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On a Initial-Boundary Value Problem for Fourth-Order Partial Differential Equations with Non-Classical Boundary Conditions

Maryam H. Jafarova

Abstract. In this paper we study some initial-boundary value problem for partial differential equation of fourth order subject the nonclassical boundary conditions. We show the existence, uniqueness and stability of the classical solution of this problem.

Key Words and Phrases: initial-boundary value problem, classical solution, nonclassical boundary conditions, Fourier method

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1. Introduction

Let $T \in \mathbb{R}$ be the positive constant and $D_T = \{(x,t) \in \mathbb{R}^2 : 0 < x < 1, 0 < t < T\}$.

We consider the following initial-boundary value problem for partial differential equation

$$(p\,(x)u_{x,x}(x,t))_{x,x} - (q(x)u_x(x,t))_x + r(x)u_{tt}(x,t) = f(x,t),\ (x,t) \in D_T, \tag{1}$$

subject the non-local conditions

$$u(x,0) + \delta_1 u(x,T) = \varphi(x),\ \ u_t(x,0) + \delta_2 u(x,T) = \psi(x),\ \ 0 \le x \le 1, \tag{2}$$

and non-classical boundary conditions

$$u(0,t) = \mu_1(t),\ \ 0 \le t \le T, \tag{3}$$

$$u_{xx}(0,t) = \mu_2(t),\ \ 0 \le t \le T, \tag{4}$$

$$p\,(1)u_{xx}(1,t) + u_x(1,t) = \mu_3(t),\ \ 0 \le t \le T, \tag{5}$$

$$(p\,(x)u_{xx}(x,t))_x|_{x=1} - q(1)\,u_x(1,t) - r(1)u_{tt}(1,t) = \mu_4(t),\ \ 0 \le t \le T, \tag{6}$$

where δ_i, $i = 1, 2$, are nonnegative constants, $p \in C^2([0,1];(0,+\infty))$, $q \in C^1([0,1];(0,+\infty))$, $r \in C([0,1];(0,+\infty))$, $\mu_i \in C^1([0,T];\mathbb{R})$, $i = 1, 2, 3, 4$, $\varphi, \psi \in C^4([0,1];\mathbb{R})$, $f \in C^{0,1}(D_T)$ is a given function and $u(x,t)$ is the desired function. Moreover, the following conditions hold:

$$\mu_1(0) + \delta_1\mu_1(T) = \varphi(0),\ \mu_1'(0) + \delta_2\mu_1'(T) = \psi(0),$$
$$\mu_2(0) + \delta_1\mu_2(T) = \varphi''(0),\ \mu_2'(0) + \delta_2\mu_2'(T) = \psi''(0),$$
$$\mu_3(0) + \delta_1\mu_3(T) = p\,(1)\varphi''(1) + \varphi'(1),\ \mu_3'(0) + \delta_2\mu_3'(T) = p\,(1)\psi''(1) + \psi'(1),$$
$$\mu_4(0) + \delta_1\mu_4(T) = (p\,(x)\varphi''(x))'\,|_{x=1} - q(1)\varphi'(1) - (f(1,0) + \delta_1 f(1,T)) +$$
$$\left((p\,(x)\varphi''(x))'' - (q(x)\varphi'(x))'\,|_{x=1}\right.,$$
$$\mu_4'(0) + \delta_2\mu_4'(T) = (p\,(x)\psi''(x))'\,|_{x=1} - q(1)\psi'(1) + (f_t(1,0) + \ \delta_2 f_t(1,T)) +$$
$$+\left((p\,(x)\psi''(x))'' - (q(x)\psi'(x))'\,|_{x=1}\right..$$

Problem (1)-(6) describes the small bending vibrations of a non-homogeneous rod, the left end of which is elastically fixed, and at the right end the mass is concentrated (see, for example, [6, 9]).

For studying the classical solution of boundary value problems and initial-boundary value problems for partial differential equations one of the main methods is the Fourier method. The justification of this method is traditionally based on the uniform convergence of the series representing the formal solution of the problem and the series obtained by its term-by-term differentiation the required number of times (see, for example, [3-7, 9, 10, 12-14]). Uniform convergence of the series representing the formal solution of the problem and obtained from it by term-by-term differentiation is proved using the basic properties of the corresponding spectral problems.

In this work, using the Fourier method, we prove the existence of a classical solution to problem (1)-(6), and also prove the uniqueness and stability of this solution.

2. Uniqueness of the solution of the initial-boundary value problem (1)-(6)

In this section, we prove the uniqueness of the classical solution to the initial-boundary value problem (1)-(6).

Let

$$C^{4,2}(\overline{D}_T) = \left\{u(x,t) : u(x,t) \in C^2(\overline{D}_T),\ u_{xxxx}(x,t) \in C(\overline{D}_T)\right\}.$$

The classical solution of problem (1)-(6) is called the function $u(x,t) \in C^{4,2}(\overline{D}_T)$ satisfying equation (1) in D_T, conditions (2) in $[0,1]$ and conditions (3)-(6) in $[0,T]$ in the usual sense (see, e.g., [9, 10]).

Theorem 1. *Suppose that* $\delta_1^2 + \delta_2^2 < 1$. *Then problem* (1)-(6) *cannot have more than one classical solution, i.e. if this problem has a classical solution* $u(x,t)$, *then it is unique.*

Proof. Suppose that there are two classical solutions $u_1(x,t)$ and $u_2(x,t)$ of problem (1)-(6) and let

$$v(x,t) = u_1(x,t) - u_2(x,t),\ (x,t) \in \overline{D_T}.$$

Obviously, the function $v(x,t)$, satisfies the following homogeneous equation

$$(p(x)v_{xx}(x,t))_{xx} - (q(x)v_x(x,t))_x + r(x)v_{tt}(x,t) = 0,\ (x,t) \in D_T, \tag{7}$$

and conditions

$$v(x,0) + \delta_1 v(x,T) = 0,\ v_t(x,0) + \delta_2 v_t(x,T) = 0,\ 0 \le x \le 1, \tag{8}$$

$$v(0,t) = 0,\ 0 \le t \le T, \tag{9}$$

$$v_{xx}(0,t) = 0,\ 0 \le t \le T, \tag{10}$$

$$p\,(1)u_{xx}(1,t) + u_x(1,t) = 0,\ 0 \le t \le T, \tag{11}$$

$$(p(x)v_{xx}(x,t))_x\,|_{x=1} - q(1)v_x(1,t) - r(1)v_{tt}(1,t) = 0,\ 0 \le t \le T. \tag{12}$$

Multiplying (7) by the function $2v_t(x,t)$ and integrating the resulting equality in the range from 0 to 1, we obtain

$$\begin{gathered}2\int_0^1 (p\,(x)v_{xx}(x,t))_{xx}v_t(x,t)dx - 2\int_0^1 (q(x)v_x(x,t))_x v_t(x,t)dx + \\ 2\int_0^1 r(x)v_{tt}(x,t)v_t(x,t)dx = 0.\end{gathered} \tag{13}$$

Note that

$$2\int_0^1 r(x)v_{tt}(x,t)v_t(x,t)dx = \frac{d}{dt}\int_0^1 r(x)v_t^2(x,t)dx,\ 0 \le t \le T. \tag{14}$$

Using the formula for the integration by parts and taking into account conditions (9)-(12) we get the following relations

$$\begin{gathered}2\int_0^1 (p\,(x)v_{xx}(x,t))_{xx}v_t(x,t)dx = 2(p(x)v_{xx}(x,t))_x\,|_{x=1}\,v_t(1,t) - \\ 2(p\,(x)v_{xx}(x,t))_x\,|_{x=0}\,v_t(0,t) - 2\int_0^1 (p\,(x)v_{xx}(x,t))_x v_{tx}(x,t)dx = \\ 2(p\,(x)v_{xx}(x,t))_x\,|_{x=1}\,v_t(1,t) - 2\int_0^1 (p\,(x)v_{xx}(x,t))_x v_{tx}(x,t)dx = \\ 2(p\,(x)v_{xx}(x,t))_x\,|_{x=1}\,v_t(1,t) - 2p\,(1)v_{xx}(1,t)v_{tx}(x,1) + \ 2p\,(0)v_{xx}(0,t)v_{tx}(0,t) + \\ 2\int_0^1 p(x)v_{xx}(x,t)v_{txx}(x,t)dx = 2(p\,(x)v_{xx}(x,t))_x\,|_{x=1}\,v_t(1,t) - \\ 2p\,(1)v_{xx}(1,t)v_{tx}(x,1) + \tfrac{d}{dt}\int_0^1 p\,(x)v_{xx}^2(x,t)dx, 0 \le t \le T.\end{gathered} \tag{15}$$

$$2\int\limits_0^1 (q(x)v_x(x,t))_x v_t(x,t)dx = 2q(1)v_x(1,t)v_t(1,t) - 2q(0)v_x(0,t)v_t(0,t)- \\ -2\int\limits_0^1 q(x)v_x(x,t)v_{tx}(x,t)dx = 2q(1)v_x(1,t)v_t(1,t) - \frac{d}{dt}\int\limits_0^1 q(x)v_x^2(x,t)dx, \tag{16}$$

Then by (14)-(16) it follows from (13) that

$$\frac{d}{dt}\int\limits_0^1 p(x)v_{xx}^2(x,t)dx + \frac{d}{dt}\int\limits_0^1 q(x)v_x^2(x,t)dx + \frac{d}{dt}\int\limits_0^1 r(x)v_t^2(x,t)dx - \\ 2q(1)v_{xx}(1,t)v_{tx}(1,t) + 2((p(x)v_{xx}(x,t))_x - q(x)v_x(x,t))\,|_{x=1}\, v_t(1,t) = 0,$$

$$\frac{d}{dt}\int\limits_0^1 p(x)v_{xx}^2(x,t)dx + \frac{d}{dt}\int\limits_0^1 q(x)v_x^2(x,t)dx + \frac{d}{dt}\int\limits_0^1 r(x)v_t^2(x,t)dx + \\ 2v_x(1,t)v_{tx}(1,t) + 2r(1)v_{tt}(1,t)v_t(1,t) = 0,\ \ 0 \le t \le T.$$

which implies that

$$\frac{d}{dt}\left(\int\limits_0^1 \left(p(x)v_{xx}^2(x,t) + q(x)v_x^2(x,t) + r(x)v_t^2(x,t)\right)dx + v_x^2(1,t) + r(1)v_t^2(1,t)\right) = 0. \tag{17}$$

Let

$$z(t) = \int\limits_0^1 \left(p(x)v_{xx}^2(x,t) + q(x)v_x^2(x,t) + r(x)v_t^2(x,t)\right)dx + \\ v_x^2(1,t) + r(1)v_t^2(1,t),\ \ 0 \le t \le T. \tag{18}$$

then it follows from (17) that

$$z'(t) = 0,\ \ t \in [0,T],$$

and consequently,

$$z(t) = C,\ \ t \in [0,T] \tag{19}$$

where C is some positive constant.

By (8) we get

$$z(0) - (\delta_1^2 + \delta_2^2)z(T) = \int\limits_0^1 r(x)(v_t^2(x,0) - (\delta_1^2+\delta_2^2)v_t^2(x,T))dx + \\ \int\limits_0^1 q(x)(v_x^2(x,0) - (\delta_1^2+\delta_2^2)v_x^2(x,T))dx + \int\limits_0^1 p(x)(v_{xx}^2(x,0) - (\delta_1^2+\delta_2^2)v_{xx}^2(x,T))dx +$$

$$r(1)(v_t^2(1,0) - (\delta_1^2+\delta_2^2)v_t^2(1,T)) + v_x^2(1,0) - (\delta_1^2+\delta_2^2)v_x^2(1,T) =$$

$$-\delta_1^2\int\limits_0^1 r(x)v_t^2(x,T))dx - \delta_1^2\int\limits_0^1 q(x)v_x^2(x,T))dx - \delta_2^2\int\limits_0^1 p(x)v_{xx}^2(x,T))dx -$$

$$-\,r(1)\delta_1^2 v_t^2(1,T) - \delta_2^2 v_x^2(1,T) = C(1 - (\delta_1^2+\delta_2^2)) \le 0.$$

whence, by relations $\delta_1^2+\delta_2^2<1$ and $C\geq 0$, implies that $C=0$. Then in view of (19), by (18), we obtain

$$\int\limits_0^1 \left(p\,(x)v_{xx}^2(x,t)+q(x)v_x^2(x,t)+r(x)v_t^2(x,t)\right)dx+v_x^2(1,t)+r(1)v_t^2(1,t)\equiv 0.$$

Therefore, it follows from last relation that

$$v_t(x,t)\equiv 0,\ \ v_x(x,t)\equiv 0,\ \ v_{xx}(x,t)\equiv 0,$$

and consequently,

$$v_t(x,t)=B,\ \ (x,t)\in\overline{D_T},$$

where B is some constant.

In view of (8) we have

$$v(x,0)+\delta_1 v(x,T)=B(1+\delta_1)=0,$$

which, by $\delta_1\geq 0$, we get $B=0$, i.e.,

$$v(x,t)\equiv 0 \text{ in } \overline{D_T}.$$

The proof of this theorem is complete.

3. Stability of the solution of the initial-boundary value problem (1)-(6)

In this section we prove the stability of the classical solution of the initial-boundary value problem (1)-(6).

Theorem 2. *Let $\delta_1=\delta_2=0$, $\mu_i\equiv 0$, $i=1, 2, 3, 4$, and let the function $u(x,t)\in C^{4,2}(\overline{D_T})$ solves problem (1)-(6). Then for this function the following inequality holds*

$$\begin{gathered}\int\limits_0^1 \left(p\,(x)u_{xx}^2(x,t)+q(x)u_x^2(x,t)+r(x)u_t^2(x,t)\right)dx+u_x^2(1,t)+r(1)u_t^2(1,t)\leq\\ \leq e^{r_0T}\left\{\int\limits_0^1 \left(r(x)\psi^2(x)+q(x)[\varphi'(x)]^2+p\,(x)[\varphi''(x)]^2\right)dx+[\varphi'(1)]^2+\right.\\ \left.+\,r(1)\psi^2(1)+\int\limits_0^T\int\limits_0^1 f^2(x,t)dtdx\right\}.\end{gathered}\tag{20}$$

Proof. Multiplying both parts of (1) by the function $2u_t(x,t)$ and integrating the resulting equality by x in the range from 0 to 1, we obtain

$$\begin{gathered}2\int\limits_0^1 (p\,(x)u_{xx}(x,t))_{xx}u_t(x,t)dx-2\int\limits_0^1 (q(x)u_x(x,t))_x u_t(x,t)dx+\\ 2\int\limits_0^1 r(x)u_{tt}(x,t)u_t(x,t)dx=2\int\limits_0^1 f(x,t)u_t(x,t)dx\end{gathered}\tag{21}$$

It is obvious that

$$2\int_0^1 f(x,t)u_t(x,t)dx \le \int_0^1 f^2(x,t)dx + \int_0^1 u_t^2(x,t)dx. \tag{22}$$

By (17) and (22) we get

$$\begin{gathered}
\frac{d}{dt}\left(\int_0^1 \left(p\,(x)u_{xx}^2(x,t)+q(x)u_x^2(x,t)+r(x)u_t^2(x,t)\right)dx+u_x^2(1,t)+r(1)u_t^2(1,t)\right)\le \\
\int_0^1 f^2(x,t)dx+\int_0^1 u_t^2(x,t)dx\le \\
\int_0^1 f^2(x,t)dx+\int_0^1 \frac{1}{r(x)}\left(p\,(x)u_{xx}^2(x,t)+q(x)u_x^2(x,t)+r(x)u_t^2(x,t)\right)dx\le \\
\int_0^1 f^2(x,t)dx+\frac{1}{r_0}\int_0^1 \left(p\,(x)u_{xx}^2(x,t)+q(x)u_x^2(x,t)+r(x)u_t^2(x,t)\right)dx,\ t\in[0,T],
\end{gathered} \tag{23}$$

where $r_0 = \min\limits_{x\in[0,1]} r(x)$.

