

**ON THE DIFFERENCE PROPERTIES OF TRACE OF THE FUNCTIONS IN SOBOLEV
 TYPE SPACES**

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Abstract. In this paper, the difference traces of function $D^\nu(f - \Phi_{l-1})(X)$ are studied. Using the theorems of generalized functions $D^\nu(f - \Phi_{l-1})(X)$ in the spaces

$$\bigcap_{k=1}^n \Lambda_{p_k, \theta_k}^{(n^k, N^k)}(G, \varphi^k),$$

which are called continuation spaces, studied in previous works by A.S. Platonov, particularly in paper [5]. Without changing the order of derivation, several theorems and lemmas are proved, their results being applicable to functions reduced by the corresponding homothety Φ_{l-1} , and new results are obtained for traces of the function $D^\nu(f - \Phi_{l-1})$. In this paper, properties of traces of the function $D^\nu(f - \Phi_{l-1})$ in Γ_S are studied. Theorem 1 in this paper is a new result for the function $D^\nu(f - \Phi_{l-1})$.

Keywords. Sobolev type function spaces, weak derivatives, polynomial of several variables, integral representation.

INTRODUCTION

Theorem 1. Let

$$f \in \bigcap_{k=1}^n \Lambda_{p_k, \theta_k}^{(m, N^k)}(G, \varphi^k) \tag{1}$$

and 1)-4) Theorem 1 [3] hold.

Let the given " integral valued non --negative vectors"

$$\begin{cases} v = (v_1, \dots, v_n), \\ \alpha = (\alpha_1, \dots, \alpha_s, 0, \dots) = (\alpha_1, \dots, \alpha_s) \quad (1 \leq s \leq n - 1) \end{cases} \tag{2}$$

be such that the following subordination conditions are fulfilled:

$$\left. \begin{aligned} \alpha_j + v_j &\geq m_j^k + N_j^k \quad (j \neq k) \\ \alpha_k + v_k &< m_k^k + N_k^k \end{aligned} \right\} (k = 1, 2, \dots, n) \tag{3}$$

We are give an integral-valued non-negative vector

$$N = (N_1, \dots, N_s, 0 \dots 0) = (N_1, \dots, N_s) \quad (4)$$

with a non-empty support

$$E_N = \text{supp}N \subset \{1, 2, \dots, S\} \quad (5)$$

and such a vector-function

$$\varphi(t) = (\varphi_1(t_1), \dots, \varphi_s(t_s)) \quad (6)$$

that its coordinate -functions

$$\left. \begin{aligned} \varphi_j = \varphi_j(t_j) > 0 & \quad \text{for } t_j \in E_1 \setminus \{0\} \\ \varphi_j(t_j) \downarrow 0 & \quad \text{for } |t_j| \downarrow 0 \end{aligned} \right\} \quad (7)$$

for and

$$\left\{ \begin{aligned} \frac{\varphi_j(\xi)}{\xi^{N_j}} \left(\int_0^\xi \left(\frac{t_j^{N_j}}{\varphi_j(t_j)} \right) \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}} & \leq \text{const} < \infty \\ \varphi_j(\xi) \left(\int_0^\xi \left(\frac{1}{\varphi_j(t_j)} \right) \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}} & \leq \text{const} < \infty \end{aligned} \right. \quad (8)$$

for all $j \in E_N = \text{supp}N$, where $\theta \geq \theta_k$ ($k = \overline{1, n}$).

Let the following conditions hold

$$\begin{aligned} H_{k,S}(1) &= \int_0^1 \prod_{j=1}^n \left(a_j(v)^{m_j^k - v_j - \alpha_j - \frac{1}{p_k}} \right) \prod_{j=1}^S a_j(v)^{\frac{1}{q}} \times \\ &\times \frac{\prod_{j \in E_{N^k}} \varphi_j^k(a_j(v)) da_k(v)}{\prod_{j \in E_N} \varphi_j(a_j(v)) a_k(v)} < \text{const} < \infty \end{aligned} \quad (9)$$

for all $k = 1, 2, \dots, n$.

