -ay(+0) \\ U_{41}(y) = y'(-0), & U_{42}(y) = -by'(+0) \end{array}\right\} \tag{4}$$

After these denotations, problem (1)-(3) can be rewritten in the following form:

$$y'' + \rho^2 \omega(x) y = 0, \quad x \in (-1,0) \cup (0,1), \tag{5}$$

$$\left.\begin{array}{l} U_1(y) = U_{11}(y) + U_{12}(y) = 0 \\ U_2(y) = U_{21}(y) + U_{22}(y) = 0 \\ U_3(y) = U_{31}(y) + U_{32}(y) = 0 \\ U_4(y) = U_{41}(y) + U_{42}(y) = 0 \end{array}\right\} \tag{6}$$

It is known that equation (4) has a fundamental system of solutions $y_{11}(x) = e^{\alpha\rho x}$, $y_{12}(x) = e^{-\alpha\rho x}$, on the interval $(-1,0)$, and $y_{21}(x) = e^{i\rho x}$, $y_{22}(x) = e^{-i\rho x}$ on the interval $(0,1)$. Then the general solution of equation (1) (or (4)) has the form

$$y(x) = \begin{cases} c_{11}y_{11}(x) + c_{12}y_{12}(x), x \in (-1,0) \\ \\ c_{21}y_{21}(x) + c_{22}y_{22}(x), \ x \in (0,1) \end{cases}$$

Let us choose the constants c_{ik} so that the function $y(x)$ satisfies the boundary conditions (5). Then, to find the numbers c_{ik} we get the following system of equations:

$$\left.\begin{array}{l} c_{11}U_{11}(y_{11}) + c_{12}U_{11}(y_{12}) + c_{21}U_{12}(y_{21}) + c_{22}U_{12}(y_{22}) = 0 \\ c_{11}U_{21}(y_{11}) + c_{12}U_{21}(y_{12}) + c_{21}U_{22}(y_{21}) + c_{22}U_{22}(y_{22}) = 0 \\ c_{11}U_{31}(y_{11}) + c_{12}U_{31}(y_{12}) + c_{21}U_{32}(y_{21}) + c_{22}U_{32}(y_{22}) = 0 \\ c_{11}U_{41}(y_{11}) + c_{12}U_{41}(y_{12}) + c_{21}U_{42}(y_{21}) + c_{22}U_{42}(y_{22}) = 0 \end{array}\right\}$$

This system of equations has a nontrivial solution if and only if the main determinant (characteristic determinant) $\Delta(\rho) = det\|U_{\nu i}(y_{ik})\|_{\nu=\overline{1,4};i,k=1,2}$ of this system is zero. Thus, the number $\lambda = \rho^2$ is an eigenvalue of the spectral problem (1)-(3) if and only if the number ρ is a solution of the following equation

$$\Delta(\rho) = \begin{vmatrix} e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ 0 & 0 & e^{i\rho} & e^{-i\rho} \\ 1 & 1 & -a & -a \\ \alpha\rho & -\alpha\rho & -bi\rho & bi\rho \end{vmatrix} = 4i\Delta_0(\rho) = 0,$$

where

$$\Delta_0(\rho) = \alpha a \ sin\rho \ ch\alpha\rho + b \ cos\rho \ sh\alpha\rho.$$

Let us divide the complex ρ-plane into the following sectors:

$$S_k = \left\{ \rho = re^{i\theta} : \frac{(k-1)\,\pi}{2} \leq \theta \leq \frac{k\pi}{2} \right\}, \ k = 0, 1, 2, 3.$$

We also denote by Q_δ the domain of the ρ-plane, obtained from it by throwing out circles with the same radius $\delta > 0$ and with centers at zeros $\Delta(\rho)$. The following theorem is true.

Theorem 1. *The characteristic determinant $\Delta(\rho)$ of the spectral problem* (1)-(3) *has the following properties:*

1) *There exists a positive number M_δ such that in the domain $S_k \cap Q_\delta$ for sufficiently large $|\rho|$ the inequality*

$$|\Delta(\rho)| \geq M_\delta \, |\rho| \, e^{\pm r sin\theta} e^{\pm \alpha r cos\theta}; \tag{7}$$

is satisfied, where the constant M_δ is independent of ρ, but depends only on the number $\delta > 0$; in addition, the signs in the exponents on the right side of this inequality are chosen depending on the sectors S_k as follows: "+", "+" for $\rho \in S_0$; "+", "-" for $\rho \in S_1$; "-", "-" for $\rho \in S_2$; "-", "+" for $\rho \in S_3$.

2) *The zeros of the function $\Delta(\rho)$ are asymptotically simple and have the following asymptotics*

$$\rho_{1n} = \pi n - \gamma + O\left(e^{-2\pi n\alpha}\right), \ n \to \infty,$$

$$\rho_{2n} = -\frac{i}{\alpha}\left(\pi n + \gamma + \frac{\pi}{2} + O\left(e^{-\alpha\pi n}\right)\right), \ n \to \infty.$$

Proof. **1)** Let us estimate the function $\Delta_0(\rho)$ in each sector S_k. Let $\rho \in S_0$. Then the inequalities

$$Re(\ i\rho) \leq 0 \leq Re(-\ i\rho), Re(\ -\alpha\rho) \leq 0 \leq Re(\alpha\ \rho),$$

hold. Let us reduce the function $\Delta_0(\rho)$ to the following form:

$$\Delta_0(\rho) = e^{-i\rho} e^{\alpha\rho} \left(\alpha a \left(1 - e^{-2i\rho}\right)\left(1 + e^{-2\alpha\rho}\right) + b\left(1 + e^{-2i\rho}\right)\left(1 - e^{-2\alpha\rho}\right)\right).$$

All exponents inside the brackets on the right side of this equality have a non-positive real part in the exponent, therefore they are bounded. Moreover, if $\rho \in Q_\delta$, then the expression in brackets is bounded from below by some positive number M_δ in absolute value. Therefore we have

$$|\Delta_0(\rho)| \geq M_\delta \left|e^{-i\rho} e^{\alpha\rho}\right| = M_\delta e^{r sin\theta} e^{\alpha r cos\theta}.$$

Hence we obtain the validity of inequality (7) for $\rho \in S_0 \cap Q_\delta$. Other cases are considered in a similar way.

2) Define the number γ as follows:

$$cos\gamma = \frac{\alpha a}{\sqrt{\alpha^2 a^2 + b^2}}, \ sin\gamma = \frac{b}{\sqrt{\alpha^2 a^2 + b^2}}, \ Re\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Then the function $\Delta_0(\rho)$ with the help of elementary transformations can be represented in the following form:

$$\Delta_0(\rho) = \frac{1}{2}\sqrt{\alpha^2 a^2 + b^2} e^{\alpha\rho} \left(sin(\rho + \gamma) + e^{-2\alpha\rho} sin(\rho - \gamma)\right). \tag{8}$$

Based on the Rouché's theorem, we obtain that the zeros of the function $\Delta_0(\rho)$, situated in the strip $|Im\rho| \le h$ are asymptotically situated in a small neighborhood of the zeros of the function $sin(\rho + \gamma)$, and for large values of $|\rho|$ near each zero of the function $sin(\rho + \gamma)$ there is one zero of the function $\Delta_0(\rho)$. Hence we obtain the asymptotics of the zeros $\Delta_0(\rho)$, situated in the strip $|Im\rho| \le h$:

$$\rho_{1n} = \pi n - \gamma + O\left(e^{-2\alpha\pi n}\right), \ n \to \infty.$$

On the other hand, replacing ρ by $i\rho$ in formula (8), we obtain

$$\Delta_0(i\rho) = \frac{i}{2}\sqrt{\alpha^2 a^2 + b^2} e^{\rho} \left(cos(\alpha\rho - \gamma) - e^{-2\rho} cos(\alpha\rho + \gamma)\right). \tag{9}$$

Applying the Rouché's theorem again, from formula (9) we obtain that the zeros of the function $\Delta_0(\rho)$, situated in the strip $|Re\rho| \le h$ are asymptotically situated in a small neighborhood of the zeros of the function $cos(\alpha\rho - \gamma)$, and for large values of $|\rho|$ near each zero of the function $cos(\alpha\rho - \gamma)$ there is one zero of the function $\Delta_0(\rho)$. Hence we obtain the asymptotics of the zeros $\Delta_0(\rho)$, situated in the strip $|Re\rho| \le h$ are:

$$\rho_{2n} = -\frac{i}{\alpha}\left(\pi n + \gamma + \frac{\pi}{2} + O\left(e^{-\alpha\pi n}\right)\right), \ n \to \infty.$$

◄

3. Construction of the Green's function of the spectral problem

The Green's function of problem (1)-(3) is defined as the kernel of the integral representation of the solution of the nonhomogeneous equation

$$-y''(x) - \rho^2 \omega(x) y(x) = f(x), \tag{10}$$

Satisfying the boundary conditions (2),(3). Let us look for a solution of the problem (10),(2),(3) in the form

$$y(x) = \begin{cases} y_1(x), & x \in [-1, 0], \\ y_2(x), & x \in [0, 1], \end{cases} \tag{11}$$

where

$$\begin{cases} y_1(x) = c_{11}y_{11}(x) + c_{12}y_{12}(x) + \int_{-1}^{0} g_1(x,\xi,\rho)f(\xi)d\xi, \\ y_2(x) = c_{21}y_{21}(x) + c_{22}y_{22}(x) + \int_{0}^{1} g_2(x,\xi,\rho)f(\xi)d\xi. \end{cases} \tag{12}$$

$$g_1(x,\xi,\rho) = \begin{cases} -\frac{1}{4\alpha\rho}(e^{\alpha\rho(x-\xi)} - e^{-\alpha\rho(x-\xi)}), & -1 \le x < \xi \le 0, \\ \frac{1}{4\alpha\rho}(e^{\alpha\rho(x-\xi)} - e^{-\alpha\rho(x-\xi)}), & -1 \le \xi < x \le 0, \end{cases} \tag{13}$$

$$g_2(x,\xi,\rho) = \begin{cases} \frac{1}{2i\rho}(e^{i\rho(x-\xi)} - e^{-i\rho(x-\xi)}), & 0 \le x < \xi \le 1, \\ -\frac{1}{2i\rho}(e^{i\rho(x-\xi)} - e^{-i\rho(x-\xi)}), & 0 \le \xi < x \le 1. \end{cases} \tag{14}$$

We require that the function $y(x)$, defined by formulas (11)-(14), satisfies the boundary conditions (2) and conjugation conditions (3). Then, to determine the numbers c_{jk} we obtain the following system of equations:

$$\begin{cases} U_\nu\left(y\right) = \sum_{j,k=1}^{2} c_{jk}U_{\nu j}(y_{jk}) + \int_{-1}^{0} U_{\nu 1}(g_1)f(\xi)d\xi + \int_{0}^{1} U_{\nu 2}(g_2)f(\xi)d\xi = 0, \\ \nu = \overline{1,4}. \end{cases} \tag{15}$$

Having determined the numbers c_{jk} from system (15) and substituting their values into (12), for the solution of equation (10) that satisfies (2), (3), we obtain the following formula:

$$y(x) = \begin{cases} y_1(x) = \int_{-1}^{0} G_{11}(x,\xi,\rho)f(\xi)d\xi + \int_{0}^{1} G_{12}(x,\xi,\rho)f(\xi)d\xi, & x \in [-1,0], \\ y_2(x) = \int_{-1}^{0} G_{21}(x,\xi,\rho)f(\xi)d\xi + \int_{0}^{1} G_{22}(x,\xi,\rho)f(\xi)d\xi, & x \in [0,1], \end{cases} \tag{16}$$

Here

$$G_{ik}(x,\xi,\rho) = \frac{1}{\Delta\left(\rho\right)}H_{ik}(x,\xi,\rho), \ \ i,k = 1,2, \tag{17}$$

$$H_{ik}(x,\xi,\rho) = \begin{vmatrix} \delta_{ik}g_k\left(x,\xi\right) & \delta_{1k}y_{11}\left(x\right) & \delta_{1k}y_{12}\left(x\right) & \delta_{2k}y_{21}\left(x\right) & \delta_{2k}y_{22}\left(x\right) \\ U_{1k}\left(g_k\right)\left(\xi\right) & U_{11}\left(y_{11}\right) & U_{11}\left(y_{12}\right) & U_{12}\left(y_{21}\right) & U_{12}\left(y_{22}\right) \\ U_{2k}\left(g_k\right)\left(\xi\right) & U_{21}\left(y_{11}\right) & U_{21}\left(y_{12}\right) & U_{22}\left(y_{21}\right) & U_{22}\left(y_{22}\right) \\ U_{3k}\left(g_k\right)\left(\xi\right) & U_{31}\left(y_{11}\right) & U_{31}\left(y_{12}\right) & U_{32}\left(y_{21}\right) & U_{32}\left(y_{22}\right) \\ U_{4k}\left(g_k\right)\left(\xi\right) & U_{41}\left(y_{11}\right) & U_{41}\left(y_{12}\right) & U_{42}\left(y_{21}\right) & U_{42}\left(y_{22}\right) \end{vmatrix},$$

δ_{ik} is the Kronecker symbol. Denote $I_1 = (-1,0)$, $I_2 = (0,1)$ and let $\chi_1\left(x\right),\ \chi_2\left(x\right)$ be the characteristic functions of these intervals, respectively. The Green's function of problem

(1)-(3) is defined as follows:

$$G(x,\xi,\rho)=\sum_{i,k=1}^{2}\chi_i\,(x)\,\chi_k\,(\xi)\,G_{ik}(x,\xi,\rho). \tag{18}$$

Then the solution of equation (10) that satisfies conditions (2), (3) can be represented as

$$y(x)=\int_{-1}^{1}G(x,\xi,\rho)f(\xi)d\xi. \tag{19}$$

According to denotation (4) and formulas (13), (14) we have

$$\begin{array}{ll}
U_{11}(g_1)=-\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho(1+\xi)}-e^{\alpha\rho(1+\xi)}\right), & U_{12}(g_2)=0,\\
U_{21}(g_1)=0, & U_{22}(g_2)=-\frac{1}{4i\rho}\left(e^{i\rho(1-\xi)}-e^{-i\rho(1-\xi)}\right),\\
U_{31}(g_1)=\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho\xi}-e^{\alpha\rho\xi}\right), & U_{32}(g_2)=\frac{a}{4i\rho}\left(e^{-i\rho\xi}-e^{i\rho\xi}\right),\\
U_{41}(g_1)=\frac{1}{4}\left(e^{-\alpha\rho\xi}+e^{\alpha\rho\xi}\right), & U_{42}(g_2)=-\frac{b}{4}\left(e^{-i\rho\xi}+e^{i\rho\xi}\right).
\end{array}$$