In view of (18), by (23) we obtain

$$z'(t)\le \int_0^1 f^2(x,t)dx + r_0 z(t),\ t\in[0,T],$$

or

$$\frac{d}{dt}\left(z(t)e^{-r_0 t}\right)\le e^{-r_0 t}\int_0^1 f^2(x,t)dx,\ t\in[0,T].$$

It follows from last relation that

$$z(t)\le e^{r_0 T}\left\{z(0)+\int_0^T\int_0^1 f^2(x,t)dxdt\right\},\ t\in[0,T]. \tag{24}$$

By initial conditions (2) we have the following relation

$$\begin{gathered}
z(0)=\int_0^1 \left(p\,(x)u_{xx}^2(x,0)+q(x)u_x^2(x,0)+r(x)u_t^2(x,0)\right)dx+u_x^2(1,0)+ \\
r(1)u_t^2(1,0)=\int_0^1 \left(p(x)\varphi''^2(x)+q(x)\varphi'^2(x)+r(x)\psi^2(x)\right)dx+\varphi'^2(1)+ \\
r(1)\psi^2(1)+\int_0^T\int_0^1 f^2(x,t)dtdx
\end{gathered} \tag{25}$$

Using (25) from (24) we obtain (20). The proof of this theorem is complete.

Corollary 1. *Let $q(x) > 0$ for $x \in [0,1]$ and let the conditions of Theorem 2 be satisfied. Then the following inequality holds:*

$$|u(x,t)|^2 \leq M \left\{ \int_0^1 \left(p\,(x)\varphi''^2(x) + q(x)\varphi'^2(x) + r(x)\psi^2(x)\right) dx + \varphi'^2(1) + \right.$$
$$\left. + r(1)\psi^2(1) + \int_0^T\int_0^1 f^2(x,t)dtdx \right\},\ (x,t) \in \overline{D_T},$$

where $M = e^{r_0 T}\left(\int_0^1 \frac{dx}{q(x)}\right)^{\frac{1}{2}}$.

Remark 1. If the function q takes zero values, then we have the following inequality:

$$|u_x(x,t)|^2 \leq \tilde{D} \left\{ \int_0^1 \left(p\,(x)\varphi''^2(x) + q(x)\varphi'^2(x) + r(x)\psi^2(x)\right) dx + \varphi'^2(1) + \right.$$
$$\left. r(1)\psi^2(1) + \int_0^T\int_0^1 f^2(x,t)dtdx \right\},$$

4. The existence of a classical solution to problem (1)-(6)

Suppose that $f \equiv 0$ in $\overline{D}_T$ and $\mu_i \equiv 0$ in $[0,T]$ for $i = 1, 2, 3, 4$. In order to solve problem (1)-(6) we apply the method of separation of variables. We will sought for a non-trivial particular solution of equation (1) that satisfies the boundary conditions (3)-(6) in the following form

$$u(x,t) = y(x)\vartheta(t),\ x \in [0,1],\ t \in [0,T]. \tag{26}$$

Taking (26) into account from (1) we obtain

$$(p\,(x)y''(x))''\vartheta(t) - (q(x)y'(x))'\vartheta(t) + r(x)y(x)\vartheta''(t) = 0 \tag{27}$$

which implies that

$$\frac{(p(x)y''(x))'' - (q(x)y'(x))'}{r(x)y(x)} = -\frac{\vartheta''(t)}{\vartheta(t)} = \lambda,\ \lambda \in \mathbb{C}. \tag{28}$$

Then the functions $y(x)$ and $\vartheta(t)$ will satisfy the following ordinary differential equations

$$(p\,(x)y''(x))'' - (q(x)y'(x))' = \lambda r(x)y(x),\ 0 < x < 1, \tag{29}$$

and

$$\vartheta''(t) + \lambda\vartheta(t) = 0,\ 0 < t < T, \tag{30}$$

respectively.

By (26) and (28) it follows from (3)-(6) (with the use of conditions $\mu_i \equiv 0$ in $[0,T]$ for $i = 1, 2, 3, 4$) that

$$y(0) = 0,\ y''(0) = 0,\ p\,(1)y''(1) + y'(1) = 0,\ \mathcal{T}y(1) + \lambda r(1)y(1) = 0,$$

where

$$\mathcal{T}y \equiv (py'')' - qy'.$$

Thus, problem (1), (3)-(6) is reduced by the change of variables (26) to the spectral problem

$$(p\,(x)y''(x))'' - (q(x)y'(x))' = \lambda r(x)y(x),\ 0 < x < 1, \tag{31}$$

$$y(0) = y''(0) = y'(1) + p\,(1)y''(1) = 0, \tag{32}$$

$$\mathcal{T}y(1) + \lambda r(1)y(1) = 0. \tag{33}$$

A more general form of the spectral problem (31)-(33) was considered in [8] (see also [1]), where the oscillatory properties of eigenfunctions and the basis properties of subsystems of eigenfunctions in the space $L_p\,(0,1)$, $1 < p < \infty$, were considered.

Remark 2. By [8, Lemma 2.2 and Theorem 2.2] the eigenvalues of problem (31)-(33) are real and simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$. Moreover, multiplying both parts of (31) by y and integrating the resulting relation in the range from 0 to 1 (using integration by parts) and taking the boundary conditions (32), (33) into account we obtain

$$\int\limits_0^1 \left\{p\,(x)y''^2(x) + q(x)y'^2(x)\right\} dx + \frac{1}{p\,(1)}\, y'^2(1) = \lambda \left\{ \int\limits_0^1 r(x)y^2(x)dx + r(1)y^2(1) \right\}$$

whence, by the first condition in (32), implies that the eigenvalues of problem (31)-(33) are positive, i.e., $\lambda_k > 0$ for any $k \in \mathbb{N}$.

Remark 3. It follows from [8, formulas (3.3) and (3.4)] that

$$\sqrt[4]{\lambda_k} = \frac{(k-1)\pi}{\gamma} + O\left(\frac{1}{k}\right), \tag{34}$$

$$\begin{gathered} y_k(x) = \sin \tfrac{(k-1)\,\pi x}{\gamma} - \cos\, \tfrac{(k-1)\,\pi x}{\gamma} - e^{-\frac{(k-1)\,\pi x}{\gamma}} + \\ (-1)^k e^{\frac{(k-1)\,\pi\,(x-1)}{\gamma}} + O\left(\tfrac{1}{k}\right), \end{gathered} \tag{35}$$

where relation (35) holds uniformly for $x \in [0,1]$ and

$$\gamma = \int\limits_0^1 \left(\frac{r(x)}{p\,(x)}\right)^{1/4} dx.$$

Remark 4. Let s be an arbitrary fixed natural number. Then, by [8, Theorem 5.1], the system $\{y_k\}_{k=1,\,k\neq s}^{\infty}$ of eigenfunctions of problem (31)-(33) forms a basis in the space $L_p\,((0,1);r)$, $1 < p < \infty$, which is an unconditional basis in $L_2((0,1);r)$. Moreover, it follows from the proof of [8, formula (4.3)] that each element v_k of the system $\{v_k\}_{k=1,\,k\neq s}^{\infty}$ conjugate to the system $\{y_k\}_{k=1,\,k\neq s}^{\infty}$ is defined as follows:

$$v_k(x) = \delta_k^{-1} \left\{ y_k - \frac{y_k(1)}{y_s(1)}\, y_s(x) \right\}, \tag{36}$$

where

$$\delta_k = \int_0^1 r(x)y_k^2(x)dx + r(1)y_k^2(1) > 0.$$

Remark 5. In view of [8, Lemma 4.1 and relations (4.11)] we have the following relation

$$v_k(x) = y_k(x) + O\left(\frac{1}{k}\right). \tag{37}$$

$$||y_k||_{2,r}^2 = 1 + O\left(\frac{1}{k}\right) \text{ and } y_k(1) = O\left(\frac{1}{k}\right), \tag{38}$$

where $||\cdot||_{2,r}$ is the norm in $L_2((0,1);r)$.

Let $H = L_2((0,1);r) \oplus \mathbb{C}$ be the Hilbert space with inner product

$$(\hat{u},\hat{v})_H = (\{y,m\},\{v,n\})_H = \int_0^1 r(x)y(x)\overline{v(x)}\,dx + \ r(1)^{-1}m\bar{s}\,, \tag{39}$$

We define the linear operator $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ as follows:

$$\mathcal{L}\hat{y} = \mathcal{L}\{y,m\} = \left\{\frac{1}{r(x)}\left(\mathcal{T}y(x)\right)', -\mathcal{T}y(1)\right\},$$

where

$$D(L) = \{\{y\ (x),\ m\}\ :\ y \in W_2^4(0,1),\ \tfrac{1}{r(x)}\left(\mathcal{T}y(x)\right)' \in L_2(0,1),$$
$$y(0) = y''(0) = y'(0) + p\,(1)y''(1) = 0,\ m = r(1)y(1)\}$$

which is everywhere in H (see [1]). Then problem (31)-(33) is equivalent to the spectral problem

$$L\hat{y} = \lambda\hat{y},\ \hat{y} \in D(L), \tag{40}$$

i.e., the eigenvalues λ_k, $k \in \mathbb{N}$, of problems (31)-(33) and (39) coincide (counting multiplicities), and there exists a one-to-one correspondence between the their eigenfunctions,

$$y_k(x) \leftrightarrow\ \{y_k(x),\, m_k\},\ \ m_k = r(1)y_k(1).$$

Since r is positive on $[0,1]$, the operator L is a self-adjoint discrete lower-semibounded in H and hence the system of eigenvectors $\{\hat{y}_k\}_{k=1}^{\infty}$ of this operator forms an orthogonal basis in H (see [1]).

For any $k,n \in \mathbb{N}$, $k \neq n$, we have

$$(\hat{y}_k, \hat{y}_n) = 0,$$

and consequently,

$$\int_0^1 r(x)y_k(x)y_n(x)dx + r(1)y_k(1)y_n(1) = 0 \text{ for any } k, \in \mathbb{N},\ k \neq n. \tag{41}$$

Note that $y_k(1) \neq 0$ for any $k \in \mathbb{N}$. Indeed, if $y_k(1) = 0$ for some $k \in \mathbb{N}$, then it follows from (33) that $Ty_k(1) = 0$. Moreover, due to the third condition in (32) we have $y'(1)y''(1) < 0$. Then by the second part of Lemma 2.1 of [2] we get $y'(0)y''(0) < 0$ in contradiction the second condition in (32).

Let k_0 be the arbitrary fixed positive integer. Then by (41) we have

$$\int_0^1 r(x)y_k(x)y_{k_0}(x)dx + r(1)y_k(1)y_{k_0}(1) = 0 \text{ for any } k \in \mathbb{N},\ k \neq k_0,$$

which implies that

$$r(1)y_k(1) + \frac{}{y_{k_0}(1)} \int_0^1 r(x)y_k(x)y_{k_0}(x)dx = 0 \text{ for any } k \in \mathbb{N},\ k \neq k_0, \tag{42}$$

Thus, by (42), λ_k, $k \in \mathbb{N}$, $k \neq k_0$, are eigenvalues and y_k, $k \in \mathbb{N}$, $k \neq k_0$, are corresponding eigenfunctions of the following spectral problem

$$\begin{cases} (p(x)y''(x))'' - (q(x)y'(x))' = \lambda r(x)y(x),\ 0 < x < 1, \\ y(0) = y''(0) = y'(1) + p(1)y''(1) = 0, \\ r(1)y(1) + \frac{1}{y_{k_0}(1)} \int_0^1 r(x)y(x)y_{k_0}(x)dx = 0. \end{cases} \tag{43}$$

Note that, unlike problem (31)-(33), problem (43) does not contain a spectral parameter in the boundary conditions.

By first relation of (38), without loss of generality, we can assume that the functions y_k, $k \in \mathbb{N}$, are normalized in $L_2((0,1);r)$. Then, by Remark 4.3, the system $\{y_k(x)\}_{k=1,\, k\neq k_0}^{\infty}$ forms a Riesz basis in the space $L_2((0,1);r)$. In this case the system $\{v_k(x)\}_{k=1,\, k\neq k_0}^{\infty}$, where

$$v_k(x) = \delta_k^{-1}\left\{y_k - \frac{y_k(1)}{y_{k_0}(1)}\, y_{k_0}(x)\right\},$$

is conjugate to the system $\{y_k(x)\}_{k=1,\, k\neq k_0}^{\infty}$. Hence for any function $g \in L_2((0,1);r)$ we have

$$g = \sum_{k=1,k\neq k_0}^{\infty} g_k y_k(x), \tag{44}$$

where

$$\begin{aligned} g_k = \int_0^1 r(x)g(x)v_k(x)dx = \delta_k^{-1} \int_0^1 r(x)g(x)y_k(x)dx \\ - \delta_k^{-1} \frac{y_k(1)}{y_{k_0}(1)} \int_0^1 r(x)g(x)y_{k_0}(x)dx. \end{aligned} \tag{45}$$

Let the following conditions hold:

$$g(x),\ g'(x),\ g''(x),\ \mathcal{T}g(x)\in C[0,1],\ g(0)=0,\ g''(0)=0,\ g'(1)+p\,(1)g''(1)=0,$$

$$J(g)=r(1)g(1)+\frac{1}{y_{k_0}(1)}\int\limits_0^1 r(x)g(x)y_{k_0}(x)dx=0 \text{ and } \frac{1}{r(x)}(\mathcal{T}g(x))'\in L_2(0,1).$$

For any $g\in D(\mathcal{L})$ we have

$$(L\hat{y}_k,\overline{\hat{g}})=\lambda_k(\hat{y}_k,\overline{\hat{g}}),\ k\in\mathbb{N},$$

whence, by (39), we get

$$\lambda_k\int\limits_0^1 r(x)y_k(x)g(x)dx\ +\lambda_k r(1)y_k(1)g(1)=\lambda_k(\hat{y}_k,\overline{\hat{g}})_H=(L\hat{y}_k,\overline{\hat{g}})_H=$$

$$\left(\hat{y}_k,L\overline{\hat{g}}\right)_H=\int\limits_0^1 y_k(x)(Tg(x))'dx-y_k(1)Tg(1),\ k\in\mathbb{N}.$$

Thus, for any $g\in D(\mathcal{L})$ we obtain

$$\lambda_k\int\limits_0^1 r(x)y_k(x)g(x)dx\ =-\,\lambda_k r(1)y_k(1)g(1)-y_k(1)Tg(1)+\int\limits_0^1 y_k(x)(Tg(x))'dx,\ k\in\mathbb{N}.$$

whence implies that

$$\lambda_{k_0}\frac{y_k(1)}{y_{k_0}(1)}\int\limits_0^1 r(x)y_{k_0}(x)g(x)dx\ =-\,\lambda_{k_0}r(1)y_k(1)g(1)-\ y_k(1)Tg(1)+$$

$$\frac{y_k(1)}{y_{k_0}(1)}\int\limits_0^1 r(x)y_{k_0}(x)(Tg(x))'dx,\ k\in\mathbb{N}.$$