Then there exists such a polynomial

$$\Phi_{l-1} = \Phi_{l-1}(x, f) \quad (10)$$

of degree $S_k - 1 = (S_k = m_k^k + N_k^k)$ for each of the variables X_k ($k = \overline{1, n}$) that in the intersection

$$\Gamma_S = G \cap \{X_{S+1} = X_{S+1}^0 = \text{const}, X_n = X_n^0 = \text{const}\} \quad (11)$$

we have the inclusion

$$D^v(f - \Phi_{l-1})|_{\Gamma_S} \in \Lambda_{q,\theta}^{(\alpha,N)}(\Gamma_S, \varphi) \quad (12)$$

and the following integral inequalities are valid

$$\|D^v(f - \Phi_{l-1})|_{\Gamma_S}\|_{\Lambda_{q,\theta}^{(\alpha,N)}} \leq c \sum_{k=1}^n H_{k,S}(1) \|f\|_{\Lambda_{p_k,\theta_k}^{(n^k,N^k)}(G,\varphi^k)} \quad (13)$$

where $1 \leq \theta < \infty$ c is a constant independent of the function $f = f(x)$.

Note that the proof of this theorem is conducted step by step by evaluating appropriate integral operators generated by new integral representations of functions at the points $x \in E_n$ and we prove new integral inequalities distinguished as separate lemmas. These auxiliary integral inequalities are necessary for proving main integral inequalities (13) of this theorem.

Step 1. General reasoning's. The domain

$$G \subset C_\varepsilon(a; 1), \quad (14)$$

consequently there exist subdomains

$$\Omega_1, \Omega_2, \dots, \Omega_M \subset G, \quad (15)$$

satisfying the horn condition and overlapping the domain $G \subset E_n$, i.e.

$$G \subset \bigcup_{i=1}^M \Omega_i \quad (16)$$

As the same time

$$G \subset \bigcup_{i=1}^M \Omega_{i,\varepsilon} \quad (17)$$

Where

$$\Omega_{i,\varepsilon} = \{X \in \Omega_i; \rho(X; G \setminus \Omega_i) > \varepsilon\} \quad (18)$$

is a set of points $X \in \Omega_i$ distant from $G \setminus \Omega_i$ at a distance $\rho > \varepsilon > 0$. We introduce a set of auxiliary functions by means of the equality (2) from [2];

$$B_i(t) = B_i(D^v(f - \Phi_{l-1}); \Omega_i + R_{\delta i}) =$$

$$\begin{aligned}
 &= \sum_{k=1}^n (-1)^{|m^{k+v+a}|} C_k \int_0^1 \omega_k(v) \frac{da_k(v)}{a_k(v)} \int_{E_{|E_{N^k}|}} dz \times \\
 &\times \int_{E_n} \left\{ \Delta^{N^k} \left(\frac{z}{N^k}; \Omega_i + R_{\delta^i} \right) D^{m^k} f(x+y) \right\} M_{k,\delta^i} \left(\frac{y}{a(v)}; \frac{z}{a(v)} \right) dy = \\
 &= \sum_{k=1}^n M_{i,\delta^i}(f) \quad \text{for all } i = \overline{1, M}
 \end{aligned} \tag{19}$$

where

$$E_{N^k} = \sup pN^k, \tag{20}$$

$$\omega_k(v) = \prod_{j \in E_{N^k}} (a_j(v))^{-1} \prod_{j=1}^n (a_j(v))^{m_j^k - \nu_j - \alpha_j - 1} \tag{21}$$

for all $k \in l_n = \{1, 2, \dots, n\}$.

Reasoning as in the proof of theorem 5.1 from [4], we obtain a chain of integral inequalities

$$\begin{aligned}
 \|D^\nu(f - \Phi_{l-1})|_{\Gamma_S}\|_{\Lambda_{q,\theta}^{(\alpha,N)}} &\leq c \sum_{k=1}^n \|B_i(f)\|_{\Lambda_{q,\theta}^{(0,N)}(E_S,\varphi)} \leq \\
 &\leq c \sum_{i=1}^M \sum_{k=1}^n \|N_{k,\delta^i}\|_{\Lambda_{q,\theta}^{(0,N)}(E_S,\varphi)}.
 \end{aligned} \tag{22}$$

Here

$$N_{k,\delta^i}(f) = N_{k,\delta^i}^*(f) + N_{k,\delta^i}^{**}(f) \tag{23}$$

and this equality is understood as

$$\begin{aligned}
 N_{k,\delta^i}(f) &= c \left(\int_0^{g(t)} + \int_{g(t)}^1 \right) \bar{\omega}(v) \frac{da_k(v)}{a_k(v)} \times \\
 &\times \int_{\bar{0}}^{\bar{a}(v)} \prod_{j \in E_{N^k}} \frac{dz_j}{p_k} \int_{\bar{0}}^{\bar{a}(v)} d\bar{y}^S \int_{X_{S+1}^0}^{X_{S+1}^0 + a_{S+1}(v)} dy_{S+1}^* \dots
 \end{aligned}$$

$$\dots \int_{X_n^0}^{X_n^0 + a_n(v)} \left| \frac{\Delta^{N^k} \left(\frac{Z}{N^k}; \Omega_i + R_{\delta^i} \right) D^{m^k} f(\bar{X}^S \bar{y}^S y_{S+1}^*, \dots, y_S^*)}{\prod_{j \in E_{N^k}} \varphi_j^k(z_j)} \right| dy^n \quad (24)$$