Taking into account these values, as well as the values $U_{\nu s}\,(y_{sk})$ in formulas $H_{kj}(x,\xi,\rho)$, we obtain

$$H_{11}(x,\xi,\rho)=\begin{vmatrix}
\frac{\pm 1}{4\alpha\rho}\left(e^{\alpha\rho(x-\xi)}-e^{-\alpha\rho(x-\xi)}\right) & e^{\alpha\rho\,x} & e^{-\alpha\rho\,x} & 0 & 0\\
-\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho(1+\xi)}-e^{\alpha\rho(1+\xi)}\right) & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0\\
0 & 0 & 0 & e^{i\rho} & e^{-i\rho}\\
\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho\xi}-e^{\alpha\rho\xi}\right) & 1 & 1 & -a & -a\\
\frac{1}{4}\left(e^{-\alpha\rho\xi}+e^{\alpha\rho\xi}\right) & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho
\end{vmatrix},\quad x,\ \xi\in I_1,$$

here the sign " + " is taken in the case of $-1\le\xi<x\le 0$, and the sign " − " in the case of $-1\le x<\xi\le 0$;

$$H_{12}(x,\xi,\rho)=\begin{vmatrix}
0 & e^{\alpha\rho x} & e^{-\alpha\rho x} & 0 & 0\\
0 & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0\\
\frac{1}{4i\rho}\left(e^{i\rho(1-\xi)}-e^{-i\rho(1-\xi)}\right) & 0 & 0 & e^{i\rho} & e^{-i\rho}\\
\frac{a}{4i\rho}\left(e^{-i\rho\xi}-e^{i\rho\xi}\right) & 1 & 1 & -a & -a\\
\frac{b}{4}\left(e^{-i\rho\xi}+e^{i\rho\xi}\right) & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho
\end{vmatrix},\quad x\in I_1,\quad \xi\in I_2;$$

$$H_{21}(x,\xi,\rho)=\begin{vmatrix} 0 & 0 & 0 & e^{i\rho x} & e^{-i\rho x} \\ -\frac{1}{4\alpha\rho}\left(e^{-\alpha\rho(1+\xi)}-e^{\alpha\rho(1+\xi)}\right) & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ 0 & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ \frac{1}{4\alpha\rho}\left(e^{-\alpha\rho\xi}-e^{\alpha\rho\xi}\right) & 1 & 1 & -a & -a \\ \frac{1}{4}\left(e^{-\alpha\rho\xi}+e^{\alpha\rho\xi}\right) & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho \end{vmatrix},\ x\in I_2,\ \ \xi\in I_1;$$

$$H_{22}(x,\xi,\rho)=\begin{vmatrix} \frac{\pm 1}{4i\rho}\left(e^{i\rho(x-\xi)}-e^{-i\rho(x-\xi)}\right) & 0 & 0 & e^{i\rho x} & e^{-i\rho x} \\ 0 & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ \frac{1}{4i\rho}\left(e^{i\rho(1-\xi)}-e^{-i\rho(1-\xi)}\right) & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ \frac{a}{4i\rho}\left(e^{-i\rho\xi}-e^{i\rho\xi}\right) & 1 & 1 & -a & -a \\ \frac{b}{4}\left(e^{-i\rho\xi}+e^{i\rho\xi}\right) & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho \end{vmatrix},\quad x,\ \xi\in I_2,$$

here the sign " + " is taken in the case of $-1\le\xi<x\le 0$, and the sign " − " in the case of $-1\le x<\xi\le 0$;

Theorem 2. *For the components $G_{ik}(x,\xi,\rho)$ of the Green's function of problem (1)-(3) in the domain Q_δ for sufficiently large values $|\rho|$ uniformly in the variables $x\in_i$, $\xi\in_k$ the estimate*

$$|G_{ik}(x,\xi,\rho)|\le\frac{C_\delta}{|\rho|},\tag{20}$$

is true, where the positive number C_δ is independent of ρ, but depends only on the number δ.

Proof. Let us perform the following transformations on the determinants $H_{ik}(x,\xi,\rho)$: in the determinant $H_{11}(x,\xi,\rho)$ in the case $-1\le\xi<x\le 0$ multiply the second and third columns by $-\frac{1}{4\alpha\rho}e^{-\alpha\rho\xi}$, $-\frac{1}{4\alpha\rho}e^{\alpha\rho\xi}$ respectively and add to the first column, then we get

$$H_{11}(x,\xi,\rho)=\begin{vmatrix} -\frac{1}{2\alpha\rho}e^{-\alpha\rho(x-\xi)} & e^{\alpha\rho x} & e^{-\alpha\rho x} & 0 & 0 \\ -\frac{1}{2\alpha\rho}e^{-\alpha\rho(1+\xi)} & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ 0 & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ -\frac{1}{2\alpha\rho}e^{\alpha\rho\xi} & 1 & 1 & -a & -a \\ \frac{1}{2}e^{\alpha\rho\xi} & \alpha\rho & -\alpha\rho & -bi\rho & bi\rho \end{vmatrix}=$$

$$= \frac{1}{2}e^{-i\rho}e^{\alpha\rho}\begin{vmatrix} -e^{-\alpha\rho(x-\xi)} & e^{\alpha\rho x} & e^{-\alpha\rho(1+x)} & 0 & 0 \\ -e^{-\alpha\rho(1+\xi)} & e^{-\alpha\rho} & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{2i\rho} & 1 \\ -e^{\alpha\rho\xi} & 1 & e^{-\alpha\rho} & -a & -a \\ e^{\alpha\rho\xi} & 1 & -e^{-\alpha\rho} & -\frac{b}{\alpha}i & \frac{b}{\alpha}i \end{vmatrix}$$

(in the case $-1 \le x < \xi \le 0$ similar actions are performed by multiplying the second and third columns by $\frac{1}{4\alpha\rho}e^{-\alpha\rho\xi}$, $\frac{1}{4\alpha\rho}e^{\alpha\rho\xi}$ respectively); in the determinant $H_{12}(x,\xi,\rho)$ multiply the fourth and fifth columns by $\frac{1}{4i\rho}e^{-i\rho\xi}$, $\frac{1}{4i\rho}e^{i\rho\xi}$ respectively and add to the first column, then we get

$$H_{12}(x,\xi,\rho) = \frac{1}{2}\begin{vmatrix} 0 & e^{\alpha\rho x} & e^{-\alpha\rho x} & 0 & 0 \\ 0 & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ e^{i\rho(1-\xi)} & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ -e^{i\rho\xi} & 1 & 1 & -a & -a \\ e^{i\rho\xi} & -\alpha i & \alpha i & -b & b \end{vmatrix} =$$

$$= \frac{1}{2}e^{-i\rho}e^{\alpha\rho}\begin{vmatrix} 0 & e^{\alpha\rho x} & e^{-\alpha\rho(1+x)} & 0 & 0 \\ 0 & e^{-\alpha\rho} & 1 & 0 & 0 \\ e^{i\rho(1-\xi)} & 0 & 0 & e^{i\rho} & 1 \\ -e^{i\rho\xi} & 1 & e^{-\alpha\rho} & -a & -ae^{i\rho} \\ e^{i\rho\xi} & 1 & \alpha i e^{-\alpha\rho} & -b & be^{i\rho} \end{vmatrix};$$

in the determinant $H_{21}(x,\xi,\rho)$ multiply the second and third columns by $-\frac{1}{4\alpha\rho}e^{-\alpha\rho\xi}$, $-\frac{1}{4\alpha\rho}e^{\alpha\rho\xi}$

respectively and add to the first column, then we get

$$H_{21}(x,\xi,\rho)=\frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & e^{i\rho x} & e^{-i\rho x} \\ -e^{-\alpha\rho(1+\xi)} & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ 0 & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ -e^{\alpha\rho\xi} & 1 & 1 & -a & -a \\ e^{\alpha\rho\xi} & 1 & -1 & -\frac{b}{\alpha}i & \frac{b}{\alpha}i \end{vmatrix}=$$

$$=\frac{1}{2}e^{-i\rho}e^{\alpha\rho}\begin{vmatrix} 0 & 0 & 0 & e^{i\rho x} & e^{i\rho(1-x)} \\ -e^{-\alpha\rho(1+\xi)} & e^{-\alpha\rho} & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{i\rho} & 1 \\ -e^{\alpha\rho\xi} & 1 & e^{-\alpha\rho} & -a & -ae^{i\rho} \\ e^{\alpha\rho\xi} & 1 & -e^{-\alpha\rho} & -\frac{b}{\alpha}i & \frac{b}{\alpha}ie^{i\rho} \end{vmatrix};$$

in the determinant $H_{22}(x,\xi,\rho)$ in the case of $0\le\xi<x\le1$ multiply the fourth and fifth columns by $\frac{1}{4i\rho}e^{-i\rho\xi}$, $\frac{1}{4i\rho}e^{i\rho\xi}$, respectively and add to the first column, then we get

$$H_{22}(x,\xi,\rho)=$$

$$=\frac{1}{2}\begin{vmatrix} e^{i\rho(x-\xi)} & 0 & 0 & e^{i\rho x} & e^{-i\rho x} \\ 0 & e^{-\alpha\rho} & e^{\alpha\rho} & 0 & 0 \\ e^{i\rho(1-\xi)} & 0 & 0 & e^{i\rho} & e^{-i\rho} \\ -e^{i\rho\xi} & 1 & 1 & -a & -a \\ e^{i\rho\xi} & -\alpha i & \alpha i & -b & b \end{vmatrix}=\frac{1}{2}e^{-i\rho}e^{\alpha\rho}\begin{vmatrix} e^{i\rho(x-\xi)} & 0 & 0 & e^{i\rho x} & e^{i\rho(1-x)} \\ 0 & e^{-2\alpha\rho} & 1 & 0 & 0 \\ e^{i\rho(1-\xi)} & 0 & 0 & e^{i\rho} & 1 \\ -e^{i\rho\xi} & 1 & 1 & -a & -ae^{i\rho} \\ e^{i\rho\xi} & -\alpha i & \alpha i & -b & be^{i\rho} \end{vmatrix},$$

(in the case $0\le x<\xi\le1$ similar actions are performed by multiplying the second and third columns by $-\frac{1}{4i\rho}e^{-i\rho\xi}$, $-\frac{1}{4i\rho}e^{i\rho\xi}$, respectively)

Thus, in the formulas obtained for $H_{ik}(x,\xi,\rho)$ in all determinants on the right side of the last equalities, all exponents have a non-positive real part in the exponent, these determinants for $\rho\in S_0$ are uniformly bounded in variables $x\in I_i,\xi\in I_k$. A similar property is established in other sectors S_k. It follows that for functions $H_{ik}(x,\xi,\rho)$ for sufficiently large values of $|\rho|$ uniformly in the variables $x\in I_i,\xi\in I_k$ the estimate

$$|H_{ik}(x,\xi,\rho)|\le Ce^{\pm r\sin\theta}e^{\pm\alpha r\cos\theta}, \tag{21}$$

holds, the signs here are taken in accordance with the rule specified in Theorem 1. Now, taking into account inequalities (7) and (21) in formula (17), we obtain the validity of inequality (20). ◄

4. Completeness and minimality of eigenfunctions in the space L_p

Recall that a system $\{u_n\}_{n\in N}$ of a Banach space X is called complete in X, if the closure of the linear span of this system coincides with the entire space X, and minimal if no element of this system is included in the closed linear span of the remaining elements of this system. Recall also that a system is complete in X if and only if there is no nonzero linear continuous functional that annihilates all elements of this system. A system is minimal in X if and only if it has a biorthogonal system.

Denote by $W_p^2(-1,\,0)\cup(0,1)$ the space of functions from $L_p(-1,\,1)$, whose restrictions to each of the intervals $(-1,\,0)$ and $(0,\,1)$ belong to the Sobolev spaces $W_p^2(-1,\,0)$ and $W_p^2(0,\,1)$ respectively. Let us define an operator L in space $L_p(-1,\,1)$ as follows:

$$D(L)=\left\{y\in W_p^2(-1,\,0)\cup(0,1):y(-1)-y(1)=\right.$$

$$\left.y(-0)-ay(+0)=y'(-0)-by'(+0)=0\right\}$$

and for $y\in D(L)$

$$Ly=-\frac{1}{\omega(x)}y''.$$

Obviously, L is a densely defined closed operator in $L_p(-1,\,1)$ with a compact resolvent. The eigenvalues of the operator L are the numbers $\lambda_{in}=(\rho_{in})^2\,,i=1,2;n\in N$. Denote by $\{y_{in}\}_{i=1,2;n\in N}$ the system of corresponding eigenfunctions and associated functions.

Theorem 3. *System $\{y_{in}\}_{i=1,2;n\in N}$ of eigenfunctions and associated functions of the operator L is complete in space $L_p(-1,\,1)$ $1<p<\infty$.*

Proof. To prove the completeness of the system $\{y_{in}\}_{i=1,2;n\in N}$ in $L_p(-1,1)\,,\ 1<p<\infty$, let us estimate the norms of the resolvent of the operator L for sufficiently large values of $|\rho|\,.$

Let $\rho\in Q_\delta,\ |\rho|\geq r_0$. Then, taking into account inequalities (20) in formula (18), we obtain that the Green's function uniformly in variables $x,\xi\in[-1,1]$ satisfies the inequality

$$|G(x,\xi,\rho)|\leq\frac{C_\delta}{|\rho|},\ \rho\in Q_\delta,\ |\rho|\geq r_0.$$

Taking into account this estimate in formula (19) for the function $y(x)$, we obtain the following estimate:

$$|y(x)|\leq\frac{C_\delta}{|\rho|}\,\|f\|_{L_p}\,,\ \rho\in Q_\delta,\ |\rho|\geq r_0.$$

Moreover, this inequality is satisfied uniformly in $x \in [-1, 1]$. As a consequence, hence we get

$$\|y\|_{L_p} \le \frac{C_\delta}{|\rho|} \|f\|_{L_p}, \ \rho \in Q_\delta, \ |\rho| \ge r_0.$$

The last inequality means that the for the resolvent $R(\lambda) = (L - \lambda I)^{-1}$ of the operator L the following estimate

$$\left\|R\left(\rho^2\right)\right\| \le \frac{C_\delta}{|\rho|}, \ \rho \in Q_\delta, \ |\rho| \ge r_0, \tag{22}$$

holds. Now suppose that the system of root functions of the operator L is not complete in $L_p(-1, 1)$. Then there exists a function $g \in L_q(-1, 1)$, $q = p/(p-1)$, orthogonal to all root subspaces of the operator L, i.e.

$$\langle Q_{in} f, g \rangle = 0, \forall f \in L_p(-1, 1), \ i = 1, 2; n \in N.$$

Hence it follows that $Q_{in}^* g = 0$, $i = 1, 2; n \in N$; here Q_{in} denotes the Riesz projectors of the operator L, i.e.

$$Q_{in} = \frac{1}{2\pi i} \oint_{\gamma_{in}(\delta)} R(\lambda)\, d\lambda,$$

where $\gamma_{in}(\delta)$ are the images of the circles $\gamma_{in}(\delta) = \{\rho : |\rho - \rho_{in}| = \delta\}$ under the mapping $\lambda = \rho^2$. In this case it is obvious that Q_{in}^*, $i = 1, 2; n \in N$, are the Riesz projectors of the adjoint operator L^*. This implies that $R(\lambda, L^*) g$ is an entire function in the λ-plane. On the other hand, according to estimate (22), the inequality

$$\|R(\lambda, L^*)\| \le \frac{C_\delta}{|\lambda|^{\frac{1}{2}}}, \ \lambda \in \Omega_\delta, \ |\lambda| \ge R_0, \tag{23}$$

is true, where Ω_δ denotes the image of the set Q_δ under the mapping $\lambda = \rho^2$. Then, according to the maximum principle, inequality (23) is satisfied in the entire λ-plane and in turn, we obtain $R(\lambda, L^*) g \to 0, |\lambda| \to \infty$. The latter, by Liouville's theorem, the entire function $R(\lambda, L^*) g$ is constant. Then, differentiating this function and taking into account the formula $\frac{d}{d\lambda} R(\lambda, L^*) = R(\lambda, L^*)^2$ we obtain that $R(\lambda, L^*)^2 g = 0$. But, since for $\lambda \in \rho(L^*)$ the operator $R(\lambda, L^*)$ is unique, then we obtain that $g = 0$. And this means that the system $\{y_{in}\}_{i=1,2; n \in N}$ of eigenfunctions and associated functions of the operator L is complete in $L_p(-1, 1)$.