It follows from two last relations that

$$\lambda_k\int\limits_0^1 r(x)g(x)\left\{y_k(x)-\frac{y_k(1)}{y_{k_0}(1)}\,y_{k_0}(x)\right\}dx\ +(\lambda_k-\lambda_{k_0})\frac{y_k(1)}{y_{k_0}(1)}\int\limits_0^1 r(x)y_{k_0}(x)g(x)dx=$$

$$-(\lambda_k-\lambda_{k_0})r(1)y_k(1)g(1)+\int\limits_0^1\left\{y_k(x)-\frac{y_k(1)}{y_{k_0}(1)}\,y_{k_0}(x)\right\}(\mathcal{T}g)'(x)dx,$$

and consequently,

$$\lambda_k \int_0^1 r(x)g(x)\left\{y_k(x)-\frac{y_k(1)}{y_{k_0}(1)}\,y_{k_0}(x)\right\}dx=$$

$$-(\lambda_k-\lambda_{k_0})y_k(1)\left\{r(1)g(1)+\frac{1}{y_{k_0}(1)}\int_0^1 r(x)y_{k_0}(x)g(x)dx\right\}+$$

$$\int_0^1\left\{y_k(x)-\frac{y_k(1)}{y_{k_0}(1)}\,y_{k_0}(x)\right\}(\mathcal{T}g)'(x)dx.$$

Since $J(g)=0$ we have the following relation

$$\int_0^1 r(x)g(x)v_k(x)dx=\frac{1}{\lambda_k}\int_0^1(\mathcal{T}g)'(x)v_k(x)dx.$$

Lemma 1. *Let the conditions $g\in C^3[0,1]$, $g\in W_2^4(0,1)$, $g(0)=g''(0)=g'(1)+p\,(1)g''(1)=0$ and $J(g)=0$ be satisfied. Then the following relation holds:*

$$g_k=\lambda_k^{-1}g_{k,1},$$

where

$$g_{k,1}=\int_0^1 r(x)G(x)v_k(x)dx,\ G(x)=\frac{(\mathcal{T}g)'(x)}{r(x)},\ x\in[0,1].$$

Corollary 2. Let the conditions of Lemma 4.1 be satisfied. Then one has the relation

$$\sum_{k=1,\,k\neq k_0}^{\infty}\lambda_k^2g_k^2=\sum_{k=1,k\neq k_0}^{\infty}g_{k,1}^2\le\int_0^1\frac{(Tg(x))'^2}{r(x)}\,dx\,.$$

Lemma 2. *Let $g_1=(Tg)'$ and the following conditions hold: $p\in C^4[0,1]$, $q\in C^2[0,1]$, $g\in C^7[0,1]$, $g\in W_2^8(0,1)$, $g(0)=g''(0)=g'(1)+p\,(1)g''(1)=0$, $J(g)=0$ and $g_1(0)=g_1''(0)=g_1'(1)+p\,(1)g_1''(1)=0$, $J(g_1)=0$. Then we have the following relation:*

$$g_k=\lambda_k^{-2}g_{k,2},\ k\in\mathbb{N},\ k\neq k_0,$$

where

$$g_{k,2}=\int_0^1 r(x)g_1(x)v_k(x)dx.$$

Corollary 3. Let the conditions of Lemma 4.2 hold. Then one has the relation

$$\sum_{k=1,\,k\neq k_0}^{\infty}\lambda_k^4 g_k^2=\sum_{k=1,k\neq k_0}^{\infty} g_{k,2}^2\leq\int_0^1\frac{(Tg(x))'^2}{r(x)}\,dx\,.$$

We will seek the solution to problem (1)-(6) in the form

$$u(x,t)=\sum_{k=1,\,k\neq k_0}^{\infty}u_k(t)y_k(x), \tag{46}$$

where

$$u_k(t)=\int_0^1 r(x)u(x,t)v_k(x)dx,$$

$$v_k(x)=\delta_k^{-1}\left(y_k(x)-\frac{y_k(1)}{y_{k_0}(1)}\,y_{k_0}(x)\right),\ k\in\mathbb{N},\ k\neq k_0.$$

We apply the method of separation of variables to determine the desired functions $u_k(t)$, $k\in\mathbb{N}$, $k\neq k_0$. Then from (1) we obtain

$$u_k''(t)+\lambda_k u_k(t)=0,\ \ k\in\mathbb{N},\ \ k\neq k_0,\ \ t\in[0,T], \tag{47}$$

$$u_k(0)+\delta_1 u_k(T)=\varphi_k,\ \ u_k'(0)+\delta_2 u_k'(T)=\psi_k,\ \ k\in\mathbb{N},\ \ k\neq k_0, \tag{48}$$

where

$$\varphi_k=\int_0^1 r(x)\varphi(x)v_k(x)dx,\ \ \psi_k=\int_0^1 r(x)\psi(x)v_k(x)dx,\ \ k\in\mathbb{N},\ \ k\neq k_0.$$

Solving problem (47), (48) by using Remark 4.1 we get

$$u_k(t)=\frac{1}{\varrho_k(T)}\left[\varphi_k(\cos\rho_k t+\delta_2\cos\rho_k(T-t))+\ \frac{\psi_k}{\rho_k}\left(\sin\rho_k t-\delta_1\sin\rho_k(T-t)\right)\right],$$

where

$$\rho_k=\sqrt{\lambda_k},\ \ \varrho_k(T)=1+(\delta_1+\delta_2)\cos\rho_k T+\delta_1\delta_2.$$

The following theorem is the main result of this paper.

Theorem 3. *Let the following conditions hold:*

(i) $1+\delta_1\delta_2\geq\delta_1+\delta_2$,

(ii) $\mu_i\equiv 0,\ i=1,\,2,\,3,\,4,\ p\in C^4([0,1];(0,+\infty)),\ q\in C^2([0,1];[0,+\infty))$,

(iii) $\varphi\in C^7([0,1];\mathbb{R}),\ \phi\in W_2^8(0,1),\ \varphi(0)=\varphi''(0)=\varphi'(1)+p\,(1)\varphi''(1)=0,\ J(\varphi)=0$

and $\phi(0) = \phi''(0) = \phi'(1) + p\,(1)\phi''(1) = 0,\ J(\phi) = 0,$ *where* $\phi = \frac{1}{r}\,(T\varphi)'$,

(iv) $\psi \in C^3([0,1];\mathbb{R}),\ \psi \in W_2^4(0,1),\ \psi''(0) = \psi'(1) + p\,(1)\psi''(1) = 0.$
Then the function

$$u(x,t) = \sum_{k=1,\,k\neq k_0}^{\infty} \frac{1}{\varrho_k(T)}\,[\varphi_k(\cos\rho_k t + \delta_2\cos\rho_k(T-t)) +$$

$$\left.\frac{\psi_k}{\rho_k}(\sin\rho_k t - \delta_1\sin\rho_k(T-t))\right]\,y_k(x)$$

is a classical solution of problem (1)-(6).

The proof of this theorem is similar to the proof of the justification of the Fourier method in [10, § 23.5] (see also [9]) with the use of Lemmas 1, 2 and Corollaries 2, 3.

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Maryam H. Jafarova
Baku State University, Baku, Azerbaijan
E-mail: maryam.bzade20@mail.ru

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Construction of a Basis in L_p From Root Functions of a Differential Operator With Non-strongly Regular Boundary Conditions

R.J. Taghiyeva*, I.G. Feyzullayev

Abstract. We study a spectral problem for an ordinary differential equation of the second order with non-strengthened regular boundary conditions on a finite interval [0,1]. Such problems arise when solving a non-local boundary value problem for partial differential equations by the Fourier method. They arise, for example, when solving non-stationary diffusion problems with boundary conditions of the Samarskii-Ionkin type, or when solving a stationary diffusion problem with opposite flows on a part of an interval. The boundary conditions of this problem are regular, but not strengthened regular in the sense of Birkhoff. The system of eigenfunctions of such a problem is complete and minimal, but does not form a basis in the space $L_p\,[0,1]$. In this case, direct application of the Fourier method is impossible. Based on these eigenfunctions, a new system of functions is constructed, which already forms a basis in $L_p\,[0,1]$.

Key Words and Phrases: non-strongly regular boundary conditions, eigenfunctions, almost normalized system, uniform minimality, basis.
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1. Introduction

The solution of some elliptic equations with nonlocal boundary conditions using the Fourier method leads to spectral problems with boundary conditions that are regular but not strongly regular. For this reason, the root functions of these problems do not generally form a basis in the corresponding function space. In such a case, direct application of the Fourier method is impossible. Based on these eigenfunctions, a new system of functions is constructed consisting of linear combinations of root functions, which already forms a basis in $L_p\,[0,1]$. However, the resulting system is not a system of eigenfunctions of the spectral problem. Nevertheless, this system is used to solve the equation under consideration by the Fourier method. One of such problems is the following initial-boundary value problem for the parabolic equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},\ 0 < x < 1,\ t > 0,$$

*Corresponding author.

with initial condition

$$U(x,0)=\varphi(x),\ 0\le x\le 1,$$

and boundary conditions

$$U(0,t)=0,\ \frac{\partial U}{\partial x}(0,t)=\frac{\partial U}{\partial x}(1,t)+\alpha U(1,t),\ t\ge 0.$$

This problem leads to the following spectral problem

$$\left.\begin{array}{c}-u''(x)=\lambda\, u(x),\ 0<x<1,\\ u(0)=0,\ \ u'(0)-u'(1)+\alpha\, u(1)=0.\end{array}\right\} \tag{1}$$

The boundary conditions of this spectral problem are regular, but not strongly regular. A number of works by the authors [1-5] are devoted to the study of such problems in the Lebesgue space $L_2[0,1]$. It should be noted that issues related to this topic in the case of $\alpha=0$ were also considered in works [6-11]. All these spectral problems are not self-adjoint. The case of $\alpha=0$ differs from the case of $\alpha\neq 0$ in that in the first case all eigenvalues are double and they correspond to one eigenfunction and one associated function, and together they form a basis in $L_2[0,1]$. In the second case, all eigenvalues are simple, but the corresponding eigenfunctions are not a basis in $L_2[0,1]$. One of the methods for constructing a basis, based on the system of eigenfunctions of problem (1) in the case of $\alpha>0$ was proposed in [1]. Using the eigenfunctions of this problem, a special system of functions is constructed, which will form a basis in $L_2[0,1]$. And this fact is applied to solve a nonlocal initial-boundary value problem for the heat equation. It is used in [3] to solve an inverse nonlocal boundary value problem for the heat equation, and in [4] to solve a nonlocal boundary value problem for the Helmholtz operator in a semicircle. A similar method was used in [5] to study the classical solvability of one nonlocal boundary value problem for the Laplace equation in a semicircle.

The aim of this work is to construct a basis in $L_p[0,1]$ from the system of eigenfunctions of problem (1) for any complex value of the parameter α.

2. Preliminaries

Let us present briefly the main definitions and facts which will be used in what follows. Let X be a Banach space. A system$\{x_n\}_{n\in N}$ of elements X is said to be complete in X if $\overline{L\left(\{x_n\}_{n\in N}\right)}=X$; that is, any element of the space X can be approximated by a linear combination of elements of this system with any accuracy in the norm of the space X.

A system $\{x_n\}_{n\in N}$ of elements X is said to be minimal in X if $x_n\notin\overline{L\left(\{x_k\}_{k\neq n}\right)}$. It is well known that a system $\{x_n\}_{n\in N}$ is minimal if and only if there exists a biorthogonal system which is dual to it, that is, a system of linear functionals $\{x_n^*\}_{n\in N}$ from X^* such that $\langle x_n,x_k^*\rangle=\delta_{nk}$ for all $n,\ k\in N$. Moreover, if the initial system is complete and minimal in X, then the biorthogonal system is uniquely defined.

We say that a system $\{x_n\}_{n\in N}$ is uniformly minimal in X, if there exists $\gamma > 0$ such that for all $n \in N$,

$$\mathrm{dist}\,(x_n, X_n) \geq \gamma \|x_n\|_X \ ,$$

where $X_n = \overline{L\left[\{x_k\}_{k\neq n}\right]}$. It is also well known that a complete and minimal system $\{x_n\}_{n\in N}$ is uniformly minimal in X if and only if:

$$\sup_{n\in N} \|x_n\|_X \|x_n^*\|_{X^*} < \infty \ .$$

A system $\{x_n\}_{n\in N}$ forms a basis of the space X if, for any element $x \in X$, there exists a unique expansion into a series

$$x = \sum_{n=1}^{\infty} c_n x_n$$

converging in the norm of the space X.

Two systems $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ of a Banach space X are called equivalent if there exists an automorphism $T : X \to X$ that maps one of these systems to the other: $Tx_n = y_n$, $\forall n \in N$. A system equivalent to a basis is itself a basis in the same space.

A system in a Hilbert space that is equivalent to an orthonormal basis is called a Riesz basis. A Riesz basis is also an unconditional basis, i.e. it remains a basis under any permutation of its elements.

A system $\{x_n\}_{n\in N}$ is called a basis with brackets in a Banach space X if there exists a sequence $\{n_k\}_{k\in N}$ of positive integers such that $n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$, and for any $x \in X$ there is a unique expansion into a series

$$x = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i x_i, \quad (n_0 = 0)$$

converging in the norm of the space X. In the case of a Hilbert space, an unconditional basis with brackets is also called a Riesz basis with brackets.

We say that a system $\{x_n\}_{n\in N}$ is almost normalized in X, if

$$0 < \inf_{n\in N} \|x_n\| \leq \sup_{n\in N} \|x_n\| < \infty \ .$$

A uniformly minimal system is almost normalized if and only if its bioorthogonal system is almost normalized.

Statement 1. *Let $\{x_n\}_{n\in N}$ be a minimal system in a Banach space X, $\{x_n^*\}_{n\in N}$ be its biorthogonal system. If the system $\{x_n\}_{n\in N}$ has two asymptotically close subsystems, i.e. there exist subsystems $\{x_{n_k}\}_{k\in N}$ and $\left\{x_{n'_k}\right\}_{k\in N}$ such that*

$$\lim_{k\to\infty} \left\|x_{n_k} - x_{n'_k}\right\|_X = 0, \qquad (2)$$

then the system $\{x_n^*\}_{n\in N}$ is not almost normalized.

Proof. By $\{x^*_{n_k}\}_{k\in N}$ and $\left\{x^*_{n'_k}\right\}_{k\in N}$ we denote the corresponding subsystems of the biorthogonal system $\{x^*_n\}_{n\in N}$. Then from the biorthonormality conditions we have $\langle\langle x_{n_k}, x^*_{n_k}\rangle = 1$, $\left\langle x_{n'_k}, x^*_{n_k}\right\rangle = 0.$. From here we get $\langle\left\langle x_{n_k} - x_{n'_k}, x^*_{n_k}\right\rangle = 1$. Then

$$1 = \left|\left\langle x_{n_k} - x_{n'_k}, x^*_{n_k}\right\rangle\right| \le \left\|x_{n_k} - x_{n'_k}\right\|_X \left\|x^*_{n_k}\right\|_{X^*}$$

or

$$\left\|x^*_{n_k}\right\|_{X^*} \ge \left(\left\|x_{n_k} - x_{n'_k}\right\|_X\right)^{-1}.$$

Then from condition (2) it follows that

$$\lim_{k\to\infty} \left\|x^*_{n_k}\right\|_{X^*} = \infty. \tag{3}$$

Similarly, it is established that $\lim_{k\to\infty}\left\|x^*_{n'_k}\right\|_{X^*} = \infty$. Consequently, the system $\{x^*_n\}_{n\in N}$ is not almost normalized.