Note that

$$\begin{aligned} \vec{X}^S &= (X_1, \dots, X_S), \quad \vec{y}^S = (y_1, \dots, y_S), \quad d\vec{y}^S = dy_1, \dots, dy_S \\ \overrightarrow{a(v)} &= (a_1(v), \dots, a_n(v)) \quad \vec{0} = (0, \dots, 0) \\ \bar{\omega}_k(v) &= \omega_k(v) \left\{ \prod_{j \in E_{N^k}} \varphi_j^k(v) (a_j(v))^{\frac{1}{p_k} - \gamma_j} \right\} \end{aligned} \quad (25)$$

where $\omega_k(v)$ are determined by the equalities () from [3], and the numbers $\gamma_j > 0$ are rather small numbers.

The function

$$g(t) = \prod_{j=1}^s (a^{-1}(t_j)) \quad (26)$$

therewith, $a^{-1}(t_j)$ is an inverse function with respect to $t_j = a_j(v)$, i.e.

$$\begin{cases} a_j^{-1}(a_j(v)) = v \\ a_j(a_j^{-1}(t_j)) = t_j \end{cases} \quad (27)$$

Step 2. The estimation of integral operators

$$\|N_{k,\delta^i}\|_{\Lambda_{q,\theta}^{(0,N)}(E_S,\varphi)} \quad \text{for } k \in l_n, \text{ and } i = 1, 2, \dots, M.$$

Retaining the denotations from [4], after some simple transformation, we have

$$\begin{aligned} \|N_{k,\delta^i}\|_{\Lambda_{q,\theta}^{(0,N)}(E_S,\varphi)} &\leq c \sum_{e \subseteq E_n} \left(\int_{E_{|E_n|}^{\pm e}} \left\| \frac{N^N \left(\frac{t}{N} \right) N_{k,\delta^i}(f)}{\prod_{j \in E_N} (\varphi_j(t_j))} \right\|_{L_q(E_S)}^\theta \prod_{j \in E_N} \frac{dt_j}{t_j} \right)^{1/\theta} \\ &\leq c \sum_{e \subseteq E_n} \sum_{e^* \subseteq e} A_{k,\delta^i,e^*}(f). \end{aligned} \quad (28)$$

Here

$$N = (N_1, \dots, N_S), \quad E_N = \sup pN \quad (29)$$

$$E_{|E_N|} = \{t \in E_S; t_j = 0 \quad j \in \{1, 2, \dots, s\} \setminus E_N\} \quad (30)$$

$$E_{|E_N|}^{\pm e} = \left\{ t \in E_{|E_N|}; \begin{array}{ll} 0 < t_j < \infty & j \in E_N \\ -\infty < t_j < 0 & j \in E_N \setminus e \end{array} \right\} \quad (31)$$

therewith

$$\left. \begin{array}{l} E_{|E_N|}^+ = E_{|E_N|}^{\pm E_N} \\ E_{|E_N|}^- = E_{|E_N|}^{\pm \Phi} \end{array} \right\} \quad (32)$$

The integral expression $A_{k, \delta^i, e^*}(f)$ qala as the right-hand side of inequalities (28) are determined by the equations

$$A_{k, \delta^i, e^*}(f) = \left(\int_{\bar{0}}^{\overline{a(1)}} \prod_{j \in e^*} \frac{dt_j}{t_j} \int_{\frac{a(1)}{a(1)}}^{\bar{\infty}} \left\| \frac{\Delta^N \left(\frac{t}{N} \right) N_{k, \delta^i}(f)}{\prod_{j \in E_N} (\varphi_j(t_j))} \right\|_{L_q(E_S)}^\theta \prod_{j \in E_N \setminus e^*} \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}} \quad (33)$$

for $k \in l_n = \{1, 2, \dots, n\}$, $i = 1, 2, \dots, M$ and for every subset e^* from $e \subset E_N$.