Theorem is proved. ◄

Denote by $\{z_{in}\}_{i=1,2; n \in N}$ the system of eigenfunctions and associated functions of the adjoint operator L^*. The operator L^* is the operator generated by the adjoint spectral problem

$$z'' + \lambda \omega(x) z = 0, \quad x \in (-1, 0) \cup (0, 1)$$

$$\left. \begin{array}{l} z(-1) = z(1) = 0, \\ z(-0) = -\frac{\alpha^2}{\bar{b}} z(+0), \\ z'(-0) = -\frac{\alpha^2}{\bar{a}} z'(+0). \end{array} \right\}$$

Then the system $\{z_{in}\}_{i=1,2;n\in N}$ (after appropriate normalization) is biorthogonal to the system $\{y_{in}\}_{i=1,2;n\in N}$. Taking this fact into account, we obtain the following corollaries from Theorem 3.

Corollary 1. *System $\{y_{in}\}_{i=1,2;n\in N}$ of eigenfunctions and associated functions of the operator L is complete and minimal in $L_p(-1,1),\ 1<p<\infty$.*

Corollary 2. *System $\{z_{in}\}_{i=1,2;n\in N}$ of eigenfunctions and associated functions of the operator L^* is complete and minimal in $L_p(-1,1),\ 1<p<\infty$.*

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On Some Embedding Theorems of Besov-Morrey Spaces with Dominant Mixed Derivatives

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Abstract. In this paper introduced and studied view embedding theory some differential properties of functions from Besov-Morrey spaces with dominant mixed derivatives.

Key Words and Phrases: Besov-Morrey spaces with dominant mixed derivatives, embedding theorems, Hölder condition.
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1. Introduction

The fact some mixed derivatives of f entering the definition of the norm of W_p^l, H_p^l and $B_{p,\theta}^l$ leads to the necessity of consideration of the function spaces of another type in which the key role is played by mixed derivatives.

In this paper introduced and studied the Besov-Morrey spaces with dominant mixed derivatives.

$$S_{p,\theta,\varphi,\beta}^l B(G_\varphi)$$

and help of method of integral representation differential and difference-differential properties of functions from this space.

Here $G \subset R^n$, $1 \le p < \infty$, $1 \le \theta \le \infty$, $\varphi = (\varphi_1(t_1), \varphi_2(t_2), \ldots, \varphi_n(t_n))$, $\varphi_j(t_j) > 0$, $\varphi_j'(t_j) > 0$, $(t_j > 0)$ be continuously differentiable functions, $\lim\limits_{t_j \to +0} \varphi_j(t_j) = 0$, $\lim\limits_{t_j \to +\infty} \varphi_j(t_j) = K_j \le \infty$, $j \in e_n = \{1, 2, ..., n\}$. We denote the set of such vector-functions φ by Ψ.

Note that the spaces with parameters constructed and studied in C.B. Morrey's papers [6], and after these results were developed and generalized in the papers of V.P. Il'in [4], Y.V.Netrusov [12], A. Mazzucato [5], V.S. Guliyev [3], A.M. Najafov [7-11] and other mathematicians.

For any $x \in R^n$ we assume

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t_j), j \in e_n \right\},$$

and let $m_j > 0$, $k_j \ge 0$ are integers and $m_j > l_j - k_j > 0$, $l_j > 0$, $j \in e_n$.

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Definition 1. *Denote by* $S^l_{p,\theta,\varphi,\beta}B(G_\varphi)$ *the Banach space of locally summable functions on* G *with finite norm*

$$\|f\|_{S^l_{p,\theta,\varphi,\beta}B(G_\varphi)} = \sum_{e\subseteq e_n}\left\{\int\limits_{0^e}^{t_0^e}\left[\frac{\|\triangle^{m^e}(\varphi(t),G_{\varphi(t)}))D^{k^e}f\|_{p,\varphi,\beta}}{\prod\limits_{j\in e}(\varphi_j(t_j))^{(l_j-k_j)}}\right]^\theta \prod_{j\in e}\frac{d\varphi_j(t_j)}{\varphi_j(t_j)}\right\}^{\frac{1}{\theta}}, \quad (1)$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{\substack{x\in G,\\ t_j>0, j\in e_n}}\left(|\varphi([t]_1)|^{-\beta}\|f\|_{p,G_{\varphi(t)}(x)}\right), \quad (2)$$

$|\varphi([t]_1)|^{-\beta} = \prod\limits_{j\in e_n}\varphi_j([t_j]_1)^{-\beta_j}$, $\beta_j \in [0,1]$, $[t_j]_1 = \min\{1,t_j\}$, $1\le\theta\le\infty$, $l^e = (l_1^e, l_2^e, \dots, l_n^e)$, $l_j^e = l_j (j\in e)$, $l_j^e = 0$ $(j\in e_n - e = e')$,

$$\triangle^{m^e}(\varphi(t))f(x) = \left(\prod_{j\in e}\triangle_j^{m_j}(\varphi_j(t_j))\right)f(x),$$

and $t_0 = (t_{01},\dots,t_{0n})$ *is a fixed positive vector,* $t_0^e = (t_{01}^e, t_{02}^e, \dots, t_{0n}^e)$, $t_{0j}^e = t_{0j}$ $(j\in e)$, $t_{0j}^e = 0$ $(j\in e')$, *and*

$$\int\limits_{a^e}^{b^e} f(x)dx^e = \left(\prod_{j\in e}\int\limits_{a_j}^{b_j}dx_j\right)f(x),$$

i.e., integration is carried out only with respect to the variables x_j *whose indices belong to* e.

The spaces $S^l_{p,\theta,\varphi,\beta}B(G_\varphi)$ in case $\varphi_j(t_j) = t_j^{\varkappa_j}, \beta_j = \frac{a_j}{p}$ $(j\in e_n)$, coincides with the space $S^l_{p,\theta,a,\varkappa}B(G)$ introduced and studied in [11], in the case $\beta_j = 0$ $(j\in e_n)$, coincides with the space $S^l_{p,\theta}B(G)$ introduced and studied by A.J. Dzhabrailov [2], in the case $\theta=\infty$, coincides with the space Nikolskii-Morrey with dominant mixed derivatives $S^l_{p,\varphi,\beta}H(G_\varphi)$.

In the case for any $t_j > 0$ $(j\in e_n)$, there exists a constant $C>0$ it holds the embedding

$$L_{p,\varphi,\beta}(G)\hookrightarrow L_p(G), \qquad S^l_{p,\theta,\varphi,\beta}B(G_\varphi)\hookrightarrow S^l_{p,\theta}B(G_\varphi),$$

i.e.,

$$\|f\|_{p,G}\le C\|f\|_{p,\varphi,\beta;G}, \ \ \|f\|_{S^l_{p,\theta}B(G_\varphi)}\le C\|f\|_{S^l_{p,\theta,\varphi,\beta}B(G_\varphi)}. \quad (3)$$

Definition 2. [10] *An open set* $G\subset R^n$ *is said to satisfy condition of flexible* φ*-horn type, if for some* $\omega\in(0,1]^n, T\in(0,\infty)^n$ *for any* $x\in G$ *there exists a vector -function*

$$\rho(\varphi(t),x) = (\rho_1(\varphi_1(t_1),x),\dots,\rho_n(\varphi_n(t_n),x)),\ 0\le t_j\le T_j,\ (j\in e_n)$$

with the following properties:

1) for all $j \in e_n$, $\rho_j(\varphi_j(t_j), x)$ is absolutely continuous on $[0, T_j]$, $|\rho_j(\varphi_j(t_j), x)| \le 1$ for almost all $t_j \in [0, T_j]$, $j \in e_n$,

2) $\rho_j(0, x) = 0$;

$$x + V(x, \omega) = x + \bigcup_{\substack{0 \le t_j \le T_j, \\ j \in e_n}} [\rho_j(\varphi_j(t_j), x) + \varphi_j(t_j)\omega_j T_j] \subset G.$$

In particular, $\varphi_j = t_j$ $(j \in e_n)$ is the set $V(x, \omega)$ and $x + V(x, \omega)$ will be said to be a set of flexible horn type introduced in [9], if $t_j = t$, $(j \in e_n)$, $\varphi(t) = t^\lambda$ $(t^\lambda = t^{\lambda_1}, t^{\lambda_2}, \ldots, t^{\lambda_n})$ is the set $V(x, \omega)$ and $x + V(x, \omega)$ will be said to be a set of flexible λ-horn type introduced in [1].

Theorem 1. *Let $1 \le p < \infty$, $1 \le \theta \le \infty$, $f \in S^l_{p,\theta}B(G_\varphi)$, $\varphi \in \Psi$. Then one can construct the sequence $h_s = h_s(x)$ $(s = 1, 2, \ldots)$ of infinitely differentiable finite functions in R^n such that*

$$\lim_{s \to \infty} \|f - h_s\|_{S^l_{p,\theta}B(G)} = 0.$$

Lemma 1. *Let $1 \le p \le q \le r \le \infty$, $0 < \eta_j, t_j \le T_j \le 1$ $(j \in e_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_j \ge 0$ $(j \in e_n)$ are integers, $\triangle^{m^e}(\varphi(t)) f \in L_{p,\varphi,\beta}(G)$ and let*

$$\mu_j = l_j - \nu_j - (1 - \beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right),$$

$$B^e_\eta(x) = \prod_{j \in e'} (\varphi_j(T_j))^{-2-\nu_j} \int_{0^e}^{\eta^e} L_e(x,t) \prod_{j \in e} (\varphi_j(t_j))^{-\nu_j - 2} \prod_{j \in e} \frac{\varphi'_j(t_j)}{\varphi_j(t_j)} dt^e \tag{4}$$

$$B^e_{\eta,T}(x) = \prod_{j \in e'} (\varphi_j(T_j))^{-2-\nu_j} \int_{\eta^e}^{T^e} L_e(x,t) \prod_{j \in e} (\varphi_j(t_j))^{-\nu_j - 2} \prod_{j \in e} \frac{\varphi'_j(t_j)}{\varphi_j(t_j)} dt^e, \tag{5}$$

$$\begin{aligned} L_e(x,t) = \int_{R^n} \int_{-\infty^e}^{+\infty^e} M_e\left(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho(\varphi(t^e + T^e), x)}{\varphi(t^e + T^{e'})}\right) \times \\ \times J_e\left(\frac{U}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t), x)\right) \times \\ \times \triangle^{m^e}(\varphi(\delta)u) f(x + y + u^e) du^e dy. \end{aligned} \tag{6}$$

Then for any $\overline{x} \in U$ the following inequalities are valid

$$\sup_{\overline{x} \in U} \left\|B^e_\eta\right\|_{q, U_{\psi(\xi)}(\overline{x})} \le C_1 \left\| \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t)) f \right\|_{p,\varphi,\beta;G} \times$$

$$\times \prod_{j\in e'} \left(\varphi_j\left(T_j\right)\right)^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)-1} \prod_{j\in e_n} \left(\psi_j\left([\xi_j]_1\right)\right)^{\beta_j \frac{p}{q}} \prod_{j\in e} \left(\varphi_j\left(\eta_j\right)\right)^{\mu_j} (\mu_j > 0), \qquad (7)$$

$$\sup_{\overline{x}\in U} \left\| B^e_{\eta,T} \right\|_{q,U_{\psi(\xi)}(\overline{x})} \le C_2 \left\| \prod_{j\in e} (\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t)) f \right\|_{p,\varphi,\beta;G}$$

$$\times \prod_{j\in e'} \left(\varphi_j\left(T_j\right)\right)^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)-1} \prod_{j\in e_n} \left(\psi_j\left([\xi_j]_1\right)\right)^{\beta_j \frac{p}{q}} \times$$

$$\times \begin{cases} \prod_{j\in e} (\varphi_j(T_j))^{\mu_j}, & for\ \mu_j > 0 \\ \prod_{j\in e} \ln \frac{\varphi_j(T_j)}{\varphi_j(\eta_j)}, & for\ \mu_j = 0, \\ \prod_{j\in e} \left(\varphi_j\left(\eta_j\right)\right)^{\mu_j}, & for\ \mu_j < 0, \end{cases} \qquad (8)$$

here $U_{\psi(\xi)}\left(\overline{x}\right) = \left\{x : |x_j - \overline{x}_j| < \frac{1}{2}\psi_j\left(\xi_j\right), j \in e_n\right\}$, *and* $\psi \in \Psi$, C_1 *and* C_2 *are constants independent of* f, ξ, η *and* T.

Proof. Applying sequentially the Minkowskii generalized inequality for any $\overline{x} \in U$

$$\left\| B^e_{\eta} \right\|_{q,U_{\psi(\xi)}(\overline{x})} \le \prod_{j\in e'} (\varphi_j(T_j))^{-2-\nu_j} \int_{0^e}^{\eta^e} \left\| L_e\left(\cdot, t^e + T^{e'}\right) \right\|_{q,U_{\psi(\xi)}(\overline{x})} \times$$

$$\times \prod_{j\in e} \left(\varphi_j\left(t_j\right)\right)^{-2-\nu_j} \prod_{j\in e} \frac{\varphi_j'\left(t_j\right)}{\varphi_j\left(t_j\right)} dt^e. \qquad (9)$$

From the Hölder inequality ($q \le r$) we have

$$\left\| L_e\left(\cdot, t^e + T^{e'}\right) \right\|_{q,U_{\psi(\xi)}(\overline{x})} \le \left\| L_e\left(\cdot, t^e + T^{e'}\right) \right\|_{r,U_{\psi(\xi)}(\overline{x})} \prod_{j\in e_n} \left(\psi_j\left(\xi_j\right)\right)^{\frac{1}{q}-\frac{1}{r}}. \qquad (10)$$

Further, we will assume that there exists a function $|M_e(x,y)| \le C|M^1_e(x)|$, for all $y \in R^n$. Let χ be a characteristic function of the set $S\left(M_e\right)$. Again applying the Hölder inequality ($\frac{1}{r} + (\frac{1}{p} - \frac{1}{r}) + (\frac{1}{s} - \frac{1}{r}) = 1$) for representing function in the form (6) in the case $1 \le p \le r \le \infty$, $s \le r$, $s \le r$ ($\frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{r}$), we get

$$\left\| L_e\left(\cdot, t^e + T^{e'}\right) \right\|_{r,U_{\psi(\xi)}(\overline{x})} \le$$

$$\le \sup_{x\in U_{\psi(\xi)}(\overline{x})} \left(\int_{R^n} \left| \int_{-\infty^e}^{+\infty^e} |J_e||\triangle^{m^e} f(x+y+u^e) du^e \right|^p \chi\left(\frac{y}{\varphi\left(t^e + T^{e'}\right)}\right) dy \right)^{\frac{1}{p}-\frac{1}{r}} \times$$

$$\times \sup_{y \in V} \left(\int\limits_{U_{\psi(\xi)}(\overline{x})} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e||\triangle^{m^e} f(y+u^e)| du^e \right|^p dx \right)^{\frac{1}{r}} \times$$

$$\times \left(\int\limits_{R^n} \| M_e^1 \left(\frac{y}{\varphi(t^e + T^{e'})} \right) \|^S dy \right)^{\frac{1}{s}}. \tag{11}$$