Statement 2. *If the system $\{x_n\}_{n\in N} \subset X$ is almost normalized and has two asymptotically close subsystems, then it is not uniformly minimal and, moreover, cannot be a basis in X.*

Proof. Let $\{x_{n_k}\}_{k\in N}$ and $\left\{x_{n'_k}\right\}_{k\in N}$be asymptotically close subsystems of $\{x_n\}_{n\in N}$, and $\{x^*_{n_k}\}_{k\in N}$ and $\left\{x^*_{n'_k}\right\}_{k\in N}$ be the corresponding subsystems of the biorthogonal system $\{x^*_n\}_{n\in N}$. Then, from the condition of almost normalization of the system $\{x_n\}_{n\in N}$, we have: $\exists m > 0 :\ \|x_{n_k}\|_X > m,\ \forall k \in N$. Taking into account (3), we obtain

$$\lim_{k\to\infty} \|x_{n_k}\|_X \left\|x^*_{n_k}\right\|_{X^*} = \infty.$$

The latter means that the system $\{x_n\}_{n\in N}$ is not uniformly minimal.

Any basis is a complete and minimal system in X, and, therefore, we can uniquely find its biorthogonal dual system $\{x^*_n\}_{n\in N}$ and hence the expansion of any element $x \in X$ with respect to the basis $\{x\}_{n\in N}$ coincides with its biorthogonal expansion, that is, $c_n = \langle x, x^*_n\rangle$ for all $n \in N$.

We will use also some facts about p-closure bases. Concerning these facts more details one can see the works [12, 13].

Systems $\{x_n\}_{n\in N}$, $\{y_n\}_{n\in N} \subset X$ in Banach space X are called p-closure if

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^p_X < \infty.$$

The minimal system $\{x_n\}_{n\in N} \subset X$ with biorthogonal system $\{x^*_n\}_{n\in N} \subset X^*$ is called p- besselian, if for any $x \in X$

$$\left(\sum_{n=1}^{\infty} |\langle x, x^*_n\rangle|^p\right)^{\frac{1}{p}} \le M\|x\|_X.$$

If the basis $\{x_n\}_{n\in N}$ for X is p- besselian, then we call it as p-basis.

It is valid the following

Theorem 1. *[12, 13] Let the system $\{x_n\}_{n\in N}$ is p -basis for Banach space X and the system $\{y_n\}_{n\in N}\subset X$ is p'– clouser to it, $1<p<\infty$. Then the following assertions are equivalent:*

1. *$\{y_n\}_{n\in N}$ is complete in X;*
2. *$\{y_n\}_{n\in N}$ is minimal in X;*
3. *$\{y_n\}_{n\in N}$ is isomorphic to $\{x_n\}_{n\in N}$ basis for X.*

It is valid the following

Statement 3. *[14, 15] Let system $\{x_n\}_{n\in N}$ forms a basis with parentheses for Banach space X. If the system $\{x_n\}_{n\in N}$ is uniformly minimal and condition*

$$\sup_{k\in N}(n_{k+1}-n_k)\ <\infty \tag{4}$$

hold, then the system $\{x_n\}_{n\in N}$ forms a basis for X.

Statement 4. *[15] Let system $\{x_n\}_{n\in N}$ forms a Riesz basis with parentheses for Hilbert space X. If the system $\{x_n\}_{n\in N}$ is almost normalized, uniformly minimal and condition* (4) *hold, then it forms a basis Riesz for X.*

3. Study of the Spectral Problem

In this section we will study the properties of the eigenvalues and eigenfunctions of the following spectral problem

$$-u''(x)=\lambda u(x)\,,\ 0<x<1,\ u(0)=0,\ u'(0)=u'(1)+\alpha\, u(1)\,, \tag{5}$$

where the parameter α can take any complex value. In the case $\alpha\neq 0$, the eigenvalues of the spectral problem can be divided into two series, which have the form

$$\lambda_{2k-1}=(\rho_{2k-1})^2,\ k\in N,\quad \lambda_{2k}=(\rho_{2k})^2,\ k\in Z^+, \tag{6}$$

where $Z^+=\{0\}\cup N$, $\rho_{2k-1}=2\pi k$, and ρ_{2k} are the roots of the equation

$$tg\ \frac{\rho}{2}\ =\frac{\alpha}{\rho}. \tag{7}$$

Using the standard method we obtain that (see [16]) the following is true

Lemma 1. *Equation* (7) *for any complex α has a countable number of solutions that are asymptotically simple and have the asymptotics*

$$\rho_{2k}=2\pi k+\frac{\alpha}{2\pi k}+O\left(\frac{1}{k^3}\right). \tag{8}$$

Each eigenvalue of problem (5) corresponds to a unique eigenfunction up to a non-zero factor. Using the numbering introduced by equalities (6), the set of eigenfunctions can be represented as

$$u_{2k-1}(x) = \sin 2\pi kx, \quad k \in N; \;\; u_{2k}(x) = \sin \rho_{k2} x, \quad k \in Z^+; \tag{9}$$

The problem conjugate to (5) is defined by the equality

$$-\vartheta''(x) = \lambda \vartheta(x), \; 0 < x < 1, \vartheta(0) = \vartheta(1), \vartheta'(1) + \alpha\, \vartheta(1) \;=\; 0. \tag{10}$$

It has the same eigenvalues (6) as problem (5). The corresponding eigenfunctions have the form

$$\vartheta_{2k-1}(x) = C_{2k-1}\left(\cos 2\pi kx \;-\; \frac{\alpha}{2\pi k}\sin 2\pi kx\right), \; k \in N; \tag{11}$$

$$\vartheta_{2k}(x) = C_{2k}\left(\cos \rho_{2k} x \;+\; \frac{\alpha}{\rho_{2k}}\sin \rho_{2k} x\right), \; k \in Z^+,$$

where

$$C_{2k-1} = -\frac{4\pi k}{\alpha}\;, \quad C_{2k} = \frac{4\pi k}{\alpha} + O\left(\frac{1}{k}\right).$$

The systems of eigenfunctions of problems (5) and (10) are numbered in such a way that $\langle u_n, \vartheta_m \rangle = \delta_{nm}$. The constants C_n are chosen so that $\langle u_n, \vartheta_n \rangle = 1, \;\; n \in Z^+$.

Let's show that the system $\{u_n(x)\}_{n \in Z^+}$ is not uniformly minimal in $L_p(0,1)$.

Theorem 2. *The system of eigenfunctions $\{u_n(x)\}_{n \in Z^+}$ of problem (5) is complete, minimal and almost normalized, but is not uniformly minimal in $L_p(0,1), 1 < p < \infty$.*

Proof. The spectral problem (5) is regular, but not strongly regular in the sense of Birkhoff (see [16]). From the results of [17], in particular, it follows that the eigenfunctions and associated functions of problem (5) form a basis with brackets in $L_p(0,1), 1 < p < \infty$. From this, in particular, follows the completeness of the system $\{u_n(x)\}_{n \in Z^+}$ in the space $L_p(0,1), \; 1 < p < \infty$. The system $\{\vartheta_n(x)\}_{n \in N}$ is a biorthogonal to $\{u_n(x)\}_{n \in Z^+}$ system regarding the space $L_p(0,1), \; 1 < p < \infty$, and therefore the system $\{u_n(x)\}_{n \in Z^+}$ is minimal in $L_p(0,1)$.

Let us show the almost normalized nature of the system $\{u_n(x)\}_{n \in Z^+}$. Let $1 < p < \infty$. We denote $2\delta_k = \rho_{2k} - 2\pi k$. Then from (8) we have $2\delta_k = \frac{\alpha}{2\pi k} + O\left(\frac{1}{k^3}\right)$ or $\delta_k = O\left(\frac{1}{k}\right)$. From here for the eigenfunctions $u_{2k}(x)$ we obtain

$$\sin \rho_{2k} x \;= \sin\left(2\pi k + 2\; delta_k\right) x \;= \sin 2\pi kx \;+ O\left(\frac{1}{k}\right). \tag{12}$$

Let's estimate the norms of eigenfunctions:

$$\|u_{2k-1}\|_{L_p} = \left(\int_0^1 |\sin 2\pi kx \;|^p dx\right)^{\frac{1}{p}} \le 1;$$

$$\|u_{2k}\|_{L_p} = \left(\int_0^1 |\sin(2\pi k + 2\delta_k)\, x\ |^p dx\right)^{\frac{1}{p}} \leq 1 + O\left(\frac{1}{k}\right).$$

From this we get

$$\overline{\lim_{k\to\infty}} \|u_{2k}\|_{L_p} \leq 1\ . \tag{13}$$

For the lower bound, we first consider the case $1 < p \leq 2$. Then for $u_{2k-1}(x)$ we have

$$\|u_{2k-1}\|_{L_p}^p = \int_0^1 |\sin 2\pi kx\ |^p dx \geq \int_0^1 \sin^2 2\pi kx\ dx = \frac{1}{2}.$$

It follows from this

$$\|u_{2k-1}\|_{L_p} \geq \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

Similarly, for large values of k for the functions $u_{2k-1}(x)$ we obtain

$$\|u_{2k}\|_{L_p} = \left(\int_0^1 |\sin(2\pi k + 2\delta_k)\ x|^p dx\right)^{\frac{1}{p}} \geq$$

$$\geq \left(\int_0^1 |sin 2\pi kx\ |^p dx\right)^{\frac{1}{p}} - O\left(\frac{1}{k}\right)$$

$$\geq \left(\frac{1}{2}\right)^{\frac{1}{p}} - O\left(\frac{1}{k}\right) \to \left(\frac{1}{2}\right)^{\frac{1}{p}}, k \to \infty.$$

Hence,

$$\underline{\lim}_{k\to\infty} \|u_{2k}\|_{L_p} \geq \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

From here, taking into account (13), we obtain the almost normalized nature of the system $\{u_n(x)\}_{n\in Z^+}$ for $1 < p \leq 2$.

Now let $p > 2$. Then we have a continuous embedding $L_p(0,1) \subset L_2(0,1)$ and

$$\|u_{2k-1}\|_{L_p} \geq \|u_{2k-1}\|_{L_2} = \left(\frac{1}{2}\right)^{\frac{1}{2}};$$

and also for large values of k

$$\|u_{2k}\|_{L_p} \geq \|u_{2k}\|_{L_2} \geq \|u_{2k-1}\|_{L_2} - O\left(\frac{1}{k}\right) \geq \left(\frac{1}{2}\right)^{\frac{1}{2}} - O\left(\frac{1}{k}\right).$$

From this we have

$$\underline{\lim}_{n\to\infty} \|u_{2k}\|_{L_p}\ \geq \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

Thus, for all $p \in (1,\infty)$

$$0 < \inf_{n\in N} \|u_n\|_{L_p} \ \le \sup_{n\in N} \|u_n\|_{L_p} \ < \infty,$$

i.e. the system $\{ \{u_n(x)\}_{n\in Z^+}$ is almost normalized in $L_p(0,1)$.

Let us now proceed to the proof of the last statement of the lemma. From the asymptotics (12) we have

$$u_{2k}(x) - u_{2k-1}(x) = O\left(\frac{1}{k}\right).$$

Hence$\|u_{2k} - u_{2k-1}\|_{L_p} = O\left(\frac{1}{k}\right)$, i.e. the subsystems $\{u_{2k-1}(x)\}_{k\in N}$ and $\{u_{2k}(x)\}_{k\in N}$ are asymptotically close. Then it follows from Statement 1 that

$$\lim_{k\to\infty} \|\vartheta_{2k-1}\|_{L_{p'}} \ = \lim_{k\to\infty} \|\vartheta_{2k}\|_{L_{p'}} = \infty. \tag{14}$$

On the other hand, the system $\{u_n(x)\}_{n\in Z^+}$ is almost normalized in $L_p(0,1)$, so from Statement 1.2 we obtain that the system $\{u_n(x)\}_{n\in Z^+}$ is not uniformly minimal. Note that the validity of relations (14) can also be obtained directly from the explicit formulas (11) for the functions $\vartheta_n(x)$. The lemma is proven.

From this lemma follows

Corollary 1. *The system $\{u_n(x)\}_{n\in Z^+}$ does not form a basis for $L_p(0,1)$, $1 < p < \infty$.*

4. Main results

Let us consider the case $\alpha = 0$ separately. In this case, the spectral problem will take the form

$$-w''(x) = \lambda w(x), \ 0 < x < 1, \ w(0) = w'(0) - w'(1) = 0. \tag{15}$$

In obtaining the main results we essentially will use the basicity in $L_p \ (0,\pi)$ the system $\{w_n(x)\}_{n\in Z^+}$ where

$$w_0(x) = x, \quad w_{2k-1}(x) = \sin 2\pi kx, \quad w_{2k}(x) = x\cos 2\pi kx \ , \ k \in N,$$

which is a collection of root functions of the spectral problem (15).

It is valid

Theorem 3. *The system $\{w_n(x)\}_{n\in N}$ forms a $q-$basis for $L_p \ (0,1)$, $\ 1 < p < +\infty$, where $q = \max\{p, \ p'\}$. In the case $p = 2$ this system is a Riesz basis for $L_2 \ (0,1)$.*

Proof. As in the case of spectral problem (5), spectral problem (15) is also not strongly regular, and from the results of [17] it follows that the system $\{w_n(x)\}_{n\in Z^+}$of eigen and associated functions of this problem forms a basis with brackets in $L_p \ (0,1)$, $\ 1 < p < \infty$, and in brackets you need to combine pairs of terms corresponding to w_{2k-1} and w_{2k}, that is $n_{k+1} - n_k = 2$.

The problem conjugate to (15) has the form

$$-z''(x) = \lambda \ z(x), \ \ 0 < x < 1, \ \ z'(1) = z(0) - z(1) = 0. \tag{16}$$

The system of eigen and associated functions of the spectral problem (16) is the system$\{z_n(x)\}_{n\in Z^+}$, where

$$z_0(x)=2,\ z_{2k-1}(x)=4(1-x)\sin 2\pi kx,\quad z_{2k}(x)=4\cos 2\pi kx,\quad k\in N.$$

The systems $\{w_n(x)\}_{n\in Z^+}$ and $\{z_n(x)\}_{n\in Z^+}$ are biorthonormal, i.e.

$$\langle w_n, z_m\rangle=\delta_{nm},\quad \forall n,m\in Z^+.$$

From the formulas for $w_n(x)$ and $z_n(x)$ it is obvious that

$$\sup_{n\in Z^+}\|w_n\|_{L_p(0,1)}\ \|z_n\|_{L_{p'}(0,1)}<+\infty.$$

Thus, all the conditions of Statement 3 are satisfied, according to which the system $\{w_n(x)\}_{n\in Z^+}$ forms a basis in the space $L_p(0,1)$, $1<p<\infty$. Let us show that the system $\{w_n(x)\}_{n\in Z^+}$ is also a $q-$basis in this space, where $q=\max\{p,\ \ p'\}$. Let $p\in(1,2]$, then $q=p'$ and, as follows from the Hausdorff-Young inequality (see [18]), for any function $f(x)$ from $L_p(0,1)$ we have

$$\left(\sum_{k=0}^{\infty}|\langle f, z_{2k}\rangle|^{p'}\right)^{\frac{1}{p'}}\le C\|f\|_{L_p};$$

$$\left(\sum_{k=1}^{\infty}|\langle f, z_{2k-1}\rangle|^{p'}\right)^{\frac{1}{p'}}=\left(\sum_{k=1}^{\infty}\left|\int_0^1 f(x)\,4(1-x)\sin 2\pi kx\ \ dx\right|^{p'}\right)^{\frac{1}{p'}}\le$$

$$\le 4\left(\sum_{k=1}^{\infty}\left|\int_0^1 \tilde f(x)\sin 2\pi kx\ \ dx\right|^{p'}\right)^{\frac{1}{p'}}\le 4C\left\|\tilde f\right\|_{L_p}\le 4C\|f\|_{L_p},$$

where$\tilde f(x)=(1-x)f(x)$ is denoted. Hence the system $\{w_n(x)\}_{n\in Z^+}$ is a $p'-$basis in $L_p(0,1)$.