Note that

$$\begin{aligned} A_{k, \delta^i, e^*}(f) &\leq c \prod_{j \in E_N \setminus e^*} \varphi_j(a_j(1)) \times \\ &\times \left(\int_{\bar{0}}^{\overline{a(1)}} \prod_{j \in e^*} \frac{dt_j}{t_j} \int_{\frac{a(1)}{a(1)}}^{\bar{\infty}} \left\| \frac{\Delta^N \left(\frac{t}{N} \right) N_{k, \delta^i}(f)}{\prod_{j \in e^*} (\varphi_j(t_j))} \right\|_{L_q(E_S)}^\theta \prod_{j \in e^*} \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}} = \\ &= \left\{ c \prod_{j \in E_N \setminus e^*} \varphi_j(a_j(1)) \right\} J_{k, \delta^i, e^*}(f). \end{aligned} \quad (34)$$

This last inequality means that the estimation of integral operators are reduced to the estimation of the operators

$$J_{k, \delta^i, e^*}(f) = \left(\int_{\bar{0}}^{\overline{a(1)}} \prod_{j \in e^*} \frac{dt_j}{t_j} \int_{\frac{a(1)}{a(1)}}^{\bar{\infty}} \left\| \frac{\Delta^N \left(\frac{t}{N} \right) N_{k, \delta^i}(f)}{\prod_{j \in e^*} (\varphi_j(t_j))} \right\|_{L_q(E_S)}^\theta \prod_{j \in e^*} \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}} \quad (35)$$

for all $i = 1, 2, \dots, M$ $\left. \begin{matrix} k = 1, 2, \dots, n \\ e^* \subset e \subset E_N \end{matrix} \right\}$, herewith it suffices to reduce the estimation in the case $e^* = \Phi$ and $e^* = E_N$.

Step 3. The estimation of integral operators

$$J_{k,\delta^i,e^*}(f) = \left(\begin{matrix} k = 1, 2, \dots, n \\ i = 1, 2, \dots, M \end{matrix} \right) \quad (\text{case } e^* = \Phi)$$

Lemma 1. The following estimations are valid in the conditions of theorem 1

$$J_{k,\delta^i,\Phi}(f) \leq cH_{k,S}^* \|f\|_{\Lambda_{p_k, \theta_k}^{(n^k, N^k)}(\Omega_i + R_{\delta^i}, \varphi^k)} \quad (36)$$

therewith,

$$\left\{ \prod_{j \in E_N} \varphi_j(a_j(1)) \right\} H_{k,S}^*(1) \leq cH_{k,S} \quad (37)$$

where $H_{k,S}(1)$ are determined by the equations (9).

Proof. Note that (see 35))

$$J_{k,\delta^i,\Phi}(f) \leq \|N_{k,\delta^i}(t)\|_{L_q(E_S)} \quad (38)$$

on the other hand

$$\begin{aligned} |N_{k,\delta^i}(t)| &\leq c \int_0^1 \bar{\omega}_k(v) \frac{da_k(v)}{a_k(v)} \int_{\vec{0}}^{\vec{a}(v)} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{1/p_k - \gamma_j}} \times \\ &\times \int_{\vec{0}}^{\vec{a}(v)} d\vec{y}^S \int_{x_{S+1}^0}^{x_{S+1}^0 + a_{S+1}(v)} dy_{S+1}^* \dots \int_{x_n^0}^{x_n^0 + a_n(v)} dy_{S+1}^* \dots | \dots | dy_n^* \end{aligned} \quad (39)$$

where

$$| \dots | = \left| \frac{\Delta^{N^k} \left(\frac{z}{N^k}; \Omega_i + R_{\delta^i} \right) D^{m^k} f(\bar{X}^S + \bar{y}^S; y_{S+1}^*, \dots, y_n^*)}{\prod_{j \in E_{N^k}} \varphi_j(t_j)} \right| \quad (40)$$

while $\bar{\omega}_k(v)$ are determined by the equalities (25).

From (40), repeating the reasoning given in the proof of theorem 1 from [1], we obtain the estimates

$$|N_{k,\delta^i}(f)|_{L_q(E_S)} \leq cH_{k,S}^*(1) \|f\|_{\Lambda_{p_k,\theta_k}^{(n^k,N^k)}(\Omega_i+R_{\delta^i,\varphi^k})} \quad (41)$$

where

$$H_{k,S}^*(1) = \int_0^1 \prod_{j=1}^n \left(a_j(v)^{m_j^k - v_j - \alpha_j - \frac{1}{p_k}} \prod_{j=1}^S a_j(v) \prod_{j \in E_{N^k}} \varphi_j^k a_j(v) \frac{da_k(v)}{a_k(v)} \right) \quad (42)$$

From the inequality (8) and the properties imposed on the functions $\varphi_j = \varphi_j(t_j) (j \in E_N = \text{sup } pN)$ we have the estimations

$$\left(\prod_{j \in E_N} \varphi_j(a_j(1)) \right) H_{k,S}^*(1) \leq cH_{k,S}(1) \quad (43)$$

that together with (41), complete the proof of lemma 1.