For any $x \in U$ we have

$$\int\limits_{R^n} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e||\triangle^{m^e} f(x+y+u^e)| du^e \right|^p \chi \left(\frac{y}{\varphi\left(t^e + T^{e'}\right)} \right) dy \leq$$

$$\leq \int\limits_{(U+V)_{\varphi(t^e+T^{e'})}(\overline{x})} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e||\triangle^{m^e} f(y+u^e)| du^e \right|^p dy \leq$$

$$\leq \int\limits_{G_{\varphi(t^e+T^{e'})}(\overline{x})} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e||\triangle^{m^e} f(y+u^e)| du^e \right|^p dy \leq$$

$$\leq \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{l_j p} \int\limits_{G_{\varphi(t^e+T^{e'})}(\overline{x})} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e|| \prod_{j \in e} \left(\varphi_j\left(t_j\right)^{-l_j} \triangle^{m^e} f(y+u^e) \right| du^e \right|^p dy \leq$$

$$\leq \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{l_j p} \left\| \int\limits_{-\infty^e}^{+\infty^e} |J_e|| \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{-l_j} \triangle^{m^e} f(y+u^e)| du^e \right\|^p_{p, G_{\varphi(t^e+T^{e'})}(\overline{x})} \leq$$

$$\leq \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{l_j p} \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{p} \| \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{-l_j} \triangle^{m^e} f \|^p_{p, G_{\varphi(t^e+T^{e'})}(\overline{x})} \leq$$

$$\leq C_1 \prod_{j \in e'} \left(\varphi_j\left(T_j\right)\right)^{\beta_j p} \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{p} \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{l_j p} \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{\beta_j p}$$

$$\times \| \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{-l_j} \triangle^{m^e} \left(\varphi(t)\right) f \|_{p, \varphi, \beta} \cdot \prod_{j \in e} \left(\varphi_j\left(t_j\right)\right)^{\beta_j p}. \tag{12}$$

For $y \in V \ \left(\varphi_j\left(t_j\right) \leq \Psi_j\left(t_j\right), j \in e_n\right)$

$$\int\limits_{U_{\psi(\xi)}} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e||\triangle^{m^e} f(x+y+u^e)| du^e \right|^p dx \leq$$

$$\leq \int\limits_{G_{\varphi(\xi)}} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e||\triangle^{m^e}(\varphi(\delta)u)f(x+u^e)|du^e \right|^p dx \leq$$

$$\leq \prod_{j\in e}(\varphi_j(t_j))^{l_j p} \int\limits_{G_{\varphi(\xi)}} \left| \int\limits_{-\infty^e}^{+\infty^e} |J_e|| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(\delta)u)f(x+u^e)|du^e \right|^p dx \leq$$

$$\leq \prod_{j\in e}(\varphi_j(t_j))^{l_j p} \| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f\|^p_{p,G_{\Psi(\xi)}(\overline{x})} \leq$$

$$\leq C_2 \prod_{j\in e}(\varphi_j(t_j))^{pl_j} \prod_{j\in e}(\varphi_j(t_j))^p \prod_{j\in e_n}(\varphi_j([\xi_j]_1))^{\beta_j p} \| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f\|^p_{p,G_{\Psi(\xi)}(\overline{x})} \leq$$

$$\leq C_1 \prod_{j\in e}(\varphi_j(t_j))^{pl_j} \prod_{j\in e}(\varphi_j(t_j))^p \prod_{j\in e_n}(\Psi_j([\xi_j]_1))^{\beta_j p} \| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f\|^p_{p,G_{\Psi(\xi)}(\overline{x})} \tag{13}$$

and

$$\int\limits_{R^n} \left| M_e^1 \left(\frac{y}{\varphi(t^e + T^{e'})} \right) \right|^s dy = \left\| M_e^1 \right\|_s \prod_{j\in e} \varphi_j(t_j) \prod_{j\in e'} \varphi_j(T_j). \tag{14}$$

From inequalities (10)-(14) it follows that

$$\|L_e\|_{q,U_{\psi(\xi)}(\overline{x})} \leq C_1 \left\| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f \right\|_{p,\varphi,\beta} \times$$

$$\times \prod_{j\in e'} (\varphi_j(T_j))^{1-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j\in e} (\varphi_j(t_j))^{1-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)+l_j} \times$$

$$\times \prod_{j\in e_n} (\psi_j([\xi_j]_1))^{\left(\frac{1}{q}-\frac{1}{r}\right)} \prod_{j\in e_n} (\psi_j([\xi_j]_1))^{\frac{\beta_j p}{q}}. \tag{15}$$

Substituting inequalities in (9) for $(r = q)$, for $\mu_j > 0$ $(j \in e)$ we obtain (7). Inequality (8) is proved in the same way.

Corollary 1. *From inequality* (7) *for* $\beta_j^1 = \frac{\beta_j p}{q}$, $j \in e_n$ *it follows that:*

$$\left\| B_\eta^e \right\|_{q,\psi,\beta^1;U} \leq C_2 \left\| \prod_{j\in e}(\varphi_j(t_j))^{-l_j} \triangle^{m^e}(\varphi(t))f \right\|_{p,\varphi,\beta;G}, \tag{16}$$

C_2 *is the constant independent of* f.

2. Main results

We prove two theorems on the properties of functions from the space $S^l_{p,\theta,\varphi,\beta}B\left(G_\varphi\right)$.

Theorem 2. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn [10], $1 \leq p \leq q \leq \infty$ and let $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \geq 0$ be entire $j \in e_n$, $\mu_j > 0$ $(j \in e_n)$, and let $f \in S^l_{p,\theta,\varphi,\beta}B(G_\varphi)$.*

Then the following embedding holds

$$D^\nu : S^l_{p,\theta_1,\varphi,\beta}B(G_\varphi) \hookrightarrow L_{q,\psi,\beta^1}(G)$$

i.e., for $f \in S^l_{p,\theta,\varphi,\beta}B\left(G_\varphi\right)$ there exists a generalized derivatives $D^\nu f$ and the following inequalities are true

$$\left\|D^\nu f\right\|_{q,G} \leq$$

$$\leq C_1 \sum_{e \subseteq e_n} \prod_{j \in e_n} (\varphi_j(T_j))^{s_{e,j}} \left\{ \int\limits_{0^e}^{t_0^e} \left[\frac{\|\triangle^{m^e}\left(\varphi(t), G_{\varphi(t)}\right) D^{k^e} f\|_{p,\alpha,\beta}}{\prod\limits_{j \in e} (\varphi_j(t_j))^{l_j - k_j}} \right]^\theta \prod_{j \in e} \frac{d\varphi_j(t_j)}{\varphi_j(t_j)} \right\}^{\frac{1}{\theta}}, \tag{17}$$

$$\left\|D^\nu f\right\|_{q,\psi^1,\beta;G} \leq C^2 \left\|f\right\|_{S^l_{p,\theta,\varphi,\beta}B(G_\varphi)}, \; p \leq q < \infty. \tag{18}$$

In particular, if

$$\mu_{j,0} = l_j - \nu_j - (1 - \beta_j p)\,\frac{1}{p} > 0, \;\; (j \in e_n),$$

then $D^\nu f\left(x\right)$ is continuous in the domain G, and

$$\sup_{x \in G} \left|D^\nu f(x)\right| \leq$$

$$\leq C^2 \sum_{e \subseteq e_n} \prod_{j \in e_n} (\varphi_j(T_j))^{s_{e,j,0}} \left\{ \int\limits_{0^e}^{t_0^e} \left[\frac{\|\triangle^{m^e}\left(\varphi(t), G_{\varphi(t)}\right) D^{k^e} f\|_{p,\alpha,\beta}}{\prod\limits_{j \in e} (\varphi_j(t_j))^{l_j - k_j}} \right]^\theta \prod_{j \in e} \frac{d\varphi_j(t_j)}{\varphi_j(t_j)} \right\}^{\frac{1}{\theta}}, \tag{19}$$

where

$$s_{e,j,0} = \begin{cases} \mu_{j,0}, & j \in e, \\ -\nu_j - (1 - \beta_j p)\frac{1}{p}, & j \in e' \end{cases}$$

$0 \leq T_j \leq \min\left\{1, t_{oj}\right\}$ $(j \in e_n)$, and C_1, C_2 are the constants indepent of f, C^1 independent of $T = (T_1, T_2, \ldots, T_n)$.

Proof. Under the conditions of our theorem, there exist generalized derivatives $D^\nu f$. Indeed, if $\mu_j > 0$, $\{j \in e_n\}$, then for $f \in S^l_{p,\theta,\varphi,\beta}B(G_\varphi) \to S^l_{p,\theta}B(G_\varphi)$ there exist generalized derivatives $D^\nu f \in L_p(G)$, and for almost each point $x \in G$ the integral representation [13]

$$D^\nu f(x) = \sum_{e \subseteq e_n} \prod_{j \in e'} (\varphi_j(T_j))^{\nu_j - 2} \int\limits_{0^e}^{T^e} \int\limits_{-\infty^e}^{+\infty^e} \int\limits_{R^n} \prod_{j \in e'} (\varphi_j(T_j))^{-\nu_j - 2}$$

$$\times M_e^{(\nu)}\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'},x\right)\right)}{\varphi\left(t^e+T^{e'}\right)}\right)J_e\left(\frac{u}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)},\frac{1}{2}\rho'(\varphi(t),x)\right)\times$$

$$\times\triangle^{m^e}(\varphi(\delta)u)f(x+y+u^e)du^e dydt \tag{20}$$

with the kernels is valid and $0\le T_j\le\min\{1,t_{j,0}\}$, $j\in e_n$, $M_e(\cdot,y)\in C_0^\infty(R^n)$, $\xi_e(\cdot,y,z)\in C_0^\infty(R^{|e|})$, where $R^{|e|}=R_1^e\times R_2^e\times R_n^e$, where $R_j^e=R=(-\infty,+\infty)$, $j\in e$; $R_j^e=1$ $j\in e'$.

Based on Minkowski inequality we have

$$\|D^\nu f\|_{q,G}\le\sum_{e\subseteq e_n}\|B_T^e\|_{q,G}\,. \tag{21}$$

By means of inequalities (7) for $U=G$, $\eta_j=T_j,(j\in e)$, $p\le\theta$ we get inequality (17).

By means are inequalities (8) for $\eta_j=T_j,(j\in e)$, and (6), $p\le\theta$ we get inequality (18).

Now let conditions $\mu_{j,0}=\mu_j(q=\infty)>0$, $(j\in e_n)$, then based around identity (20), for $q=\infty$, $p\le\theta$ we get

$$\left\|D^\nu f-f_{\varphi(T)}^{(\nu)}\right\|_{\infty,G}\le$$

$$\le C\sum_{\varnothing\ne e\subseteq e_n}\prod_{j\in e}(\varphi_j(T_j))^{s_{e,j,0}}\left\{\int\limits_{0^e}^{t_0^e}\left[\frac{\|\triangle^{m^e}(\varphi(t))\,D^{k^e}f\|_{p,\varphi,\beta}}{\prod\limits_{j\in e}(\varphi_j(t_j))^{l_j-k_j}}\right]^\theta\prod_{j\in e}\frac{d\varphi_j(t_j)}{\varphi_j(t_j)}\right\}^{\frac{1}{\theta}}.$$

As $T_j\to 0$, $j\in e$, then $\left\|D^\nu f-f_{\varphi(T)}^{(\nu)}\right\|_{\infty,G}\to 0$. Since $f_{\varphi(T)}^{(\nu)}(x)$ is continuous on G the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f(x)$ is continuous on G. Theorem 2 is proved.

Let γ be an n-dimensional vector.

Theorem 3. *Let all the conditions of Theorem 2 be satisfied. Then for $\mu_j>0$ $(j\in e_n)$ the generalized derivatives $D^\nu f$ satisfies on G the generalized Hölder condition, i.e. the following inequality is valid:*

$$\|\Delta(\gamma,G)\,D^\nu f\|_{q,G}\le C\,\|f\|_{S^l_{p,\varphi,\beta}B(G_\varphi)}\prod_{j\in e_n}(\sigma_j(|\gamma_j|)), \tag{22}$$

where

$$\sigma_j(|\gamma_j|)=\begin{cases}\max\left\{\left(\varphi_j(|\gamma_j^*|)\right)^{\mu_j},\left(\varphi_j(|\gamma_j^*|)\right)^{\mu_{j-1}}\right\}, & \text{for } j\in e,\\ (\varphi_j(T_j))^{\mu_j-l_j}, & \text{for } j\in e',\end{cases}$$

If $\mu_{j,0}>0$ $(j\in e_n)$, then

$$\sup_{x\in G}|\Delta(\gamma,G)D^\nu f(x)|\le C\|f\|_{S^l_{p,\varphi,\beta}B(G)}\prod_{j\in e_n}(\sigma_{j,0}(|\gamma_j|)). \tag{23}$$

where $\sigma_{j,0}$ satisfies the same conditions as σ_j, but with μ_j replaced $\mu_{j,0}$.

Proof. By Lemma 8.6 from [1] there exists a domain $G_\omega \subset G\,(\omega = (\omega_1, \omega_2, \dots, \omega_n)$, $\omega_j = \lambda_j \rho(x)$, $\lambda_j > 0$ $(j \in e_n)$, $\rho(x) = dist\,(x, \partial G)$, $x \in G)$.