If $p\in(2;+\infty)$, then $p'\in(1;2)$ and $q=p$, and again applying the Hausdorff-Young inequality and taking into account the embedding $L_p(0,1)\subset L_{p'}(0,1)$, we obtain

$$\left(\sum_{n=0}^{\infty}|\langle f, z_n\rangle|^{p}\right)^{\frac{1}{p}}\le C\|f\|_{L_{p'}}\le C\|f\|_{L_p},$$

i.e. the system $\{w_n(x)\}_{n\in Z^+}$ is a p -basis in $L_p(0,1)$**.**

Consider the case $p=2$. According to the results of [19], the system $\{\ \{w_n(x)\}_{n\in Z^+}$ forms a Riesz basis with brackets in $L_2(0,1)$, where the lengths of the brackets are uniformly bounded ($n_{k+1}-n_k=2,\ \forall k\in N$). In addition, it follows from the previous reasoning that this system is almost normalized and uniformly minimal in $L_2(0,1)$. Thus, all the conditions of Statement 1.4 are satisfied, according to which the system $\{w_n(x)\}_{n\in Z^+}$ forms a Riesz basis in the space $L_2(0,1)$. Theorem is proved.

Let us return to the case $\alpha \neq 0$. As shown above, in this case the eigenfunctions of the spectral problem (5) do not form a basis in any space $L_p(0,1)$, $1 < p < \infty$. However, from the linear combinations of the elements of this system, it is possible to compose a new system, which will already be a basis in $L_p(0,1)$, and, accordingly, a Riesz basis in $L_2(0,1)$.

Following the work [1] we introduce to the consideration the following system

$$\varphi_{2k-1}(x) = u_{2k-1}(x);\varphi_{2k}(x) = (u_{2k}(x) - u_{2k-1}(x))(2\delta_k)^{-1}, \forall k \in N, \tag{17}$$

which is a linear combination of the system $\{u_n(x)\}_{n\in N}$. It is valid the following

Theorem 4. *The system $\{\varphi_n\}_{n\in Z^+}$ forms an equivalent to the system $\{w_n\}_{n\in Z^+}$ basis for $L_p(0,1)$, $1 < p < \infty$, with biorthogonal system $\{\psi_n\}_{n\in Z^+}$ where*

$$\psi_{2k-1} = \vartheta_{2k} + \vartheta_{2k-1}, \psi_{2k} = 2\delta_k\vartheta_{2k}, \forall k \in N. \tag{18}$$

In particular, for $p = 2$ the system $\{\varphi_n\}_{n\in Z^+}$ forms a Riesz basis in $L_2(0,1)$.

Proof. Let us show that the system of functions $\{\varphi_n\}_{n\in Z^+}$ forms a basis in $L_p(0,1)$, $1 < p < \infty$. It is obvious that it is complete and minimal in this space. Completeness follows from the completeness of the system $\{u_n\}_{n\in Z^+}$ in $L_p(0,1)$. The minimality of this system follows from the fact that it has a biorthogonal system $\{\psi_n\}_{n\in Z^+}$, defined by formula (18), which is verified directly.

From formulas (17) we have

$$\varphi_{2k-1}(x) - w_{2k-1}(x) = 0;$$

$$\varphi_{2k}(x) = \frac{1}{2\delta_k}\left(\sin\left((2\pi k + 2\delta_k)x\right) - \sin 2\pi kx\right) =$$

$$= \frac{\sin\delta_k x}{\delta_k x}\cdot x\cos\left((2\pi k + \delta_k)x\right) = (1 + O(\delta_k))\,x\cos 2\pi kx\,\left(1 + O\left(\delta_k^2\right)\right) =$$

$$= x\cos 2\pi kx + O(\delta_k) = w_{2k}(x) + O\left(\frac{1}{k}\right),$$

or

$$\varphi_{2k}(x) - w_{2k}(x) = O\left(\frac{1}{k}\right).$$

As a result, we obtain that for any $s,\ p \in (1, +\infty)$ we have

$$\sum_{n=0}^{\infty} \|\varphi_n - w_n\|_{L_p}^s < +\infty, \tag{19}$$

i.e. the systems $\{\varphi_n\}_{n\in Z^+}$ and $\{w_n\}_{n\in Z^+}$ are s -close in the space $L_p(0,1)$.

On the other hand, according to Theorem 3.1, the system $\{w_n\}_{n\in Z^+}$ is a $q-$ basis in $L_p(0,1)$, where $q = \max\{p,\ p'\}$. Choosing $s = q'$ in (19), we obtain that the systems $\{\varphi_n\}_{n\in Z^+}$ and $\{w_n\}_{n\in Z^+}$ are $q'-$close. Thus, all the conditions of Theorem 1.1 are satisfied and therefore the system $\{\varphi_n\}_{n\in Z^+}$ forms a basis in $L_p(0,1)$, equivalent to the basis $\{w_n\}_{n\in Z^+}$.

The second part of the theorem, which concerns the case $p=2$, follows from the fact that according to Theorem 3.1 in this case the system $\{w_n\}_{n\in Z^+}$ is a Riesz basis in $L_2(0,1)$, and the system equivalent to the Riesz basis is itself a Riesz basis. The theorem is proved.

Corollary 2. *The system* $\{\varphi_n\}_{n\in Z^+}$ *is a* $q-$*basis in* $L_p(0,1)$, $1<p<\infty$, *where* $q=max\{p,\ p'\}$.

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Reyhan J. Taghiyeva
Baku State University, Baku, Azerbaijan
National Aviation Academy, Baku, Azerbaijan
E-mail: reyhanabasli2015@gmail.com

Ilkin Q.Feyzullayev
Institute of Mathematics and Mechanics, The Ministry of Science and Education, Baku, Azerbaijan
E-mail: ilkin0704@mail.ru

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Asymptotics of the Eigenvalues and Eigenfunctions of a Differential Operator with a conjugation conditions and a Summable Potential

A.Q.Ahmadov

Abstract. In this paper is studied the spectral problem for a discontinuous second order differential operator with a summabl potential function and a spectral parameter in conjugation conditions, that arises by solving the problem on vibrations of a loaded string with free ends. In the case of a summable potential function, asymptotic formulas for the eigenvalues and eigenfunctions of the spectral problem are obtained.

Key Words and Phrases: Eigenvalue, eigenfunction, asymptotic formulas.

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Consider following spectral problem:

$$l\,(y) = -y^{''}(x) + q\,(x)\,y = \lambda y,\ x \in \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right), \tag{1}$$

$$\left.\begin{array}{c} y^{'}(0) = y^{'}(1) = 0, \\ y(\frac{1}{3} - 0) = y(\frac{1}{3} + 0), \\ y^{'}(\frac{1}{3} - 0) - y^{'}(\frac{1}{3} + 0) = m\lambda y\left(\frac{1}{3}\right), \end{array}\right\} \tag{2}$$

here, λ is spectral parameter, $q\,(x)$ is a complex-valued function summing over the interval $(0,1)$, m is complex nuber, and $m \neq 0$. Such spectral problems arise when the problem of vibrations of a loaded string with fixed ends is solved by applying the Fourier method [1-3]. The case of boundary conditions corresponding to a string with fixed ends (i.e. when instead of the boundary conditions $y^{'}(0) = y^{'}(1) = 0$ in (2) $y(0) = y(1) = 0$ are taken), is investigated in [4-10]. In [11], the asymptotic expressions for the eigenvalues and eigenfunctions of problem (1)–(2) in the case $q\,(x)$ were obtained, a linearization operator was constructed, and theorems on completeness and minimality were proved. Furthermore, [12,13] in the case $q\,(x) = 0$ investigated the basis properties of the eigenfunctions of this problem in the spaces $L_p(0,1) \bigoplus C$ and Morrey spaces, respectively.

1. The asymptotic of eigenvalues

Let us denote $\lambda = \rho^2, \mathrm{Im}\rho = \tau$. Also, let us denote by $y_1(x,\rho)$ the solution of equation (1) that satisfies the initial condition

$$\left.\begin{array}{l} y_1(0,\rho) = 1 \\ y_1'(0,\rho) = 0 \end{array}\right\} \tag{3}$$

in the segment $\left[0,\ \frac{1}{3}\right]$. Similarly let $y_2(x,\rho)$ be the solution that satisfies the initial condition

$$\left.\begin{array}{l} y_2(1,\rho) = 1 \\ y_2'(1,\rho) = 0 \end{array}\right\} \tag{4}$$

in the segment $\left[\ \frac{1}{3}, 1\right]$ of the same equation.

Lemma 1. The following formulas are true for the solutions $y_1(x,\rho)$ and $y_2(x,\rho)$ of the equation of (1) and their derivatives with respect to x.

$$y_1(x,\rho) = \cos\rho x + \frac{1}{\rho}\int_0^x q(t)\, y_1(t,\rho)\sin\rho(x-t)\,dt,\ 0 < x < \frac{1}{3}, \tag{5}$$

$$y_1'(x,\rho) = -\rho\sin\rho x + \int_0^x q(t)\, y_1(t,\rho)\cos\rho(x-t)\,dt,\ 0 < x < \frac{1}{3}, \tag{6}$$

$$y_2(x,\rho) = \cos\rho(1-x) - \frac{1}{\rho}\int_x^1 q(t)\, y_2(t,\rho)\sin\rho(x-t)\,dt, \qquad \frac{1}{3} < x < 1, \tag{7}$$

$$y_2'(x,\rho) = \rho\sin\rho(1-x) - \int_x^1 q(t)\, y_2(t,\rho)\cos\rho(x-t)\,dt, \qquad \frac{1}{3} < x < 1. \tag{8}$$

Proof. Since the function $y_1(x,\rho)$ is a solution of equation (1)

$$\int_0^x q(t)\, y_1(t,\rho)\ \sin\rho(x-t)\ dt =$$

$$= \int_0^x \sin\rho(x-t)\ y_1''(t,\rho)\,dt + \rho^2\int_0^x \sin\rho(x-t)\, y_1(t,\rho)\,dt. \tag{9}$$

If we intergarate the first integral on the right-hand side of the last equation twice by patrs and consider (3), we obtain following

$$\int_0^x q(t)\, y_1(t,\rho)\sin\rho(x-t)\,dt = \rho y_1(x,\rho) - \rho\cos\rho x. \tag{10}$$

That is, (5) is true. To get the equation (6), it is enough to differentiate the equation (5). Equations (7) and (8) are obtained by making similar calculations.

Lemma 2. When $\rho \to \infty$, the following asymptotic formulas hold true:

$$y_1(x,\rho) = \cos\rho\, x + O\left(\frac{e^{|\tau|x}}{|\rho|}\right),\quad x \in \left[0, \frac{1}{3}\right], \tag{11}$$

$$y_2(x,\rho) = cos\rho(1- x) + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right), x \in \left[\frac{1}{3}, 1\right]. \tag{12}$$

Let us introduce the following functions to express the subsequent results:

$$q_1(x) = \frac{1}{2}\int_0^x q(t)\,dt, q_2(x) = \frac{1}{2}\int_x^1 q(t)\,dt \tag{13}$$

Theorem 1. The eigenvalues of problem (1)-(2) are asymptotically simple and consist of two series: $\lambda_{i,n} = \rho_{i,n}^2, i = 1, 2;\ n \in Z^+,\ Z^+ = N \cup \{\emptyset\}$ and the following asymptotic expressions hold for $\rho_{i,n}$.

$$\left.\begin{array}{l} \rho_{1,n} = 3\pi n + \frac{3\pi}{2} + O\left(\frac{1}{n}\right), \\ \rho_{2,n} = \frac{3\pi n}{2} + \frac{3\pi}{4} + O\left(\frac{1}{n}\right). \end{array}\right\} \tag{14}$$

Proof. If we substitute the asymptotic expression of $y_1(x,\rho)$ from (11) into the right-hand side of (5), we obtain following:

$$y_1(x,\rho) = \cos\rho x + \frac{1}{\rho}\int_0^x q(t)\sin\rho(x-t)\left[\cos\rho t + O\left(\frac{e^{|\tau|}}{\rho}\right)\right]dt =$$

$$= \cos\rho x + \frac{1}{\rho}\int_0^x q(t)\sin\rho(x-t)\cos\rho t dt +$$

$$+ \frac{1}{\rho^2}\int_0^x q(t)\sin\rho(x-t)\ O\left(e^{|\tau|t}\right)dt =$$

$$= \cos\rho x + \frac{1}{2\rho}\int_0^x q(t)\left[\sin\rho(x-2t) + \sin\rho x\right]dt +$$

$$+ \frac{1}{\rho^2}\int_0^x q(t)\sin\rho(x-t)\ O\left(e^{|\tau|t}\right) =$$

$$= \cos\rho x + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt + \frac{\sin\rho x}{2\rho}\int_0^x q(t)\,dt +$$

$$+ \frac{1}{\rho^2}\int_0^x q(t)\sin\rho(x-t)\ O\left(e^{|\tau|t}\right)dt = \cos\rho x + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt +$$

$$+ \frac{1}{\rho}\sin\rho x\left(\frac{1}{2}\int_0^x q(t)\,dt\right) + \frac{e^{|\tau|x}}{\rho^2}\int_0^x \frac{\sin\rho(x-t)}{e^{|\tau|(x-t)}}dt.$$

Therefor,

$$y_1(x,\rho) = \cos\rho x + \frac{1}{\rho}q_1(x)\sin\rho x +$$

$$+ \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt + O\left(\frac{e^{|\tau|x}}{|\rho|^2}\right) \tag{15}$$

is true.