Step 4. The estimation of integral expression

$$\begin{aligned} J_{k,\delta^i,\Phi}(f) &= \left(\int_{\bar{0}}^{\overline{a(1)}} \left\| \frac{\Delta^N \left(\frac{t}{N} \right) N_{k,\delta^i}(f)}{\prod_{j \in e^*} \varphi_j(t_j)} \right\|_{L_q(E_S)}^\theta \prod_{j \in E_N} \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}} \leq \\ &\leq c \left(\int_{\bar{0}}^{\overline{a(1)}} \left\| \frac{\Delta^N \left(\frac{t}{N} \right) N_{k,\delta^i}^*(f)}{\prod_{j \in e^*} \varphi_j(t_j)} \right\|_{L_q(E_S)}^\theta \prod_{j \in E_N} \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}} + \\ &+ c \left(\int_{\bar{0}}^{\overline{a(1)}} \left\| \frac{\Delta^N \left(\frac{t}{N} \right) N_{k,\delta^i}^{**}(f)}{\prod_{j \in e^*} \varphi_j(t_j)} \right\|_{L_q(E_S)}^\theta \prod_{j \in E_N} \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}}. \end{aligned} \quad (44)$$

Integral expression $N_{k,\delta^i}^*(f)$, $N_{k,\delta^i}^{**}(f)$ is determined by the equations (24).

Lemma 2. The following integral inequalities are valid in the conditions of Theorem 1

$$c \left(\int_{\bar{0}}^{\overline{a(1)}} \left\| \frac{\Delta^N \left(\frac{t}{N} \right) N_{k,\delta^i}^*(f)}{\prod_{j \in e^*} \varphi_j(t_j)} \right\|_{L_q(E_S)}^\theta \prod_{j \in E_N} \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}} \leq cH_{k,S}(1) \|f\|_{\Lambda_{p_k,\theta_k}^{(n^k,N^k)}(\Omega_i+R_{\delta^i,\varphi^k})} \quad (45)$$

where $H_{k,S}(1)$ are determined by the equations (9), where c is a constant independent of the function $f = f(x)$.

Proof. Recall that

$$\begin{aligned}
 |N_{k,\delta^i}^*(t)| &\leq c \int_0^{g(t)} \bar{\omega}_k(v) \frac{da_k(v)}{a_k(v)} \int_0^{\bar{a}(v)} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{1/p_k - \gamma_j}} \times \\
 &\times \int_0^{\bar{a}(v)} d\vec{y}^S \int_{x_{S+1}^0}^{x_{S+1}^0 + a_{S+1}(v)} dy_{S+1}^* \dots \int_{x_n^0}^{x_n^0 + a_n(v)} dy_{S+1}^* \dots | \dots | dy_n^*
 \end{aligned} \tag{46}$$

where $|\dots|$ is determined by the equality (40), while $\bar{\omega}_k(v)$ are determined by the equations (25).

Now do the right hand side of the inequality (46) we apply the Holder inequality with the exponents

$$\lambda_1 = p_k, \lambda_2 = p_k' = \frac{p_k}{p_k - 1} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1 \right), \tag{47}$$

whence after some simple transformations and calculations we have the inequalities

$$|N_{k,\delta^i}^*(t)| \leq c \int_0^{g(t)} \tilde{\omega}(v) \frac{da_k(v)}{a_k(v)} \left(\int_0^{\bar{a}(v)} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{1/p_k - \gamma_j}} \int_0^{\bar{a}(v)} \|\dots\|_{L_{p_k}(E_n \setminus S)}^{p_k} \right)^{\frac{1}{p_k}} \tag{48}$$

we obtain

$$\begin{aligned}
 &\|\dots\|_{L_{p_k}(E_n \setminus S)}^{p_k} = \\
 &= \left\{ \int_{E_n \setminus S} \left| \frac{\Delta^{N^k} \left(\frac{z}{N^k}; \Omega_i + R_{\delta^i} \right) D^{m^k} f(\vec{X}^S + \vec{y}^S; y_{S+1}^*, \dots, y_n^*)}{\prod_{j \in E_{N^k}} \varphi_j(t_j)} \right| dy_{S+1}^*, \dots, dy_n^* \right\}^{\frac{1}{p_k}} \tag{49}
 \end{aligned}$$

$$\tilde{\omega}(v) = \prod_{j=1}^n (a_j(v))^{m_j^k - v_j - \alpha_j - \frac{1}{p_k}} \prod_{j \in E_{N^k}} \varphi_j(v) (a_j(v))^{\gamma_j}. \tag{50}$$