Suppose that $|\gamma_j| < \omega_j$, $j \in e_n$, then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identity (20) with the same kernels are valid. After same transformations, from (20) we get

$$|\Delta\,(\gamma, G)\, D^\nu f\,(x)| \le C_1 \sum_{e \subseteq e_n} \prod_{j \in e'} (\varphi_j\,(T_j))^{-\nu_j - 2} \times$$

$$\int_0^{|\gamma_1^e|} \cdots \int_0^{|\gamma_n^e|} \prod_{j \in e} (\varphi_j\,(t_j))^{-\nu_j - 2} \prod_{j \in e} \frac{\varphi_j'(t_j)}{\varphi_j(t_j)} \times$$

$$\int_{-\infty^e}^{+\infty^e} \int_{R^n} \left| M_e^{(\nu)} \left(\frac{y}{\varphi\,(t^e + T^{e'})}, \frac{\rho\left(\varphi\left(t^e + T^{e'}, x\right)\right)}{\varphi\,(t^e + T^{e'})} \right) \times \right.$$

$$\left. \times J_e \left(\frac{u}{\varphi(t)}, \frac{\rho\,(\varphi\,(t)\,, x)}{\varphi\,(t)}, \frac{1}{2} \rho'\,(\varphi\,(t)\,, x) \right) \right| \times$$

$$\left| \Delta\,(\gamma, G)\, \triangle^{m^e} (\varphi(\delta) u) f(x + y + u^e) \right| du^e dy dt + \prod_{j \in e'} (\varphi_j\,(T_j))^{-\nu_j - 3} \times$$

$$\prod_{j \in e_n} |\gamma_j| \int_{|\gamma_1^e|}^{T_1^e} \cdots \int_{|\gamma_n^e|}^{T_n^e} \prod_{j \in e} (\varphi_j\,(t_j))^{-\nu_j - 3} \prod_{j \in e} \frac{\varphi_j'(t_j)}{\varphi_j(t_j)} \times$$

$$\int_{-\infty^e}^{+\infty^e} \int_{R^n} \left| M_e^{(\nu)} \left(\frac{y}{\varphi\,(t^e + T^{e'})}, \frac{\rho\left(\varphi\left(t^e + T^{e'}, x\right)\right)}{\varphi\,(t^e + T^{e'})} \right) \right| \times$$

$$\times J_e \left(\frac{u}{\varphi(t)}, \frac{\rho\,(\varphi\,(t)\,, x)}{\varphi\,(t)}, \frac{1}{2} \rho'\,(\varphi\,(t)\,, x) \right) \Bigg| \times$$

$$\int_0^1 \cdots \int_0^1 \left| \triangle^{m^e} (\varphi(\delta) u) f(x + y + u^e + \gamma v) \right| dv dy du^e dt =$$

$$= C_1 \sum_{e \subseteq e_n} \left(B_e^1(x, \gamma) + B_e^2(x, \gamma) \right), \tag{24}$$

where $|\gamma_j^e| = |\gamma|$ $(j \in e)$, $0 < T_j \le t_{0,j}$ $j \in e_n$. We also assume that $|\gamma_j| < T_j (j \in e_n)$, and consequently, $|\gamma_j| < \min\,(\omega_j, T_j) (j \in e_n)$. If $x \in G \setminus G_\omega$, then

$$\Delta\,(\gamma, G)\, D^\nu f\,(x) = 0.$$

Based around (24) we have

$$\|\Delta(\gamma,G)\,D^{\nu}f\|_{q,G}\le C^{1}\sum_{e\subseteq e_n}\Big(\left\|B_e^1(\cdot,\gamma)\right\|_{q,G_\omega}+$$

$$+\left\|B_e^2(\cdot,\gamma)\right\|_{q,G_\omega}\Big) \tag{25}$$

By means of inequality (7), for $U=G$, $\eta_j=|\gamma_j|$ $(j\in e)$ we have

$$\left\|B_e^1(\cdot,\gamma)\right\|_{q,G_\omega}\le C_1\left\|\prod_{j\in e}(\varphi_j(t_j))^{-l_j}\triangle^{m^e}(\varphi(t))f\right\|_{p,\varphi,\beta;G}\prod_{j\in e'}(\varphi_j(T_j))^{\mu_j-l_j}. \tag{26}$$

and by means of inequality (8) for $U=G$, $\eta_j=|\gamma_j|$ $(j\in e_n)$ we have

$$\left\|B_e^2(\cdot,\gamma)\right\|_{q,G_\omega}\le C_2\left\|\prod_{j\in e}(\varphi_j(t_j))^{-l_j}\triangle^{m^e}(\varphi(t))f\right\|_{p,\varphi,\beta;G}\prod_{j\in e'}\varphi_j(T_j)^{\mu_j-l_j}\times$$

$$\times\prod_{j\in e}(\varphi_j(|\gamma_j|))^{\mu_j-1}. \tag{27}$$

Now suppose that $|\gamma_j|\ge\min(\omega_j,T_j)\,,(j\in e_n)$, then

$$\|\Delta(\gamma,G)\,D^{\nu}f\|_{q,G}\le 2\,\|D^{\nu}f\|_{q,G}\le C(\omega,T)\,\|D^{\nu}f\|\prod_{j\in e_n}(\sigma_j(|\gamma_j|)).$$

Estimating for $\|D^{\nu}f\|_{q,G}$ by means of inequality (17), in this case, we again get the required inequality. Theorem 3 is proved.

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Basic Property of Eigenfunctions of the Eigenvalue Problem for Fourth-Order Ordinary Differential Equations with a Spectral Parameter Contained in ll Boundary Conditions

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Abstract. This paper considers the spectral problem for fourth-order ordinary differential equations, all boundary conditions of which contain a spectral parameter. This problem describes small bending vibrations of an Euler-Bernoulli beam in the cross sections of which a longitudinal force acts, at both ends of which follower forces act, and also loads are attached to these ends using weightless rods, which are kept in balance by elastic springs. The basis properties of the system of eigenfunctions of the problem under consideration in the space L_p, $1 < p < \infty$, are studied.

Key Words and Phrases: Euler-Bernoulli beam, spectral parameter, eigenvalue, eigenfunction, basis property

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1. Introduction

We consider the following eigenvalue problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < 1, \tag{1.1}$$

$$y''(0) - a\lambda y'(0) = 0, \tag{1.2}$$

$$Ty(0) - b\lambda y(0) = 0, \tag{1.3}$$

$$y''(1) - c\lambda y'(1) = 0, \tag{1.4}$$

$$Ty(1) - d\lambda y(1) = 0, \tag{1.5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, q is a positive absolutely continuous function on $[0, 1]$, $a,\ b,\ c,\ d$ are real constants such that $a > 0$, $b > 0$, $c > 0$ and $d < 0$.

Note that since the second half of the last century, boundary value problems for the Sturm-Liouville equations of the second and fourth orders with boundary conditions depending on the spectral parameter have been intensively studied (see [1-10, 13, 16-25]). This is due to the fact that these problems describe small longitudinal, torsional and transverse vibrations of a beam, at the ends of which either loads or inertial loads are concentrated, or tracking forces act (see [14-16, 18, 24]). For example, the spectral problem (1.1)-(1.5) we are studying arises when describing small bending vibrations of an elastic homogeneous cantilever beam, in the cross sections of which a longitudinal force acts, loads are attached to the ends using weightless rods, which are kept in balance by elastic springs, as well as both of them are subject to tracking forces [14]. It should be noted that to study this problem of mechanics, we need to study the convergence of expansions in the system of root functions of problem (1.1)-(1.5) in various function spaces.

The spectral properties of second-order Sturm-Liouville problems with a spectral parameter in boundary conditions and fourth-order Sturm-Liouville problems with a spectral parameter in boundary conditions (but not in all boundary conditions) were studied in works [1-10, 13, 16-25] (see also their bibliography).

The purpose of this article is to study the spectral properties, including the basic properties of the system of root functions in the space L_p, $1 < p < \infty$ of the spectral problem (1.1)-(1.5).

2. Some properties of solutions to the initial boundary value problem (1.1), (1.3)-(1.5)

In this section we consider the initial boundary value problem (1.1), (1.3)-(1.5). For the study of this problem we introduce the following boundary condition

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0,\ \ \alpha \in [0, \pi/2]. \tag{2.1}$$

Following the corresponding reasoning carried out in [8], we can prove the following oscillatory theorem for the eigenvalue problem (1.1), (2.1), (1.3)-(1.5).

Theorem 2.1. *For each α the spectrum of problem (1.1), (2.1), (1.3)-(1.5) consists real and simple eigenvalues forming an infinitely increasing sequence $\{\lambda_k(\alpha)\}_{k=1}^{\infty}$ such that $\lambda_1(\alpha) = 0$ and $\lambda_k(\alpha) > 0$ for $k \geq 2$. $\{\lambda_k(\alpha)\}_{k=1}^{\infty}$ such that $\lambda_1(\alpha) = 0$ and $\lambda_k(\alpha) > 0$ for $k \geq 2$. Moreover, for each $k \in \mathbb{N}$ the eigenfunction $y_{k,\alpha}(x)$ corresponding to the eigenvalue $\lambda_{k,\alpha}(x)$ and its derivative has the following oscillatory property:*

(i) the eigenfunction $y_{k,\alpha}(x)$for $k \geq 3$ has either $k-2$ or $k-1$ simple zeros in $(0, 1)$, while the function $y_{1,\alpha}(x)$ has no zeros and $y_{2,\alpha}(x)$ has one simple zero in $(0, 1)$;

(ii) the function $y'_{k,\alpha}(x)$ for $k \geq 2$ has exactly $k-2$ simple zeros in the interval $(0, 1)$.

By the maximum-minimum property of eigenvalues (see [15, Ch. 6, § 1, p. 405]), it follows from Theorem 2.1 that the following relation holds:

$$\lambda_1(\pi/2) = \lambda_1(\alpha) = \lambda_1(0) = 0 < \lambda_2(\pi/2) < \lambda_2(\alpha) < \lambda_2(0) < \lambda_3(\pi/2) < \lambda_3(\alpha) < \lambda_3(0) < \ \dots \ . \tag{2.2}$$

Theorem 2.2. *For each fixed $\lambda \in \mathbb{C}\backslash\{0\}$ there is a nontrivial solution $y(x,\lambda)$, unique up to a constant factor, of problem (1.1), (1.3)-(1.5). The function $y(x,\lambda)$ for each fixed $x \in [0,l]$ is an entire function of parameter λ, which has the following representation:*

$$y(x,\lambda) = -D_2(\lambda)\{\psi_1(x,\lambda) + d\lambda\psi_4(x,\lambda)\} + D_1(\lambda)\{\psi_2(x,\lambda) + c\lambda\psi_3(x,\lambda)\}, \tag{2.3}$$

where $\varphi_k(x,\lambda), k = \overline{1,4}$, is solutions of equation (1.1) satisfying the Cauchy conditions (normalized for $x = 1$)

$$\psi_k^{(s-1)}(1,\lambda) = \delta_{ks},\ s = 1, 2, 3,\ T\psi_k(1,\lambda) = \delta_{k4}, \tag{2.4}$$

δ_{ks} is the Kronecker delta, and

$$D_1(\lambda) = T\psi_1(0,\lambda) + d\lambda T\psi_4(0,\lambda) - b\lambda\{\psi_1(0,\lambda) + d\lambda\psi_4(0,\lambda)\},$$

and

$$D_2(\lambda) = T\psi_2(0,\lambda) + c\lambda T\psi_3(0,\lambda) - b\lambda\{\psi_2(0,\lambda) + c\lambda\psi_3(0,\lambda)\},$$

.

The proof of this theorem is similar to the proof of Theorem 2.2 of [8] with regard to Theorem 2.1.

Remark 2.1. Let $y(x,\lambda)$, $\lambda \in \mathbb{R}\backslash\{0\}$, be the nontrivial solution of the spectral problem (1.1), (1.3)-(1.5). Then this function can be normalized using the condition

$$y'(1,\lambda) = 1, \tag{2.5}$$

if $\lambda > 0$, and using the condition

$$y'(0,\lambda) = 1, \tag{2.6}$$

if $\lambda < 0$, in view of [12, Theorems 2.1 and 2.2].

By the second part of [12, Lemma 2.1] we get $D_1(\lambda) < 0$. Therefore, by Remark 2.1 and formula (2.3), without loss of generality, we can represent the function $y(x,\lambda)$ for $\lambda > 0$ in the form

$$y(x,\lambda) = -\tfrac{D_2(\lambda)}{D_1(\lambda)}\{\psi_1(x,\lambda) + d\lambda\psi_4(x,\lambda)\} + \psi_2(x,\lambda) + c\lambda\psi_3(x,\lambda). \tag{2.7}$$

Remark 2.2. If $\lambda = 0$, then according to relations (2.4) problem (1.1), (1.3)-(1.5) has two linearly independent solutions $y_1(x,0) = \psi_1(x,0) = 1$ and $y_2(x,0) = \psi_2(x,0)$, $x \in [0,1]$.

Remark 2.3. By [8, formulas (2.10) and (2.11)] (with replacement $x = 0$ by $x = 1$) from (2.4) for $y(x,\ \lambda)$ we obtain the following representation

$$y(x,\lambda) = A(\lambda)\{\psi_1(x,\lambda) + d\lambda\psi_4(x,\lambda)\} + \psi_2(x,\lambda) + c\lambda\psi_3(x,\lambda), \tag{2.8}$$

where

$$A(\lambda) = -\frac{\int\limits_0^1 \psi_2(t,\lambda)\,dt \,+\, c\lambda\int\limits_0^1 \psi_3(t,\lambda)\,dt \,+\, b\,\{\psi_2(0,\lambda) \,+\, c\lambda\psi_3(0,\lambda)\}}{\int\limits_0^1 \psi_1(t,\lambda)\,dt \,-\, d \,+\, d\lambda\int\limits_0^1 \varphi_4(t,\lambda)\,dt \,+\, b\,\{\psi_1(0,\lambda) \,+\, d\lambda\psi_4(0,\lambda)\}}. \tag{2.9}$$

Then passing to the limit as $\lambda \to 0$ in (2.8) we get

$$\lim_{\lambda\to 0} y(x,\lambda) = \psi_2(x,0) - \frac{\int\limits_0^1 \psi_2(t,0)\,dt + b\psi_2(0,0)}{1+b-d}.$$

Therefore, if we put

$$y(x,0) = \psi_2(x,0) - \frac{\int\limits_0^1 \psi_2(t,0)\,dt + b\psi_2(0,0)}{1+b-d}. \tag{2.10}$$

then the solution $y(x,\lambda)$ to problem (1.1), (1.3)-(1.5) will be defined everywhere on $[0,1]\times\mathbb{C}$.

We consider the function

$$G(\lambda) = \frac{y''(0,\lambda)}{y'(0,\lambda)}$$

which is well defined on

$$\mathcal{M} \equiv (\mathbb{C}\backslash\mathbb{R}) \cup (-\infty,\,\lambda_2(0)) \cup \left(\bigcup_{k=3}^{\infty} (\lambda_{k-1}(0), \lambda_k(0))\right).$$

It follows from Theorems 2.1 and 2.2 that $G(\lambda)$ is a meromorphic function of finite order and the eigenvalues $\lambda_k\,(\pi/2)$ and $\lambda_k(0)$, $k = 2, 3, \ldots$, of problem (1.1), (2.1), (1.3)-(1.5) for $\alpha = \pi/2$ and $\alpha = 0$ are zeros and poles of this function, respectively.

Lemma 2.1 *One has the following relations:*

$$\frac{dG(\lambda)}{d\lambda} = \frac{1}{y'^2(0,\lambda)}\left\{\int\limits_0^1 y^2(x,\lambda)\,dx + by^2(0,\lambda) + cy'^2(1,\lambda) - dy^2(1,\lambda)\right\},\ \lambda\in\mathcal{M}. \tag{2.11}$$

$$\lim_{\lambda\to-\infty} G(\lambda) = -\infty \tag{2.12}$$

The proof of formulas (2.11) and (2.12) is similar to that of [8, formula (2.19)] and [6, formula (3.8)], respectively.