Also, if we substitute the asymptotic expression of $y_1(x,\rho)$ from (11) into the right-hand side of the equation (6), we obtain:

$$y_1^{'}(x,\rho) = -\rho\sin\rho x + \int_0^x q(t)\, y_1(t,\rho)\cos\rho(x-t)\,dt =$$

$$= -\rho\sin\rho x + \int_0^x q(t)\cos\rho(x-t)\left[\cos\rho t + O\left(\frac{e^{|\tau|t}}{|\rho|}\right)\right]dt =$$

$$= -\rho\sin\rho x + \frac{1}{2}\int_0^x q(t)\left[\cos\rho x + \cos\rho(2t-x)\right]dt +$$

$$+ \int_0^x q(t)\cos\rho(x-t)\;O\left(\frac{e^{|\tau|t}}{|\rho|}\right)dt = -\rho\sin\rho x +$$

$$+\frac{1}{2}\int_0^x q(t)\cos\rho x dt + \frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)\,dt +$$

$$+ \int_0^x q(t)\cos\rho(x-t)\;O\left(\frac{e^{|\tau|t}}{|\rho|}\right)dt = -\rho\sin\rho x + q_1(x)\cos\rho x +$$

$$+\frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)\,dt + O\left(\frac{e^{|\tau|x}}{|\rho|}\right)\int_0^x q(t)\,\frac{\cos\rho(x-t)}{e^{|\tau|(x-t)}}\;O(1)\,dt =$$

$$= -\rho\sin\rho x + q_1(x)\cos\rho x + \frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)\,dt + O\left(\frac{e^{|\tau|x}}{|\rho|}\right).$$

Thus,

$$y_1^{'}(x,\rho) = -\rho\sin\rho x + q_1(x)\cos\rho x +$$

$$+\frac{1}{2}\int_0^x \cos\rho(x-2t)\cdot q(t)\,dt + O\left(\frac{e^{|\tau|x}}{|\rho|}\right). \tag{16}$$

By similar calculations, we obtain the following asymptotic equalities for $y_2(x,\rho)$ and $y_2^{'}(x,\rho)$:

$$y_2(x,\rho) = \cos\rho(1-x) + \frac{1}{\rho}\cdot q_2(x)\sin\rho(1-x) +$$

$$+\frac{1}{2\rho}\int_x^1 \sin\rho(2t-x-1)\cdot q(t)\,dt + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|^2}\right), \tag{17}$$

$$y_2^{'}(x,\rho) = \rho\sin\rho(1-x) - q_2(x)\cdot\cos\rho(1-x) -$$

$$-\frac{1}{2}\int_x^1 \cos\rho(2t-x-1)\cdot q(t)\,dt + O\left(\frac{e^{|\tau|(1-x)}}{|\rho|}\right). \tag{18}$$

The solution $y(x,\rho)$ of problem (1)-(2) should be in the following form:

$$y(x,\rho)=\begin{cases} C_1y_1(x,\rho), & 0<x<\frac{1}{3}, \\ C_2y_2(x,\rho), & \frac{1}{3}<x<1, \end{cases} \tag{19}$$

Here C_1 and C_2 are complex numbers. Therefore, the function satisfies the conditions given in (2). Now, let us impose the requirement that it also satisfies the conditions in (3) and (4). In this case, to determine the coefficients C_1 and C_2 we obtain the following system:

$$\begin{cases} C_1y_1\left(\frac{1}{3},\rho\right)-C_2y_2\left(\frac{1}{3},\rho\right)=0 \\ C_1{y'}_1\left(\frac{1}{3},\rho\right)-C_2{y'}_2\left(\frac{1}{3},\rho\right)=C_1\rho^2my_1\left(\frac{1}{3},\rho\right) \end{cases} \tag{20}$$

Taking into account the expressions (15), (16), (17), and (18) in (20), we obtain:

$$\begin{cases} C_1\left(\cos\frac{1}{3}\rho+\frac{1}{\rho}q_1\sin\frac{1}{3}\rho+\frac{1}{2\rho}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt+O\left(\frac{e^{1/3|\tau|}}{|\rho|^2}\right)\right)- \\ -C_2\left(\cos\frac{2}{3}\rho-\frac{1}{\rho}q_2\sin\frac{2}{3}\rho+\frac{1}{2\rho}\int_{1/3}^{1}\sin\rho\left(2t-\frac{4}{3}\right)q(t)\,dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right)\right)=0 \\ C_1\left(-\rho\sin\frac{1}{3}\rho+q_1\cos\frac{1}{3}\rho+\frac{1}{2}\int_0^{1/3}\cos\rho\left(2t-\frac{1}{3}\right)q(t)\,dt+O\left(\frac{e^{|\tau|/3}}{|\rho|}\right)\right)- \\ -C_2\left(\rho\sin\frac{2}{3}\rho-q_2\cos\frac{2}{3}\rho-\frac{1}{2}\int_{1/3}^{1}\cos\rho\left(2t-\frac{4}{3}\right)q(t)\,dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)\right)= \\ =C_1\rho^2m\left(\cos\frac{1}{3}\rho+\frac{1}{\rho}q_1\sin\frac{1}{3}\rho+\frac{1}{2\rho}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt+O\left(\frac{e^{1/3|\tau|}}{|\rho|^2}\right)\right). \end{cases}$$

Here,

$$q_1=q_1\left(\frac{1}{3}\right),q_2=q_2\left(\frac{1}{3}\right).$$

For the determination of the eigenvalues, we obtain the following equality:

$$\Delta\left(\rho^2\right)=\begin{vmatrix} a_{11}(\rho) & a_{12}(\rho) \\ a_{21}(\rho) & a_{22}(\rho) \end{vmatrix}=0,$$

Here

$$a_{11}\rho=\cos\frac{1}{3}\rho+\frac{1}{\rho}q_1\sin\frac{1}{3}\rho+\frac{1}{2\rho}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q(t)\,dt+O\left(\frac{e^{1/3|\tau|}}{|\rho|^2}\right)$$

$$a_{12}(\rho)=-\cos\frac{2}{3}\rho+\frac{1}{\rho}q_2\sin\frac{2}{3}\rho-\frac{1}{2\rho}\int_{1/3}^{1}\sin\rho\left(2t-\frac{4}{3}\right)q(t)\,dt-O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right)$$

$$a_{21}(\rho)=\left(-\rho\sin\frac{1}{3}\rho-\rho^2m\cos\frac{1}{3}\rho\right)+\left(q_1\cos\frac{1}{3}\rho-\rho mq_1\sin\frac{1}{3}\rho\right)+$$

$$\left(\frac{1}{2}\int_0^{1/3}\cos\rho\left(2t-\frac{1}{3}\right)q\left(t\right)dt-\frac{\rho m}{2}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q\left(t\right)dt\right)+O\left(e^{1/3|\tau|}\right)$$

$$a_{22}\left(\rho\right)=-\rho\sin\frac{2}{3}\rho+q_2\cos\frac{2}{3}\rho+\frac{1}{2}\int_{1/3}^{1}\cos\rho\left(\frac{4}{3}-2t\right)q\left(t\right)dt-O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right).$$

For any arbitrary complex number z, by utilizing the inequalities

$$|\sin z|\leq e^{|\mathrm{Im}z|},\ |\cos z|\leq e^{|\mathrm{Im}z|},$$

the following results can be derived:

$$\begin{array}{ll}\left|\cos\rho\left(\frac{1}{3}-2t\right)\right|\leq e^{1/3|\tau|}, & 0\leq t\leq\frac{1}{3},\\ \left|\cos\rho\left(2t-\frac{4}{3}\right)\right|\leq e^{2/3|\tau|}, & \frac{1}{3}\leq t\leq 1,\\ \left|\sin\rho\left(2t-\frac{1}{3}\right)\right|\leq e^{1/3|\tau|}, & 0\leq t\leq\frac{1}{3},\\ \left|\sin\rho\left(\frac{4}{3}-2t\right)\right|\leq e^{2/3|\tau|}, & \frac{1}{3}\leq t\leq 1,\end{array}$$

Here Im$\rho\ =\tau$ is denoted. As $|\rho|\to\infty$, by applying the previously mentioned inequalities, the following is obtained:

$$\int_0^{1/3}q\left(t\right)\cos\rho\ \left(\frac{1}{3}-2t\right)dt=O\left(e^{1/3|\tau|}\right),$$

$$\int_{1/3}^{1}q\left(t\right)\cos\rho\ \left(2t-\frac{4}{3}\right)dt=O\left(e^{2/3|\tau|}\right),$$

$$\int_0^{1/3}q\left(t\right)\sin\rho\ \left(2t-\frac{1}{3}\right)dt=O\left(e^{1/3|\tau|}\right),$$

$$\int_{1/3}^{1}q\left(t\right)\sin\rho\left(\frac{4}{3}-2t\right)\ dt=O\left(e^{2/3|\tau|}\right).$$

By applying the asymptotic formulas above, $\Delta\left(\rho^2\right)$ can be expressed as follows:

$$\Delta\left(\rho^2\right)=\begin{vmatrix}\cos\frac{1}{3}\rho & -\cos\frac{2}{3}\rho\\ -\rho\sin\frac{1}{3}\rho-\rho^2m\cos\frac{1}{3}\rho & -\rho\sin\frac{2}{3}\rho\end{vmatrix}+$$

$$+\begin{vmatrix}\cos\frac{1}{3}\rho & \frac{1}{\rho}q_2\sin\frac{2}{3}\rho\\ -\rho\sin\frac{1}{3}\rho-\rho^2m\cos\frac{1}{3}\rho & q_2\cos\frac{2}{3}\rho\end{vmatrix}+$$

$$+\begin{vmatrix}\cos\frac{1}{3}\rho & -\frac{1}{2\rho}\int_{\frac{1}{3}}^{1}\sin\rho\left(2t-\frac{4}{3}\right)\cdot q\left(t\right)dt\\ -\rho\sin\frac{1}{3}\rho-\rho^2m\cos\frac{1}{3}\rho & \frac{1}{2}\int_{\frac{1}{3}}^{1}\cos\rho\left(\frac{4}{3}-2t\right)\cdot q\left(t\right)dt\end{vmatrix}+$$

$$+\begin{vmatrix} \cos\frac{1}{3}\rho & O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right) \\ -\rho\sin\frac{1}{3}\rho-\rho^2 m\cos\frac{1}{3}\rho & O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)\end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{\rho}q_1\sin\frac{1}{3}\rho & -\cos\frac{2}{3}\rho \\ q_1\cos\frac{1}{3}\rho-\rho m q_1\sin\frac{1}{3}\rho & -\rho\sin\frac{2}{3}\rho\end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{\rho}q_1\sin\frac{1}{3}\rho & \frac{1}{\rho}q_2\sin\frac{2}{3}\rho \\ q_1\cos\frac{1}{3}\rho-\rho m q_1\sin\frac{1}{3}\rho & q_2\cos\frac{2}{3}\rho\end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{\rho}q_1\sin\frac{1}{3}\rho & -\frac{1}{2\rho}\int_{\frac{1}{3}}^{1}\sin\rho\left(2t-\frac{4}{3}\right)\cdot q\left(t\right)dt \\ q_1\cos\frac{1}{3}-\rho m q_1\sin\frac{1}{3}\rho & \frac{1}{2}\int_{\frac{1}{3}}^{1}\cos\left(\frac{4}{3}-2t\right)\cdot q\left(t\right)dt\end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{\rho}q_1\sin\frac{1}{3}\rho & O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right) \\ q_1\cos\frac{1}{3}\rho-\rho m q_1\sin\frac{1}{3}\rho & O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)\end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int_0^{\frac{1}{3}}\sin\rho\left(\frac{1}{3}-2t\right)q\left(t\right)dt & -\cos\frac{2}{3}\rho \\ \frac{1}{2}\int_0^{\frac{1}{3}}\cos\rho\left(2t-\frac{1}{3}\right)q\left(t\right)dt-\frac{\rho m}{2}\int_0^{1/3}\sin\rho\left(\frac{1}{3}-2t\right)q\left(t\right)dt & -\rho\sin\frac{2}{3}\rho\end{vmatrix}+$$

$$+\begin{vmatrix} \frac{1}{2\rho}\int_0^{\frac{1}{3}}\sin\rho\left(\frac{1}{3}-2t\right)q\left(t\right)dt & \frac{1}{\rho}q_2\sin\frac{2}{3}\rho \\ \frac{1}{2}\int_0^{\frac{1}{3}}\cos\rho\left(2t-\frac{1}{3}\right)q\left(t\right)dt-\frac{\rho m}{2}\int_0^{\frac{1}{3}}\sin\rho\left(\frac{1}{3}-2t\right)q\left(t\right)dt & q_2\cos\frac{2}{3}\rho\end{vmatrix}+$$

$$+\begin{vmatrix} O\left(\frac{e^{1/3|\tau|}}{|\rho|^2}\right) & -\cos\frac{2}{3}\rho \\ O\left(e^{1/3|\tau|}\right) & \rho\sin\frac{2}{3}\rho\end{vmatrix}+O\left(\frac{e^{|\tau|}}{|\rho|}\right).$$

In the final expression, after expanding all the determinants and performing the corresponding calculations, the following expression for $\Delta\left(\rho^2\right)$ is obtained:

$$\Delta\left(\rho^2\right)=\cos^3\frac{1}{3}\rho\left(-2\rho^2 m+4q_1-2mq_1q_2\right)+$$

$$+\sin^3\frac{1}{3}\rho\left(4\rho-2\rho m q_2+2\rho m q_1\right)+$$

$$+\sin\frac{1}{3}\rho\left(-3\rho+2\rho m q_2-\rho m q_1-\frac{1}{\rho}q_1q_2\right)+$$

$$+\cos\frac{1}{3}\rho\left(\rho^2m-3q_1+q_2+2mq_1q_2\right)+\sin\frac{1}{3}\rho\times$$

$$\times\left(\begin{array}{c}\frac{1}{2}\int_{\frac{1}{3}}^{1}q(t)\sin\rho\left(\frac{4}{3}-2t\right)dt-\frac{mq_1}{2}\int_{\frac{1}{3}}^{1}q(t)\sin\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)+\\-\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right)+\frac{1}{2\rho}q_1\int_{\frac{1}{3}}^{1}q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)\end{array}\right)+$$

$$+\cos\frac{1}{3}\rho\left(\begin{array}{c}\frac{1}{2}\int_{\frac{1}{3}}^{1}q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt-O\left(\frac{e^{2/3|\tau|}}{|\rho|}\right)+\frac{\rho m}{2}\int_{\frac{1}{3}}^{1}q(t)\sin\rho\left(\frac{4}{3}-2t\right)dt+\\+\frac{q_1}{2\rho}\int_{\frac{1}{3}}^{1}q(t)\cos\rho\left(2t-\frac{4}{3}\right)dt+O\left(\frac{e^{2/3|\tau|}}{|\rho|^2}\right)+O\left(e^{\frac{2}{3}|\tau|}\right)\end{array}\right)+$$

$$+\sin\frac{1}{3}\rho\cos\frac{1}{3}\rho\left(\begin{array}{c}\int_0^{\frac{1}{3}}q(t)\sin\rho\left(2t-\frac{1}{3}\right)dt+q_2m\int_0^{\frac{1}{3}}q(t)\sin\rho\left(\frac{1}{3}-2t\right)dt-\\-\frac{1}{\rho}q_2\int_0^{\frac{1}{3}}q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt+O\left(\frac{e^{1/3|\tau|}}{|\rho|}\right)\end{array}\right)+$$

$$+\cos\frac{2}{3}\rho\left(\begin{array}{c}\frac{1}{2}\int_0^{\frac{1}{3}}q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt+\frac{1}{2\rho}q_2\int_0^{\frac{1}{3}}q(t)\sin\rho\left(\frac{1}{3}-2t\right)dt-\\-\frac{\rho m}{2}\int_0^{\frac{1}{3}}q(t)\sin\rho\left(\frac{1}{3}-2t\right)dt+O\left(e^{\frac{1}{3}|\tau|}\right)\end{array}\right)+$$

$$+O\left(\frac{e^{|\tau|}}{|\rho|}\right)\qquad(21)$$

Subsequently, we will consider that the parameter ρ varies within the strip $|Im\rho\ |\leq\alpha$. Under this condition, as $|\rho|\to+\infty$, the following asymptotic equalities hold:

$$\left.\begin{array}{c}O\left(\frac{e^{|\tau|}}{\rho}\right)=O\left(\frac{e^{1/3|\tau|}}{\rho}\right)=O\left(\frac{e^{2/3|\tau|}}{\rho}\right)=O\left(\frac{1}{\rho}\right)\\O\left(\frac{e^{2/3|t|}}{\rho^2}\right)=O\left(\frac{1}{\rho^2}\right),\\O\left(e^{|\tau|}\right)=O(1)\end{array}\right\}\qquad(22)$$

On the other hand,as $|\rho|\to+\infty$ within the strip $|Im\rho\ |\leq\alpha$ the following relations hold:

$$\left.\begin{array}{c}\int_0^{1/3}q(t)\cos\rho\left(\frac{1}{3}-2t\right)dt=o(1),\\\int_{1/3}^{1}q(t)\cos\rho\ \ (2t-4/3)\,dt=o(1),\\\int_0^{1/3}q(t)\sin\rho\ \ \left(2t-\frac{1}{3}\right)dt=o(1),\\\int_{1/3}^{1}q(t)\sin\left(\frac{4}{3}-2t\right)\ dt=o(1),\end{array}\right\}\qquad(23)$$

Taking (22) and (23) into account in (21), we obtain the following:

$$\Delta\left(\rho^2\right)=\rho\sin\frac{\rho}{3}\ \left(\beta_1+\beta_2\sin^2\frac{\rho}{3}\ +O\left(\frac{1}{\rho}\right)\right)+\rho^2 m\cos\frac{\rho}{3}-2\rho^2 m\cos^3\frac{\rho}{3}+O\left(\frac{1}{\rho}\right). \quad (24)$$

Here,

$$\beta_1=-3+2mq_1-mq_2,\beta_2=4+2mq_1-2mq_2$$

is denoted. From the resulting expression (23), based on Rouche's theorem, it is clear that the function $\Delta\left(\rho^2\right)$ has two series of zeros, $\rho_{1,n}$ and $\rho_{2,n}$ which are asymptotically close to the zeros of the functions $cos\frac{\rho}{3}$ and $cos\frac{2\rho}{3}$, respectively. Thus, the following asymptotic formulas hold for, $\rho_{1,n}$ and $\rho_{2,n}$:

$$\rho_{1,n}=3\pi n+\frac{3\pi}{2}+O\left(\frac{1}{n}\right),\ \rho_{2,n}=\frac{3\pi n}{2}+\frac{3\pi}{4}+O\left(\frac{1}{n}\right).$$

The estimate of the remainder term of the asymptotics in these formulas is obtained by the standard method (see [14]).