In the case, $q = p_k$, from (48) we have

$$\|N_{k,\delta^i}^*(f)\|_{L_{p_k}(E_S)} \leq c \int_0^{g(t)} \tilde{\omega}(v) \frac{da_k(v)}{a_k(v)} \times$$

$$\times \left\{ \int_{\vec{0}}^{\overline{a(v)}} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{\frac{1}{p_k} - \gamma_j}} \int_{\vec{0}}^{\overline{a(v)}} d\vec{y}^S \int_{E_S} \|\dots\|_{L_{p_k}(E_n \setminus S)}^{p_k} d\vec{x}^S \right\}^{\frac{1}{p_k}}. \quad (51)$$

Then, making a change of variables

$$X_j + y_j = X_j^* \quad (j = 1, 2, \dots, S) \quad (52)$$

we get

$$\begin{aligned} \|N_{k, \delta^i}^*(f)\|_{L_{p_k}(E_S)} &\leq c \int_0^{g(t)} \tilde{\omega}(v) \prod_{j=1}^S a_j(v) \frac{da_k(v)}{a_k(v)} \times \\ &\times \left\{ \int_{\vec{0}}^{\overline{a(v)}} \left\| \frac{\Delta^{N^k} \left(\frac{z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j(t_j)} \right\|_{L_{p_k}(E_n)}^{p_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{\frac{1}{p_k} - \gamma_j}} \right\}^{\frac{1}{p_k}} \end{aligned} \quad (53)$$

Consequently in the case $p_k = \theta_k$, we have

$$\begin{aligned} \|N_{k, \delta^i}^*(f)\|_{L_{p_k}(E_S)} &\leq c \int_0^{g(t)} \prod_{j=1}^n (a_j(v))^{m_j^k - \nu_j - \alpha_j - \frac{1}{p_k}} \prod_{j=1}^S (a_j(v))^{\frac{1}{p_k}} \times \\ &\times \prod_{j \in E_{N^k}} \varphi_j^k(a_j(v)) \frac{da_k(v)}{a_k(v)} \times \\ &\times \left\{ \int_{\vec{0}}^{\overline{a(v)}} \left\| \frac{\Delta^{N^k} \left(\frac{z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j(t_j)} \right\|_{L_{p_k}(E_n)}^{p_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{\frac{1}{p_k} - \gamma_j}} \right\}^{\frac{1}{p_k}}. \end{aligned} \quad (54)$$

Let now

$$p_k = q < \theta_k. \quad (55)$$

Then, to integral expressions in the braces, in the right hand side of the inequality (53), we apply the Holder inequality with the exponents

$$\lambda_1 = \frac{\theta_k}{p_k} > 1, \lambda_2 = \frac{\theta_k}{\theta_k - p_k} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1 \right), \quad (56)$$

whence we have

$$\left(\int_{\frac{0}{\theta}}^{\overline{a(v)}} \left\| \frac{\Delta^{N^k} \left(\frac{z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j(t_j)} \right\|_{L_{p_k}(E_n)}^{p_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{\frac{1}{p_k} - \gamma_j}} \right)^{\frac{1}{p_k}} \leq$$

$$\leq c \prod_{j \in E_{N^k}} (a_j(v))^{\gamma_j} \times$$

$$\times \left(\int_{\frac{0}{\theta}}^{\overline{a(v)}} \left\| \frac{\Delta^{N^k} \left(\frac{z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j(t_j)} \right\|_{L_{p_k}(E_n)}^{p_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{\frac{1}{p_k} - \gamma_j}} \right)^{\frac{1}{\theta_k}} \quad (58)$$

for

$$q = p_k = \theta_k. \quad (59)$$

where

$$\omega_k^*(v) = \prod_{j=1}^n (a_j(v))^{m_j^k - \nu_j - \alpha_j - \frac{1}{p_k}} \prod_{j=1}^m (a_j(v))^{\frac{1}{p_k}} \prod_{j \in E_{N^k}} \varphi_j^k(a_j(v)). \quad (60)$$