Remark 2.4 By conditions $b > 0$, $c > 0$ and $d < 0$ it follows from (2.11) that

$$\frac{dG(\lambda)}{d\lambda} > 0 \text{ for } \lambda\in\mathcal{M}. \tag{2.13}$$

Remark 2.5 By (2.3), it follows from the second part of [12, Lemma 2.1] that

$$\psi_2(x,\lambda)<0,\ \psi_2'(x,\lambda)>0,\ \psi_2''(x,\lambda)<0 \text{ and } T\psi_2(x,\lambda)>0 \text{ for } x\in[0,1) \text{ and } \lambda>0.$$

Hence we have

$$\psi_2''(x,0)\le 0 \text{ and } T\psi_2(x,0)\ge 0 \text{ for } x\in[0,1]. \tag{2.14}$$

In view of (2.14) we get

$$\psi_2'''(x,0)\ge q(0)\,\psi'(x,0)>0 \text{ for } x\in[0,1].$$

By the relation $\psi_2''(1,0)=0$ it follows from last relation that $\psi_2''(0,0)<0$, and consequently, we have the following relation

$$G(0)=\frac{\psi_2''(0,0)}{\psi_2'(0,0)}<0. \tag{2.15}$$

3. The properties of eigenvalues of the eigenvalue problem (1.1)-(1.5)

Lemma 3.1 *The non-zero eigenvalues of problem (1.1)-(1.5) are real and simple.*

Proof. It is obvious that the non-zero eigenvalues of problem (1.1)-(1.5) are the roots of the equation

$$y''(0,\lambda)-a\lambda y'(0,\lambda)=0. \tag{3.1}$$

If λ is a non-real eigenvalue of problem (1.1)-(1.5), then, due to the realness of the coefficients q, a, b, c and d from (1.1)-(1.5) it follows that $\overline{\lambda}$ is also its eigenvalue. Note that in this case to the eigenvalue $\overline{\lambda}$ corresponds the eigenfunction $y(x,\overline{\lambda})=\overline{y(x,\lambda)}$, therefore (3.1) also holds for $\overline{\lambda}$.

By (1.1) for any λ, $\mu\in\mathbb{C}$ we have

$$(Ty(x,\mu))'\,y(x,\lambda)-(Ty(x,\lambda))'\,y(x,\mu)=(\mu-\lambda)y(x,\mu)y(x,\lambda). \tag{3.2}$$

Integrating equality (3.2) from 0 to 1, using the formula integration by parts to this resulting equality, and taking into account boundary conditions (1.3)-(1.5) we get

$$\begin{aligned} y''(0,\mu)\,y'(0,\lambda)-y''(0,\lambda)\,y'(0,\mu)\ &=(\mu-\lambda)\left\{\int_0^1 y(x,\mu)\,y(x,\lambda)dx+\right.\\ &\left. by(0,\mu)\,y(0,\lambda)+cy'(1,\mu)\,y'(1,\lambda)-dy(1,\mu)\,y(1,\lambda)\right\}. \end{aligned} \tag{3.3}$$

Setting $\mu=\overline{\lambda}$ in (3.3), using (3.1) and the relation $\lambda\ne\overline{\lambda}$ we obtain

$$\int_0^1 |\,y(x,\lambda)|^2dx\ -a|y'(0,\lambda)|^2+b\,|y(0,\lambda)|^2+c\,|y'(1,\lambda)|^2-d\,|y(1,\lambda)|^2=0. \tag{3.4}$$

On the other hand multiplying (1.1) by $\overline{y(x,\lambda)}$, integrating resulting equality from 0 to 1, using the formula integration by parts, and taking into account boundary conditions (1.2)-(1.5) we get

$$\int_0^1 \left\{ | \, |y''(x,\lambda)|^2 + q(x)|y'(x,\lambda)|^2 \right\} dx =$$
$$\lambda \left\{ \int_0^1 |\, y(x,\lambda)|^2 dx \, - a|y'(0,\lambda)|^2 + b\,|y(0,\lambda)|^2 + c\,|y'(1,\lambda)|^2 - d\,|y(1,\lambda)|^2 \right\}, \tag{3.5}$$

which, by (3.4), implies that

$$\int_0^1 \left\{ | \, |y''(x,\lambda)|^2 + q(x)|y'(x,\lambda)|^2 \right\} dx = 0.$$

Since q is a positive continuous function on $[0,1]$ it follows from last relation that $y'(x,\lambda) \equiv 0$, which contradicts equality (1.1). The proof of this lemma is complete.

Remark 3.1 Note that the function on the left side of the equation (3.1) is entire and, by Lemma 3.1, does not have zero values for non-real λ. Hence, this function does not vanish identically. Consequently, the zeros of this function form a countable set without a finite limit point.

Lemma 3.2 *The non-zero eigenvalues of problem (1.1)-(1.5) are simple.*

Proof. If $\lambda \neq 0$ is an eigenvalue of (1.1)-(1.5) such that $y'(0,\lambda) = 0$, then it follows from (3.1) that $y''(0,\lambda) = 0$ in contradiction with relation (2.2). Hence by (3.1) non-zero eigenvalues of problem (1.1)-(1.5) are the roots of the equation

$$G(\lambda) = a\lambda. \tag{3.6}$$

Let $\tilde{\lambda} \neq 0$ be the double eigenvalue of problem (1.1)-(1.5). Then we have

$$G(\tilde{\lambda}) = a\tilde{\lambda} \text{ and } G'(\tilde{\lambda}) = a. \tag{3.7}$$

By the second relation on (3.7) from (2.11) we obtain

$$\int_0^1 y^2(x,\tilde{\lambda})dx \, - ay'^2(0,\tilde{\lambda}) + b\,y^2(0,\tilde{\lambda}) + c\,y'^2(1,\tilde{\lambda}) - d\,y^2(1,\tilde{\lambda}) = 0. \tag{3.8}$$

Since $\tilde{\lambda}$ is real by (3.8) it follows from (3.5) that

$$\int_0^1 \left\{ y''^2(x,\tilde{\lambda}) + q(x)y'^2(x,\tilde{\lambda}) \right\} dx = 0. \tag{3.9}$$

which implies that $y'(x,\tilde{\lambda}) \equiv 0$, in contradiction with equality (1.1). The proof of this lemma is complete.

Remark 3.2 For $\lambda = 0$, the general solution to problem (1.1), (1.3)-(1.5) has the form

$$v(x) = \tau_1 + \tau_2 \psi_2(x, 0),\; x \in [0, 1].$$

Then it follows from (3.1) that $\tau_2 \psi_2''(0, 0) = 0$. Hence by Remark 2.5 we have $\psi_2''(0, 0) \neq 0$, and consequently, $\tau_2 = 0$. Therefore, $\lambda = 0$ is a simple eigenvalue of the spectral problem (1.1)-(1.5) and without loss of generality we can assume that this eigenvalue has an eigenfunction $v(x) \equiv 1$.

Lemma 3.3 *In each of the intervals* $(-\infty, 0)$*,* $(0, \lambda_2(0)$*,* $(\lambda_{k-1}(0), \lambda_k(0)),\; k = 3, 4, \dots$ *, equation (3.6) cannot have more than one solution.*

Proof. Let $\lambda^* \in (-\infty, 0)$ be a solution of problem (3.6). Then by Lemma 3.2 we have $G'(\lambda) - a \neq 0$. Since $\lambda^* \in (-\infty, 0)$ it follows from (3.5) (with $\tilde{\lambda}$ replaced by λ^*) that

$$\int_0^1 y^2(x, \lambda^*)dx - a y'^2(0, \lambda^*) + b\, y^2(0, \lambda^*) + c\, y'^2(1, \lambda^*)^2 - d\, y^2(1, \lambda^*) < 0. \tag{3.10}$$

and consequently, $G'(\lambda^*) - a < 0$. Therefore, the function $G(\lambda) - a\lambda$ except λ^* cannot have another solution in the interval $(-\infty, 0)$.

The remaining cases are considered similarly. The proof of this lemma is complete.

Theorem 3.1 *The eigenvalues of problem (1.1)-(1.5) form an infinitely increasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *such that*

$$\lambda_1 \in (-\infty, 0),\; \lambda_2 = 0,\; \lambda_3 \in (\lambda_2(\pi/2), \lambda_2(0)),\; \dots,\; \lambda_k \in (\lambda_{k-1}(\pi/2), \lambda_{k-1}(0)),\; \dots\; . \tag{3.11}$$

Proof. Following the corresponding reasoning carried out in the proof of Lemma 3.3 of [7], we can verify that for the function $G(\lambda)$ the following representation holds

$$G(\lambda) = G(0) + \sum_{k=2}^{\infty} \frac{c_k \lambda}{\lambda_k(0)(\lambda - \lambda_k(0))},\; \lambda \in \mathcal{M}, \tag{3.12}$$

where $c_k = \underset{\lambda=\lambda_k(0)}{res} G(\lambda) < 0,\; k = 2, 3, \dots$. Hence it follows from (3.12) that

$$G''(\lambda) = 2 \sum_{k=2}^{\infty} \frac{c_k}{(\lambda - \lambda_k(0))^3},\; \lambda \in \mathcal{M}. \tag{3.13}$$

By (3.13) we have $G''(\lambda) > 0$ for $\lambda \in (-\infty, \lambda_2(0))$, i.e. the function $G(\lambda)$ is convex on $(-\infty, \lambda_2(0))$.

In view of Lemma 2.1 and representation (3.12) we get the following relations:

$$\lim_{\lambda \to \lambda_k(0) - 0} G(\lambda) = +\infty,\; \lim_{\lambda \to \lambda_k(0) + 0} G(\lambda) = -\infty,\; k = 2, 3, \dots\; . \tag{3.14}$$

Since the function $G(\lambda)$ is increasing (see Remark 2.4) and convex in the interval $(-\infty, \lambda_2(0))$ and $G(0) < 0$, and the function $a\lambda$ is increasing in the same interval, the straight line $a\lambda$ intersects the graph of the function $G(\lambda)$ in the interval $(-\infty, \lambda_2(0))$ at two points, one of which lies in $(-\infty, 0)$, and the other lies in the interval $(\lambda_2(\pi/2), \lambda_2(0))$. Thus, by Remark 3.2, problem (1.1)-(1.5) in the interval $(-\infty, \lambda_2(0))$ has three simple eigenvalues λ_1, λ_2 and λ_3 such that

$$\lambda_1 \in (-\infty, 0),\ \ \lambda_2 = 0 \text{ and } \lambda_3 \in (\lambda_2(\pi/2), \lambda_2(0)).$$

Next, by relations (2.11), (3.14) and Lemma 3.3, for each $k \in \mathbb{N}$, $k \geq 3$, the straight line $a\lambda$ intersects the graph of the function $G(\lambda)$ in the interval $(\lambda_{k-1}(0), \lambda_k(0))$ at one point which lies in $(\lambda_k(\pi/2), \lambda_k(0))$. Therefore, problem (1.1)-(1.5) in the interval $(\lambda_{k-1}(0), \lambda_k(0))$, $k = 3, 4, \ldots$, has one simple eigenvalues λ_{k+1} such that

$$\lambda_{k+1} \in (\lambda_k(\pi/2), \lambda_k(0)).$$

The proof of this theorem is complete.

Theorem 3.1 *For the eigenvalues and eigenfunctions of problem the following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = (k - 7/2)\,\pi + O\,(1/k)\,, \tag{3.15}$$

$$\begin{gathered} y_k(x) = -\tfrac{c\sqrt{\lambda_k}}{2}\left\{\sin\,(k-7/2)\,\pi(x-1) + \cos\,(k-7/2)\,\pi(x-1) + \right. \\ \left. (-1)^k e^{-\,(k-7/2)\pi x} - e^{(k-7/2)\pi(x-1)} + O\,(1/k)\right\}, \end{gathered} \tag{3.16}$$

where relation (3.16) holds uniformly for $x \in [\,0, 1]$.

The proof of this theorem is similar to that of [21, Theorem 3.1] with the use of [8, Theorem 3.2] and (3.11).

4. Operator interpretation of the eigenvalue problem (1.1)-(1.5)

It is known (see, for example, [6, 8]) that the spectral problem (1.1)-(1.5) reduces to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0, 1) \oplus \mathbb{C}^4$, equipped with scalar product

$$\begin{gathered} (\hat{y}, \hat{v})_H = (\{y, m, n, \varrho, \sigma\}, \{v, s, t, \varsigma, \tau\})_H = \\ \int_0^1 y(x)\,\overline{v(x)}\,dx + |\,a|^{-1} m\bar{s} + |\,b|^{-1} n\bar{t} + |\,c|^{-1} \varrho\bar{\sigma} + |\,d|^{-1} \varsigma\bar{\tau}, \end{gathered} \tag{4.1}$$

where operator L define by

$$L\hat{y} = L\{y, m, n, \tau, \sigma\} = \{\ell(y),\ y''(0),\ Ty(0),\ y''(1),\ Ty(1)\}$$

on the domain

$$\begin{gathered} D(L) = \{\{y\ (x),\ m,\ n\} \in H\ :\ y \in W_2^4(0, 1),\ \ell(y) \in L_2(0, 1), \\ m = ay'(0), n = by(0),\ \tau = cy'(1),\ \sigma = dy(1)\}. \end{gathered}$$

which is dense everywhere in H. Then problem (1.1)-(1.5) is equivalent to the spectral problem

$$L\hat{y} = \lambda\hat{y},\ \hat{y} \in D(L),$$

i.e., the eigenvalues λ_k, $k \in \mathbb{N}$, of problem (1.1)-(1.5) and the operator L coincide and between the eigenvectors, there is a one-to-one correspondence

$$y_k(x) \leftrightarrow\ \hat{y}_k = \{y_k(x),\, m_k,\, n_k, \tau_k, \sigma_k\},\ m_k = ay_k'(0),$$

$$n_k = by_k\,(0),\ \varrho_k = cy_k'(1),\ \sigma_k = dy_k\,(1),\ k \in \mathbb{N}.$$

If $a < 0$, then L is a positive, self-adjoint and discrete operator in H, and consequently, the system of eigenvectors $\{y_k(x), m_k, n_k, \varrho_k, \sigma_k\}_{k=1}^{\infty}$ of this operator forms an orthogonal basis in H.

If $a > 0$, then L is a closed (nonself-adjoint) and discrete operator in H.

Let J be the linear operator defined in H by

$$J\{y, m, n, \tau, \sigma\} = \{y, -m, n, \tau, \sigma\}$$

Note that J is a unitary and symmetric operator in H spectrum of which consists of two eigenvalues: -1 with multiplicity 1 and $+1$ with infinite multiplicity (see [13, Lemma 2.1]). Hence this operator generates the Pontryagin space $\Pi_1 = L_2(0,1) \oplus \mathbb{C}^4$ equipped with inner product (or more precisely J-metric) [11]

$$\begin{gathered}(\hat{y}, \hat{v})_{\Pi_1} = (\hat{y}, J\hat{v})_H = (\{y, m, n, \varrho, \sigma\}, \{v, s, t, \varsigma, \tau\})_{\Pi_1} = \\ \int_0^1 y(x)\,\overline{v(x)}\,dx - a^{-1}m\bar{s} + b^{-1}n\bar{t} + c^{-1}\varrho\bar{\sigma} - d^{-1}\varsigma\bar{\tau}.\end{gathered} \tag{4.2}$$

Theorem 4.1 *L is a J-self-adjoint operator in Π_1.*

The proof of this Theorem is similar to that of [13, Theorem 2.2] with the use of [11, Propostions 1^o and 2^o].

Theorem 4.2 *If L is the adjoint operator of L in H, then $L = JLJ$. Moreover, the system of eigenvectors $\{\hat{y}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k, m_k, n_k, \varrho_k, \sigma_k\}$, of the operator L forms an unconditional basis in H.*

The first statement of this theorem follows from [11, § 3, Propostion 5] and the second statement follows from [11, § 4, Theorem 4.2].