2. The asymptotic of eigenfunctions

We now proceed to determine the asymptotic formulas for the eigenfunctions associated with the problem (1)-(2).

Theorem 2. Suppose that the function $q(x)$ satisfies the conditions of Theorem 1. Then, for the eigenvalues $\lambda_{i,n}=(\rho_{1,n})^2, i=1,2; n\in N$, the corresponding eigenfunctions $y_{i,n}(x)$ satisfy the following asymptotic formulas:

$$y_{2,n}(x)=\begin{cases} cos\left(3\pi n+\frac{3\pi}{2}\right)x+O\left(\frac{1}{n}\right), & x\in\left[0,\frac{1}{3}\right], \\ O\left(\frac{1}{n}\right), & x\in\left[\frac{1}{3},1\right], \end{cases} \quad (25)$$

$$y_{2,n}(x)=\begin{cases} O\left(\frac{1}{n}\right), & x\in\left[0,\frac{1}{3}\right], \\ cos\left(\frac{3\pi n}{2}+\frac{3\pi}{4}\right)(1-x)+O\left(\frac{1}{n}\right), & x\in\left[\frac{1}{3},1\right], \end{cases} \quad (26)$$

Proof. First, let us determine the eigenfunction corresponding to the eigenvalue $\lambda_{1,n}$. To this end, let us substitute $\rho=\rho_{1,n}$ into equation (20) and choose:

$$C_{1,n}=-y_2\left(\frac{1}{3},\rho_{1,n}\right)$$

$$C_{1,n}=-\left(\cos\frac{2}{3}\ \rho_{1,n}+\frac{1}{\rho_{1,n}}q_2\sin\frac{2}{3}\ \rho_{1,n}\right)+O\left(\frac{1}{n}\right)=$$

$$=-cos\left(\pi+2\pi n+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=1+O\left(\frac{1}{n}\right),$$

$$C_{2,n}=-y_1\left(\frac{1}{3},\rho_{1,n}\right)=-\left(\cos\frac{1}{3}\ \rho_{1,n}+\frac{1}{\rho_{1,n}}q_1\sin\frac{1}{3}\ \rho_{1,n}\right)+O\left(\frac{1}{n}\right)=$$

$$=-cos\left(\frac{\pi}{2}+\pi n+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=O\left(\frac{1}{n}\right).$$

Consequently, we obtain:

$$y_{1,n}(x)=\begin{cases}\left(1+O\left(\frac{1}{n}\right)\right)y_1\left(x,\rho_{1,n}\right), & x\in\left[0,\frac{1}{3}\right]\\ O\left(\frac{1}{n}\right)y_2\left(x,\rho_{1,n}\right), & x\in\left[\frac{1}{3},1\right]\end{cases}=$$

$$=\begin{cases}cos\left(3\pi n+\frac{3\pi p}{2}\right)x+O\left(\frac{1}{n}\right), & x\in\left[0,\frac{1}{3}\right]\\ O\left(\frac{1}{n}\right), & x\in\left[\frac{1}{3},1\right]\end{cases}.$$

Now, let us determine the eigenfunction corresponding to the eigenvalue $\lambda_{2,n}$. To this end, we substitute $\rho=\rho_{2,n}$ into equation (20) and define

$$C_{1,n}=(-1)^n y_2\left(\frac{1}{3},\rho_{2,n}\right),\ C_{2,n}=(-1)^n y_1\left(\frac{1}{3},\rho_{2,n}\right).$$

Then, we obtain:

$$C_{1,n}=(-1)^n y_2\left(\frac{1}{3},\rho_{2,n}\right)=(-1)^n\left(\cos\frac{2}{3}\ \rho_{2,n}+\frac{1}{\rho_{1,n}}q_2\sin\frac{2}{3}\ \rho_{2,n}\right)+O\left(\frac{1}{n}\right)=$$

$$=(-1)^n cos\left(\pi n+\pi+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=O\left(\frac{1}{n}\right),$$

$$C_{2,n}=(-1)^n y_1\left(\frac{1}{3},\rho_{2,n}\right)=(-1)^n\left(\cos\frac{1}{3}\ \rho_{2,n}+\frac{1}{\rho_{1,n}}q_2\sin\frac{1}{3}\ \rho_{2,n}\right)+O\left(\frac{1}{n}\right)=$$

$$=(-1)^n cos\left(\pi n+\frac{\pi}{2}+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n}\right)=(-1)^n+O\left(\frac{1}{n}\right).$$

Consequently, we obtain:

$$y_{2,n}(x)=\begin{cases}O\left(\frac{1}{n}\right), & x\in\left[0,\frac{1}{3}\right],\\ cos\left(\frac{3pn}{2}+\frac{3p}{4}\right)(1-x)+O\left(\frac{1}{n}\right), & x\in\left[\frac{1}{3},1\right].\end{cases}$$

The theorem is proven.

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A.Q.Ahmadov
Karabakh University, Khankendi, Azerbaijan
Baku State University, Baku, Azerbaijan
E-mail:alirza.ahmadov@karabakh.edu.az

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Nodal solutions of nondifferentiable perturbations of some fourth-order half-linear boundary value problem

M.M. Mammadova

Abstract. In this paper, we consider nondifferentiable perturbations of a certain half-linear boundary value problem for ordinary differential equations of the fourth order. Using the results of global bifurcation for the corresponding nonlinear half-eigenvalue problems, we show the existence of nodal solutions of the considered problem.

Key Words and Phrases: nondifferentiable perturbation, half-linear problem, half-eigenvalue, global bifurcation, nodal solution

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1. Introduction

Consider the following nonlinear boundary value problem

$$\ell y \equiv (p\,(x)\,y'')'' - (q(x)y')' + r(x)y = \chi\tau(x)\,h(y) + \alpha(x)y^+ + \beta(x)y^-,\ x \in (0,l), \tag{1.1}$$

$$y(0) = y'(0) = y(l) = y'(l) = 0, \tag{1.2}$$

where $p\,(x)$ is a positive twice continuously differentiable function on $[0,l]$, $q(x)$ is a nonnegative continuously differentiable function on $[0,l]$, $r(x)$ is a real-valued continuous function on $[0,l]$, $\tau(x)$ is a positive continuous function on $[0,l]$, $\alpha(x)$ and $\beta(x)$ are real-valued continuous functions on $[0,l]$ such that $\alpha(x) \not\equiv -\beta(x)$. The functions h has the form $h = f + g$, where the real-valued functions f and g are continuous on $\mathbb{R}$ and satisfy the following conditions: there exists a positive constant M such that

$$\frac{|f(s)|}{|s|} \le M,\ s \in \mathbb{R},\ s \ne 0; \tag{1.3}$$

there exists positive constants g_0 and g_∞ such that

$$\lim_{|s|\to 0+} \frac{g(s)}{s} = g_0 \ \text{ and } \ \lim_{|s|\to +\infty} \frac{g(s)}{s} = g_\infty. \tag{1.4}$$

Nonlinear boundary value problems for ordinary differential equations of fourth order arise in the mathematical modeling of various processes in mechanics, physics, and other areas of natural science. Note that problem (1.1), (1.2) describes small bending vibrations of an inhomogeneous beam, in the cross sections of which a longitudinal force acts and both ends of which are rigidly fixed (see, e.g., [15]).

Note that in the papers of many authors the existence of nodal solutions to nonlinear boundary value problems for ordinary differential equations of the second and fourth orders was investigated (see [2, 3, 5-12, 14, 16] and references therein). Using various methods, they established the conditions under which exist solutions with a fixed oscillation count of the nonlinear problems under consideration. Should be noted that in [4, 6, 14] established the existence of nodal solutions of nonlinear perturbations of half-linear boundary value problems.

In this paper, we consider the question of the existence of nodal solutions to problem (1.1), (1.2), depending on the parameter χ. Under some additional conditions on the data of this problem, using the bifurcation technique, we establish intervals of this parameter in which there are solutions to problem (1.1), (1.2), contained in classes of functions with a fixed number of simple nodal zeros.

2. Preliminary

Let $(b.c.)$ be the set of functions $y \in C^1[0,l]$ satisfying the boundary conditions (2).

By E we denote the Banach space $C^3[0,l] \cup (b.c.)$ with the norm $||y||_3 = \sum\limits_{j=0}^{3} ||y^{(j)}||_\infty$, where $||y||_\infty = \max\limits_{x\in[0,l]} |y(x)|$.

From on ν we will denote either $+$ or $-$; $-\nu$ we will denote the opposite sign to ν.

For each $k \in \mathbb{N}$ and each ν let S_k^ν be the set of functions of the space E constructed in [1, § 3] using the Prüfer-type transformation. Note that these classes consist of functions having the oscillatory properties of eigenfunctions (and their derivatives) of the linear spectral problem which obtained from the half-linear problem

$$\begin{cases} \ell(y) \equiv \lambda\tau(x)y + \alpha(x)y^+ + \beta(x)y^-, x \in (0,l), \\ y \in (b.c.). \end{cases} \tag{2.1}$$

by setting $\alpha \equiv \beta \equiv 0$.

We have the following oscillation theorem for problem (2.1).

Theorem 2.1 [6, Theorem 2.1] (see also [14, Theorem 3.3]. *There exist two unbounded sequences* $\{\lambda_k^+\}_{k=1}^\infty$ *and* $\{\lambda_k^-\}_{k=1}^\infty$ *of simple half-eigenvalues of problem (2.1) such that*

$$\lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ < \ldots \text{ and } \lambda_1^- < \lambda_2^- < \ldots < \lambda_k^- < \ldots ;$$

the half-eigenfucntions y_k^+ *and* y_k^- *corresponding to the half-eigenvalues* λ_k^+ *and* λ_k^- *lie in* S_k^+ *and* S_k^-*, respectively. Furthermore, aside from solutions on the collection of the* *half-lines* $\{(\lambda_k^+, ty_k^+) : t > 0\}$ *and* $\{(\lambda_k^-, ty_k^-) : t > 0\}$ *and trivial ones, problem (2.1) has no other solutions.*

By (1.4) for the function g we have the following representations:

$$g(s) = g_0 s + s\,\xi(s) \text{ and } g(s) = g_\infty s + s\zeta(s), s \in \mathbb{R}, s \neq 0, \tag{2.2}$$

where

$$\lim_{|s|\to 0+} \xi(s) = 0 \text{ and } \lim_{|s|\to +\infty} \zeta(s) = 0. \tag{2.3}$$

Remark 2.1. We can extend ξ to $s = 0$ by $\xi(0) = 0$, and consequently, $\xi \in C(\mathbb{R})$.

Let

$$\varphi(s) = s\xi(s) \text{ and } \phi(s) = s\zeta(s), \; s \in \mathbb{R}. \tag{2.4}$$

Then it follows from (2.3) that

$$\lim_{|s|\to 0+} \frac{\varphi(s)}{s} = 0 \text{ and } \lim_{|s|\to +\infty} \frac{\phi(s)}{s} = 0. \tag{2.5}$$

Remark 2.2. By Remark 2.1 we have $\varphi \in C(\mathbb{R})$ and $\varphi(0) = 0$. In other hand by (2.2) and (2.4) we get

$$\phi(s) = g_0 s - g_\infty s + \varphi(s), \; s \in \mathbb{R},$$

which implies that $\phi \in C(\mathbb{R})$ and $\phi(0) = 0$.

To establish the existence of nodal solutions to problem (1.1), (1.2) we need the following result.

Lemma 2.1. *The following relations hold:*

$$||\varphi(u)||_\infty = o(||u||_3) \text{ as } ||u||_3 \to 0 \; (u \in E); \tag{2.6}$$

$$||\phi(u)||_\infty = o(||u||_3) \text{ as } ||u||_3 \to 0 \; (u \in E); \tag{2.7}$$

$$||f(u)||_\infty \leq M||u||_\infty \text{ for any } u \in E. \tag{2.8}$$

Proof. We define the continuous functions

$$\tilde{\varphi} : [0, +\infty) \to [0, +\infty) \text{ and } \tilde{\phi} : [0, +\infty) \to [0, +\infty)$$

as follows:

$$\tilde{\varphi}(t) = \max_{0\leq|s|\leq t} |\varphi(s)| \text{ and } \tilde{\phi}(t) = \max_{0\leq|s|\leq t} |\phi(s)|. \tag{2.9}$$

Obviously, the functions $\tilde{\varphi}$ and $\tilde{\phi}$ are nondecreasing on the half-interval $[0, +\infty)$. Hence for any $t \in (0, +\infty)$ there exists $s^*(t) \in (-t, t)$, $s^*(t) \neq 0$, such that

$$\tilde{\varphi}(t) = \max_{0\leq|s|\leq t} |\varphi(s)| = |\varphi(s^*(t))|,$$

and consequently,

$$\frac{\tilde{\varphi}(t)}{t} = \frac{|\varphi(s^*(t))|}{|s^*(t)|}\frac{|s^*(t)|}{t} \leq \frac{|\varphi(s^*(t))|}{|s^*(t)|}. \tag{2.10}$$

Since $|s^*(t)| \le t$, by (2.5), it follows from (2.10) that

$$\lim_{t\to 0+} \frac{\tilde{\varphi}(t)}{t} = 0. \tag{2.11}$$

By the first relation of (2.9) for any $u \in E$ we get

$$\frac{|\varphi(u)|}{||u||_3} = \frac{\tilde{\varphi}(|u|)|}{||u||_3} \le \frac{\tilde{\varphi}(||u||_\infty)}{||u||_3} \le \frac{\tilde{\varphi}(||u||_3)}{||u||_3},$$

whence implies that

$$\frac{||\varphi(u)||_\infty}{||u||_3} \le \frac{\tilde{\varphi}(||u||_3)}{||u||_3}. \tag{2.12}$$

By (2.11) from (2.12) we obtain (2.6).