We now assume

$$q > p_k. \quad (61)$$

In this case (61), we apply the Holder inequality to the right hand side of (48) with the exponents

$$\lambda_1 = \frac{qp_k}{q - p_k}, \lambda_2 = q, \lambda_3 = p_k, \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 1 \quad (62)$$

and therewith we know that

$$p_k \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) = 1, \quad (63)$$

$$\frac{1}{p_k} - \gamma_j = (1 - \gamma_j p_k) \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right), \quad (64)$$

consequently, we have the following inequalities

$$\begin{aligned}
 \|N_{k,\delta^i}^*(f)\|_{L_{p_k}(E_S)} &\leq c \int_0^{g(t)} \prod_{j=1}^n (a_j(v))^{m_j^k - \nu_j - \alpha_j - \frac{1}{p_k}} \times \\
 &\times \prod_{j \in E_{N^k}} \varphi_j^k(a_j(v)) (a_j(v))^{-\nu_j} \frac{da_k(v)}{a_k(v)} \times \\
 &\times \left\{ \int_{\vec{0}}^{\overline{a(v)}} \left\| \frac{\Delta^{N^k} \left(\frac{z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j^k(t_j)} \right\|_{L_{p_k}(E_n)}^{\theta_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{\frac{1}{p_k} - \nu_j}} \right\}^{\frac{1}{p_k} \frac{1}{q}} \times \\
 &\times \left\{ \int_{E_S} d\vec{x}^S \int_{\vec{0}}^{\overline{a(v)}} \times \right. \\
 &\times \left. \left(\int_{\vec{0}}^{\overline{a(v)}} \left\| \frac{\Delta^{N^k} \left(\frac{z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j^k(t_j)} \right\|_{L_{p_k}(E_n)}^{\theta_k} d\vec{y}^S \right) \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{\frac{1}{p_k} - \nu_j}} \right\}^{\frac{1}{q}}. \quad (65)
 \end{aligned}$$

In the integral expression standing on braces in the right hand side of (65), at first we change the integration order and then make a change of variables

$$X_j + y_j = X_j^* \quad (j = 1, 2, \dots, S) \quad (66)$$

where after some transformation we have the estimations

$$\begin{aligned}
 \|N_{k,\delta^i}^*(f)\|_{L_q(E_S)} &\leq c \int_0^{g(t)} \prod_{j=1}^n (a_j(v))^{m_j^k - \nu_j - \alpha_j - \frac{1}{p_k}} \times \\
 &\times \prod_{j=1}^n (a_j(v))^{\frac{1}{q}} \prod_{j \in E_{N^k}} \varphi_j^k(a_j(v)) (a_j(v))^{-\nu_j} \frac{da_k(v)}{a_k(v)} \times \\
 &\times \left\{ \int_{\vec{0}}^{\overline{a(v)}} \left\| \frac{\Delta^{N^k} \left(\frac{z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j^k(t_j)} \right\|_{L_{p_k}(E_n)}^{p_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j^{\frac{1}{p_k} - \nu_j}} \right\}^{\frac{1}{p_k}}. \quad (67)
 \end{aligned}$$

From the last inequality (67) assuming

$$q > p_k = \theta_k \quad (68)$$

it follows that

$$\begin{aligned} & \|N_{k,\delta^i}^*(f)\|_{L_q(E_S)} \leq c \int_0^{g(t)} \prod_{j=1}^n (a_j(v))^{m_j^k - \nu_j - \alpha_j - \frac{1}{p_k}} \times \\ & \times \prod_{j=1}^n (a_j(v))^{\frac{1}{q}} \prod_{j \in E_{N^k}} \varphi_j^k(a_j(v)) (a_j(v))^{-\nu_j} \frac{da_k(v)}{a_k(v)} \times \\ & \times \left(\int_{\vec{0}}^{\vec{a}(v)} \left\| \frac{\Delta^{N^k} \left(\frac{Z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j^k(t_j)} \right\|_{L_{p_k}(E_n)}^{p_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j} \right)^{\frac{1}{p_k}}. \end{aligned} \quad (69)$$

Let now

$$\theta_k > p_k. \quad (70)$$

In this case, to the integral expression standing in the braces, in the right hand side of (69), we apply the Holder inequality with the exponents

$$\lambda_1 = \frac{\theta_k}{p_k} > 1, \quad \lambda_2 = \frac{\theta_k}{\theta_k - p_k} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1 \right) \quad (71)$$

whence after some transformations we have the following inequalities

$$\begin{aligned} & \|N_{k,\delta^i}^*(f)\|_{L_q(E_S)} \leq c \int_0^{g(t)} Q_{k,S}^*(v) \frac{da_k(v)}{a_k(v)} \times \\ & \times \left\{ \int_{\vec{0}}^{\vec{a}(v)} \left\| \frac{\Delta^{N^k} \left(\frac{Z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j^k(t_j)} \right\|_{L_{p_k}(E_n)}^{\theta_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j} \right\}^{\frac{1}{\theta_k}}, \end{aligned} \quad (72)$$

where

$$Q_{k,S}^*(v) = \prod_{j=1}^n (a_j(v))^{m_j^k - \nu_j - \alpha_j - \frac{1}{p_k}} \prod_{j=1}^n (a_j(v))^{\frac{1}{q}} \prod_{j \in E_{N^k}} \varphi_j^k(a_j(v)). \quad (73)$$