By Theorem 3.1 we get

$$L\hat{y}_k = \lambda_k\hat{y}_k,\ k \in \mathbb{N}. \tag{4.3}$$

Let $\{\hat{v}_k^*\}_{k=1}^{\infty}$, $\hat{v}_k^* = \{v_k, s_k, t_k, \varsigma_k, \tau_k\}$, be the system of eigenvectors of operator L^*. Then, view of Theorem 3.1 and (4.3), we have

$$L^*\hat{v}_k = \lambda_k\hat{v}_k,\ k \in \mathbb{N}. \tag{4.4}$$

On the base of first part of Theorem 4.2 it follows from (4.3) and (4.4) that

$$\hat{v}_k^* = J\hat{y}_k,\ k \in \mathbb{N}. \tag{4.5}$$

By (4.1), (4.2), (4.5) and Theorem 4.1 for any k, $l \in \mathbb{N}$, $k \neq l$, we get

$$(\hat{y}_k, \hat{v}_l)_H = (\hat{y}_k, \hat{y}_l)_{\Pi_1} = 0. \tag{4.6}$$

By Lemma 3.2 and Remark 3.2 we have $G'(\lambda_k) - a \neq 0$ which, by (2.1), implies that

$$(\hat{y}_k, \hat{v}_k)_H = (\hat{y}_k, \hat{y}_k)_{\Pi_1} = \int_0^1 y_k^2(x)dx - a y_k'^2(0) + b\, y_k^2(0) + c\, y_k'^2(1)^2 - d\, y_k^2(1) \neq 0. \tag{4.7}$$

Theorem 4.3 *Let $\delta_k = (\hat{y}_k, \hat{y}_k)_{\Pi_1}$. Then each element $\hat{v}_k = \{v_k, s_k, t_k, \varsigma_k, \tau_k\}$, $k \in \mathbb{N}$, of the system $\{\hat{v}_k\}_{k=1}^{\infty}$ adjoint to the system $\{\hat{y}_k\}_{k=1}^{\infty}$ is defined as follows:*

$$\hat{v}_k = \delta_k^{-1} \hat{y}_k. \tag{4.8}$$

The proof of this theorem follows from (4.6) and (4.7).

5. Basis property of subsystems of the system of eigenfunctions of the spectral problem (1.1)-(1.5)

Let i, j, r and l be different arbitrary fixed natural numbers and

$$\Delta_{i,j,r,l} = \begin{vmatrix} s_i & t_i & \varsigma_i & \tau_i \\ s_j & t_j & \varsigma_j & \tau_j \\ s_r & t_r & \varsigma_r & \tau_r \\ s_l & t_l & \varsigma_l & \tau_l \end{vmatrix}. \tag{5.1}$$

Theorem 4.3 *If $\Delta_{i,j,r,l} \neq 0$, then the system $\{y_k(x)\}_{k=1,\, k\neq i,j,r,l}^{\infty}$ is a basis in $L_p(0,1), 1 < p < \infty$ (and even an unconditional basis for $p = 2$). If $\Delta_{i,j,r,l} = 0$, then the system $\{y_k(x)\}_{k=1,\, k\neq i,j,r,l}^{\infty}$ is neither complete nor minimal in $L_p(0,1)$, $1 < p < \infty$.*

The proof of this theorem is similar to that of [5, Theorem 4.1] with the use of (3.15) and (3.16).

By (4.8) from (5.1) we obtain

$$\Delta_{i,j,r,l} = \delta_i^{-1} \delta_j^{-1} \delta_r^{-1} \delta_l^{-1} \tilde{\Delta}_{i,j,r,l}, \tag{5.2}$$

where

$$\tilde{\Delta}_{i,j,r,l} = \begin{vmatrix} y_i'(0) & y_i(0) & y_i'(1) & y_i(1) \\ y_j'(0) & y_j(0) & y_j'(1) & y_j(1) \\ y_r'(0) & y_r(0) & y_r'(1) & y_r(1) \\ y_l'(0) & y_l(0) & y_l'(1) & y_l(1) \end{vmatrix}.$$

Since $\delta_k \neq 0$ for any $k \in \mathbb{N}$ it follows from (5.1) and (5.2) that

$$\Delta_{i,j,r,l} \neq 0 \Longleftrightarrow \tilde{\Delta}_{i,j,r,l} \neq 0. \tag{5.3}$$

By refining the asymptotic formulas for eigenvalues and eigenfunctions of problem (1.1)-(1.5) and applying Theorems 5.1, it is possible to establish sufficient conditions for the system $\{y_k(x)\}_{k=1,\,k\neq i,\,j,\,r,\,l}^{\infty}$ to form a basis in $L_p(0,1), 1<p<\infty$.

Theorem 4.3 *Let $i=2$ and $j,\ r,\ l,\ j<r<l$, be arbitrary sufficiently large fixed natural numbers, two of which are even and the third odd. Then the system $\{y_k(x)\}_{k=1,\,k\neq i,\,j,\,r,\,l}^{\infty}$ is a basis in $L_p(0,1), 1<p<\infty$, which is an unconditional basis in $L_2(0,1)$.*

Proof. Following the corresponding reasoning carried out in the proof of formulas (5.15) and (5.17) of Theorem 5.4 in [7] we obtain the following asymptotic formulas

$$\begin{gathered}\varrho_k=\left(k-\tfrac{7}{2}\right)\pi+O\left(\tfrac{1}{k}\right),\ y_k'(0)=(-1)^k\tfrac{c}{a}\left(1+O\left(\tfrac{1}{\varrho_k^2}\right)\right),\\ y_k(0)=(-1)^k\,\tfrac{c}{b}\,\varrho_k\left(1+O\left(\tfrac{1}{\varrho_k^2}\right)\right),\ y_k(1)=\tfrac{c}{d}\,\varrho_k\left(1+\tfrac{1}{d\varrho_k}+O\left(\tfrac{1}{\varrho_k^2}\right)\right),\end{gathered}\tag{5.4}$$

where $\varrho_k=\sqrt[4]{\lambda_k}$.

Let $i=2,\ j,\ l,\ r,\ j<r<l$, be arbitrary fixed sufficiently large natural numbers such that j and r be even, and l be odd. Then, by (5.4) we have

$$\tilde{\Delta}_{1,\,j,\,r,\,l}=\frac{c}{a}\begin{vmatrix}0&1&0&1\\1&\frac{c}{b}\varrho_j&1&\frac{c}{d}\varrho_j+\frac{c}{d^2}\\1&\frac{c}{b}\varrho_r&1&\frac{c}{d}\varrho_r+\frac{c}{d^2}\\-1&-\frac{c}{b}\varrho_l&1&\frac{c}{d}\varrho_l+\frac{c}{d^2}\end{vmatrix}+O\left(\frac{1}{\varrho_j}\right)=$$

$$\frac{2c}{a}\begin{vmatrix}0&1&1\\1&\frac{c}{b}\varrho_j&\frac{c}{d}\varrho_j+\frac{c}{d^2}\\1&\frac{c}{b}\varrho_r&\frac{c}{d}\varrho_r+\frac{c}{d^2}\end{vmatrix}+O\left(\frac{1}{\varrho_j}\right)=\frac{2c}{a}\begin{vmatrix}0&1&1\\0&\frac{c}{b}(\varrho_j-\varrho_r)&\frac{c}{d}(\varrho_j-\varrho_r)\\1&\frac{c}{b}\varrho_r&\frac{c}{d}\varrho_r+\frac{c}{d^2}\end{vmatrix}+$$

$$O\left(\frac{1}{\varrho_j}\right)=\frac{2c^2}{a}\left(\frac{1}{d}-\frac{1}{b}\right)(\varrho_j-\varrho_r)+O\left(\frac{1}{\varrho_j}\right)>0.$$

Hence bin view of (5.2) and (5.3) it follows from Theorem 5.1 that the system $\{y_k(x)\}_{k=1,\,k\neq i,\,j,\,r,\,l}^{\infty}$ is a basis in $L_p(0,1), 1<p<\infty$, and for $p=2$ this basis is an unconditional basis.

Other cases are considered similarly. The proof of this theorem is complete.

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Transformation Operators for one Second-Order Differential Equation with Increasing Coefficient

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Abstract. The Stark equation with a step-like perturbed potential is considered. Using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. Estimates for the kernels of the transformation operators are obtained.
Key Words and Phrases: Stark equation, transformation operator, Airy functions, triangular representation.
2010 Mathematics Subject Classifications: Primary 34A55, 34B20, 34L05

1. Introduction

The Stark effect is the shifting and splitting of spectral lines of atoms and molecules due to presence of an external electric field. The effect is named after Stark, who discovered it in 1913. The Stark effect has been of marginal benefit in the analysis of atomic spectra, but has been a major tool for molecular rotational spectra. The perturbation theory of the Stark effect is of particular interest. The application of transformation operators to the perturbation theory of linear operators is well known (see [1], [2] and the references therein). These operators arose from the general ideas of the theory of generalized shift operators created by Delsarte [3].

For arbitrary Sturm–Liouville equations, transformation operators were constructed by Povzner [4]. Marchenko [5] used transformation operators for studying inverse spectral problems and the asymptotic behavior of the spectral function of the singular Sturm–Liouville operator. Levin [6] introduced transformation operators of a new form that preserve the asymptotic expansions of solutions at infinity. Marchenko [5] used them to solve the inverse problem of scattering theory. Similar problems for the Schrodinger equation with unbounded potentials were considered in [7]- [9].

We consider the differential equation

$$-y'' + xy + p(x)\,y + q(x)\,y = \lambda y, \ -\infty < x < \infty, \quad \lambda \in C. \tag{1}$$

where real potentials $p(x)$ and $q(x)$ satisfy the conditions

$$p(x) = \begin{cases} \alpha_+, x \geq 0, \\ \alpha_-, \, x < 0, \end{cases} \tag{2}$$

$$q(x) \in C(-\infty, +\infty), \int_{-\infty}^{\infty} |xq(x)|\, dx < \infty. \tag{3}$$

In the present paper, using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. The results obtained can be used to solve inverse spectral problems for an equation (1). Some questions of the spectral theory of the one-dimensional Stark equation were studied in [10]-[13].

2. The transformation operators

In what follows, we deal with special functions satisfying the Airy equation

$$-y'' + zy = 0.$$

It is well known (e.g., see [14]) that this equation has two linearly independent solutions $Ai(z)$ and $Bi(z)$ with the initial conditions

$$Ai(0) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)},\ Ai'(0) = \frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)},$$
$$Bi(0) = \frac{1}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)},\ Bi'(0) = \frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)}.$$

The Wronskian $\{Ai(z), Bi(z)\}$ of these functions satisfies

$$\{Ai(z), Bi(z)\} = Ai(z)\, Bi'(z) - Ai'(z)\, Bi(z) = \pi^{-1}.$$

Both functions are entire functions of order $\frac{3}{2}$ and type $\frac{2}{3}$. Note that the functions $Ai(x-\lambda)$, $Ai(x-\lambda) - iBi(x-\lambda)$ satisfy the relations (see [2]) $Ai(x-\lambda) \in L_2(0, +\infty)$, $Ai(x-\lambda) - iBi(x-\lambda) \in L_2(-\infty, 0)$ for $Im\lambda \geq 0$.
In what follows we will need special solutions of the unperturbed equation

$$-y'' + xy + p(x)\, y = \lambda y,\ -\infty < x < \infty, \quad \lambda \in C. \tag{4}$$

Lemma 1. *For any λ from the complex plane, equation (4) has solutions $\psi_\pm(x,\lambda)$ in the form*

$$\psi_+(x,\lambda) =$$

$$= \begin{cases} Ai(x+\alpha_+ - \lambda), x \geq 0, \\ \pi\left[Ai(\alpha_+ - \lambda)\, Bi'(\alpha_- - \lambda) - Ai'(\alpha_+ - \lambda)\, Bi(\alpha_- - \lambda)\right] Ai(x + \alpha_- - \lambda) + \\ +\pi\left[Ai(\alpha_- - \lambda)\, Ai'(\alpha_+ - \lambda) - Ai(\alpha_+ - \lambda)\, Ai'(\alpha_- - \lambda)\right] Bi(x+\alpha_- - \lambda), x < 0, \end{cases} \tag{5}$$

$$\psi_-(x,\lambda)=\begin{cases}\pi\left\{Bi'(\alpha_+-\lambda)\left[Ai(\alpha_--\lambda)-iBi(\alpha_--\lambda)\right]-\right.\\ \left.Bi(\alpha_+-\lambda)\left[Ai'(\alpha_--\lambda)-iBi'(\alpha_--\lambda)\right]\right\}Ai(x+\alpha_+-\lambda)+\\ \pi\left\{Ai(\alpha_+-\lambda)\left[Ai(\alpha_--\lambda)-iBi(\alpha_--\lambda)\right]-\right.\\ \left.Ai'(\alpha_+-\lambda)\left[Ai'(\alpha_--\lambda)-iBi'(\alpha_--\lambda)\right]\right\}Bi(x+\alpha_+-\lambda),\ x\geq 0\\ Ai(x+\alpha_--\lambda)-iBi(x+\alpha_--\lambda),\ x<0.\end{cases} \tag{6}$$

Proof. Obviously, when $x\geq 0$ one of the solutions of equation (4) is function $Ai(x+\alpha_+-\lambda)$. On the other hand, for $x\leq 0$ any solution of equation (4) can be represented as

$$CAi(x+\alpha_--\lambda)+DBi(x+\alpha_--\lambda).$$

If we glue these solutions at a point $x=0$, we get

$$C=\pi\left[Ai(\alpha_+-\lambda)Bi'(\alpha_--\lambda)-Ai'(\alpha_+-\lambda)Bi(\alpha_--\lambda)\right],$$

$$D=\pi\left[Ai(\alpha_--\lambda)Ai'(\alpha_+-\lambda)-Ai(\alpha_+-\lambda)Ai'(\alpha_--\lambda)\right].$$

Thus, formula (5) is established. Formula (6) is derived similarly.

The lemma is proved.

We shall use the following notation

$$\sigma_\pm(x)=\pm\int_x^{\pm\infty}\left|p(t)-\alpha_\pm+q(t)\right|dt.$$

In the following theorem the representation of solution from the equation (1) is found by means of transformation operator.