For any $t \in (0, +\infty)$ there exists $s^\bullet(t) \in (-t, t)$, $s^\bullet(t) \neq 0$, such that

$$\tilde{\phi}(t) = \max_{0\le|s|\le t} |\phi(s)| = |\phi\,(s^\bullet(t))|.$$

Then by the second relation of (2.9) we get

$$\frac{\tilde{\phi}(t)}{t} = \frac{|\phi(s^\bullet(t))|}{t} = \frac{|\phi(s^\bullet(t))|}{|s^\bullet(t)|}\frac{|s^\bullet(t)|}{t} \le \frac{|\phi(s^\bullet(t))|}{|s^\bullet(t)|}. \tag{2.13}$$

If $t \to +\infty$, then either

(a) $|s^\bullet(t)| \to 0$, or

(b) $|s^\bullet(t)| \to +\infty$, or

(c) there exist positive constants κ_0 and κ_∞ such that $\kappa_0 \le |s^\bullet(t)| \le \kappa_\infty$.

By Remark 2.2 we have $\phi \in C(\mathbb{R})$, and consequently, there exists a positive constant K such that

$$|\phi(s)| \le K \text{ for any } s \in \mathbb{R},\ \kappa_0 \le |s| \le \kappa_\infty. \tag{2.14}$$

In the case (a) by Remark 2.1 it follows from (2.13) that

$$\frac{\tilde{\phi}(t)}{t} = \frac{|\phi(s^\bullet(t))|}{t} \to 0 \text{ as } t \to +\infty;$$

in the case (b) by the second relation of (2.5) from (2.13) we obtain

$$\frac{\tilde{\phi}(t)}{t} \le \frac{|\phi(s^\bullet(t))|}{|s^\bullet(t)|} \to 0 \text{ as } t \to +\infty;$$

in the case (c) by (2.14) we get

$$\frac{\tilde{\phi}(t)}{t} = \frac{|\phi(s^\bullet(t))|}{|s^\bullet(t)|}\frac{|s^\bullet(t)|}{t} \le \frac{K}{\kappa_0}\frac{\kappa_1}{t} \to 0 \text{ as } t \to +\infty.$$

Thus we show that

$$\frac{\tilde{\phi}(t)}{t} \to 0 \text{ as } t \to +\infty. \tag{2.15}$$

Since the function $\tilde{\phi}$ is nondecreasing on $(0, +\infty)$ for any $u \in E$, $u \neq 0$, we have the following relation

$$\frac{|\phi(u)|}{||u||_3} \leq \frac{\tilde{\phi}(|u|)}{||u||_3} \leq \frac{\tilde{\phi}(||u||_\infty)}{||u||_3} \leq \frac{\tilde{\phi}(|_3|u||_3)}{||u||_3}.$$

From the last relation we obtain

$$\frac{||\phi(u)||_\infty}{||u||_3} \leq \frac{\tilde{\phi}(|_3|u||_3)}{||u||_3},$$

whence, by relation (2.13), implies (2.7).

Finally, due to (1.3) we get inequality (2.8). The proof of this lemma is complete.

3. Behavior of global continua of nontrivial solutions bifurcating from zero and infinity of an auxiliary nonlinear half-eigenvalue problem

To investigate the existence of nodal solutions to problem (1.1), (1.2), we consider the following nonlinear half-eigenvalue problem

$$\begin{cases} \ell(y) = \lambda\chi g_0\tau(x)y + \alpha(x)y^+ + \beta(x)y^- + \chi\tau(x)f(y) + \chi\tau(x)\varphi(y),\ x \in (0,l), \\ y \in (b.c.). \end{cases} \tag{3.1}$$

Remark 3.1. Let $\chi \in \mathbb{R}$, $\chi \neq 0$, be fixed. Then the first relation of (2.5) shows that (3.1) is a bifurcation from zero problem. Due to relations (2.6) and (2.8) of Lemma 2.1, we can apply the results of Sections 2 and 3 of [7] to problem (3.1). Then, by Lemma 2.2 and Theorem 3.1 of [7], for each $k \in \mathbb{N}$ and each ν, there exists a component C_k^ν of the set of nontrivial solutions of problem (3.1) which bifurcates from $I_k \times \{0\}$, is contained in $\mathbb{R} \times S_k^\nu$ and is unbounded in $\mathbb{R} \times E$ (in this case either C_k^ν meet (λ, ∞) for some $\lambda \in \mathbb{R}$ or the projection of C_k^ν onto $\mathbb{R} \times \{0\}$ is unbounded), where

$$I_k^\nu = \left[\tilde{\lambda}_k^\nu - \frac{N_\alpha + N_\beta}{\chi\tilde{\tau}_0} - \frac{M}{g_0},\ \tilde{\lambda}_k^\nu + \frac{N_\alpha + N_\beta}{\chi\tilde{\tau}_0} + \frac{M}{g_0}\right], \tag{3.2}$$

$\tilde{\lambda}_k^+$ and $\tilde{\lambda}_k^-$ are k-th half-eigenvalues of the half-linear problem

$$\begin{cases} \ell(y) = \lambda\chi g_0\tau(x)y + \alpha(x)y^+ + \beta(x)y^-, x \in (0,l), \\ y \in (b.c.). \end{cases} \tag{3.3}$$

$$\tilde{\tau}_0 = g_0\tau_0, \tau_0 = \min_{x\in[0,l]} \tau(x), N_\alpha = \max_{x\in[0,l]} |\alpha(x)|, N_\beta = \max_{x\in[0,l]} |\beta(x)|.$$

By (3.3) it follows from (2.1) that

$$\lambda_k^\nu = \tilde{\lambda}_k^\nu \chi g_0 \text{ for each } k \in \mathbb{N} \text{ and each } \nu, \tag{3.4}$$

where λ_k^+ and λ_k^- are k-th half-eigenvalues of the half-linear problem (2.1). Then, by (3.4), from (3.2) we get

$$I_k^\nu = \left[\frac{\lambda_k^+}{\chi g_0} - \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{M}{g_0}, \frac{\lambda_k^\nu}{\chi g_0} + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M}{g_0}\right]. \tag{3.5}$$

Remark 3.2. By the second relations of (2.2) and (2.4), we rewrite problem (3.1) in the following form

$$\begin{cases} \ell(y) = \left(\lambda + \frac{g_\infty}{g_0} - 1\right) \chi g_0 \tau(x) y + \alpha(x) y^+ + \beta(x) y^- + \chi \tau(x) f(y) + \\ + \chi \tau(x) \phi(y), \ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{3.6}$$

The second relation of (2.5) shows that problem (3.6) is a bifurcation at infinity problem. By the relations (2.6)-(2.8) of Lemma 2.1, we can apply the results of [6, Section 3] and [8, Section 3] to problem (3.6). Then, by [8, Theorem 3.1 and Theorem 3.2], for each $k \in \mathbb{N}$ and each ν, there exists a component D_k^ν of the set of nontrivial solutions of problem (3.6) which emanates from $J_k \times \{\infty\}$, is contained in $\mathbb{R} \times S_k^\nu$ and either meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$ or its projection onto $\mathbb{R} \times \{0\}$ is unbounded, where

$$J_k^\nu = \left[\bar{\lambda}_k^\nu - \frac{N_\alpha + N_\beta}{\tilde{\tau}_0} - \frac{M}{g_0}, \bar{\lambda}_k^\nu + \frac{N_\alpha + N_\beta}{\tilde{\tau}_0} + \frac{M}{g_0}\right], \tag{3.7}$$

$\bar{\lambda}_k^+$ and $\bar{\lambda}_k^-$ are k-th half-eigenvalues of the half-linear problem

$$\begin{cases} \ell(y) = \left(\lambda + \frac{g_\infty}{g_0} - 1\right) \chi g_0 \tau(x) y + \alpha(x) y^+ + \beta(x) y^-, x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{3.8}$$

By (3.8) it follows from (2.1) that for each $k \in \mathbb{N}$ and each ν the relation

$$\lambda_k^\nu = \left(\bar{\lambda}_k^\nu + \frac{g_\infty}{g_0} - 1\right) \chi g_0$$

holds. Then it follows from last relation that

$$\bar{\lambda}_k^\nu = \frac{\lambda_k^\nu}{\chi g_0} - \frac{g_\infty}{g_0} + 1.$$

Consequently, from (3.7) we obtain

$$J_k^\nu = \left[\frac{\lambda_k^\nu}{\chi g_0} - \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{g_\infty + M}{g_0} + 1, \frac{\lambda_k^\nu}{\chi g_0} + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{g_\infty - M}{g_0} + 1\right]. \tag{3.9}$$

We have the following result.

Lemma 3.2. *If C_k^ν meets (λ, ∞) for some $\lambda \in \mathbb{R}$, then $\lambda \in J_k$, and if D_k^ν meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, then $\lambda \in I_k$.*

The proof of this lemma follows from [7, Lemma 2.2] and [8, Remark 3.2] (see also [13, Theorem 3.3]) due to the above arguments.

Lemma 3.2. *For each $k \in \mathbb{N}$ and each ν, the projection of C_k^ν and D_k^ν onto $\mathbb{R} \times \{0\}$ are bounded.*

Proof. Let $k \in \mathbb{N}$ and ν are arbitrary fixed and let $(\hat{\lambda}, \hat{y}) \in \mathbb{R} \times S_k^\nu$ be the solution of problem (2.15), where $|\lambda|$ is large enough.

We introduce the following notations:

$$\hat{\psi}(x) = \begin{cases} \frac{f(\hat{y}(x))}{\hat{y}(x)} & \text{if } \ \hat{y}(x) \neq 0, \\ 0 & \text{if } \ \hat{y}(x) = 0, \end{cases}, \ \text{and } \ \hat{\xi}(x) = \xi(\hat{y}(x)), x \in [0, l]. \tag{3.10}$$

Then $\lambda = \hat{\lambda}_k^\nu$, where $\hat{\lambda}_k^\nu$ is the k-th half-eigenvalue of the half-linear problem

$$\begin{cases} \ell(y) = \lambda \chi g_0 \tau(x) y + a(x) y^+ + b(x) y^- + \chi \tau(x)(\hat{\psi}(x) + \hat{\zeta}(x)) y, \ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{3.11}$$

In view of condition (1.3) by the first notation of (3.10) we obtain

$$|\hat{\psi}(x)| \leq M, \ x \in [0, l]. \tag{3.12}$$

It follows from relations (2.2) and (2.4) that

$$\xi(s) = g_\infty - g_0 + \zeta(s), \ s \in \mathbb{R}.$$

Since $\zeta \in C(\mathbb{R})$ by (2.3) there exists a positive constant L such that

$$|\zeta(s)| \leq L|s|, \ s \in \mathbb{R}.$$

which, by the second relation of (3.10), implies that

$$|\hat{\xi}(x)| \leq L, \ x \in [0, l]. \tag{3.13}$$

Then, in view of (3.12) and (3.13), it follows from [3, relation (2.16)] that

$$|\hat{\lambda}_k^\nu - \lambda_k^\nu| \leq \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M + L}{g_0}. \tag{3.14}$$

Therefore, we have the following estimate

$$|\hat{\lambda}| = |\hat{\lambda}_k^\nu| \leq |\hat{\lambda}_k^\nu - \lambda_k^\nu| + |\lambda_k^\nu| \leq |\lambda_k^\nu| + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M + L}{g_0},$$

which contradicts the fact that $|\hat{\lambda}|$ is large enough.

Thus, we have shown that the projection of C_k^ν onto $\mathbb{R} \times \{0\}$ is bounded. In a similar way it can be shown that the projection of D_k^ν onto $\mathbb{R} \times \{0\}$ is also bounded. The proof of this lemma is complete.

Corollary 3.1. *For each $k \in \mathbb{N}$ and each ν the components C_k^ν and D_k^ν of the set of nontrivial solutions of problem (3.1) coincide.*

Thus, by Corollary 3.1, we have the following result.

Theorem 3.1. *For each $k \in \mathbb{N}$ and each ν the component C_k^ν of the set of nontrivial solutions to problem (3.1) is contained in $\mathbb{R} \times S_k^\nu$ and meets the intervals $I_k \times \{0\}$ and $J_k \times \{\infty\}$.*

4. Existence of nodal solutions to problem (1.1), (1.2)

The following theorem is the main result of this paper.

Theorem 4.1. *Let the following conditions hold:*

(i) $g_0 > M$ and $g_\infty > M$;

(ii) for some $k \in \mathbb{N}$ and some ν, $\lambda_k^\nu - \frac{N_a+N_b}{\tau_0} > 0$, and either

$$\frac{\lambda_k^\nu}{g_0 - M} + \frac{N_\alpha + N_\beta}{\tau_0(g_0 - M)} < \chi < \frac{\lambda_k^\nu}{g_\infty + M} - \frac{N_\alpha + N_\beta}{\tau_0(g_\infty + M)}, \tag{3.15}$$

or

$$\frac{\lambda_k^\nu}{g_\infty - M} + \frac{N_\alpha + N_\beta}{\tau_0(g_\infty - M)} < \chi < \frac{\lambda_k^\nu}{g_0 + M} - \frac{N_\alpha + N_\beta}{\tau_0(g_0 + M)}. \tag{3.16}$$

Then there exists a solution υ_k^ν of problem (1.1), (1.2) such that $\upsilon_k^\nu \in S_k^\nu$, i.e. the function υ_k^ν has exactly $k-1$ simple nodal zeros in the interval $(0,l)$.

Proof. It is obvious that any nontrivial solution $(\lambda, y) \in \mathbb{R} \times E$ with $\lambda = 1$ of problem (3.1) is a nontrivial solution of problem (1.1), (1.2). Then, according to Theorem 3.1, if for some $k \in \mathbb{N}$ the right end of the interval I_k is to the left of 1 and the left end of the interval J_k is to the right of 1 on the real axis, or the right end of the interval J_k is to the left of 1 and the left end of the interval I_k is to the right of 1 on the real axis, then problem (1.1), (1.2) will have a solution that is contained in the class S_k^ν.

Let conditions (i) and (ii) of this theorem be satisfied. If (3.15) holds, then we have the following relations

$$\frac{\lambda_k^\nu}{g_0 - M} + \frac{N_\alpha + N_\beta}{\tau_0(g_0 - M)} < \chi \ \text{ and } \ \chi < \frac{\lambda_k^\nu}{g_0 + M} - \frac{N_\alpha + N_\beta}{\tau_0(g_0 + M)},$$

which implies that

$$\frac{\lambda_k^\nu}{\chi g_0} + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M}{g_0} < 1 \ \text{ and } \ 0 < \frac{\lambda_k^\nu}{\chi g_0} - \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{g_\infty + M}{g_0}. \tag{3.17}$$

From (3.17) we obtain

$$\frac{\lambda_k^\nu}{\chi g_0} + \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} + \frac{M}{g_0} < 1 < \frac{\lambda_k^\nu}{\chi g_0} - \frac{N_\alpha + N_\beta}{\chi g_0 \tau_0} - \frac{g_\infty + M}{g_0} + 1,$$

which show that the right end of the interval I_k is to the left of 1, and the left end of the interval J_k is to the right of 1 on the real axis.

If (3.16) is satisfied, then it can be shown in a similar way that the right end of the interval J_k is to the left of 1, and the left end of the interval I_k is to the right of 1 on the real axis. The proof of this theorem is complete.

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Masuma M. Mammadova
Baku State University, Baku, Azerbaijan
E-mail: memmedova.mesume@inbox.ru

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