It follows from inequality (54),(58), (69),(72) that the last inequality (72) holds for

$$q \geq p_k \quad \text{and} \quad p_k \leq \theta_k \quad (74)$$

by means of which we have

$$\|N_{k,\delta^i}^*(f)\|_{L_q(E_S)} \leq c \int_0^{g(t)} Q_{k,S}^*(v) \frac{da_k(v)}{a_k(v)} \|f\|_{\Lambda_{p_k, \theta_k}^{(n^k, N^k)}(\Omega_i + R_{\delta^i, \varphi^k})} \quad (75)$$

for $1 < p_k \leq \theta_k < \infty, q \geq p_k$ where $Q_{k,S}^*(v)$ are determined by the equalities (73)

$$g(t) = \prod_{j=1}^n a_j^{-1}(t_j). \quad (76)$$

Herewith

$$\left. \begin{aligned} a_j^{-1}(a_j(v)) &= v \\ a_j(a_j^{-1}(t_j)) &= t_j \end{aligned} \right\}. \quad (77)$$

From (75) it follows that

$$\left(\int_0^{\overline{a(v)}} \left\| \frac{\Delta^{N^k} \left(\frac{z}{N^k}, \Omega_i + R_{\delta^i} \right) D^{m^k} f(\cdot)}{\prod_{j \in E_{N^k}} \varphi_j^k(t_j)} \right\|_{L_{p_k}(E_n)}^{\theta_k} \prod_{j \in E_{N^k}} \frac{dz_j}{z_j} \right)^{\frac{1}{\theta}} \leq c \|f\|_{\Lambda_{p_k, \theta_k}^{(n^k, N^k)}(\Omega_i + R_{\delta^i, \varphi^k})}.$$

It is easy to see that

$$\left\{ \int_0^{\overline{a(v)}} \prod_{j \in E_N} \left(\frac{1}{\varphi_j(t_j)} \right)^\theta \frac{dt_j}{t_j} \left(\int_0^{g(t)} Q_{k,S}^*(v) \frac{da_k(v)}{a_k(v)} \right)^\theta \right\}^{\frac{1}{\theta}}. \quad (78)$$

It is easy to see that

$$\left\{ \int_0^{\overline{a(v)}} \prod_{j \in E_N} \left(\frac{1}{\varphi_j(t_j)} \right)^\theta \frac{dt_j}{t_j} \left(\int_0^{g(t)} Q_{k,S}^*(v) \frac{da_k(v)}{a_k(v)} \right)^\theta \right\}^{\frac{1}{\theta}} \leq c H_{k,S}(1). \quad (79)$$

The validity of inequalities (45) of Lemma 2 follows from the inequalities (78) and (79).

The proof of Theorem 1 is proved.

CONCLUSION

The embedding theory of spaces of differentiable functions of several variables developed as a new trend in mathematics in the 1930's as a results of the S.L. Sobolev's works. This theory studies important connections and relationships of smoothness properties of functions in different metrics. Apart from its independent interest from the point of view of the functions theory, its has numerous effective applications in the theory of partial differential equations. In this paper author estimate the semi-norm of the functions $D^\nu(f - \Phi_{l-1})(X)$ in the Sobolev's type spaces.

REFERENCES

1. Platonov, A.S. (2003). L_p estimation traces containing limit cases for $1 < p_k < q < \infty$. Collection of scientific papers on Mechanics 13 (2).
2. Jabrailov, A.S., Aliyev, M.K. (2007). Estimation of the norm of functions of appropriate polynomials, Institute of Mathematics and Mechanics of National Akademy of Sxiences of Azerbaijan 108.
3. Aliyev, M.K. (2002). Estimations of the traces of functions $D^\nu(f - \Phi_{l-1})$ in the surface Γ_S , collection of Scientific paper in Mechanics 12(2).
4. Aliyev, M.K. (1987). Embedding theorems of functions from the spaces of mixed differencial-diffelence properties, Dep. VINITI, 7(201).
5. Maksudov, F.G., Djabrailov, A.J. (2000). Integral representation method in theory spaces, Elm, Baku.