Theorem 1. *If the potentials $p(x)$ and $q(x)$ satisfy the conditions (2),* (3) *then for any λ from the closed upper half-plane equation* (1) *has a solution $f_+(x,\lambda)$ that can be represented in the form*

$$f_+(x,\lambda)=\psi_+(x,\lambda)+\int_x^\infty K_+(x,t)\ \psi_+(t,\lambda)\,dt, \tag{7}$$

where kernel $K_+(x,t)$ is continuous function and satisfies relations

$$K_+(x,t)=O\left(\sigma_+\left(\frac{x+t}{2}\right)\right),x+t\to\infty,K_+(x,x)=\frac{1}{2}\int_x^\infty\left[p(t)-\alpha_++q\ (t)\right]\,dt. \tag{8}$$

Proof. We rewrite the perturbed equation (1) in the form

$$-y''+xy+Q(x)y=(\lambda-\alpha_+)y,\ -\infty<x<\infty. \tag{9}$$

where $Q(x) = p(x) - \alpha_+ + q(x)$. Obviously, the $Q(x)$ function for all $x > a, a > -\infty$ satisfies the condition

$$Q(x) \in C(-\infty, +\infty), \int_a^\infty |xQ(x)|\, dx < \infty. \tag{10}$$

Let $f_+(x,\lambda)$ be solution of equation (10) with the asymptotic behavior $f_+(x,\lambda) = f_0(x,\lambda)(1 + o(1))$, $x \to +\infty$, where $f_0(x,\lambda) = Ai(x + \alpha_+ - \lambda)$. Subject to the conditions (11), such solution exist, is determined uniquely by its asymptotic behavior. With the aid of operator transformations, we have the representation

$$f_+(x,\lambda) = f_0(x,\lambda) + \int_x^\infty K(x,t)\; f_0(t,\lambda)\, dt, \tag{11}$$

Moreover, the kernel $K(x,t)$ is a continuous function and satisfies the following relations

$$K(x,t) = O\left(\sigma_+\left(\frac{x+t}{2}\right)\right), x + t \to \infty, \tag{12}$$

$$K(x,x) = \frac{1}{2}\int_x^\infty Q\;(t)\; dt. \tag{13}$$

In addition, rewriting the unperturbed equation (4) in the form

$$-y'' + xy + Q_0(x)\, y = (\lambda - \alpha_+)\, y,\; -\infty < x < \infty.$$

where $Q_0(x) = p(x) - \alpha_+$, we obtain

$$\psi_+(x,\lambda) = f_0(x,\lambda) + \int_x^\infty K_0(x,t)\; f_0(t,\lambda)\, dt\,. \tag{14}$$

Moreover, in this case, $K_0(x,t)$ satisfies the identity$K_0(x,t) \equiv, x \geq 0$. From the well-known properties of the transformation operators it follows that (see [5]) the function $f_0(x,\lambda)$ also admits the representation

$$f_0(x,\lambda) =\; \psi_+(x,\lambda) + \int_x^\infty \tilde{K}_0(x,t)\;\; \psi_+(t,\lambda)\, dt\,, \tag{15}$$

where the kernels$K_0(x,t)\,, \tilde{K}_0(x,t)$ are connected by the equality

$$K_0(x,t) + \tilde{K}_0(x,t) + \int_x^t \tilde{K}_0(x,u)\;\; K_0(u,t)\, du = 0. \tag{16}$$

Substituting the expression (16) from the $f_0(x,\lambda)$ in (12), we get

$$\begin{aligned} f_+(x,\lambda) &= \psi_+(x,\lambda) + \textstyle\int_x^\infty K(x,t)\left[\psi_+(t,\lambda) + \int_t^\infty \tilde{K}_0(t,u)\,\psi_+(u,\lambda)\, du\right]\; dt = \\ &= \psi_+(x,\lambda) + \textstyle\int_x^\infty K(x,t)\,\psi_+(t,\lambda)\, dt + \int_x^\infty K(x,t)\int_t^\infty \tilde{K}_0(t,u)\,\psi_+(u,\lambda)\, dudt = \\ &= \psi_+(x,\lambda) + \textstyle\int_x^\infty K(x,t)\,\psi_+(t,\lambda)\, dt + \int_x^\infty \left(\int_x^t K(x,u)\,\tilde{K}_0(u,t)\, du\right)\psi_+(t,\lambda)\, dt. \end{aligned}$$

Setting

$$K_{+}(x,t) = K(x,t) + \int_x^t K(x,u) \ \ \tilde{K}_0(u,t)\, du, \tag{17}$$

one can recast the last relation in the form

$$f_{+}(x,\lambda) = \psi_{+}(x,\lambda) + \int_x^{\infty} K_{+}(x,t) \ \ \psi_{+}(t,\lambda)\, dt.$$

Formula (8) is a straightforward consequence of (13), (17) . Taking $t = x$ in the equality (17), we find that$K_{+}(x,t) = K(x,t)$. Whence, by virtue of (15), formula (9) follows.

The theorem is proved.

The following theorem is proved in a similar way.

Theorem 2. *If the potentials* $p(x)$ *and* $q(x)$ *satisfy the conditions (2), (3), then, for any* λ *from the closed upper half-plane, equation* (1) *has a solution* $f_{-}(x,\lambda)$ *representable as*

$$f_{-}(x,\lambda) = \psi_{-}(x,\lambda) + \int_{-\infty}^{x} K_{-}(x,t) \ \ \psi_{-}(t,\lambda)\, dt.$$

where the kernel $K_{-}(x,t)$ *is continuous function and satisfy the following conditions*

$$K_{-}(x,t) = O\left(\sigma_{-}\left(\frac{x+t}{2}\right)\right), x+t \to -\infty, K_{-}(x,x) = \frac{1}{2}\int_{-\infty}^{x} \left[p(t) - \alpha_{-} + q\ (t)\right]\, dt.$$

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Problem Statement about the Unsymmetrical Oscillations of a Cylindrical cover Reinforced with rods Subjected to a Compressive Force Along the Axis with a Fluid

O. SH. Salmanov

Abstract. In the article, a physical and mathematical model was built to study the issue of asymmetric oscillations of a cylindrical cover reinforced with rods under the influence of a compressive force in the direction of the axis, together with a liquid. The frequency equation of the system was established and calculated by the asymptotic method. effect has been studied. Problem statement about the unsymmetrical oscillations of a cylindrical cover reinforced with rods subjected to a compressive force along the axis with a fluid.

Key Words and Phrases: compressive force, reinforced with rods, cylindrical cover, dances.
2010 Mathematics Subject Classifications: Primary 34A55, 34B20, 34L05

1. Introduction

A rod-reinforced cylindrical cover is a structural element consisting of a combination of cover and rods that deforms together. It is assumed that the cover and the rods interact along a certain line and the conditions of equality of displacements on their contact line are satisfied. This method was used in [1] to obtain the equilibrium equations and natural boundary conditions of a cylindrical cover reinforced with rods. Equilibrium equations for a cover reinforced with rods were obtained in [2]. The system of equilibrium equations of a cover with rods in an arbitrary position on its surface was obtained in works [3].

2. Problem solving method

The system we studied consists of a cylindrical cover reinforced with rods and a liquid that completely fills its interior. Therefore, in order to study the oscillations of such a system, we will use the system of equations of motion of a cylindrical cover reinforced with rods and the contact conditions added to them.

A rod-reinforced cylindrical cover means a cylindrical cover and a system consisting of rods rigidly attached to it along the coordinate lines (Fig. 1). It is assumed that the coordinate axes coincide with the main curvature lines of the coating, and the rods are in rigid contact with the coating along these lines. Using the Ostrogradsky-Hamilton variation principle, the system of basic equations of the mentioned system can be deduced. It is considered that the stress-strain state of the cylindrical cover is completely determined by the equations of the linear theory of covers based on the Kirchhoff-Liav hypothesis. In the calculation of rods, equations based on the Kirchhoff-Klebsch theory are used for straight-axis rods. If the cylindrical coating is under the influence of compressive stress along the axis, its potential energy is determined as follows [2]:

$$\Psi=\frac{Eh}{2\left(1-\nu^2\right)}\int_0^{\xi_1}\int_0^{2\pi}\left\{\left(\frac{\partial u}{\partial \xi}+\frac{\partial v}{\partial \theta}-w\right)^2+2\left(1-\nu\right)\left[\frac{\partial u}{\partial \xi}\left(\frac{\partial v}{\partial \theta}-w\right)-\right.\right.$$

$$\left.\left.-\frac{1}{4}\left(\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial \xi}\right)^2\right]\right\}d\xi d\theta+\frac{Eh^3}{24\left(1-\nu^2\right)R^2}\int_0^{\xi_1}\int_0^{2\pi}\left\{\left(\frac{\partial^2 w}{\partial \xi^2}+\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\right)^2-\right.$$

$$\left.-2\left(1-\nu\right)\left[\frac{\partial^2 w}{\partial \xi^2}\left(\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\right)-\left(\frac{\partial^2 w}{\partial \xi\partial \theta}+\frac{\partial v}{\partial \xi}\right)^2\right]\right\}d\xi d\theta+$$

$$+\frac{E_s}{2\mathrm{R}}\sum_{j=1}^{k_1}\int_0^{2\pi}\left[F_s\left(\frac{\partial v}{\partial \theta}-w-\frac{h_s}{R}\frac{\partial^2 w}{\partial \theta^2}\right)^2+\frac{I_{xs}}{R^2}\left(\frac{\partial^2 w}{\partial \theta^2}+w\right)^2+\frac{G_s}{R^2E_s}I_{kp.s}\times\right. \tag{1}$$

$$\left.\times\left(\frac{\partial^2 w}{\partial \xi\partial \theta}+\frac{\partial u}{\partial \theta}\right)^2\right]\Bigg|_{\xi=\xi_j}d\theta-\frac{\sigma_x h}{2}\int_0^{\xi_1}\int_0^{2\pi}\left(\frac{\partial w}{\partial \xi}\right)^2 d\xi d\theta-\frac{\sigma_x F_c}{2R}\sum_{i=1}^{k}\int_0^{\xi_1}\left(\frac{\partial w}{\partial \xi}\right)^2\Bigg|_{\theta=\theta_i}d\xi+$$

$$+\frac{E_c}{2R}\sum_{i=1}^{k_2}\int_0^{\xi_1}\left[F_c\left(\frac{\partial u}{\partial \xi}-\frac{h_c}{R}\frac{\partial^2 w}{\partial \xi^2}\right)^2+\frac{I_{us}}{R^2}\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2+\frac{G_c}{E_c}I_{kp.s}\left(\frac{\partial^2 w}{\partial \xi\partial \theta}+\frac{\partial v}{\partial \xi}\right)^2\right]\Bigg|_{\theta=\theta_i}d\xi\ .$$

In his expressions u, v, w - cover displacements, $E, \nu-$ respectively, the modulus of elasticity and Poisson's ratio of the material of the cylindrical coating, R, h- respectively, the radius and thickness of the cylindrical coating, E_c, E_s- the modulus of elasticity of the longitudinal bar and the ring, respectively, F_c, F_s - the cross-sectional areas of the longitudinal bar and the ring, respectively, I_{us}, $I_{kp.s}$ - moments of inertia of the cross section of the longitudinal bar, $I_{xs}, I_{kp.s}$ - moments of inertia of the cross section of the ring, q_x, q_θ, q_r - components of the pressure force acting on the cylindrical cover by the medium, G_c, G_s - are the shear modulus of the longitudinal bar and the ring, respectively.

$$K=\frac{Eh}{2(1-\nu^2)}\int_0^{\xi_1}\int_0^{2\pi}\left[\left(\frac{\partial u}{\partial t_1}\right)^2+\left(\frac{\partial v}{\partial t_1}\right)^2+\left(\frac{\partial w}{\partial t_1}\right)^2\right]d\xi d\theta+$$

$$+\frac{\bar{\rho}_c E_c F_c}{2R\left(1-\nu^2\right)}\sum_{i=1}^{k_2}\int_0^{\xi_1}\left[\left(\frac{\partial u}{\partial t_1}\right)^2+\left(\frac{\partial w}{\partial t_1}\right)^2\right]\Bigg|_{\theta=\theta_j}d\xi+$$

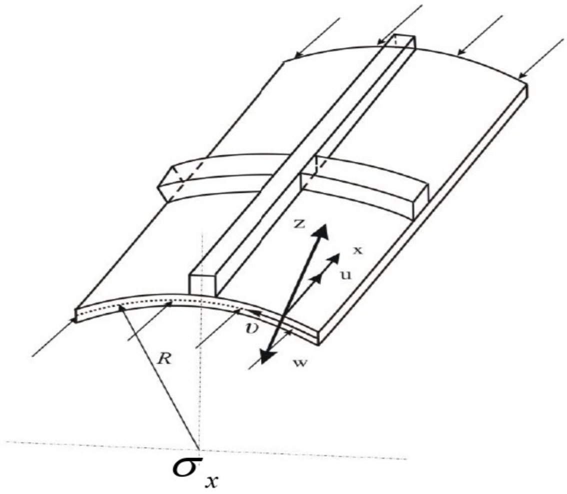

Fig 1. Cylindrical cover reinforced with rods subjected to compressive force and in contact with the environment. The kinetic energy of the rod-reinforced coating is as follo

$$+\frac{\bar{\rho}_s E_s F_s}{2R\left(1-\nu^2\right)} \sum_{i=1}^{k_1} \int_0^{2\pi} \left[\left(\frac{\partial v}{\partial t_1}\right)^2 + \left(\frac{\partial w}{\partial t_1}\right)^2\right]\Bigg|_{\xi=\xi_j} d\theta \tag{2}$$

Using the decision condition of the Ostrogradsky-Hamilton effect, the equation of motion of the cover reinforced with rods can be obtained:

$$\delta W = \delta\left(\Psi + K\right) = 0 \tag{3}$$

Expressions of potential and kinetic energy (1) and (2)- is also shown. Here $W = \int_{t_0}^{t_1} \tilde{L} dt$ Hamilton effect, $\tilde{L} = K-$. It is a lag function. (3) if we carry out the operation of taking variations in the equation and δu, δv, δw If we take into account that the variation is arbitrary and independent, we get the following system of equations of motion:

$$\begin{cases} \mathrm{L_x}(u,v,w) + q_x = 0 \\ \mathrm{L_y}(u,v,w) + q_y = 0 \\ \mathrm{L_z}(u,v,w) - (q_z - q_{zz}) = 0 \end{cases} \tag{4}$$

The propagation of small excitations in an ideal fluid is expressed by the following equation.

$$\nabla^2\Phi - \frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2} = 0. \tag{5}$$

Here Φ- potential of liq, a- is the speed of sound propagation in a liquid. In a harmonic dance (5) converts the equation to the Helmolts equation:

$$\nabla^2\Phi + \frac{\omega^2}{a^2}\Phi = 0. \tag{6}$$

When the fluid is incompressible $a^2 \to \infty$ since (6) transforms the equation into Laplace's equation:

$$\nabla^2\Phi = 0. \tag{7}$$

If the liquid is an ideal liquid with two-phase bubbles, the propagation of small perturbations in such a liquid is given by the following equation [2]:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\rho_{j>}R}{Eh}\frac{\partial^2 p}{\partial t^2} = 0. \tag{8}$$

Here, $a^2 = \frac{1}{a_{20}(1-a_{20})}\left(\frac{\rho_{10}}{\rho_{10}-\rho_{20}}\right)\frac{p_0}{p_{10}^0}$, ρ_{10}, ρ_{20}- true density of liquid and gas; p_0- static pressure; ρ_{jo}- density of the mixture; α_{20}- volume of gas bubbles; The equilibrium values of the parameters correspond to the zero index;

$$\rho_{j>} = (1-\alpha_{20})\rho_{10} + \alpha_2\rho_{20}.$$

Contact conditions are also added to the system of equations of motion of the coating (3), equations of motion of the fluid (4), (6). The normal components of velocity and pressure on the contact surface of the coating with the liquid are assumed to be equal, and the tangential stresses are equal to zero:

$$\vartheta_r = \frac{\partial w}{\partial t}, \quad q_z = -p, \quad q_x = q_y = 0 \quad (r = R) \tag{9}$$

Here q_x, q_y, q_z are the components of the pressure force exerted by the fluid on the coating. The systems of equations of motion of the rod-reinforced coating and liquid (4)-(8) together with the contact conditions (9) allow solving the problem of free oscillations of the constructive-orthotropic coating-liquid system. In other words, the study of the free oscillations of an orthotropic cylindrical coating in contact with a solid medium and a liquid is brought to the joint integration of the system of equations of the constructive-orthotropic coating and the equation of motion of the liquid within the contact conditions.

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